Assignment of Heterogeneous Agents in Trees under the Permission Value

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Assignment of Heterogeneous Agents in Trees under the Permission Value

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Abstract

We investigate assignment of heterogeneous agents in trees where the allocation rule is given by the permission value. We focus on efficient hierarchies, namely those, for which the payoff of the top agent is maximized. For additive games, such hierarchies are always cogent, namely, more productive agents occupy higher positions. The result can be extended to non-additive games with appropriate restrictions on the value function. Finally, we consider auctions where agents bid for positions in a two agent vertical hierarchy. Under simultaneous bidding, an equilibrium does not exist while sequential bidding always results in a non-cogent hierarchy.

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Keywords: permission value, hierarchies

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1 Introduction

Myerson's (1977) seminal paper introduced graph restricted cooperative games. Hierarchical permission structures were introduced by Gilles, Owen and van den Brink (1992) which in addition to the graph also incorporated a dominance structure where agents were in a superior-subordinate relationship with each other. The permission value came in two flavours. In the case of the conjunctive permission value, each agent required the permission of all his superiors in order to be productive. In the case of disjunctive permission value, each agent required the permission of at least one superior in order to be productive. Under special hierarchies called trees, the two values coincided and could be simply referred to as the permission value.

We restrict ourselves to trees in this paper so we are only dealing with the permission value. Consider a set of agents who are heterogeneous in the sense that as singletons, they all produce different values. Now, how they are assigned in the hierarchy matters to the top agent (or agent who is superior to every other agent), because his payoff would change with each assignment. We find the interesting result that for additive games, the assignment which maximizes his payoff (which we call an efficient assignment) has more productive agents occupying higher positions in the hierarchy. We call the latter a cogent assignment. The result can be extended to non-additive games albeit with severe restrictions on the value function.

Finally, we study a simple two agent vertical hierarchy where positions are auctioned. We derive the rather interesting result that in a subgame perfect Nash equilibrium, a non-cogent assignment always results.

The rest of the paper proceeds as follows. Section 2 discusses the notation and terminology. In Section 3, we derive the formula of the payoff of the top agent for additive games. Section 4 shows the equivalence of efficient and cogent assignments for additive games. Section 5 extends the result to non-additive games. Section 6 discusses auctioning of positions. Section 7 concludes.

2 Preliminaries

The hierarchy of a firm is described by a finite set of positions N and a set of directed relations. The relational structure is determined by a map $S : N \to 2^N$ which assigns to each position $p \in N$ a set of *successor* positions $S(p)$. Hence the positions $q \in S(p)$ are the successor positions of p. The positions $p \in S^{-1}(q)$ are called the *predecessor* positions of q. The collection of relational structures on the set of positions N is denoted by S^N.

For every relational structure $S \in S^N$, we introduce its transitive closure by the mapping $\widehat{S} : N \to 2^N$. Hence for every position $p \in N$, we define $q \in \widehat{S}(p)$ if there exists a sequence $(h_1, h_2, \ldots, h_z)$ with $h_1 = p$ and $h_z = q$ and for every $1 \leqslant k \leqslant z-1$, $h_{k+1} \in S(h_k)$. We call the positions in the collection $\widehat{S}(p)$ the *subordinate* positions of p in the relational structure S. The set $\widehat{S}^{-1}(p) = \{q|\ p \in \widehat{S}(q)\}$ is called the set of *superior* positions of p. Next, $B_S = \{p \in N | \ S^{-1}(p) = \emptyset\}$ is the set of *top positions*. $W_S = \{p \in N | \ S(p) = \emptyset\}$ is the set of *front positions* in the structure S. The other positions belonging to the set $N \backslash (B_S \cup W_S)$ are the intermediate positions.

Denote the cardinality of any arbitrary set A by $|A|$. A hierarchy is a *forest* if $|S^{-1}(p)| = 1$ for all $p \in N \backslash B_S$. It is a *tree* if additionally $|B_S| = 1$. For a tree, the positions at the lth level are members of a set N_l for $l = 1, \ldots, m$ such that each position has the same relational distance to the top level, namely, $\left|\widehat{S}^{-1}(p)\right| = l$ for all $p \in N_l$. The set of levels are denoted by $L = \{0, 1, 2, \ldots, m\}$. Note that $N_0 = B_S$ and $N_m \subset W_S$. The level of a certain position p is given by $L(p)$.

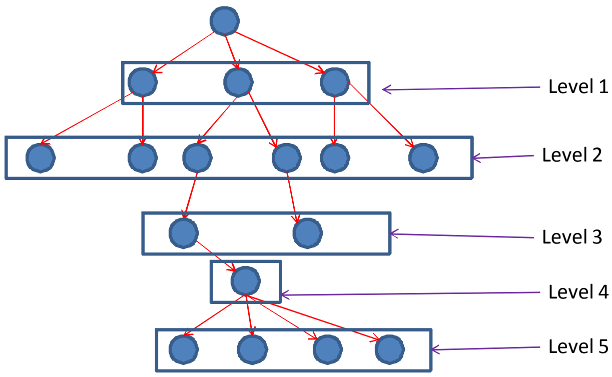


Fig 1: Levels in a tree

A special kind of tree is an (m, s) regular hierarchy, or simply an (m, s)-*hierarchy*, if there exists $m, s \in \mathbb{R}$, such that $|S(i)| = s$ for all $i \in N \backslash B_S$ and $\left|\widehat{S}^{-1}(i)\right| = m$ for all $i \in W_S$. m is called the depth of the hierarchy and s is called the span of control. If $s = 1$, we refer to the (m, s)-*hierarchy* as a *vertical hierarchy*.

We can extend these definitions to sets of positions rather than single positions. Hence, denote for all $E \subset N$,

$$\widehat{S}(E) = \bigcup_{p \in E} \widehat{S}(p).$$

Similarly,

$$\widehat{S}^{-1}(E) = \bigcup_{p \in E} \widehat{S}^{-1}(p).$$

To denote a certain position, we will sometimes use a double-index notation. The members of the set N_l are given by $\{p_{l,1}, \ldots, p_{l,|N_l|}\}$ for $l = 1, \ldots, m$ where $p_{l,1}$ indicates the position at level l to the extreme left, $p_{l,2}$ indicates the position next to $p_{l,1}$, and so forth with $p_{l,|N_l|}$ being the position at level l to the extreme right.

The number of positions in a tree is given by

$$|N| = \sum_{l=0}^{m} |N_l|$$

We denote by $\widehat{S}_l(p)$ the set of subordinate positions of p at level l, namely, $\widehat{S}_l(p) = \widehat{S}(p) \cap N_l$. For $p \in N_l$,

$$\left|\widehat{S}(p)\right| = \sum_{j=l+1}^{m} \left|\widehat{S}_j(p)\right|$$

A hierarchy is activated if some or all the positions are occupied by agents. Let M be the set of occupied positions or interchangeably the set of agents occupying positions in the hierarchy. We can define a TU cooperative game (N, v) where $v : 2^M \to \mathbb{R}$ where denotes the characteristic function. $v(E)$ is the maximum value achieved by a coalition $E \subset M$. For the rest of this paper, assume that all positions are filled, i.e., $M = N$.

For time being, we do not distinguish between agents themselves and the positions they occupy. An agent is simply identified by its position. The distinction becomes relevant only in Section 4.

Given a cooperative game, one can always define a Shapley value as a fair bargaining solution. But one has to take into account the hierarchical structure. The preferred way to incorporate the hierarchical structure in the literature is the method introduced by Gilles, Owen and van den Brink (1992) where it is assumed that each agent needs the permission of all of his superiors in order to be productive.[1] Based on this, one can define a restrictive game,

$$R_{S(v)}(E) = v(\sigma(E))$$

where $\sigma(E) = E \backslash \widehat{S}(N \backslash E)$ called the *sovereign part* of E. In other words, the sovereign part excludes all players whose superiors lie outside the coalition E. We also derive $\zeta(E) = E \cup \widehat{S}^{-1}(E)$ as the *authorizing set* of E in S. Gilles and van den Brink (1996) have shown that

$$Sh_p(R_{S(v)}) = \sum_{E \in \Gamma_p} \frac{\Delta_v(E)}{|\zeta(E)|} \tag{1}$$

where $Sh = (Sh_p)_{p \in N}$ denotes the vector of Shapley values, $\Delta_v(E)$ is the Harsanyi

[1] Strictly speaking there are two distinct approaches.The conjunctive approach introduced by Gilles, Owen and van den Brink (1992) assumes that each agent needs the permission of all his superiors in order to be productive. The disjunctive approach introduced by the same authors assumes that each agent needs the permission of at least one superior in order to be productive. For a tree, the two approaches are identical for obvious reasons.

dividend of E defined by

$$\Delta_v(E) = \sum_{F\subset E}(-1)^{|E|-|F|}v(F) \text{ and}$$
$$\Gamma_p = \{E \subset N | E \cap \left[\widehat{S}(p) \cup \{p\}\right] \neq \emptyset\}.$$

$\varphi_p(v, S) = Sh_p(R_{S(v)})$ is called the *permission value* of the agent in the position p.

