The Smart Beta Indexing Puzzle

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Lyxor Asset Management

15 July 2013

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July 2013

Abstract

In this article, we consider smart beta indexing, which is an alternative to capitalization-weighted (CW) indexing. In particular, we focus on risk-based (RB) indexing, the aim of which is to capture the equity risk premium more effectively. To achieve this, portfolios are built which are more diversified and less volatile than CW portfolios. However, RB portfolios are less liquid than CW portfolios by construction. Moreover, they also present two risks in terms of passive management: tracking difference risk and tracking error risk. Smart beta investors then have to a puzzle out the trade-off between diversification, volatility, liquidity and tracking error. This article examines the trade-off relationships. It also defines the return components of smart beta indexes.

Keywords: Smart beta, risk-based indexing, minimum variance portfolio, risk parity, equally weighted portfolio, equal risk contribution portfolio, diversification, low beta anomaly, low volatility anomaly, tracking error, liquidity.

JEL classification: G11.

1 Introduction

Smart beta is a (marketing) term used to refer to alternative-weighted indexing, which is a new form of passive management (Roncalli, 2013). An alternative-weighted index is defined as an index in which assets are weighted in a different way from those based on market capitalization. We generally distinguish between two forms of alternative-weighted indexing: fundamental indexing and risk-based indexing. The underlying idea of alternative-weighted indexes is to improve the risk-return profile of market-cap indexes. This is already the goal of active management, which is benchmarked on capitalization-weighted indexes. Nevertheless, there is a major difference compared with active management solutions, because smart beta indexes are formula-based solutions. As a result, smart beta may be offered by both asset managers and index providers.

*We are grateful to Tzee-man Chow for his helpful comments.

The success of smart beta in recent years can be explained by two main products: the RAFI indexes launched by Robert Arnott and minimum variance solutions. The former is related to the value factor of Fama and French (1992), while the latter exploits the low beta anomaly (Goltz and Martellini, 2013). However, smart beta is not limited to these two solutions, and a broad indexing universe now exists. For instance, Amenc *et al.* (2013) identify 13 portfolio allocation methods and combine them with different factors (value stocks, growth stocks, high liquid stocks, small caps, etc.). Our article is less ambitious and focuses on low volatility indexes.

Many investors would like to have a smart beta index that has low volatility, broad diversification, low tracking error (and high liquidity) and strong performance. The goal of this article is to show that there is a trade-off between these different characteristics. For instance, it is obvious that a low volatility index implies high tracking error volatility. In the same way, we can see that there is a negative relationship between volatility and diversification. Thus, it is intuitive that a diversified smart beta index offering a volatility reduction of 30% versus a capitalization-weighted index with tracking error volatility below 5% is unrealistic. The key issue is to precisely measure these trade-off relationships. This is the aim of our work.

The article is organized as follows. In section two, we propose an allocation method for building a low volatility diversified (LVD) portfolio. In particular, we show how we can target an ex-ante volatility reduction using traditional risk-based indexing and create a range of risk-based indexes. In section three, we estimate the trade-off relationships using simulations based on the S&P 500 universe. We also propose a framework for assessing the performance of LVD portfolios, and for gauging the difference in returns versus the CW portfolio. Section four offers some concluding remarks.

2 Building a low volatility diversified portfolio

2.1 Traditional risk-based indexing

We consider a universe of n assets. Let x be the portfolio weights, b the benchmark portfolio (or the index) and Σ the covariance matrix of asset returns. Demey *et al.* (2010) highlight four methods:

1. The equally weighted (EW) portfolio
   In this case, the weights are equal, meaning that:

$$x_i = x_j = \frac{1}{n}$$

   This is the least concentrated portfolio in terms of weights.

2. The minimum variance (MV) portfolio
   The goal here is to define a portfolio that minimizes ex-ante volatility:

$$x^\star = \arg\min x^\top \Sigma x$$

3. The most diversified (MDP) portfolio
   Choueifaty and Coignard (2008) define the diversification ratio as follows:

$$\mathcal{DR}(x) = \frac{x^\top \sigma}{\sqrt{x^\top \Sigma x}}$$

where σ is the volatility vector. The MDP then corresponds to the portfolio that maximizes the statistic $\mathcal{DR}(x)$:

$$x^\star = \arg\min \mathcal{DR}(x)$$

4. The equal risk contribution (ERC) portfolio
In the ERC portfolio, the risk contributions are the same for all assets (Maillard *et al.*, 2010):

$$\mathcal{RC}_i = \mathcal{RC}_j$$

where:

$$\mathcal{RC}_i = x_i \cdot \frac{\partial\, \sigma(x)}{\partial\, x_i}$$

In Table 9 (Appendix B on page 23), we set out the composition of the Eurostoxx 50 index at the end of February 2013. We also indicate the composition of the (long-only) risk-based portfolios by considering a one-year empirical covariance matrix. In Table 1, we give some statistics for these portfolios[1]. $\sigma(x)$ is the ex-ante volatility of the portfolio x, while $\sigma(x \mid b)$ is the ex-ante volatility of tracking errors between the portfolio x and the benchmark b:

$$\sigma(x \mid b) = \sqrt{(x-b)^\top \Sigma (x-b)}$$

We also compute the ex-ante risk reduction defined as follows:

$$\delta_\sigma(x \mid b) = 1 - \frac{\sigma(x)}{\sigma(b)}$$

$\beta(x \mid b)$ is the beta of the portfolio x with respect to the benchmark b. We have:

$$\beta(x \mid b) = \frac{x^\top \Sigma b}{b^\top \Sigma b}$$

$\mathcal{N}(x)$ measures the degrees of freedom of portfolio weights and corresponds to the inverse of the Herfindahl index (Roncalli, 2013). It is equal to 1 if the portfolio is concentrated in one stock. Conversely, it is equal to n for the EW portfolio, which is the least concentrated portfolio in terms of weights. Similarly, $\mathcal{N}(\mathcal{RC})$ measures the degrees of freedom of portfolio risk contributions[2]. In this case, $\mathcal{N}(\mathcal{RC})$ is equal to n for the ERC portfolio, which is the least concentrated portfolio in terms of risk budgets. The minimum variance portfolio provides a significant reduction in the volatility of the portfolio ($\delta_\sigma(x \mid b)) = 37.64\%$), but it corresponds to a highly concentrated portfolio both in terms of weights and risk concentrations. Indeed, $\mathcal{N}(x)$ and $\mathcal{N}(\mathcal{RC})$ are equal to 3.06, meaning that the MV portfolio is equivalent to an EW or ERC portfolio, despite having only three assets. Conversely, the equally weighted portfolio has a low concentration, but its volatility is higher than the volatility of the CW portfolio. The MDP and ERC portfolios are located between these two extremes. For instance, the ERC portfolio is highly diversified, but the reduction in risk is small. We also report the tracking error volatility and the beta. We notice that we face a high risk with respect to the benchmark in the case of the MV and MDP portfolios.

[1] The statistics are expressed in %, except for the beta and concentration measures.

[2] We have:

$$\mathcal{N}(x) = \frac{1}{\sum_{i=1}^n x_i^2}$$

and:

$$\mathcal{N}(\mathcal{RC}) = \frac{1}{\sum_{i=1}^n \mathcal{RC}_i^2}$$

Table 1: Statistics of risk-based portfolios (Eurostoxx 50, Feb. 2013)

Statistic	CW	EW	MV	MDP	ERC
$\sigma(x)$	21.30	22.40	13.29	17.66	19.71
$\delta_\sigma(x \mid b)$		-5.15	37.64	17.09	7.49
$\sigma(x \mid b)$		2.35	12.63	9.75	2.48
$\beta(x \mid b)$		1.05	0.52	0.74	0.92
$\mathcal{N}(x)$	36.65	50.00	3.06	10.31	44.27
$\mathcal{N}(\mathcal{RC})$	35.93	43.66	3.06	9.66	50.00
Risk reduction		No	High	Medium	Low
Diversification		High	Low	Low	High
Beta risk		Low	High	High	Low

The traditional way to obtain a portfolio with low volatility and low concentration is thus to consider a constrained minimum variance portfolio:

$$\begin{aligned} x^\star &= \arg\min x^\top \Sigma x \\ \text{u.c.} &\quad x_i^- \leq x_i \leq x_i^+ \end{aligned}$$

We can then define the lower and upper bounds in an absolute way, for example $1\% \leq x_i \leq 5\%$. We can also define them in a relative way with respect to the benchmark. In this case, we have $c_i^- b_i \leq x_i \leq c_i^+ b_i$ where b_i is the weight of asset i in the benchmark b, $c_i^- \leq 1$ and $c_i^+ \geq 1$. For instance, suppose that $b_i/2 \leq x_i \leq 2b_i$. It means that the portfolio weights may deviate from the benchmark weights by -50% and $+100\%$. We report some results[3] in Table 2. The drawback of this method is that the solution strongly depends on the choice of the bounds. Moreover, we cannot control the risk reduction with this method. We observe the same problem with the beta risk. With these methods, the beta and the tracking error volatility are observed in an ex-post analysis and depend on the design of the weight constraints.