By applying the Möbius transformation, the following property is true of the Harsanyi dividend that we shall use below.

$$v(E) = \sum_{F\subseteq E}\Delta v(F) \quad (2)$$
$$\Rightarrow \Delta v(E) = v(E) - \sum_{F\subsetneq E}\Delta v(F). \quad (3)$$

3 Additive Games

Previous studies of the permission value in hierarchies have considered homogeneous agents or hierarchies where only the lowest level of the hierarchy is productive. For instance, see Gilles, Owen and van den Brink (1992). In those models there is no assignment problem. However, if we consider heterogeneous agents (that is, agents that differ in productivity), then we have an assignment problem because the placement of the agent in a particular position affects the payoff of other agents in the hierarchy. We consider the question of assignment only from the perspective of the agent at the top position who can both decide which agent is placed in which position and he chooses these placements to maximize his payoff. The question then is what kind of hierarchy would emerge.

We begin by identifying an agent called the top agent say $p_{0,1}$, who always occupies the top position and is unproductive but has a free hand in assigning agents to positions. Let us consider an additive game where each agent $p \in N\backslash\{p_{0,1}\}$ has a productivity μ_p given by some real number. For sake of convenience, assume all these productivities are distinct, namely, $p \neq q \Rightarrow \mu_p \neq \mu_q$. Hence, a natural order exists with regard to these productivities, namely, the agents can be ranked based on their productivity.

Hence,

$$v(E) = \sum_{p\in E}\mu_p$$

with $v(\varnothing) = 0$. Since both the characteristic function v as well as the relational structure S is fixed, we shall not refer to them but denote the permission value simply as φ_p.

Proposition 1 *Consider a tree and assume all positions are filled. Then the payoff of an agent at position p at level l is given by:*

$$\varphi_p = \frac{\mu_p}{l+1} + \sum_{k=l+1}^{m} \sum_{q \in \widehat{S}_k(p)} \left(\frac{\mu_q}{k+1} \right)$$

Proof. First we claim that for an arbitrary additive game $v(E) = \sum_{p \in E} \mu_p$,

$$\Delta_v(E) = \begin{cases} \mu_p \text{ if } E = \{p\}; \\ 0 \text{ otherwise.} \end{cases}$$

We will prove this claim by induction. We begin with singleton coalitions. Obviously,

$$\Delta_v(\{p\}) = v(\{p\}) = \mu_p.$$

Now consider two agent coalitions. We get using (3),

$$\begin{aligned} \Delta_v(\{p,q\}) &= v(\{p,q\}) - v(\{p\}) - v(\{q\}) \\ &= \mu_p + \mu_q - \mu_p - \mu_q \\ &= 0. \end{aligned}$$

Suppose the result is true for all coalitions of size r. Consider a coalition of size $(r+1)$, say E. Then applying (3), we get

$$\begin{aligned} \Delta_v(E) &= v(E) - \sum_{F \subsetneq E} \Delta v(F) \\ &= v(E) - \sum_{p \in E} \Delta v(\{p\}) \\ &= \sum_{p \in E} \mu_p - \sum_{p \in E} \mu_p \\ &= 0. \end{aligned}$$

This completes the proof.

Hence, from (1), we get that

$$\begin{aligned} \varphi_p &= \sum_{\{q\} \in \Gamma_p} \frac{\Delta_v(\{q\})}{|\zeta(\{q\})|} \\ &= \sum_{\{q\} \in \Gamma_p} \frac{\mu_q}{|\zeta(\{q\})|}. \end{aligned}$$

Now, the set of singletons belonging to Γ_p includes the player p and all her subordinates $\widehat{S}(p)$. Any agent q at a level k has k superiors, namely,

$$\widehat{S}^{-1}(q) = k.$$

Therefore, $|\zeta(\{q\})| = k + 1$. Define $L(q)$ to be the level at which q exists. Hence,

$$\varphi_p = \sum_{\{q\}\in\Gamma_p} \frac{\mu_q}{L(q)+1}.$$

Given that the set of subordinates p has at a certain level k is given by $\widehat{S}_k(p)$ where $l < k \leqslant m$, we get

$$\varphi_p = \frac{\mu_p}{l+1} + \sum_{k=l+1}^{m} \sum_{q\in\widehat{S}_k(p)} \left(\frac{\mu_q}{k+1}\right).$$

■

We remark that the above proposition has been proved by Gilles, Owen and van den Brink (1992) (p. 288). We reproduce the proof in a simpler way for the benefit of the reader.

4 Cogent and Optimal Assignments

It is now necessary to distinguish an agent from the position it occupies. Agents are indexed by i, j and positions by p, q. The set of agents are denoted by N and the set of positions by N' with $|N| = |N'|$. Productivity μ_i is associated with agent i rather than a position p. Positions at a certain level l will now be denoted by N'_l.

An *assignment* is an allocation of a certain set of agents $E \subset N$ to a certain set of positions $E' \subset N'$ with one agent occupying one position, namely, $|E| = |E'|$. We denote it by $\mathcal{A}_{E',E}$. If $E = N$ and $E' = N'$, then the assignment is *complete*. Assignments will be generally denoted by calligraphic capital Latin letters. We will also follow the convention of referring to sets of agents by capital Latin letters and adding primes to denote sets of positions. Since the sets of positions as well as players are obvious in a complete assignment, we will not refer to them but simply denote the assignment by $\mathcal{A}$.

Let the agent occupying the position $p \in E'$ under an assignment $\mathcal{A}_{E',E}$ is given by $i_{\mathcal{A}_{E',E}}(p)$. For a complete assignment $\mathcal{A}$, we refer to it simply as $i_{\mathcal{A}}(p)$.

Corresponding to any assignment $\mathcal{A}_{E',E}$, for any strict subset $F' \subsetneq E'$, let $F = \{i_{\mathcal{A}_{E',E}}(p) \,|p \in F'\}$. Then the assignment that assigns agent $i_{\mathcal{A}_{E',E}}(p)$ to all positions $p \in F'$ is called a subassignment of $\mathcal{A}_{E',E}$ to F' and is denoted by $\mathcal{A}^{F'}_{E',E}$. If the initial assignment is complete, we refer to it simply as $\mathcal{A}^F$.

The top agent is denoted by T. Payoffs of all agents including the top agent are a function of the complete assignment. We next define cogent assignments and optimal assignments.

Definition 1 *A complete assignment $\mathcal{A}$ is* cogent *if more efficient agents occupy higher positions. Namely for any two levels l_1 and l_2, and two occupied positions $p \in N'_{l_1}$ and $q \in N'_{l_2}$, $l_1 > l_2 \Rightarrow \mu_{i_{\mathcal{A}(p)}} > \mu_{i_{\mathcal{A}(q)}}$.*

Definition 2 *A complete assignment* $\mathcal{A}$ *is* optimal *if no interchange of agents among a set of positions can increase the payoff of the top agent. Namely, for a set of positions* $E' \subset N'$ *and* $E = \{i_{\mathcal{A}}(p) \,|p \in E'\}$*, consider an assignment* $\mathcal{B}_{E',E} \neq \mathcal{A}^{E'}$*. Then,*

$$\varphi_T(\mathcal{A}) \geqslant \varphi_T(\mathcal{B})$$

where $\mathcal{B}$ *is the complete assignment that assigns agent* $i_{\mathcal{A}}(p)$ *to all positions* $p \in N' \backslash E'$ *and agent* $i_{\mathcal{B}_{E',E}}(p)$ *to all positions* $p \in E'$*.*

Next, we come to the following result.

Proposition 2 *A complete assignment is optimal if and only if it is cogent.*

Proof. First, we will prove that an optimal assignment is necessarily cogent. Towards a contradiction, consider a complete assignment $\mathcal{A}$ that is optimal but not cogent. Then, there must be at least two positions p and q such that $L(p) > L(q)$ but $\mu_{i_{\mathcal{A}}(p)} > \mu_{i_{\mathcal{A}}(q)}$. Consider an assignment that swaps these two agents in these positions such that agent $i_{\mathcal{A}}(p)$ is assigned to position q and agent $i_{\mathcal{A}}(q)$ is assigned to position p. Then, the payoff of the top agent changes by

$$\begin{aligned} G &= \frac{\mu_{i_{\mathcal{A}}(p)}}{L(q)+1} + \frac{\mu_{i_{\mathcal{A}}(q)}}{L(p)+1} - \frac{\mu_{i_{\mathcal{A}}(p)}}{L(p)+1} - \frac{\mu_{i_{\mathcal{A}}(q)}}{L(q)+1} \\ &= \left(\mu_{i_{\mathcal{A}}(p)} - \mu_{i_{\mathcal{A}}(q)}\right)\left[\frac{1}{L(q)+1} - \frac{1}{L(p)+1}\right] \\ &= \frac{\left(\mu_{i_{\mathcal{A}}(p)} - \mu_{i_{\mathcal{A}}(q)}\right)(L(p) - L(q))}{(L(p)+1)(L(q)+1)} > 0. \end{aligned} \tag{4}$$

This contradicts the fact that $\mathcal{A}$ is optimal.

Next, we will prove that any cogent assignment is optimal. Without loss of generality, let us rank the agents based on their productivity. Namely, $\mu_1 > \mu_2 > \mu_3 > \cdots > \mu_{|N|}$. Based on this we can identify an agent with her productivity. First note that, using the double index notation introduced above, in all cogent assignments, the set of agents $\{\mu_{\tau+1}, \mu_{\tau+2}, \ldots, \mu_{\tau+|N'_l|}\}$ occupy positions at level l given by $\{p_{l,1}, \ldots, p_{l,|N'_l|}\}$ where $\tau = \sum_{k=1}^{|N'_{l-1}|} |N'_k|$; in any particular order. In other words, the difference between two arbitrary cogent assignments only involves a re-assignment of agents among positions at the same level rather than between levels. Since such an re-assignment does not change the payoff of the top agent, all cogent assignments result in the same amount of payoff to the top agent. Given the fact that any optimal assignment is cogent, this completes the proof. ■

5 Extension to Non-Additive Games

Let us now consider the possibility of extending the above result to non-additive games. The first question is how do we define productivity in the case of non-additive games. If we simply denote it by the amount that an agent individually can produce, namely, $v(\{i\})$, then it can be shown that in the general case, the result of Proposition 2 does not hold.

Example 1 *Let* $v(\{1\}) = 5$*,* $v(\{2\}) = 3$*,* $v(\{3\}) = 1$*,* $v(\{1,2\}) = 8$*,* $v(\{1,3\}) = 6$*,* $v(\{2,3\}) = 104$*,* $v(\{1,2,3\}) = 109$ *and the hierarchy in question is a vertical hierarchy with depth* 4*. Then, the cogent assignment* $\mathcal{A}$ *is* $i_{\mathcal{A}}(p_{j,1}) = j$ *for* $j = 1,2,3$*. The payoff of the top agent in this hierarchy is* $\frac{345}{12}$*. However, consider the assignment* $\mathcal{B}$ *such that* $i_{\mathcal{B}}(p_{1,1}) = 2$*;* $i_{\mathcal{B}}(p_{2,1}) = 3$ *and* $i_{\mathcal{B}}(p_{3,1}) = 1$*. The payoff of the top agent increases to* $\frac{437}{12}$*.*

The main hindrance to the result holding successfully is that an agent's productivity may vary depending on what coalition he is part of. For instance, in Example 1, agent 2 by himself is less productive than agent 1 but he makes a far greater marginal contribution than agent 1 when in a coalition with agent 3.

Hence, we define a class of games where the notion of higher productivity remains consistent irrespective of the coalition the agent is part of. Namely, for any coalition $E \subset N \backslash \{i,j\}$, $v(\{i\}) > v(\{j\})$ implies $v(E \cup \{i\}) > v(E \cup \{j\})$. In fact, we introduce a stronger condition that implies the above.

Definition 3 *A TU cooperative game* (N,v) *belongs to the class* C^* *if* $v(\{i\}) > v(\{j\})$ *implies* $\Delta_v(E \cup \{i\}) \geqslant \Delta_v(E \cup \{j\})$ *for all* $E \subseteq N \backslash \{i,j\}$*.*

The condition is trivially satisfied for additive games. It directly implies our condition of consistent productivities.

Lemma 1 *For a TU cooperative game in* (N,v) *in* C^**, for any coalition* $E \subset N \backslash \{i,j\}$*,* $v(\{i\}) > v(\{j\})$ *implies* $v(E \cup \{i\}) > v(E \cup \{j\})$*.*

Proof. Using (2),

$$\begin{aligned} v(E \cup \{i\}) &= \sum_{F \subseteq E \cup \{i\}} \Delta v(F) \\ &= \Delta_v(\{i\}) + \sum_{F \subseteq E} \Delta_v(F \cup \{i\}) + \sum_{F \subseteq E} \Delta_v(F) \\ &= v(\{i\}) + \sum_{F \subseteq E} \Delta_v(F \cup \{i\}) + \sum_{F \subseteq E} \Delta_v(F). \end{aligned}$$

Also,

$$\begin{aligned} v(E \cup \{j\}) &= \sum_{F \subseteq E \cup \{j\}} \Delta v(F) \\ &= \Delta_v(\{j\}) + \sum_{F \subseteq E} \Delta_v(F \cup \{j\}) + \sum_{F \subseteq E} \Delta_v(F) \\ &= v(\{j\}) + \sum_{F \subseteq E} \Delta_v(F \cup \{j\}) + \sum_{F \subseteq E} \Delta_v(F). \end{aligned}$$

Given $v(\{i\}) > v(\{j\})$, and $\Delta_v(F \cup \{i\}) \geqslant \Delta_v(F \cup \{j\})$ for all $F \subseteq E$, the result follows. ■

Given that we distinguish between agents and positions, there are two sets of directed relations. One is a relation between agents and the other is a relation between positions. The former changes while the latter remains unchanged when one re-assigns agents among positions.

Let $S : N \to 2^N$ denote the relationship between agents and correspondingly $S' : N' \to 2^{N'}$ denote the relationship among positions. S' is given exogenously while S changes with every new assignment and is a function of the assignment. In the figure below, $S'(p_{0,1}) = p_{1,1}$ and $S'(p_{1,1}) = p_{2,1}$. Now there are two assignments $\mathcal{A}$ and $\mathcal{B}$. $S^{\mathcal{A}}(T) = \{1\}$ and $S^{\mathcal{A}}(1) = \{2\}$ while $S^{\mathcal{B}}(T) = \{2\}$ and $S^{\mathcal{B}}(2) = \{1\}$.

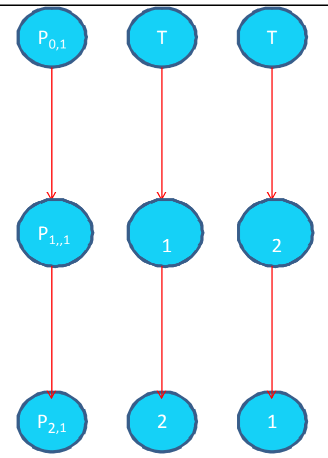


Figure 2: Assignments

A particularly neat way to avoid this rather cumbersome notation is to simply permute the agents. Let $\pi : N \to N$ be a permutation of players. Then, if S is the initial relational mapping in some assignment, this permutation automatically creates a new assignment with a corresponding relational mapping $\pi(S)$ where

$$\pi(S)(\pi(i)) = \{\pi(j) \in N | j \in S(i)\}$$

for each $i \in N$. The position of player $\pi(i) = j$ in the new assignment is same as the position of player i in the initial assignment. For instance, in going from $\mathcal{A}$ to $\mathcal{B}$ in the figure above, the relevant permutation is $\pi(1) = 2$ and $\pi(2) = 3$. Thus, if we fix an initial assignment, there is bijective relationship between the set of assignments and the set of permutations and any assignment can be represented by a permutation. In what follows, a hierarchy will be denoted by (N', S') and the relational structure in an assignment either by S (initial assignment) or by $\pi(S)$ (the new assignment created by a permutation). Other variables (such as the set ζ which is the set of agents and their superiors) will be denoted by making reference to the relational structure of the assignment rather than the assignment itself. Also, in a slight abuse of notation we shall extend the notation π from individual players to coalitions of players. Namely, for all $E \subseteq N$,

$$\pi(E) = \{\pi(i) \,|\, i \in E\}.$$

Since each assignment is associated with an unique relational structure among players (as opposed to positions), the terms assignment and relational structure will be used interchangeably.

Lemma 2 *Consider two assignments $\mathcal{A}$ and $\mathcal{B}$ with relational structures S and $\pi(S)$ respectively. Let $\widehat{N} = \{i \in N | \pi(i) = i\}$ and $\widehat{\widehat{N}} = \{i \in N | \pi(i) \neq i\}$.*
(i) If $E \subseteq \widehat{N}$, $|\zeta_S(E)| = \left|\zeta_{\pi(S)}(E)\right|$.
(ii)If $\widehat{\widehat{N}} \subseteq E$, $|\zeta_S(E)| = \left|\zeta_{\pi(S)}(E)\right|$.