Table 2: Statistics of constrained MV portfolios (Eurostoxx 50, Feb. 2013)

Lower bound	0%	0%	1%	1%	$0 \times b_i$	$0.5 \times b_i$	1%
Upper bound	10%	5%	5%	4%	$2 \times b_i$	$2 \times b_i$	$2 \times b_i$
$\sigma(x)$	14.32	15.75	18.05	18.45	16.12	17.81	18.19
$\delta_\sigma(x \mid b)$	32.79	26.09	15.27	13.38	24.33	16.40	14.63
$\sigma(x \mid b)$	10.31	8.25	4.60	4.03	7.71	4.77	4.24
$\beta(x \mid b)$	0.61	0.70	0.84	0.86	0.72	0.82	0.84
$\mathcal{N}(x)$	10.66	20.68	29.02	33.68	17.00	26.43	27.84
$\mathcal{N}(\mathcal{RC})$	10.71	20.51	37.75	41.27	16.06	33.08	36.58
Risk reduction	High	High	Medium	Medium	High	Medium	Medium
Diversification	Low	Low	Medium	High	Low	Medium	Medium
Beta risk	High	High	Medium	Medium	High	Medium	Medium

[3]Compositions are given in Table 10 on page 24

2.2 Low volatility diversified portfolio

We now explore another route to building a low volatility portfolio that has a sufficient level of diversification. Let $x^\star(c)$ be an optimized portfolio that depends on a scalar $c \in [c^-, c^+]$. We assume that the volatility $\sigma(x^\star(c))$ is a continuous decreasing function with respect to c. We also assume that $x^\star(c^-) = x_{\mathrm{mv}}$ and $x^\star(c^+) = x_{\mathrm{ew}}$, we obtain:

$$\delta^- \leq \delta_\sigma(x^\star(c) \mid b) \leq \delta^+$$

with:

$$\delta^- = 1 - \frac{\sigma(x^\star(c^+))}{\sigma(b)}$$

and:

$$\delta^+ = 1 - \frac{\sigma(x^\star(c^-))}{\sigma(b)}$$

This implies that we can achieve a risk reduction $\delta \in [\delta^-, \delta^+]$. Targeting a risk reduction $\delta^\star$ is thus equivalent to finding the optimal value $c^\star$ such that:

$$\delta_\sigma(x^\star(c^\star) \mid b) = \delta^\star$$

2.2.1 The LVD-MV portfolio

Let $\mathcal{N}(x)$ be the inverse of the Herfindahl index. We have:

$$\begin{aligned}\mathcal{N}(x) &= \mathcal{H}^{-1}(x) \\ &= \frac{1}{\sum_{i=1}^n x_i^2}\end{aligned}$$

We consider the following optimization program:

$$\begin{aligned}x^\star(c) &= \arg\min x^\top \Sigma x \\ &\text{u.c.} \quad \begin{cases}\mathcal{N}(x) \geq c \\ \mathbf{1}^\top x = 1 \\ x \geq 0\end{cases}\end{aligned}$$

where $c \in [c^-, n]$ and c^- is the effective number of stocks of the long-only MV portfolio[4]. We verify that:

$$\sigma(x_{\mathrm{mv}}) \leq \sigma(x^\star(c)) \leq \sigma(x_{\mathrm{ew}}) \tag{1}$$

We can then calibrate c such that the risk reduction is equal to $\delta^\star$.

Remark 1 *To solve the previous optimization problem, we consider the shrinkage formulation of the LVD-MV solution given in Appendix A.1.*

2.2.2 The LVD-ERC portfolio

Maillard *et al.* (2010) show that the ERC portfolio is the solution of this optimization program:

$$\begin{aligned}x^\star(c) &= \arg\min x^\top \Sigma x \\ &\text{u.c.} \quad \begin{cases}\sum_{i=1}^n \ln x_i \geq c \\ \mathbf{1}^\top x = 1 \\ x \geq 0\end{cases}\end{aligned}$$

[4] We have $c^- = \mathcal{N}(x_{\mathrm{mv}})$ and $c^+ = \mathcal{N}(x_{\mathrm{ew}}) = n$.

for a special value of the scalar c. Because we have $x^{\star}(-\infty) = x_{\mathrm{mv}}$ and $x^{\star}(n \ln n) = x_{\mathrm{ew}}$, the inequalities (1) also hold in this case. Targeting a risk reduction of $\delta^{\star}$ is equivalent to finding the value of c such that:

$$\sigma(x^{\star}(c)) = (1 - \delta^{\star})\, \sigma(x_{\mathrm{cw}})$$

Remark 2 *We can define other forms of low volatility diversified portfolios. For instance, Appendix A.2 present a LVD allocation based on MDP. We can also build a LVD-RB portfolio by using an iterative risk budgeting portfolio (see Appendix A.3).*

2.2.3 An illustration

We consider an investment universe of four assets. We assume that the CW portfolio is the equally weighted portfolio. The volatility is respectively equal to 20%, 15%, 25% and 35%, while the correlation matrix C is equal to:

$$C = \begin{pmatrix} 100\% & & & \\ 80\% & 100\% & & \\ 70\% & 50\% & 100\% & \\ 60\% & 40\% & 20\% & 100\% \end{pmatrix}$$

In Table 3, we report the weights and statistics of risk-based portfolios. Suppose that we target a risk reduction of 20%. In this case, we obtain the results given in Table 4. We notice that these are very similar for the LVD-ERC and LVD-RB portfolios. Moreover, we obtain similar results in terms of weight and risk concentrations for the different LVD portfolios, except for the MDP[5]. If we now consider a smaller risk reduction, we find that the difference between the LVD-MV, LVD-ERC and LVD-RB portfolios is smaller (see Table 5).

Table 3: Weights and statistics of risk-based portfolios

Statistic	EW	MV	MDP	ERC
x_1	25.00	0.00	0.00	12.51
x_2	25.00	0.00	15.97	20.28
x_3	25.00	67.33	46.59	37.47
x_4	25.00	32.67	37.44	29.74
$\sigma(x)$	19.91	13.08	13.91	16.70
$\delta_{\sigma}(x \mid b)$	0.00	34.32	30.13	16.12
$\sigma(x \mid b)$	0.00	10.41	7.45	3.79
$\beta(x \mid b)$	1.00	0.58	0.67	0.83
$\mathcal{N}(x)$	4.00	1.79	2.61	3.50
$\mathcal{N}(\mathcal{RC})$	3.31	1.79	2.82	4.00

2.2.4 Some results

In Table 6, we again use the Eurostoxx 50 index and report the statistics of low volatility diversified portfolios for two values of $\delta^{\star}$, while the portfolio weights are given in Table 11 on page 25. The difference between the LVD-RB and LVD-RB* portfolios comes from the initial portfolio: the former starts the algorithm with the EW portfolio whereas the latter starts the algorithm with the CW portfolio. If $\delta^{\star} = 10\%$, the diversification of the LVD

[5] In fact, the LVD-MDP does not allow both the volatility and the diversification to be managed.

Table 4: Statistics of LVD portfolios ($\delta^\star = 20\%$)

Statistic	EW	MV	MDP	ERC	RB
x_1	25.00	7.22	3.70	9.82	9.71
x_2	25.00	23.56	28.12	18.13	18.33
x_3	25.00	36.96	29.29	41.86	41.51
x_4	25.00	32.27	38.90	30.18	30.45
$\sigma(x)$	19.91	15.93	15.93	15.93	15.93
$\delta_\sigma(x \mid b)$	0.00	20.00	20.00	20.00	20.00
$\sigma(x \mid b)$	0.00	4.61	5.14	4.81	4.79
$\beta(x \mid b)$	1.00	0.79	0.79	0.79	0.79
$\mathcal{N}(x)$	4.00	3.32	3.15	3.24	3.25
$\mathcal{N}(\mathcal{RC})$	3.31	3.77	3.17	3.92	3.92

Table 5: Statistics of LVD portfolios ($\delta^\star = 10\%$)

Statistic	EW	MV	MDP	ERC	RB
x_1	25.00	16.32	12.66	16.96	17.35
x_2	25.00	24.15	27.91	22.83	22.11
x_3	25.00	30.79	21.26	31.79	32.64
x_4	25.00	28.73	38.17	28.41	27.90
$\sigma(x)$	19.91	17.92	17.92	17.92	17.92
$\delta_\sigma(x \mid b)$	0.00	10.00	10.00	10.00	10.00
$\sigma(x \mid b)$	0.00	2.27	3.42	2.29	2.32
$\beta(x \mid b)$	1.00	0.90	0.89	0.90	0.90
$\mathcal{N}(x)$	4.00	3.81	3.51	3.81	3.80
$\mathcal{N}(\mathcal{RC})$	3.31	3.85	3.55	3.86	3.85

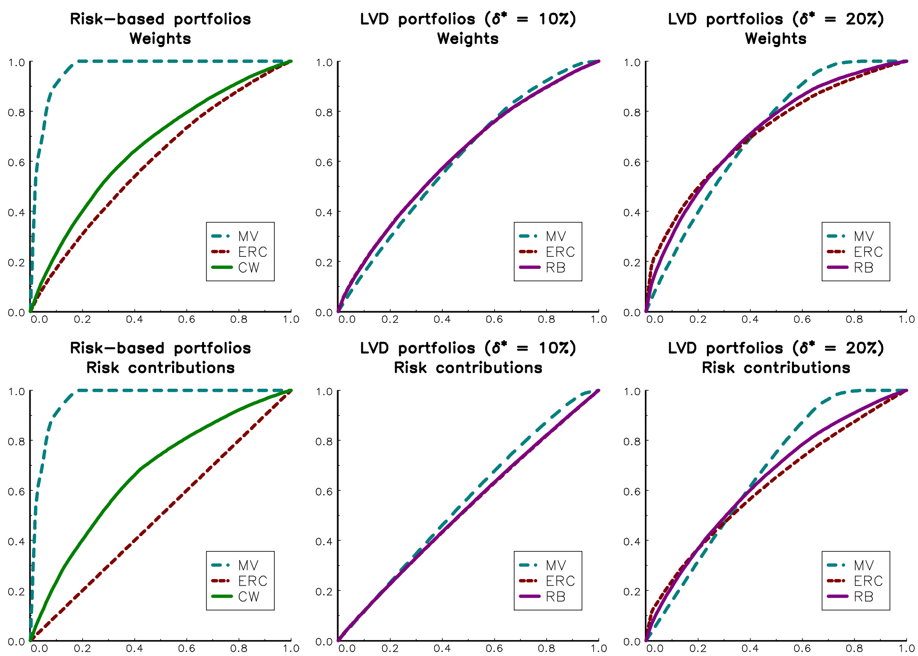
portfolios is equal to or larger than the diversification of the CW portfolio. Only the LVD-RB* portfolio has a lower weight diversification than the CW portfolio. We notice that the LVD portfolios have the same beta, but not the same tracking error volatility. If $\delta^\star = 20\%$, the diversification remains at a satisfactory level in terms of risk contributions. However, we observe some weight concentrations for the ERC and RB portfolios, because of the exposure to Unilever.