Proof. We start with (i). Let $E' = \{p \in N' | i_{\mathcal{A}}(p) \in E\}$. Now, $|\zeta_S(E)| = |\zeta_{S'}(E')|$. Given that agents in E have not changed their positions, $\{p \in N' | i_{\mathcal{B}}(p) \in E\} = E'$. Hence, $\left|\zeta_{\pi(S)}(E)\right| = |\zeta_{S'}(E')|$. Hence, $|\zeta_S(E)| = \left|\zeta_{\pi(S)}(E)\right|$.

Next, we come to (ii). Again, let $E' = \{p \in N' | i_{\mathcal{A}}(p) \in E\}$. Now, $|\zeta_S(E)| = |\zeta_{S'}(E')|$. This time while some agents in E have changed their positions, the changes do not involve positions outside E'. Hence, again $\{p \in N' | i_{\mathcal{B}}(p) \in E\} = E'$. Thus, $\left|\zeta_{\pi(S)}(E)\right| = |\zeta_{S'}(E')|$. Hence, $|\zeta_S(E)| = \left|\zeta_{\pi(S)}(E)\right|$. ■

Then, we have the following proposition as a direct consequence of Lemma 2.

Proposition 3 *Consider two assignments $\mathcal{A}$ and $\mathcal{B}$ with relational structures S and $\pi(S)$ respectively such that $\pi(i) = j$ and $\pi(j) = i$ and $\pi(k) = k$ for each $k \in N$ such that $k \neq i, j$. Then,*

$$\varphi_T(\mathcal{A}) - \varphi_T(\mathcal{B}) = \sum_{\substack{E = K \cup \{i\} \\ K \subseteq N \setminus \{i,j\}}} [\Delta_v(E) - \Delta_v(\pi(E))] \left[\frac{1}{|\zeta_{\pi(S)}(E)|} - \frac{1}{|\zeta_S(E)|}\right].$$

Proof. From Lemma 2, for each $E \subseteq N$ such that $E \supseteq \{i, j\}$, we have $|\zeta_S(E)| = |\zeta_{\pi(S)}(E)|$. For each $E \subseteq N$ such that $E \cap \{i, j\} = \emptyset$, we obtain the same result: $|\zeta_S(E)| = |\zeta_{\pi(S)}(E)|$. Then, the payoff of the top agent as a result of this permutation changes by:

$$\begin{aligned}
\varphi_T(\mathcal{A}) - \varphi_T(\mathcal{B}) &= \sum_{E\subseteq N} \frac{\Delta_v(E)}{|\zeta_{\pi(S)}(E)|} - \sum_{E\subseteq N} \frac{\Delta_v(E)}{|\zeta_S(E)|} \\
&= \sum_{\substack{E\subseteq N \\ i\in E \\ j\notin E}} \frac{\Delta_v(E)}{|\zeta_{\pi(S)}(E)|} + \sum_{\substack{E\subseteq N \\ j\in E \\ i\notin E}} \frac{\Delta_v(E)}{|\zeta_{\pi(S)}(E)|} \\
&\quad - \sum_{\substack{E\subseteq N \\ i\in E \\ j\notin E}} \frac{\Delta_v(E)}{|\zeta_S(E)|} - \sum_{\substack{E\subseteq N \\ j\in E \\ i\notin E}} \frac{\Delta_v(E)}{|\zeta_S(E)|} \\
&= \sum_{\substack{E\subseteq N \\ i\in E \\ j\notin E}} \Delta_v(E) \left[\frac{1}{|\zeta_{\pi(S)}(E)|} - \frac{1}{|\zeta_S(E)|}\right] \\
&\quad + \sum_{\substack{E\subseteq N \\ j\in E \\ i\notin E}} \Delta_v(E) \left[\frac{1}{|\zeta_{\pi(S)}(E)|} - \frac{1}{|\zeta_S(E)|}\right]
\end{aligned}$$

$$\begin{aligned}
= &\sum_{\substack{E=K\cup\{i\} \\ K\subseteq N\setminus\{i,j\}}} \Delta_v(E) \left[\frac{1}{|\zeta_{\pi(S)}(E)|} - \frac{1}{|\zeta_S(E)|}\right] \\
&+ \sum_{\substack{E=K\cup\{j\} \\ K\subseteq N\setminus\{i,j\}}} \Delta_v(E) \left[\frac{1}{|\zeta_{\pi(S)}(E)|} - \frac{1}{|\zeta_S(E)|}\right] \qquad (5)
\end{aligned}$$

Now, for any coalition $E = K \cup \{i\}$, where $K \subseteq N\setminus\{i, j\}$, replacing i by j produces $\pi(E) = K\cup\{j\}$. Hence, there is a bijective relation between the set of coalitions given by $\Omega = \{E|\, E = K \cup \{i\} \text{ with } K \subseteq N\setminus\{i, j\}\}$ and $\Psi = \{E|\, E = K \cup \{j\} \text{ with } K \subseteq N\setminus\{i, j\}\}$ given by the function $\pi : \Omega \to \Psi$. This implies we re-write (5) as follows:

$$\begin{aligned}
&\varphi_T(\mathcal{A}) - \varphi_T(\mathcal{B}) \\
= &\sum_{\substack{E=K\cup\{i\} \\ K\subseteq N\setminus\{i,j\}}} \left[\Delta_v(E)\left(\frac{1}{|\zeta_{\pi(S)}(E)|} - \frac{1}{|\zeta_S(E)|}\right) + \Delta_v(\pi(E))\left(\frac{1}{|\zeta_{\pi(S)}(\pi(E))|} - \frac{1}{|\zeta_S(\pi(E))|}\right)\right]. \quad (6)
\end{aligned}$$

Now, the set of positions occupied by players in the coalition E under the relational structure S are occupied by the players in the coalition $\pi(E)$, under the relational structure $\pi(S)$ and are given by, say, $E' = \{p \in N' | i_{\mathcal{A}}(p) \in E\} = \{p \in N' | i_{\mathcal{B}}(p) \in \pi(E)\}$. Therefore, it follows that $|\zeta_{\pi(S)}(\pi(E))| = |\zeta_S(E)| = |\zeta_{S'}(E')|$. Hence, we can rewrite

(6) as

$$\begin{aligned}
&\varphi_T(\mathcal{A}) - \varphi_T(\mathcal{B}) \\
= &\sum_{\substack{E=K\cup\{i\} \\ K\subseteq N\backslash\{i,j\}}} \left[\Delta_v(E)\left(\frac{1}{|\zeta_{\pi(S)}(E)|} - \frac{1}{|\zeta_S(E)|}\right) + \Delta_v(\pi(E))\left(\frac{1}{|\zeta_S(E)|} - \frac{1}{|\zeta_S(\pi(E))|}\right)\right]. \quad (7)
\end{aligned}$$

Next, we shall show that $|\zeta_{\pi(S)}(E)| = |\zeta_S(\pi(E))|$. $E = K \cup \{i\}$ for some $K = N\backslash\{i,j\}$. $\pi(E) = K \cup \{j\}$. First, consider the set of positions occupied by E in the relational structure $\pi(S)$. It is given by $\{p \in N' | i_{\mathcal{B}}(p) \in K \cup \{i\}\} = M'$ (say). Next, consider the set of positions occupied by $\pi(E)$ in the relational structure S. It is given by $\{p \in N' | i_{\mathcal{A}}(p) \in K \cup \{j\}\} = L'$. Since i occupies the same position in assignment $\mathcal{B}$ as j occupies in assignment $\mathcal{A}$, and the positions of players in the set K are identical in both assignments, $L' = M'$, namely the set of positions occupied by E in the relational structure $\pi(S)$ is the same as the set of positions occupied by $\pi(E)$ in the relational structure S. Therefore, $|\zeta_{\pi(S)}(E)| = |\zeta_S(\pi(E))| = |\zeta_{S'}(L')|$. Hence, we can re-write (7) as

$$\begin{aligned}
&\varphi_T(\mathcal{A}) - \varphi_T(\mathcal{B}) \\
= &\sum_{\substack{E=K\cup\{i\} \\ K\subseteq N\backslash\{i,j\}}} \Delta_v(E)\left[\left(\frac{1}{|\zeta_{\pi(S)}(E)|} - \frac{1}{|\zeta_S(E)|}\right) + \Delta_v(\pi(E))\left(\frac{1}{|\zeta_S(E)|} - \frac{1}{|\zeta_{\pi(S)}(E)|}\right)\right] \\
= &\sum_{\substack{E=K\cup\{i\} \\ K\subseteq N\backslash\{i,j\}}} [\Delta_v(E) - \Delta_v(\pi(E))]\left[\frac{1}{|\zeta_{\pi(S)}(E)|} - \frac{1}{|\zeta_S(E)|}\right]. \qquad (8)
\end{aligned}$$

■

The reader may verify that (8) morphs into (4), when we restrict ourselves to additive games since non-singleton coalitions have a Harsayni dividend of zero.

Now, consider a coalition E in (8). Observe that if $v(\{i\}) > v(\{j\})$, then the $\Delta_v(E) - \Delta_v(\pi(E))$ is strictly positive for singleton coalitions and non-negative for non-singleton coalitions if the game belongs to the class C^*. However, even if the level occupied by player i in assignment $\mathcal{A}$ is greater than that occupied by player j, the expression $\frac{1}{|\zeta_{\pi(S)}(E)|} - \frac{1}{|\zeta_S(E)|}$ is not necessarily positive (or non-negative for that matter) precluding the straight forward generalization of Proposition 2 to non-additive games, as the following example demonstrates.

Example 2 *Consider the following assignment shown in the figure below.*

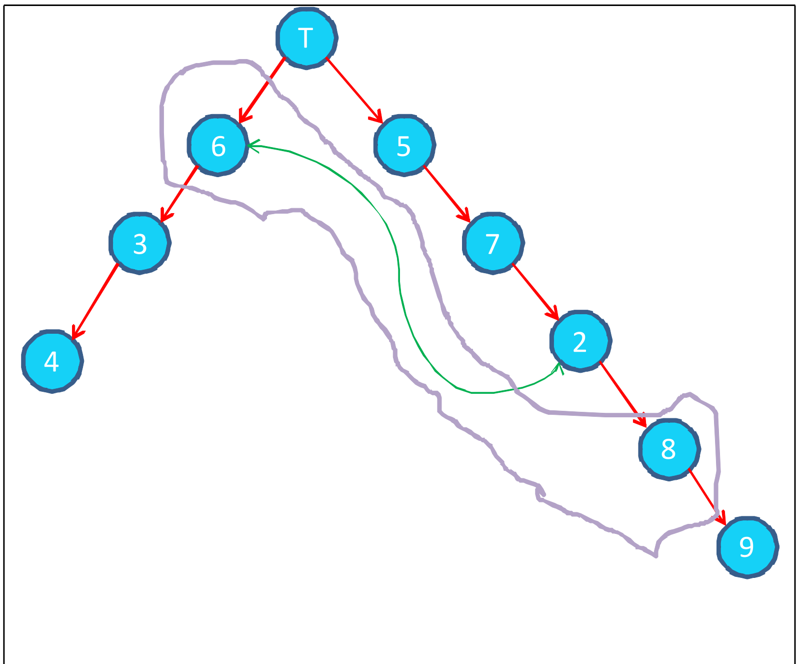


Figure 3: Switching positions

Let $\pi(2) = 6$*,* $\pi(6) = 2$ *and* $\pi(k) = k$ *for* $k \neq 2, 6$*. For* $E = \{2, 8\}$*,* $\pi(E) = \{6, 8\}$*. Hence,*

$$\frac{1}{|\zeta_{\pi(S)}(E)|} - \frac{1}{|\zeta_S(E)|}$$
$$= \frac{1}{6} - \frac{1}{5} = -\frac{1}{30}.$$

Only for special trees, namely, vertical hierarchies, is the aforesaid expression always non-negative. This is proved in the next lemma. But before that let us introduce the level of a player (as opposed to a position) in an assignment say $\mathcal{A}$. For all $j \in N$, define $L_S(j) = L(p)$ where $p \in N'$ is such that $i_{\mathcal{A}}(p) = j$ and S is the relational structure associated with $\mathcal{A}$.

Lemma 3 *Consider a vertical hierarchy and an assignment* $\mathcal{A}$ *(with associated relational structure* S*). Let* i *and* j *be such that* $L_S(i) > L_S(j)$*. Consider a permutation* π *such that* $\pi(i) = j$ *and* $\pi(j) = i$ *and* $\pi(k) = k$ *for each* $k \in N$ *such that* $k \neq i, j$*. Then, for all* $E \subseteq N$ *such that* $E = K \cup \{i\}$ *for some* $K \subseteq N \backslash \{i, j\}$*, we get*

$$\frac{1}{|\zeta_{\pi(S)}(E)|} - \frac{1}{|\zeta_S(E)|} \geqslant 0.$$

Furthermore, if $K = \emptyset$*, then,*

$$\frac{1}{|\zeta_{\pi(S)}(E)|} - \frac{1}{|\zeta_S(E)|} > 0.$$

Proof. Let us begin by considering non-empty K. Let the highest level among all members of K in assignment $\mathcal{A}$ be given by l_K. Namely,

$$l_K = \max\{L_S(i) \,|i \in K\}.$$

Since in a vertical hierarchy, no two players have the same level, $l_K \neq L_S(i) \neq L_S(j)$. We can discern three cases:

Case 1: $l_K < L_S(j) < L_S(i)$

In this case, $|\zeta_{\pi(S)}(E)| = L_S(j) + 1$ and $|\zeta_S(E)| = L_S(i) + 1$. Therefore,

$$\begin{aligned} & \frac{1}{|\zeta_{\pi(S)}(E)|} - \frac{1}{|\zeta_S(E)|} \\ = & \frac{1}{L_S(j)+1} - \frac{1}{L_S(i)+1} \\ = & \frac{(L_S(i) - L_S(j))}{(L_S(j)+1)(L_S(i)+1)} > 0. \end{aligned}$$

Case 2: $L_S(j) < l_K < L_S(i)$

In this case, $|\zeta_{\pi(S)}(E)| = l_K + 1$ and $|\zeta_S(E)| = L_S(i) + 1$. Therefore,

$$\begin{aligned} & \frac{1}{|\zeta_{\pi(S)}(E)|} - \frac{1}{|\zeta_S(E)|} \\ = & \frac{1}{l_K+1} - \frac{1}{L_S(i)+1} \\ = & \frac{(L_S(i) - l_K)}{(l_K+1)(L_S(i)+1)} > 0. \end{aligned}$$

Case 3: $L_S(j) < L_S(i) < l_K$

In this case, $|\zeta_{\pi(S)}(E)| = l_K + 1$ and $|\zeta_S(E)| = l_K + 1$. Therefore,

$$\begin{aligned} & \frac{1}{|\zeta_{\pi(S)}(E)|} - \frac{1}{|\zeta_S(E)|} \\ = & \frac{1}{l_K+1} - \frac{1}{l_K+1} = 0. \end{aligned}$$

If finally, $K = \emptyset$, $E = \{i\}$ and

$$\begin{aligned} & \frac{1}{|\zeta_{\pi(S)}(E)|} - \frac{1}{|\zeta_S(E)|} \\ = & \frac{1}{L_S(j)+1} - \frac{1}{L_S(i)+1} \\ = & \frac{(L_S(i) - L_S(j))}{(L_S(j)+1)(L_S(i)+1)} > 0. \end{aligned}$$

■

Finally, we can use Lemma 3 and Proposition 3 to arrive at the following result.

Theorem 1 *If the TU cooperative game belongs to the class C^* and the hierarchy is vertical, a complete assignment is optimal if and only if it is cogent.*

Proof. Towards a contradiction, consider a complete assignment $\mathcal{A}$ (with relational structure S) that is optimal but not cogent. Then, there must be at least two agents i and j such that $L_S(i) > L_S(j)$ but $v(i) > v(j)$. Consider an assignment that swaps these two agents in these positions, namely an assignment $\mathcal{B}$ with relational structure $\pi(S)$ where $\pi(i) = j$ and $\pi(j) = i$ and $\pi(k) = k$ for each $k \in N$ such that $k \neq i, j$. Then, the payoff of the top agent changes by

$$\varphi_T(\mathcal{A}) - \varphi_T(\mathcal{B}) = \sum_{\substack{E=K\cup\{i\} \\ K\subseteq N\setminus\{i,j\}}} [\Delta_v(E) - \Delta_v(\pi(E))] \left[\frac{1}{|\zeta_{\pi(S)}(E)|} - \frac{1}{|\zeta_S(E)|}\right]. \tag{9}$$