Table 6: Statistics of low volatility diversified portfolios (Eurostoxx 50, Feb. 2013)

		$\delta^\star = 10\%$				$\delta^\star = 20\%$			
Statistic	CW	MV	ERC	RB	RB*	MV	ERC	RB	RB*
$\sigma(x)$	21.30	19.17	19.17	19.17	19.17	17.04	17.04	17.04	17.04
$\delta_\sigma(x \mid b)$		10.00	10.00	10.00	10.00	20.00	20.00	20.00	20.00
$\sigma(x \mid b)$		3.23	3.04	3.05	2.90	6.04	5.78	5.86	6.01
$\beta(x \mid b)$		0.89	0.89	0.89	0.90	0.78	0.78	0.78	0.78
$\mathcal{N}(x)$	36.65	43.05	41.27	41.53	32.11	32.60	19.55	27.68	22.63
$\mathcal{N}(\mathcal{RC})$	35.93	47.46	49.61	49.62	36.91	36.09	33.47	38.82	29.91

If we compare the concentration using the Lorenz curves[6], we obtain the results given in Figure 1. We can see that the three approaches (MV, ERC and RB) are very similar in terms of weight and risk concentrations, especially when the risk reduction is small. If we consider a larger value of $\delta^{\star}$, we observe some differences, because the MV portfolio shrinks the weights towards zero more quickly than the ERC and RB portfolios.

Figure 1: Lorenz curve of weights and risk contributions



In Figure 2, we show the trade-off relationships computed with the LVD-MV portfolios. The first panel represents the change in tracking error volatility with respect to the risk reduction. For low values of $\delta_{\sigma}(x \mid b)$ (lower than 10%), we observe that $\sigma(x \mid b)$ is relatively constant. The relationship between $\delta_{\sigma}(x \mid b)$ and $\sigma(x \mid b)$ follows then an increasing function. In the second panel, we consider the beta $\beta(x \mid b)$ of the portfolio. It is a decreasing function of the volatility reduction. Finally, the last two panels show the impact of the volatility reduction on the weight or risk diversification[7]. If we consider LVD-ERC portfolios instead of LVD-MV portfolios, we obtain the results in Figure 3. It is remarkable that the results are so similar and show little sensitivity to the diversification definition. We only observe some small diversification differences in the region where the LDV portfolio converges to the MV portfolio.

Remark 3 *Results for the LVD-RB and LVD-RB* portfolios are given on page 26. We observe the same patterns, except in the case of the LVD-RB* portfolio. This portfolio uses the CW portfolio as the initial portfolio, whereas the LVD-RB portfolio uses the EW portfolio.*

[6]The x-axis of the curve corresponds to the percentile of the most important stocks, and the y-axis to the percentage of cumulated weights or risk contributions. For instance, if $x = 20\%$ and $y = 90\%$, it means that 90% of the weights (or the risk contributions) are concentrated in 20% of stocks.

[7]The diversification statistic is equal to the ratio between $\mathcal{N}$ and the number of stocks n. It takes the value 1 when the portfolio is perfectly diversified and should move towards 0 for a perfectly concentrated portfolio.

Figure 2: Trade-off relationships with LVD-MV portfolios (Eurostoxx 50, Feb. 2013)

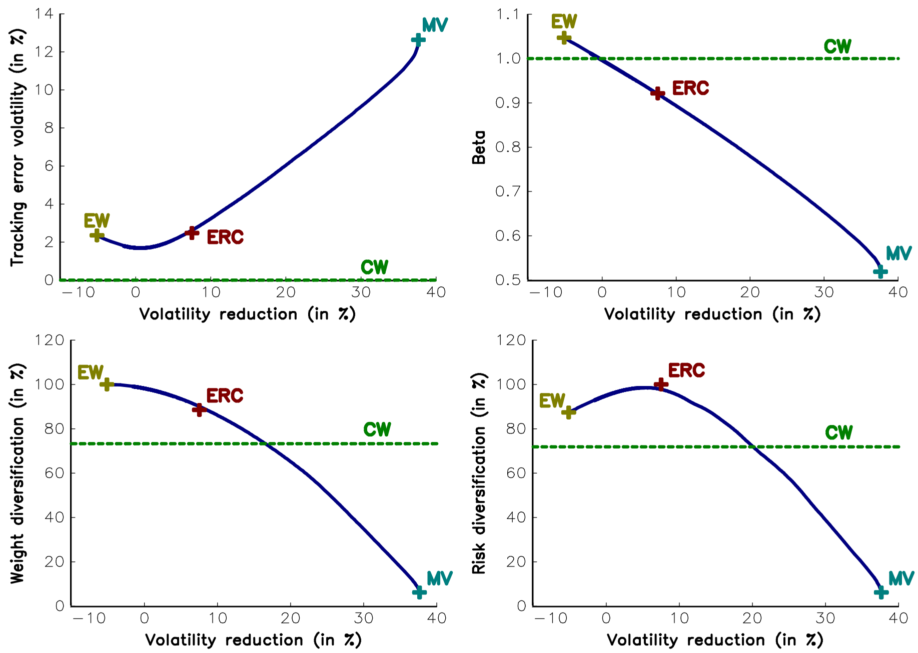
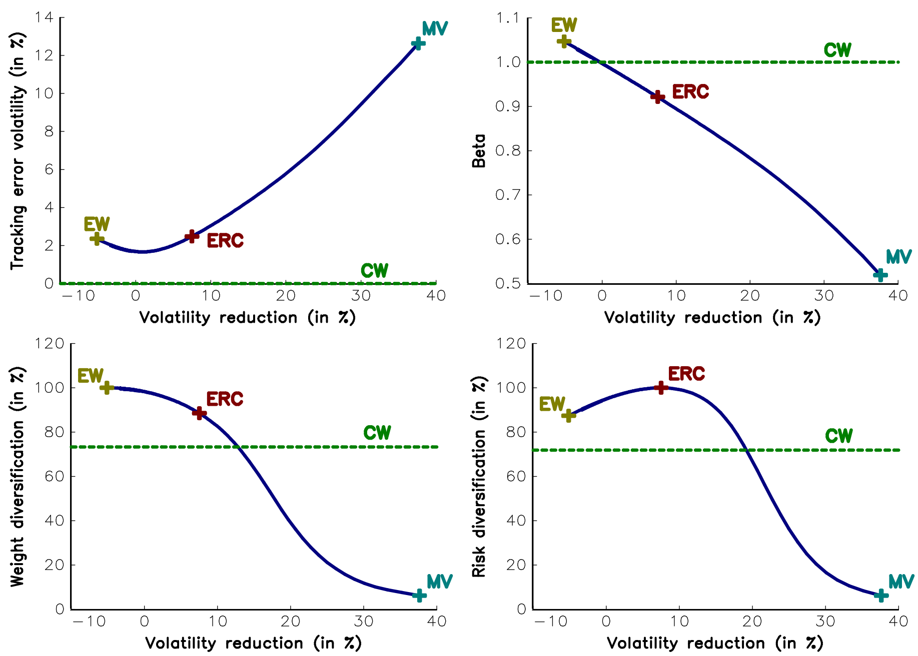


Figure 3: Trade-off relationships with LVD-ERC portfolios (Eurostoxx 50, Feb. 2013)

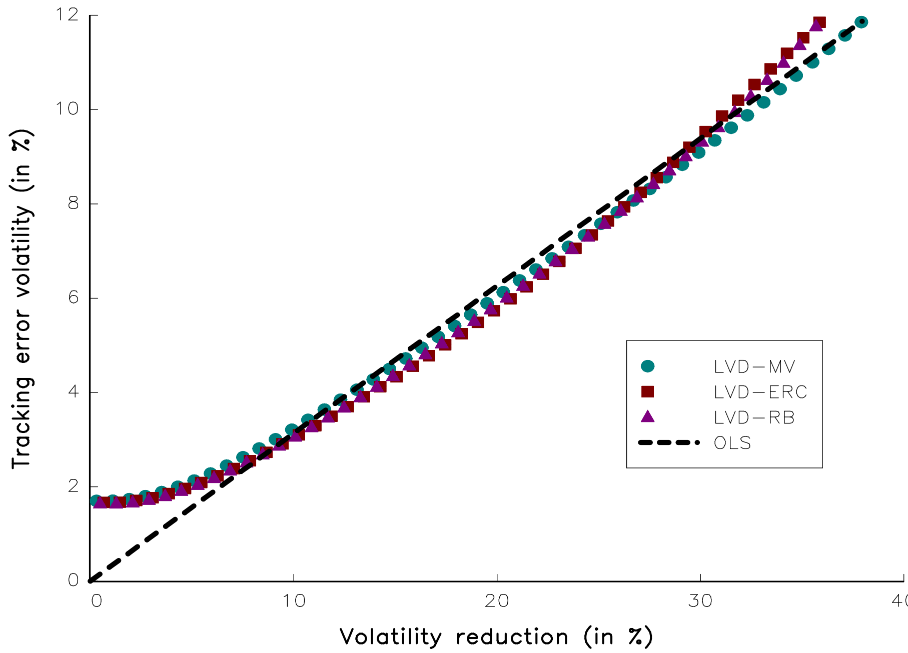


If we estimate the OLS relationship between the volatility reduction and tracking error volatility (see Figure 4), we obtain the following result:

$$\sigma(x \mid b) \simeq 0.31 \cdot \delta_\sigma(x \mid b)$$

Each percentage point of volatility reduction costs 31 bps in terms of tracking error volatility. This result depends on the rebalancing date and the asset universe, but it illustrates the puzzle of smart beta indexing. It is not possible to build a smart beta index that combines all the patterns: low volatility, high diversification and low risk with respect to the benchmark.