We shall prove that this change $\varphi_T(\mathcal{A}) - \varphi_T(\mathcal{B})$ is strictly positive. Consider $\Delta_v(E) - \Delta_v(\pi(E))$ where $E = K \cup \{i\}$ and $K \subseteq N\setminus\{i,j\}$. Now, $\Delta_v(E) - \Delta_v(\pi(E)) = \Delta_v(K \cup \{i\}) - \Delta_v(K \cup \{j\})$. Given $v(i) > v(j)$ and the cooperative game (N, v) belongs to the class C^*, it follows that $\Delta_v(K \cup \{i\}) - \Delta_v(K \cup \{j\}) \geqslant 0$. Since the hierarchy is vertical, by Lemma 3, we get

$$\frac{1}{|\zeta_{\pi(S)}(E)|} - \frac{1}{|\zeta_S(E)|} \geqslant 0.$$

Since the product of two non-negative terms is non-negative, it follows that $\varphi_T(\mathcal{A}) - \varphi_T(\mathcal{B})$ is a sum of non-negative terms and hence is non-negative. To prove that it is positive, we need only prove that one of the terms in the sum of non-negative terms is positive, namely, there exists E such that $[\Delta_v(E) - \Delta_v(\pi(E))] \left[\frac{1}{|\zeta_{\pi(S)}(E)|} - \frac{1}{|\zeta_S(E)|}\right] > 0$. Take $E = \{i\}$. Then, using Lemma 3, we get

$$\begin{aligned} & [\Delta_v(E) - \Delta_v(\pi(E))] \left[\frac{1}{|\zeta_{\pi(S)}(E)|} - \frac{1}{|\zeta_S(E)|}\right] \\ = \; & (v(\{i\}) - v(\{j\})) \left[\frac{(L_S(i) - L_S(j))}{(L_S(j)+1)(L_S(i)+1)}\right] > 0. \end{aligned}$$

Hence, $\varphi_T(\mathcal{A}) - \varphi_T(\mathcal{B}) > 0$. Hence, $\mathcal{A}$ cannot be optimal. Hence, an optimal hierarchy is cogent. The converse is obvious since a cogent hierarchy is uniquely defined. ■

6 Bidding for Positions

6.1 The Two Agent Model

In this section, we shall analyse the situation where two players bid for position in a vertical hierarchy and we arrive at some rather interesting results. Suppose we consider a vertical hierarchy with just two levels and two agents to fill these two levels. The hierarchy is given in the figure below.

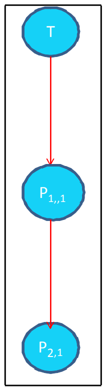
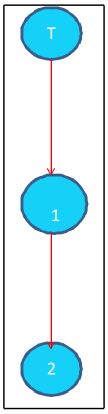


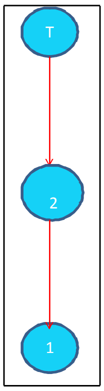
Figure 1

Let the productivity of the two agents be given by μ_1 and μ_2 where $\mu_1 > \mu_2$. We restrict ourselves for time being to the additive game. The permission value associated with position p is given by φ_p. Now there are two possible assignments which we refer to as $\mathcal{B}$ and $\mathcal{C}$. We depict them below. $\mathcal{B}$ is cogent while $\mathcal{C}$ is not.



Assignment $\mathcal{B}$

Figure 2



Assignment $\mathcal{C}$

Figure 3

In assignment $\mathcal{B}$, the permission values of the three agents are as follows:

$$\begin{aligned}\varphi_T(\mathcal{B}) &= \frac{\mu_1}{2} + \frac{\mu_2}{3};\\ \varphi_1(\mathcal{B}) &= \frac{\mu_1}{2} + \frac{\mu_2}{3};\\ \varphi_2(\mathcal{B}) &= \frac{\mu_2}{3}.\end{aligned}$$

In assignment $\mathcal{C}$, the permission values of the three agents are as follows:

$$\begin{aligned}\varphi_T(\mathcal{C}) &= \frac{\mu_1}{3} + \frac{\mu_2}{2};\\ \varphi_1(\mathcal{C}) &= \frac{\mu_1}{3};\\ \varphi_2(\mathcal{C}) &= \frac{\mu_1}{3} + \frac{\mu_2}{2}.\end{aligned}$$

While the top agent would like a cogent hierarchy, there may be an information asymmetry in the sense that he may not be aware of the productivities of the agents. If he knows which agent is more productive, he can assign them in the designated positions. Hence, for a genuine problem, we shall assume an extreme form of asymmetry namely, he is not aware of which agent is more productive in the first place. Let us start with the most simple mechanism, namely, workers submits bids and the person with the highest bid gets the higher position. If the bids are identical, then the tie is broken by tossing a coin. Hence, for an assignment $\mathcal{A}$, if b_i is the bid for agent i, then payoff of agent i is

$$\pi_i = \varphi_i(\mathcal{A}) - b_i.$$

Of course, the assignment that will result is a function of the bids. Therefore,

$$\pi_i(b_1, b_2) = \varphi_i(\mathcal{A}(b_1, b_2)) - b_i$$

if $b_1 \neq b_2$; where

$$\mathcal{A}(b_1, b_2) = \begin{cases} \mathcal{B} \text{ if } b_1 > b_2; \\ \mathcal{C} \text{ if } b_1 < b_2; \end{cases}.$$

On the other hand, if $b_1 = b_2$, then,

$$\pi_i(b_1, b_2) = \frac{1}{2}\varphi_i(\mathcal{B}) + \frac{1}{2}\varphi_i(\mathcal{C}) - b_i.$$

We shall assume that bids take place in increments of ε starting from zero. Furthermore, all the permission values constitute valid bids. Namely, for any permission value φ_i, there exists a positive integer K such that $\varphi_i = K \cdot \varepsilon$. ε is a sufficiently small positive real number.

6.2 Best Response Correspondences

We begin by showing that matching a bid is a "never best reponse" strategy.

Lemma 4 *Let the best response correspondence of player i be given by $R_i(b_{-i})$ given that the other player bids b_{-i}. Then,*

$$b_{-i} \notin R_i(b_{-i}).$$

Proof. Suppose player i bids b_{-i}. Then, his payoff is given by

$$\frac{1}{2}\varphi_i(\mathcal{B}) + \frac{1}{2}\varphi_i(\mathcal{C}) - b_{-i}. \tag{10}$$

If on the other hand, he bids $b_{-i} + \varepsilon$, his payoff is

$$\varphi_i(\mathcal{B}) - b_{-i} - \varepsilon \tag{11}$$

if $i = 1$ and

$$\varphi_i(\mathcal{C}) - b_{-i} - \varepsilon \tag{12}$$

if $i = 2$.

Start with $i = 1$. $\varphi_1(\mathcal{B}) - \varphi_1(\mathcal{C}) = \frac{\mu_1}{6} + \frac{\mu_2}{3} > 0$. Now, (11) is greater than (10) if

$$\frac{1}{2}[\varphi_1(\mathcal{B}) - \varphi_1(\mathcal{C})] - \varepsilon > 0$$

or,

$$\frac{\mu_1}{12} + \frac{\mu_2}{6} > \varepsilon.$$

Assuming ε is sufficiently small, this will be the case. The proof for $i = 2$ is similar. ■

The best response bids will fall into two categories. Either a player will bid zero or a player will bid ε higher than his rival. The next lemma proves this.

Lemma 5 *For any arbitrary b_{-i}, $R_i(b_{-i}) \subset \{0, b_{-i} + \varepsilon\}$.*

Proof. We prove the lemma for $i = 1$. The proof for $i = 2$ is similar. Given that the player is never bid b_{-i} by Lemma 4, there are two possibilities. Either $b_i > b_{-i}$ or $b_i < b_{-i}$. Consider the first case. In this case,

$$\pi_i = \varphi_i(\mathcal{B}) - b_i.$$

The payoff is strictly decreasing in b_i and hence, the maximum payoff is obtained by minimizing b_i subject to the constraint that $b_i > b_{-i}$. This is precisely $b_{-i} + \varepsilon$.

Next, consider the case $b_i < b_{-i}$. In this case,

$$\pi_i = \varphi_i(\mathcal{C}) - b_i.$$

Again the payoff is decreasing in b_i and so is maximized by choosing the minimum feasible b_i which is zero. ■

Next, we focus on which of the two bidding strategies will be chosen given a certain bid by the other player. This is shown by the next lemma.

Lemma 6 *There exists a certain critical threshold B_i for player i such that:*
(i) If $b_{-i} < B_i$, then $R_i(b_{-i}) = \{b_{-i} + \varepsilon\}$.
(ii) If $b_{-i} > B_i$, then $R_i(b_{-i}) = \{0\}$.
(iii) If $b_{-i} = B_i$, then $R_i(b_{-i}) = \{b_{-i} + \varepsilon, 0\}$.