Figure 4: Relationships between $\delta_\sigma(x \mid b)$ and $\sigma(x \mid b)$ (Eurostoxx 50, Feb. 2013)



3 Risk-return profile of LVD portfolios

3.1 Application to the S&P 500 index

We consider the universe of the S&P 500 index from December 31, 1989, to December 31, 2012. We simulate the performance of LVD portfolios using the following characteristics:

- Every month, we consider only the stocks belonging to the S&P 500 index.
- We compute the empirical covariance matrix using daily returns and a one-year rolling window.
- The LVD portfolio is rebalanced on the first trading day of the next month.
- The LVD index is computed daily as a price index.

Results are reported in Figures 5, 6 and 7. From an ex-ante point of view, we observe the same trade-off relationships as those noted with the Eurostoxx 50 index. In Figure 5, we

Figure 5: Trade-off relationships with LVD portfolios (S&P 500, 1990 – 2012)

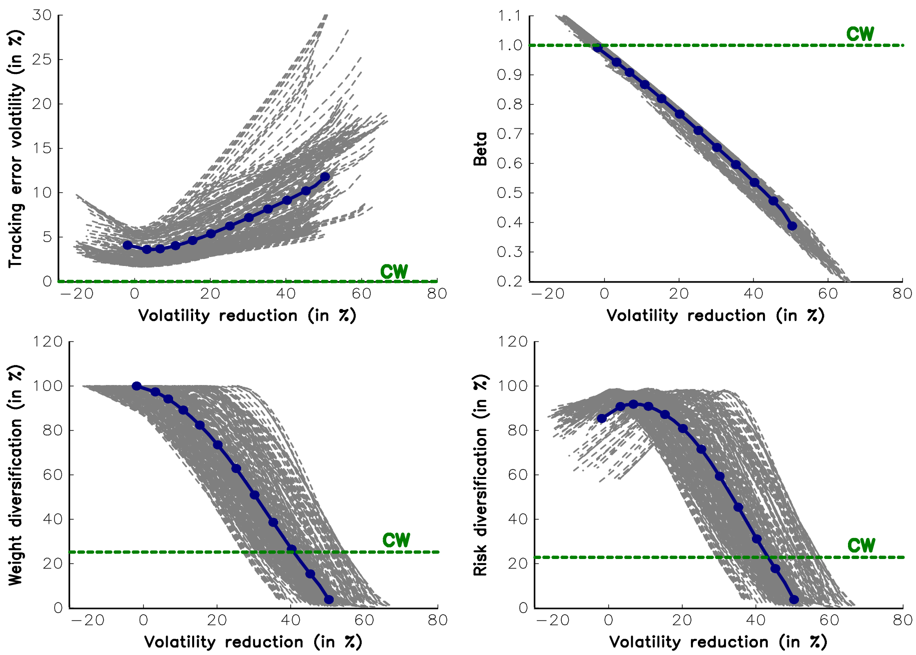


Figure 6: Trade-off relationships with LVD portfolios (S&P 500, 1990 – 2012)

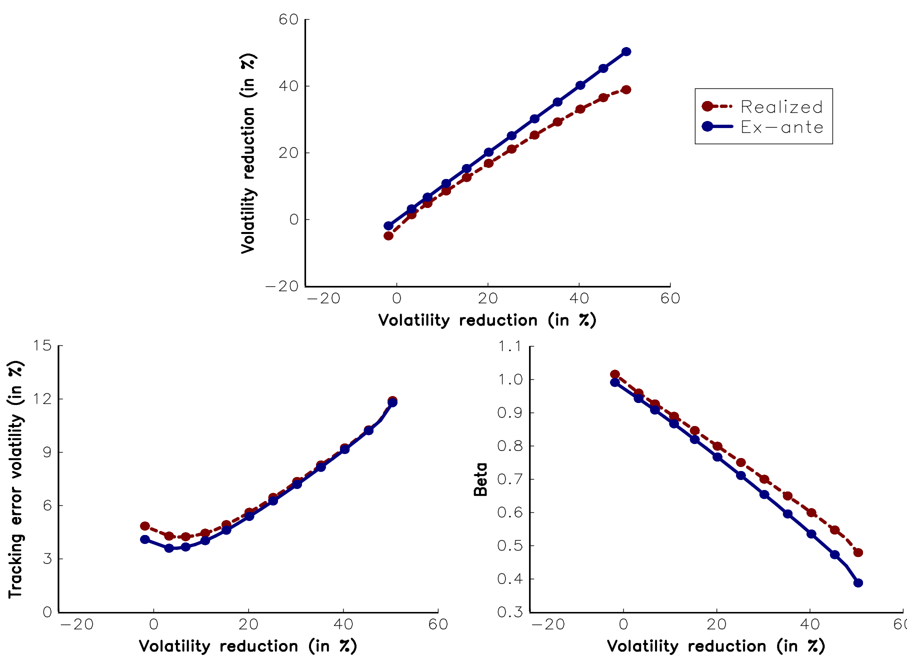
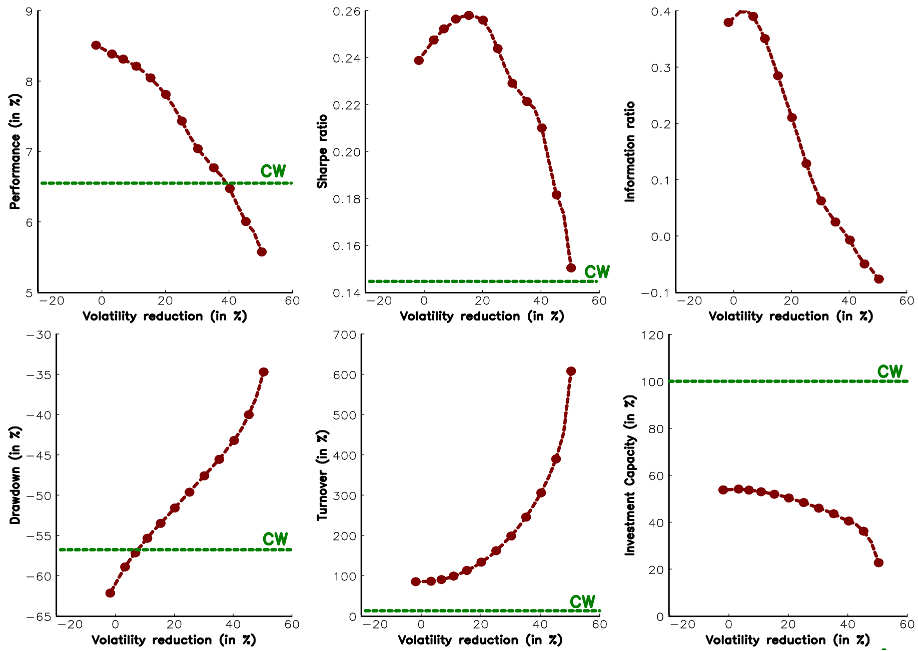


Figure 7: Trade-off relationships with LVD portfolios (S&P 500, 1990 – 2012)



represent them for the 276 rebalancing dates and also compute the average values. Figure 6 shows that the ex-post relationships are very similar to the ex-ante relationships. We only observe a small discrepancy when we target a substantial volatility reduction, of larger than 30%. We confirm that a 10% volatility reduction implies a cost of about 3% in terms of tracking error volatility. In Figure 7, we estimate some risk-return statistics by using the simulated performance[8] of LVD portfolios from December 31, 1989, to December 31, 2012. We observe a decreasing function between the annualized return and the volatility reduction, while the drawdown is reduced when we consider a greater volatility reduction. The Sharpe and information ratios are relatively stable when the volatility reduction is between 0% and 20%. We obtain the same conclusion if we consider the (two-sided) turnover. The investment capacity ratio[9] of LVD portfolios is given in the last panel in Figure 7. This measure gives some insight into capacity constraints, meaning that liquidity issues are more important if this ratio is low. For this simulation, the investment capacity ratio is close to 60% if the volatility reduction is relatively small.

[8] These results are sensitive to the universe and the period under analysis. In particular, they depend on the relative performance of the EW and MV portfolios with respect to the CW portfolio.