Proof. We prove the lemma for $i = 1$. The proof for $i = 2$ is similar. From Lemma 5, player i will either bid $b_{-i} + \varepsilon$ or, zero. The payoffs are respectively $\varphi_i(\mathcal{B}) - b_{-i} - \varepsilon$ and $\varphi_i(\mathcal{C})$. Hence,

$$\begin{aligned} \varphi_i(\mathcal{B}) - b_{-i} - \varepsilon &> \varphi_i(\mathcal{C}) \\ &\Leftrightarrow b_{-i} < \varphi_i(\mathcal{B}) - \varphi_i(\mathcal{C}) - \varepsilon. \end{aligned}$$

Similarly,

$$\begin{aligned} \varphi_i(\mathcal{B}) - b_{-i} - \varepsilon &< \varphi_i(\mathcal{C}) \\ &\Leftrightarrow b_{-i} > \varphi_i(\mathcal{B}) - \varphi_i(\mathcal{C}) - \varepsilon. \end{aligned}$$

Finally,

$$\begin{aligned} \varphi_i(\mathcal{B}) - b_{-i} - \varepsilon &= \varphi_i(\mathcal{C}) \\ &\Longleftrightarrow b_{-i} = \varphi_i(\mathcal{B}) - \varphi_i(\mathcal{C}) - \varepsilon. \end{aligned}$$

Hence, assuming $B_i = \varphi_i(\mathcal{B}) - \varphi_i(\mathcal{C}) - \varepsilon$, the lemma is proven. ■

Now,

$$\begin{aligned} B_1 &= \varphi_1(\mathcal{B}) - \varphi_1(\mathcal{C}) - \varepsilon \\ &= \frac{\mu_1}{6} + \frac{\mu_2}{3} - \varepsilon \end{aligned}$$

and

$$\begin{aligned} B_2 &= \varphi_2(\mathcal{C}) - \varphi_2(\mathcal{B}) - \varepsilon \\ &= \frac{\mu_1}{3} + \frac{\mu_2}{6} - \varepsilon. \end{aligned}$$

6.3 The Simultaneous Bidding Game

Consider the bidding game where both players bid simultaneously. We begin by showing that this game does not admit a Nash equilibrium.

Lemma 7 *The simultaneous bidding game does not admit a Nash equilibrium.*

Proof. Towards a contradiction, suppose a Nash equilibrium (b_1^*, b_2^*) exists. Then,

$$\begin{aligned} b_2^* &\in R_2(b_1^*) \text{ and} \\ b_1^* &\in R_1(b_2^*). \end{aligned}$$

We consider three separate cases.

Case 1: Suppose $b_1^* = 0$. Then, $b_2^* \in R_2(b_1^*) = R_2(0) = \{\varepsilon\}$. But $R_1(b_2^*) = R_1(\varepsilon) = \{2 \cdot \varepsilon\}$ and $0 \notin R_1(\varepsilon)$ which is a contradiction.

Case 2: Next, suppose $b_2^* = 0$. We can arrive at a contradiction by a method similar to Case 1.

Case 3: Suppose $b_1^* \neq 0$ and $b_2^* \neq 0$. By Lemma 5, $b_1^* = b_2^* + \varepsilon$ and $b_2^* = b_1^* + \varepsilon$ simultaneously which is an impossibility. ■

The diagram below given further insights into the non-existence of a Nash equilibrium. We plot the best response correspondences and find that there is no point of intersection between them.

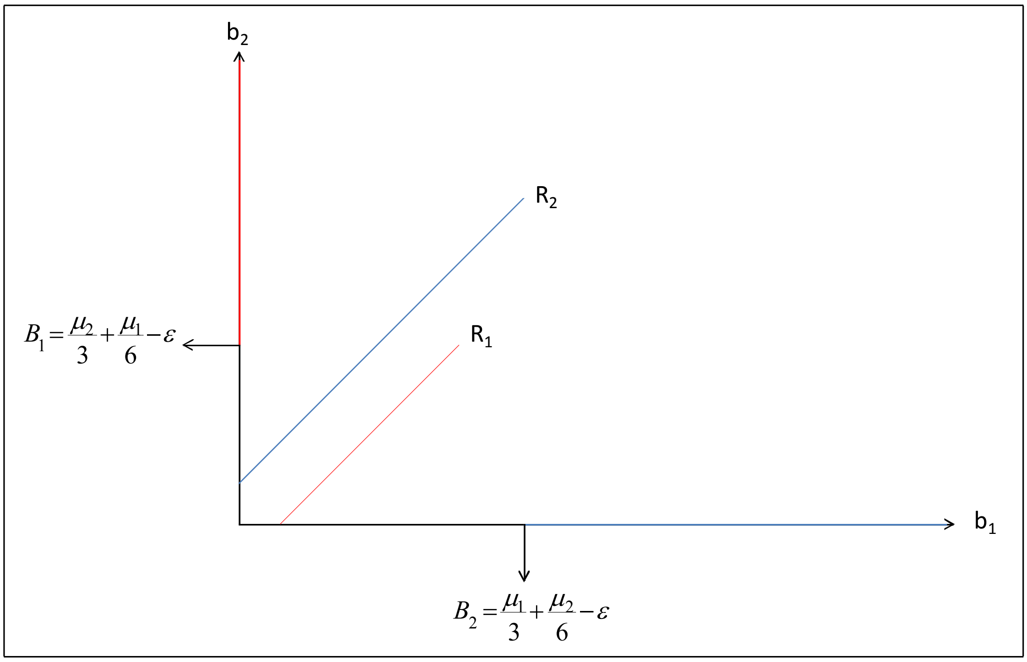


Figure 4

6.4 The Sequential Bidding Game

Next, we consider sequential bidding. Suppose 1 bids first followed by 2. First, 1's profit function is given by

$$\pi_1(b_1, b_2) = \begin{cases} \varphi_1(\mathcal{B}) - b_1 & \text{if } b_1 > b_2; \\ \frac{1}{2}\varphi_1(\mathcal{B}) + \frac{1}{2}\varphi_1(\mathcal{C}) - b_1 & \text{if } b_1 = b_2; \\ \varphi_1(\mathcal{C}) - b_1 & \text{if } b_1 < b_2. \end{cases}$$

Now, given that 1 is the first mover, he anticipates 2's moves from her reaction function, and hence we can replace b_2 by elements of $R_2(b_1)$ where $R_2(b_1)$ derived earlier was given by:

$$R_2(b_1) = \begin{cases} \{b_1 + \varepsilon\} \text{ if } b_1 < \frac{\mu_1}{3} + \frac{\mu_2}{6} - \varepsilon \\ \{b_1 + \varepsilon, 0\} \text{ if } b_1 = \frac{\mu_1}{3} + \frac{\mu_2}{6} - \varepsilon \\ \{0\} \text{ if } b_1 > \frac{\mu_1}{3} + \frac{\mu_2}{6} - \varepsilon \end{cases}.$$

Therefore, 1's profit function is given by

$$\widehat{\pi}_1(b_1) = \pi_1(b_1, R_2(b_1)) = \begin{cases} \varphi_1(\mathcal{B}) - b_1 & \text{if } b_1 > \frac{\mu_1}{3} + \frac{\mu_2}{6} - \varepsilon; \\ \pi(\varphi_1(\mathcal{B}) - b_1) + (1-\pi)(\varphi_1(\mathcal{C}) - b_1) & \text{if } b_1 = \frac{\mu_1}{3} + \frac{\mu_2}{6} - \varepsilon; \\ \varphi_1(\mathcal{C}) - b_1 & \text{if } b_1 < \frac{\mu_1}{3} + \frac{\mu_2}{6} - \varepsilon. \end{cases}$$

$$\Rightarrow \widehat{\pi}_1(b_1) = \begin{cases} \frac{\mu_1}{2} + \frac{\mu_2}{3} - b_1 & \text{if } b_1 > \frac{\mu_1}{3} + \frac{\mu_2}{6} - \varepsilon; \\ \pi\left(\frac{\mu_1}{2} + \frac{\mu_2}{3}\right) + (1-\pi)\left(\frac{\mu_1}{3}\right) - b_1 & \text{if } b_1 = \frac{\mu_1}{3} + \frac{\mu_2}{6} - \varepsilon; \\ \frac{\mu_1}{3} - b_1 & \text{if } b_1 < \frac{\mu_1}{3} + \frac{\mu_2}{6} - \varepsilon. \end{cases}$$

where π is some real number such that $0 \leqslant \pi \leqslant 1$ representing an arbitrary probability. Treating the three possibilities of (13) as three cases, in each case the payoffs are decreasing in b_1. So, we can restrict attention to only those possibilities where the value of b_1 is the minimum feasible number. This gives rise to the following payoff function.

$$\begin{aligned} \widehat{\pi}_1(b_1) &= \begin{cases} \frac{\mu_1}{2} + \frac{\mu_2}{3} - \frac{\mu_1}{3} - \frac{\mu_2}{6} & \text{if } b_1 = \frac{\mu_1}{3} + \frac{\mu_2}{6}; \\ \pi\left(\frac{\mu_1}{2} + \frac{\mu_2}{3}\right) + (1-\pi)\left(\frac{\mu_1}{3}\right) - \frac{\mu_1}{3} - \frac{\mu_2}{6} + \varepsilon & \text{if } b_1 = \frac{\mu_1}{3} + \frac{\mu_2}{6} - \varepsilon; \\ \frac{\mu_1}{3} & \text{if } b_1 = 0. \end{cases} \\ &= \begin{cases} \frac{\mu_1}{6} + \frac{\mu_2}{2} & \text{if } b_1 = \frac{\mu_1}{3} + \frac{\mu_2}{6}; \\ \pi\left(\frac{\mu_1}{6} + \frac{\mu_2}{2}\right) - (1-\pi)\left(\frac{\mu_2}{6}\right) + \varepsilon & \text{if } b_1 = \frac{\mu_1}{3} + \frac{\mu_2}{6} - \varepsilon; \\ \frac{\mu_1}{3} & \text{if } b_1 = 0. \end{cases} \end{aligned}$$

Two fact emerge from (13). First, 1 will be averse to bidding $b_1 = \frac{\mu_1}{3} + \frac{\mu_2}{6} - \varepsilon$ because bidding ε higher will almost always yield a higher (expected) payoff.[2] Second, given $\frac{\mu_1}{6} + \frac{\mu_2}{2} < \frac{\mu_1}{3}$, in equilibrium he will bid zero. 2 will bid ε. We summarize this result in the form of a lemma below.

Lemma 8 *In the bidding game where* 1 *is the first mover and* 2 *bids second, there is an unique SPNE in which* 1 *bids zero and* 2 *bids* ε. *The assignment that results is* $\mathcal{C}$. *The surplus extracted is* ε.

The next case we will handle is sequential bidding where 2 is the first mover. The reaction function of 1 derived earlier is given by:

$$R_1(b_2) = \begin{cases} \{b_2 + \varepsilon\} \text{ if } b_2 < \frac{\mu_2}{3} + \frac{\mu_1}{6} - \varepsilon \\ \{b_2 + \varepsilon, 0\} \text{ if } b_2 = \frac{\mu_2}{3} + \frac{\mu_1}{6} - \varepsilon \\ \{0\} \text{ if } b_2 > \frac{\mu_2}{3} + \frac{\mu_1}{6} - \varepsilon. \end{cases}.$$

Therefore, 2's profit function is given by (π is a real number such that $0 \leqslant \pi \leqslant 1$),

$$\begin{aligned} \widehat{\pi}_2(b_2) &= \pi_2(R_1(b_2), b_2) = \begin{cases} \varphi_2(\mathcal{C}) - b_2 & \text{if } b_2 > \frac{\mu_2}{3} + \frac{\mu_1}{6} - \varepsilon; \\ \pi(\varphi_2(\mathcal{C}) - b_2) + (1-\pi)(\varphi_2(\mathcal{B}) - b_2) & \text{if } b_2 = \frac{\mu_2}{3} + \frac{\mu_1}{6} - \varepsilon; \\ \varphi_2(\mathcal{B}) - b_2 & \text{if } b_2 < \frac{\mu_2}{3} + \frac{\mu_1}{6} - \varepsilon. \end{cases} \\ \Rightarrow \widehat{\pi}_2(b_2) &= \begin{cases} \frac{\mu_1}{3} + \frac{\mu_2}{2} - b_2 & \text{if } b_2 > \frac{\mu_2}{3} + \frac{\mu_1}{6} - \varepsilon; \\ \pi\left(\frac{\mu_1}{3} + \frac{\mu_2}{2} - b_2\right) + (1-\pi)\left(\frac{\mu_2}{3} - b_2\right) & \text{if } b_2 = \frac{\mu_2}{3} + \frac{\mu_1}{6} - \varepsilon; \\ \frac{\mu_2}{3} - b_2 & \text{if } b_2 < \frac{\mu_2}{3} + \frac{\mu_1}{6} - \varepsilon. \end{cases} \\ \Rightarrow \widehat{\pi}_2(b_2) &= \begin{cases} \frac{\mu_1}{3} + \frac{\mu_2}{2} - b_2 & \text{if } b_2 > \frac{\mu_2}{3} + \frac{\mu_1}{6} - \varepsilon; \\ \pi\left(\frac{\mu_1}{3} + \frac{\mu_2}{2}\right) + (1-\pi)\left(\frac{\mu_2}{3}\right) - b_2 & \text{if } b_2 = \frac{\mu_2}{3} + \frac{\mu_1}{6} - \varepsilon; \\ \frac{\mu_2}{3} - b_2 & \text{if } b_2 < \frac{\mu_2}{3} + \frac{\mu_1}{6} - \varepsilon. \end{cases} \end{aligned}$$

[2] There is an extreme case where player 2 actually chooses $\pi = 1$ (or sufficiently close to 1) in which case bidding $b_1 = \frac{\mu_1}{3} + \frac{\mu_2}{6} - \varepsilon$ actually gives a higher payoff. But that does not alter our overall result.

Again, since the payoffs are decreasing in b_2, we restrict ourselves to situations where b_2 is the minimum feasible number. Therefore,

$$\begin{aligned}\widehat{\pi}_2(b_2) &= \begin{cases} \frac{\mu_1}{2}+\frac{\mu_2}{6} & \text{if } b_2 = \frac{\mu_2}{3}+\frac{\mu_1}{6}; \\ \pi\left(\frac{\mu_1}{3}+\frac{\mu_2}{2}\right)+(1-\pi)\left(\frac{\mu_2}{3}\right)-\frac{\mu_2}{3}-\frac{\mu_1}{6}+\varepsilon & \text{if } b_2 = \frac{\mu_2}{3}+\frac{\mu_1}{6}-\varepsilon; \\ \frac{\mu_2}{3} & \text{if } b_2 = 0. \end{cases} \\ &= \begin{cases} \frac{\mu_1}{2}+\frac{\mu_2}{6} & \text{if } b_2 = \frac{\mu_2}{3}+\frac{\mu_1}{6}; \\ \pi\left(\frac{\mu_1}{2}+\frac{\mu_2}{6}\right)-(1-\pi)\left(\frac{\mu_1}{6}\right)+\varepsilon & \text{if } b_2 = \frac{\mu_2}{3}+\frac{\mu_1}{6}-\varepsilon; \\ \frac{\mu_2}{3} & \text{if } b_2 = 0. \end{cases}\end{aligned}$$

Since $\frac{\mu_1}{2}+\frac{\mu_2}{6} > \frac{\mu_2}{3}$, the optimal strategy for 2 is to bid $b_2 = \frac{\mu_2}{3}+\frac{\mu_1}{6}$.[3] Player 1 bids zero again. We summarize this in form of the following lemma.

Lemma 9 *In the bidding game where 2 is the first mover and 1 bids second, there is an unique SPNE in which 1 bids zero and 2 bids* $\frac{\mu_2}{3}+\frac{\mu_1}{6}$*. The assignment that results is* $\mathcal{C}$*. The surplus extracted is* $\frac{\mu_2}{3}+\frac{\mu_1}{6}$*.*

So, we get quite a perverse result in the sense that sequential bidding always results in a non-cogent hierarchy. However, there is one interesting fact. If the top agent is aware of the productivities, he can in effect still organize a sequential bidding in which 2 moves first. Even though the hierarchy that results is non-cogent, the loss (which amounts to $\frac{\mu_1}{6}-\frac{\mu_2}{6}$) is smaller than the surplus extraction as a result of the bidding.

7 Conclusion

We have therefore shown that where payoff is determined by the permission value in regular hierarchies under additive games, cogent and efficient hierarchies coincide and the result can be extended to non-additive games with appropriate restrictions on the value function. We also study auctioning these positions using a bidding mechanism in simple hierarchies and these always result in a non-cogent hierarchy.

Topics for further research include what auction mechanism can result in a cogent hierarchy and whether the result can be extended to more complicated hierarchies. Also, implications for the size of the firm may be studied. Williamson (1967) study a model in which control reduces with increasing hierarchical firm depth and this

[3] There is a possibility of winning ε more by bidding ε less but the risks are quite high and the additional return is too low to warrant this strategy.

determines the size of the firm. Ruys and van den Brink (1999) study a model similar to others where workers are also paid the permission value in regular hierarchies. However, only the workers in the front positions are productive and all other members of the hierarchy earn positional rents from those workers. The depth of the hierarchy is then determined by the reservation wage. Similar issues can be studied in our model.

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