[9] According to NBIM (1992), the investment capacity ratio of asset i corresponds to the CW weight $x_{\mathrm{cw},i}$ divided by the portfolio weight x_i:

$$\mathrm{ICR}_i = \frac{x_{\mathrm{cw},i}}{x_i}$$

We then define the investment capacity ratio of portfolio x as the weighted mean:

$$\mathrm{ICR} = \sum_{i=1}^{n} w_i \, \mathrm{ICR}_i$$

with:

$$w_i = \frac{x_{\mathrm{cw},i} \cdot \mathbb{1}\left\{x_i > x_{\mathrm{cw},i}\right\}}{\sum_{j=1}^{n} x_{\mathrm{cw},j} \cdot \mathbb{1}\left\{x_j > x_{\mathrm{cw},j}\right\}}$$

Remark 4 *The performance analysis of low volatility strategies is a difficult task. In this simulation, the performance decreases when we consider a greater volatility reduction. This implies that the EW portfolio has the higher return, while the MV portfolio has the lower return. This is not always the case if we consider another universe or analysis period. In what follows, we propose a framework for understanding the sources of performance for LVD portfolios.*

3.2 A framework for performance attribution

In this section, we break down the performance of LDV portfolios into three components:

1. Beta return
2. Diversification return
3. Alpha return

Let $R_t(x)$ be the return of the LVD portfolio x at time t. We have:

$$R_t(x) - r = R_{\beta,t} + R_{d,t} + R_{\alpha,t}$$

where r is the risk-free rate, $R_{\beta,t}$ is the beta return, $R_{d,t}$ is the diversification return and $R_{\alpha,t}$ is the alpha return.

Remark 5 *This approach to measuring the performance may seem curious, because the traditional way to gauge the performance of an equity portfolio is to consider a Fama-French breakdown. One of the problems with such approach is the relative instability of the Fama-French factors. Moreover, we think that the small caps and value factors are special cases of the low beta anomaly, which is the key aspect of the alpha return.*

3.2.1 Beta return

Let us consider the CAPM model of Sharpe (1964). We have:

$$\mathbb{E}[R_t(x)] - r = \beta(\mathbb{E}[R_{m,t}] - r)$$

where β is the beta of the portfolio and $R_{m,t}$ is the return of the market portfolio. We consider that the beta return is equal to $\beta(R_{m,t} - r)$. In this case, $R_{\beta,t}$ is an increasing function of β. In LVD portfolios, the beta is a decreasing function of the volatility reduction $\delta_\sigma(x \mid b)$. We therefore deduce that the beta return is also a decreasing function of $\delta_\sigma(x \mid b)$.

3.2.2 Diversification return

We may show that the return between the CW portfolio and the LVD portfolio also depends on the level of diversification. This diversification return may be explained by two reasons. By considering a more diversified portfolio than the CW portfolio, we capture the systematic risk more effectively. Moreover, rebalancing a portfolio may generate extra performance (Booth and Fama, 1992), particularly when the dispersion of volatilities and correlations is high[10]. If we assume that a proxy for measuring the diversification return is given by the risk diversification, we deduce that the diversification return is an increasing function of $\delta_\sigma(x \mid b)$ when the volatility reduction is small and a decreasing function of $\delta_\sigma(x \mid b)$ when the volatility reduction becomes substantial. If the portfolio is concentrated, the diversification return is negative.

[10] More precisely, we may show that the diversification return depends on the difference between the arithmetic mean and the harmonic mean of volatilities and is a decreasing function of the average correlation. Simulations show that the diversification return lies between 0% and 3% for a highly diversified portfolio, typically the ERC portfolio.

3.2.3 Alpha return

This component is linked to the low beta anomaly and presupposes that the relationship between the excess return $\pi_i = \mathbb{E}[R_{i,t}] - r$ of asset i and its beta β_i is much flatter than that predicted by the CAPM theory. This anomaly was first illustrated by empirical analysis (Black *et al*, 1972). Different explanations have been put forward to explain this stylized fact (Goltz and Martellini, 2013). The most pertinent is the impact of constraints on the CAPM model (Black, 1972). For instance, Frazzini and Pedersen (2010) have developed a model with borrowing constraints. In this case, some investors cannot leverage the tangency portfolio in order to obtain an optimal portfolio with a higher return. These investors will then invest in a portfolio that is not optimal, but which is more risky than the tangency portfolio. In this model, the CAPM relationship becomes:

$$\mathbb{E}[R_{i,t}] - r = \alpha_i + \beta_i \left(\mathbb{E}[R_{m,t}] - r\right)$$

where α_i is a linear decreasing function of β_i and depends on the borrowing constraints.

Example 1 *We consider four assets where* $\mu_1 = 5\%$, $\mu_2 = 6\%$, $\mu_3 = 8\%$, $\mu_4 = 6\%$, $\sigma_1 = 15\%$, $\sigma_2 = 20\%$, $\sigma_3 = 25\%$, $\sigma_4 = 20\%$ *and*

$$C = \begin{pmatrix} 1.00 & & & \\ 0.10 & 1.00 & & \\ 0.20 & 0.60 & 1.00 & \\ 0.40 & 0.50 & 0.50 & 1.00 \end{pmatrix}$$

The risk-free rate is set to 2%.

This example is taken from Roncalli (2013). Using the previous parameters, we computed the tangency portfolio $x^\star$ and obtained the results given in Table 7. In this case, the expected return and the volatility of the tangency portfolio are $\mu(x^\star) = 6.07\%$ and $\sigma(x^\star) = 13.77\%$. Let $\beta_i(x)$ and $\pi_i(x)$ be the beta of the asset i and its implied risk premium[11] with respect to the market portfolio x. These two statistics are also reported in Table 7. We found that the implied risk premium $\pi_i(x^\star)$ is equal to the true risk premium $\mu_i - r$.

Table 7: Tangency portfolio $x^\star$

Asset	$x_i^\star$	$\beta_i(x^\star)$	$\pi_i(x^\star)$
1	47.50%	0.74	3.00%
2	19.83%	0.98	4.00%
3	27.37%	1.47	6.00%
4	5.30%	0.98	4.00%

Let us suppose that the market includes two investors. The first investor cannot leverage his risky portfolio, whereas the second investor must hold 50% of his wealth in cash. The market portfolio[12] $\bar{x}$ is therefore different from the tangency portfolio $x^\star$ (see Table 8). It is more heavily weighted in high risk assets and less heavily weighted in low risk assets. This is why the implied risk premium is underestimated for low beta assets and overestimated

[11] We have:

$$\pi_i(x) = \beta_i(x) \cdot (\mu(x) - r)$$

where $\mu(x)$ is the expected return of the portfolio x.

[12] We have $\mu(\bar{x}) = 6.30\%$ and $\sigma(\bar{x}) = 14.66\%$.

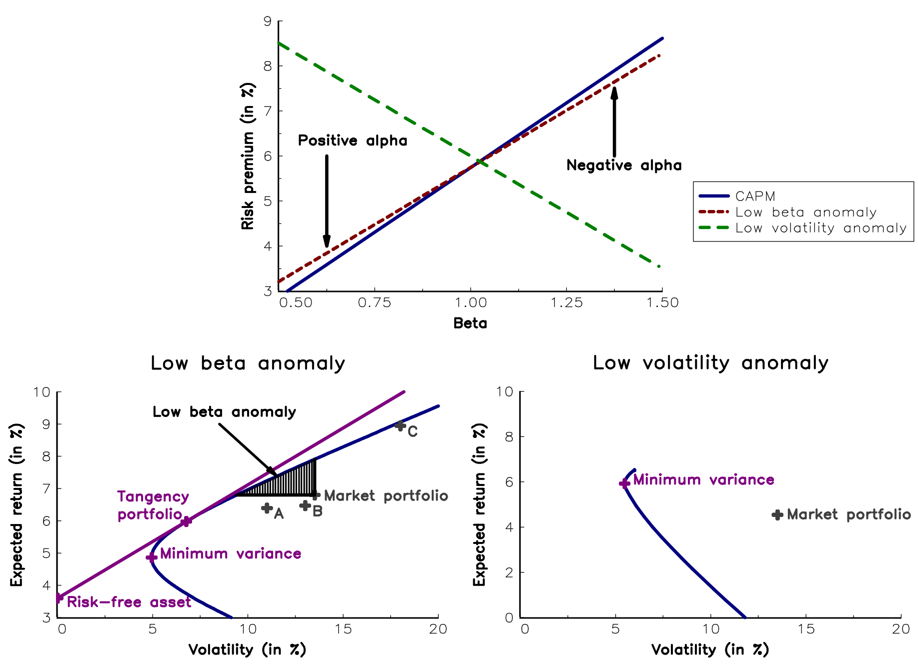
for high beta assets. For instance, the implied risk premium of the first asset is 2.68%. We can also compute the alpha. We found that it is positive for low beta assets ($\beta_i(\bar{x}) < 1$) and negative for high beta assets ($\beta_i(\bar{x}) > 1$). If we add the alpha return to the implied risk premium, we obtain the true risk premium.

Table 8: Market portfolio $\bar{x}$ with two investors

Asset	$\bar{x}_i$	α_i	$\beta_i(\bar{x})$	$\pi_i(\bar{x})$	$\alpha_i + \pi_i(\bar{x})$
1	42.21%	0.32%	0.62	2.68%	3.00%
2	15.70%	0.07%	0.91	3.93%	4.00%
3	36.31%	−0.41%	1.49	6.41%	6.00%
4	5.78%	0.07%	0.91	3.93%	4.00%

Remark 6 *There is some confusion about the low beta anomaly and the low volatility anomaly. The low beta anomaly tells us that the risk premium of low beta stocks (or high beta stocks) is underestimated (or overestimated) if we consider the CAPM model. The low volatility anomaly assumes that low volatility stocks perform better than high volatility stocks. These statements are illustrated in the first panel of Figure 8. Given the low beta anomaly, it is therefore possible to improve the risk-return profile of the market portfolio. However, the low beta anomaly does not assume that the the minimum variance portfolio outperforms the tangency portfolio. In the case of the low volatility anomaly, the minimum variance portfolio is very close to the tangency portfolio, and outperforms the market portfolio. The two anomalies can only be reconciled when we consider that the market portfolio has no risk premium or negative risk premium.*

Figure 8: Illustration of the low beta and low volatility anomalies



Let us illustrate that LVD portfolios benefit from the low beta anomaly. In the CAPM model, the covariance matrix Σ can be broken down as:

$$\Sigma = \beta\beta^\top \sigma_m^2 + D$$

where $\beta = (\beta_1, \ldots, \beta_n)$ is the vector of betas, σ_m^2 is the variance of the market portfolio and $D = \operatorname{diag}\left(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_n^2\right)$ is the diagonal matrix of specific variances. Following Scherer (2011) and Clarke *et al.* (2010), we may show that the weights of the LDV portfolio are:

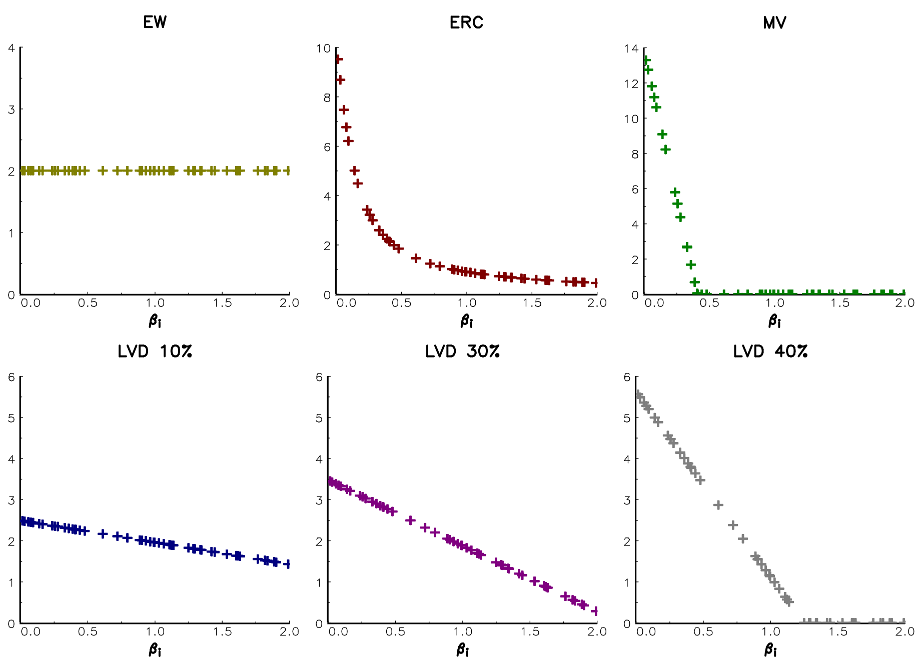
$$x_i^\star = \frac{\sigma^2(x^\star)}{\tilde{\sigma}_i^2 + \lambda} \cdot \left(1 - \frac{\beta_i}{\beta^\star}\right) \cdot \mathbf{1}\left\{\beta_i > \beta^\star\right\}$$

with:

$$\beta^\star = \frac{1 + \sigma_m^2 \sum_{\beta_i < \beta^\star} \tilde{\beta}_i \beta_i}{\sigma_m^2 \sum_{\beta_i < \beta^\star} \tilde{\beta}_i}$$

and $\tilde{\beta}_i = \beta_i / \left(\tilde{\sigma}_i^2 + \lambda\right)$. The parameter λ is the Lagrange coefficient associated with the Herfindahl constraint. In Figure 9, we show the composition of the EW, ERC, MV and LVD portfolios when asset returns follow the one-factor CAPM model[13]. We find that LVD portfolios exhibit a low beta bias. This bias is particularly marked when the volatility reduction is substantial[14].

Figure 9: The composition of LVD portfolios with respect to asset betas



[13]The idiosyncratic volatility $\tilde{\sigma}_i$ is set to 15% for all assets, while the volatility of the market portfolio σ_m is taken to be 25%. The values of beta β_i have been simulated uniformly between 0 and 2. Moreover, we assume that the benchmark is the EW portfolio.

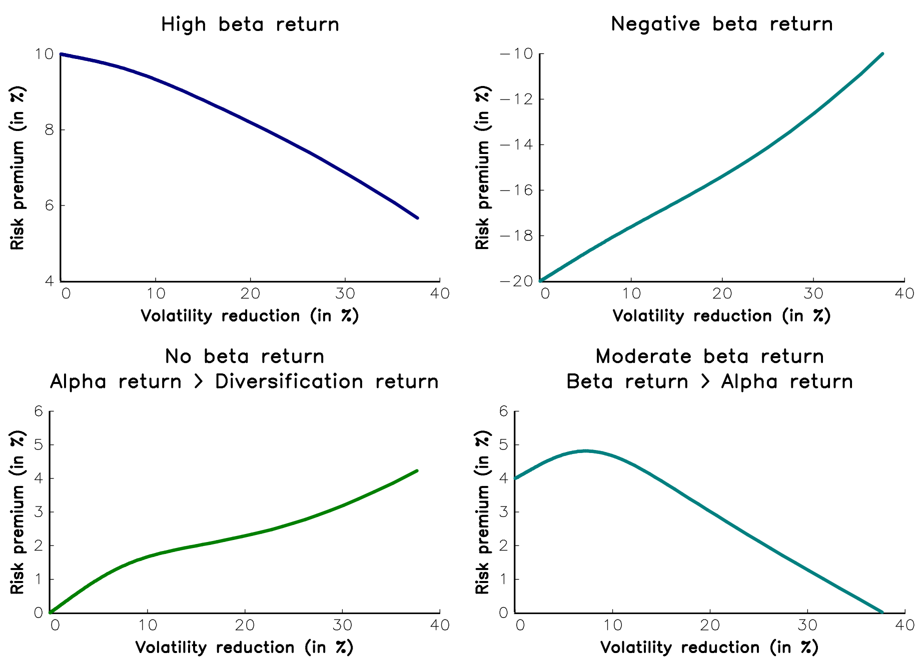
[14]See panels 5 and 6 in Figure 9 where the volatility reduction is equal to 30% and 40%.

3.2.4 Performance dilemma

We face a difficulty in terms of performance attribution. Smart beta indexing is designed to capture the equity risk premium. The relative performance of such portfolios with respect to CW portfolios must then be measured in a complete economic cycle. This is why smart beta indexing principally concerns long-term investors. According to Benartzi and Thaler (1995), we must however distinguish the *evaluation period* and the *planning horizon* of investors. Even long-term investors evaluate their portfolios frequently, typically once per year. In this context, benchmarking smart beta investments is a difficult task.

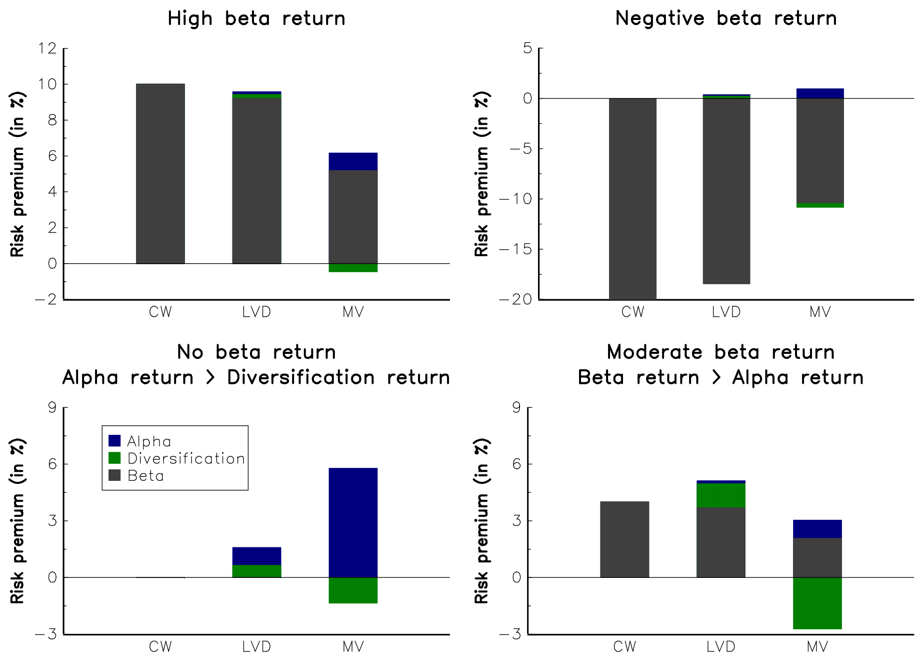
Suppose that an investor prefers a smart beta portfolio with a substantial volatility reduction. In this case, the smart beta portfolio will outperform the CW portfolio if the market is down or flat. We can explain this gain because the beta return is lower (Figure 10). Moreover, the portfolio may also benefit from the alpha return (Figure 11). A typical example of such a situation is the year 2008. Nevertheless, the level of underperformance may be high if the performance of the market is strong or moderate. For instance, if the beta of the portfolio is 50% and the market performance is 30%, the beta return is only 15%. It is highly unlikely that the alpha return due to the low beta anomaly will offset this relative loss. This explains why minimum variance strategies largely underperformed the CW indexes in 1999 and 2012.

Figure 10: Scenario-based performance analysis



Investors who evaluate their portfolios move on a yearly basis need to keep in mind the three performance components that explain the risk premium. Targeting a substantial reduction in volatility implies a negative diversification return, which is largely offset by the beta return if the market performance is significantly negative. The alpha return may be high, but it may not offset the beta return loss in a strong bull market. Targeting a small volatility reduction implies a lower tracking error with respect to the CW portfolio. It is

Figure 11: Breakdown of performance



a good solution when the market performance is positive, because these portfolios benefit from the beta return and the diversification return. However, in the case of a bear market, the loss reduction will be less significant.

4 Conclusion

Smart beta indexing is becoming increasingly popular with institutional investors and pension funds. It is perceived as a method of reducing risk and increasing performance with respect to capitalization-weighted indexing. This is particularly true with low volatility strategies, e.g. minimum variance portfolios. In this context, we observe a growing demand for smart beta solutions that explicitly target a volatility reduction.

Building an equity portfolio with 30% lower volatility than the CW portfolio and the same performance is attractive. However, everything has a cost and there is no free lunch. There is a trade-off between the volatility reduction and the risks of such solutions. In this article, we show and measure the relationships between volatility on the one hand, and diversification, tracking-error, liquidity and performance on the other. For instance, the cost of a 10% reduction in volatility is between 3% and 4% in terms of tracking error. Investors who are considering such strategies should keep these trade-off in mind.

Performance is another big issue in smart beta indexing. The return of a low volatility portfolio can be broken down into three components: the beta return, the diversification return and the alpha return. From a theoretical point of view, the beta return is a decreasing function of the volatility reduction, whereas the alpha return is an increasing function. It explains that low volatility indexing may comfortably outperform (or underperform) capitalization-weighted indexing when the performance of the equity market is low or negative (or strong). As a consequence, choosing a low volatility index is a bet on the future behavior of the equity market.

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A Technical appendix

A.1 Shrinkage formulation of the LVD-MV portfolio

It is easy to show that the LVD-MV portfolio is the solution to this optimization problem:

$$\begin{aligned} x^{\star}(\lambda) &= \arg\min x^{\top}(\Sigma + \lambda I_n)\, x \\ &\text{u.c.} \quad \begin{cases} \mathbf{1}^{\top} x = 1 \\ x \geq 0 \end{cases} \end{aligned}$$

We obtain a minimum variance portfolio when the covariance matrix is shrunk (Bruder *et al.*, 2013). In this case, we have the following correspondence:

$$c = \frac{1}{\sum_{i=1}^{n} x_i^{\star}(\lambda)^2}$$

A.2 The LVD-MDP portfolio

We consider the following optimization program:

$$\begin{aligned} x^{\star}(c) &= \arg\min x^{\top} \Sigma x \\ &\text{u.c.} \quad \begin{cases} x^{\top} \sigma \geq c \\ \mathbf{1}^{\top} x = 1 \\ x \geq 0 \end{cases} \end{aligned}$$

if $c = 0$, the optimized portfolio is equal to the MV portfolio. if $c = \sup_i \sigma_i$, the optimized portfolio is fully invested in the most risky asset. This implies that a scalar c^{+} exists such that:

$$\sigma(x_{\text{mv}}) \leq \sigma(x^{\star}(c))$$

We can then target a risk reduction of $\delta^{\star}$ in the same way as we proceed for the LVD-ERC portfolio. However, these solutions are not necessarily diversified, because they do not encompass the equally weighted portfolio.

A.3 The LVD-RB portfolio

Let $B = (B_1, \ldots, B_n)$ be a vector of budgets. We note $x(B)$ the risk budgeting (RB) portfolio such that the risk contribution of asset i is equal to the given risk budget B_i:

$$\mathcal{RC}_i = B_i$$

Bruder and Roncalli (2012) noticed that using portfolio weights B as risk budgets reduces volatility:

$$\sigma(x(B)) \leq \sigma(B)$$

Starting from the equally weighted portfolio $x^{(0)} = x_{\text{ew}}$, we consider the iterative RB portfolio. At iteration k, the portfolio $x^{(k)}$ is defined such that the risk budgets are equal to the portfolio weights of the previous iteration:

$$B = x^{(k-1)}$$

We finally obtain:

$$\sigma(x_{\text{mv}}) = \sigma\left(x^{(\infty)}\right) \leq \sigma\left(x^{(k)}\right) \leq \sigma\left(x^{(0)}\right) = \sigma(x_{\text{ew}})$$

We can therefore find a RB portfolio such that $\sigma\left(x^{(k)}\right) < (1-\delta^{\star})\,\sigma(x_{\mathrm{cw}})$. Targeting a risk reduction equal to $\delta^{\star}$ can then be achieved by carrying out an interpolation between the two consecutive RB portfolios: $x^{\star} = \alpha x^{(k)} + (1-\alpha)\,x^{(k-1)}$.

Remark 7 *If we consider that the initial portfolio $x^{(0)}$ is the capitalization-weighted portfolio x_{cw}, we obtain another solution. In this case, the lower bound δ^{-} is 0.*

B Results

Table 9: Composition of risk-based portfolios – Eurostoxx 50 universe (Feb. 2013)

Stock	CW	EW	MV	MDP	ERC
SANOFI	5.56	2.00			2.31
TOTAL	5.40	2.00			2.16
BASF	4.22	2.00			1.90
SIEMENS	4.21	2.00	13.69		2.61
BAYER	3.99	2.00			2.11
BANCO SANTANDER	3.81	2.00			1.30
SAP	3.52	2.00	2.81		2.97
ANHEUSER-BUSCH	3.39	2.00	1.48	17.29	3.62
ALLIANZ	3.01	2.00			1.96
UNILEVER	2.97	2.00	53.12		4.42
ENI	2.97	2.00			1.86
BNP PARIBAS	2.90	2.00			1.14
DAIMLER	2.85	2.00			1.84
BBVA	2.57	2.00			1.24
TELEFONICA	2.53	2.00			1.61
LVMH	2.24	2.00			1.96
DEUTSCHE BANK	2.08	2.00			1.28
SCHNEIDER ELECTRIC	2.05	2.00			1.37
DANONE	2.05	2.00	3.28	8.95	3.27
AIR LIQUIDE	1.85	2.00			2.29
L'OREAL	1.74	2.00			2.65
AXA	1.70	2.00			1.33
E ON	1.63	2.00		1.92	2.26
DEUTSCHE TELEKOM	1.56	2.00	12.88	13.90	2.93
VOLKSWAGEN	1.53	2.00		7.10	2.13
ING	1.51	2.00			1.16
SOCIETE GENERALE	1.45	2.00			1.01
BMW	1.44	2.00			1.84
MUENCHENER RUCK	1.41	2.00			2.44
INDITEX	1.38	2.00		6.26	2.22
UNICREDIT	1.34	2.00			0.98
PHILIPS	1.32	2.00	8.14	8.09	2.74
GDF SUEZ	1.31	2.00		4.88	2.30
VIVENDI	1.29	2.00		1.79	2.05
ASML HOLDING	1.24	2.00		9.24	2.65
VINCI	1.14	2.00			1.53
ENEL	1.12	2.00			1.66
IBERDROLA	1.11	2.00		0.65	1.43
INTESA SANPAOLO	1.10	2.00			0.99
ESSILOR	1.07	2.00	2.25	2.02	3.09
GENERALI	1.05	2.00			1.32
UNIBAIL-RODAMCO	1.04	2.00	2.35		2.63
FRANCE TELECOM	0.91	2.00			1.92
RWE	0.87	2.00			2.30
SAINT GOBAIN	0.86	2.00			1.45
REPSOL	0.81	2.00		2.24	1.33
CARREFOUR	0.79	2.00		0.97	1.75
CRH	0.77	2.00		5.39	1.98
ARCELORMITTAL	0.68	2.00			1.27
NOKIA	0.66	2.00		9.32	1.45

Table 10: Composition of constrained MV portfolios – Eurostoxx 50 universe (Feb. 2013)

Lower bound	0%	0%	1%	1%	$0 \times b_i$	$0.5 \times b_i$	1%
Upper bound	10%	5%	5%	4%	$2 \times b_i$	$2 \times b_i$	$2 \times b_i$
SANOFI		5.00	1.00	4.00	11.12	5.66	4.72
TOTAL		5.00	1.00	1.00	10.81	2.70	1.00
BASF			1.00	1.00		2.11	1.00
SIEMENS	10.00	5.00	5.00	4.00	8.42	8.42	8.42
BAYER			1.00	1.00	0.23	1.99	1.00
BANCO SANTANDER			1.00	1.00		1.91	1.00
SAP	10.00	5.00	5.00	4.00	7.03	7.03	7.03
ANHEUSER-BUSCH	10.00	5.00	5.00	4.00	6.78	6.78	6.78
ALLIANZ			1.00	1.00	3.76	1.50	1.00
UNILEVER	10.00	5.00	5.00	4.00	5.94	5.94	5.94
ENI			1.00	1.00		1.48	1.00
BNP PARIBAS			1.00	1.00		1.45	1.00
DAIMLER			1.00	1.00		1.43	1.00
BBVA			1.00	1.00		1.29	1.00
TELEFONICA			1.00	1.00		1.26	1.00
LVMH			1.00	1.00		1.12	1.00
DEUTSCHE BANK			1.00	1.00		1.04	1.00
SCHNEIDER ELECTRIC			1.00	1.00		1.03	1.00
DANONE	10.00	5.00	5.00	4.00	4.09	4.09	4.09
AIR LIQUIDE		5.00	1.00	4.00	3.70	3.70	3.70
L'OREAL		5.00	5.00	4.00	3.47	3.47	3.47
AXA			1.00	1.00		0.85	1.00
E ON	0.27	5.00	1.00	2.69	3.25	3.25	3.21
DEUTSCHE TELEKOM	10.00	5.00	5.00	4.00	3.12	3.12	3.12
VOLKSWAGEN		3.63	1.00	1.00	3.06	0.76	1.00
ING			1.00	1.00		0.76	1.00
SOCIETE GENERALE			1.00	1.00		0.73	1.00
BMW			1.00	1.00		0.72	1.00
MUENCHENER RUCK	3.83	5.00	5.00	4.00	2.82	2.82	2.82
INDITEX		2.94	1.00	1.31	2.77	1.08	1.00
UNICREDIT			1.00	1.00		0.67	1.00
PHILIPS	10.00	5.00	5.00	4.00	2.64	2.64	2.64
GDF SUEZ	4.70	5.00	2.54	4.00	2.62	2.62	2.62
VIVENDI		3.43	1.00	1.00	2.58	0.64	1.00
ASML HOLDING	0.81	5.00	5.00	4.00	2.47	2.47	2.47
VINCI			1.00	1.00		0.57	1.00
ENEL			1.00	1.00		0.56	1.00
IBERDROLA			1.00	1.00		0.56	1.00
INTESA SANPAOLO			1.00	1.00		0.55	1.00
ESSILOR	10.00	5.00	5.00	4.00	2.14	2.14	2.14
GENERALI			1.00	1.00		0.53	1.00
UNIBAIL-RODAMCO	10.00	5.00	5.00	4.00	2.07	2.07	2.07
FRANCE TELECOM			1.00	1.00	1.82	0.46	1.00
RWE	0.39	5.00	1.46	4.00	1.75	1.75	1.75
SAINT GOBAIN			1.00	1.00		0.43	1.00
REPSOL			1.00	1.00		0.40	1.00
CARREFOUR			1.00	1.00		0.39	1.00
CRH			1.00	1.00	1.54	0.39	1.00
ARCELORMITTAL			1.00	1.00		0.34	1.00
NOKIA			1.00	1.00		0.33	1.00

Table 11: Composition of LVD portfolios – Eurostoxx 50 universe (Feb. 2013)

Stock		$\delta^{\star} = 10\%$				$\delta^{\star} = 20\%$			
	CW	MV	ERC	RB	RB*	MV	ERC	RB	RB*
SANOFI	5.56	2.53	2.29	2.32	5.94	2.88	1.88	2.26	5.60
TOTAL	5.40	2.42	2.13	2.15	5.55	2.75	1.75	2.01	5.11
BASF	4.22	2.14	1.81	1.83	3.83	1.99	1.33	1.43	2.92
SIEMENS	4.21	2.75	2.71	2.73	5.16	3.61	2.84	3.26	6.15
BAYER	3.99	2.35	2.04	2.06	3.89	2.41	1.54	1.79	3.22
BANCO SANTANDER	3.81	1.18	1.17	1.16	2.53	0.20	0.78	0.63	1.34
SAP	3.52	2.94	3.18	3.20	4.75	3.90	3.55	4.17	6.12
ANHEUSER-BUSCH	3.39	3.20	4.11	4.07	5.37	4.39	5.15	6.03	7.89
ALLIANZ	3.01	2.22	1.88	1.90	2.84	2.28	1.45	1.59	2.34
UNILEVER	2.97	3.42	5.67	5.39	5.75	5.04	17.89	10.53	11.52
ENI	2.97	2.11	1.78	1.79	2.69	2.09	1.34	1.43	2.11
BNP PARIBAS	2.90	0.73	1.01	1.00	1.72		0.64	0.45	0.75
DAIMLER	2.85	2.06	1.74	1.75	2.53	1.81	1.27	1.33	1.89
BBVA	2.57	1.03	1.12	1.10	1.65		0.74	0.57	0.82
TELEFONICA	2.53	1.78	1.50	1.50	2.04	1.52	1.11	1.05	1.42
LVMH	2.24	2.21	1.88	1.90	2.10	2.09	1.38	1.52	1.65
DEUTSCHE BANK	2.08	1.10	1.15	1.14	1.35		0.75	0.59	0.67
SCHNEIDER ELECTRI	2.05	1.29	1.23	1.23	1.41	0.08	0.80	0.67	0.74
DANONE	2.05	3.08	3.60	3.60	3.04	4.18	4.22	5.02	4.34
AIR LIQUIDE	1.85	2.52	2.27	2.29	1.98	2.91	1.88	2.23	1.89
L'OREAL	1.74	2.76	2.72	2.75	2.12	3.38	2.39	3.03	2.33
AXA	1.70	1.22	1.20	1.20	1.16	0.06	0.80	0.65	0.61
E ON	1.63	2.49	2.25	2.27	1.80	2.90	1.98	2.26	1.86
DEUTSCHE TELEKOM	1.56	2.93	3.16	3.17	2.17	4.00	3.86	4.29	3.11
VOLKSWAGEN	1.53	2.36	2.08	2.10	1.56	2.54	1.70	1.91	1.41
ING	1.51	0.79	1.03	1.02	0.91		0.66	0.47	0.41
SOCIETE GENERALE	1.45	0.29	0.89	0.87	0.78		0.56	0.35	0.30
BMW	1.44	2.05	1.74	1.75	1.29	1.83	1.29	1.35	0.98
MUENCHENER RUCK	1.41	2.65	2.48	2.50	1.63	3.30	2.24	2.70	1.78
INDITEX	1.38	2.46	2.19	2.21	1.47	2.73	1.78	2.07	1.39
UNICREDIT	1.34	0.18	0.86	0.84	0.70		0.53	0.33	0.26
PHILIPS	1.32	2.85	2.90	2.92	1.74	3.86	3.29	3.71	2.32
GDF SUEZ	1.31	2.54	2.31	2.33	1.48	3.11	2.09	2.42	1.59
VIVENDI	1.29	2.31	2.00	2.01	1.30	2.56	1.65	1.82	1.21
ASML HOLDING	1.24	2.76	2.71	2.74	1.54	3.31	2.43	3.00	1.74
VINCI	1.14	1.63	1.41	1.41	0.88	0.91	0.97	0.89	0.54
ENEL	1.12	1.86	1.56	1.56	0.94	1.72	1.17	1.14	0.70
IBERDROLA	1.11	1.46	1.31	1.31	0.82	0.84	0.92	0.80	0.50
INTESA SANPAOLO	1.10	0.22	0.87	0.85	0.59		0.54	0.34	0.22
ESSILOR	1.07	2.99	3.34	3.35	1.52	3.95	3.67	4.41	2.06
GENERALI	1.05	1.24	1.20	1.19	0.72	0.24	0.80	0.65	0.39
UNIBAIL-RODAMCO	1.04	2.77	2.73	2.75	1.29	3.55	2.74	3.22	1.55
FRANCE TELECOM	0.91	2.19	1.85	1.86	0.87	2.30	1.48	1.57	0.75
RWE	0.87	2.54	2.31	2.33	0.98	3.03	2.05	2.37	1.04
SAINT GOBAIN	0.86	1.48	1.33	1.32	0.64	0.58	0.89	0.78	0.37
REPSOL	0.81	1.23	1.21	1.20	0.55	0.27	0.82	0.66	0.30
CARREFOUR	0.79	1.95	1.65	1.66	0.70	1.72	1.22	1.23	0.53
CRH	0.77	2.22	1.91	1.92	0.75	2.32	1.54	1.65	0.66
ARCELORMITTAL	0.68	1.05	1.14	1.13	0.44		0.75	0.58	0.22
NOKIA	0.66	1.47	1.34	1.33	0.51	0.86	0.93	0.82	0.33

Figure 12: Trade-off relationships with LVD-RB portfolios (Eurostoxx 50, Feb. 2013)

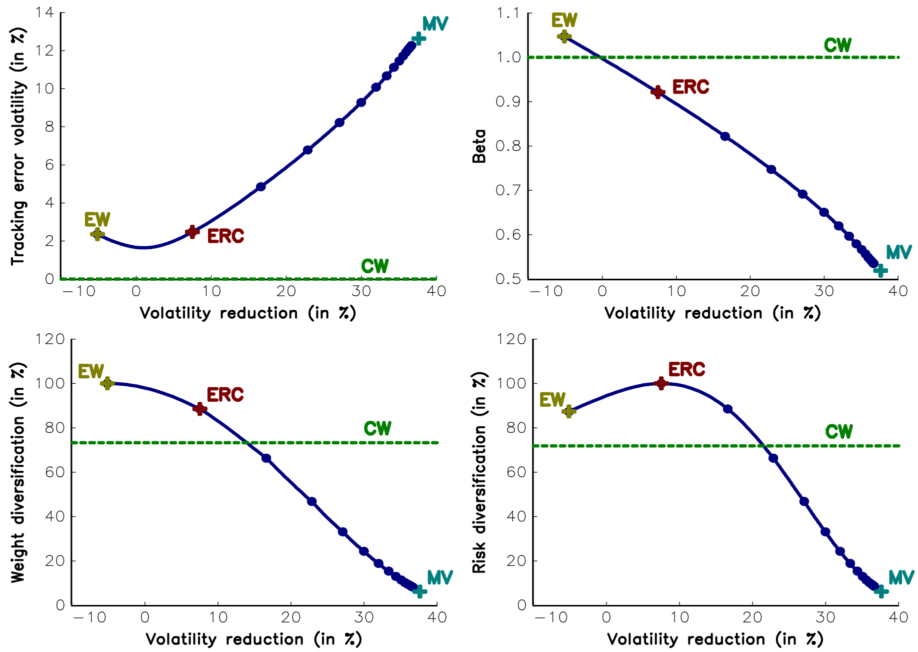

Figure 13: Trade-off relationships with LVD-RB* portfolios (Eurostoxx 50, Feb. 2013)