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Almost Stochastic Dominance and Moments

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Abstract: This paper establishes some equivalent relationships for the first three orders of the almost stochastic dominance (ASD). Using these results, we first prove formally that the ASD definition modified by Tzeng et al. (2012) does not possess any hierarchy property. Thereafter, we conclude that when the first three orders of ASD are used in the prospects comparison, investors prefer the one with positive gain, smaller variance and positive skewness. This information, in turn, enables decision makers to determine the ASD relationship among prospects when they know the moments of the prospects.

Keywords: stochastic dominance; almost stochastic dominance; risk aversion, mean, variance, skewness.

1 Introduction

Stochastic dominance (SD) theory has been well established, see, for example, Hanoch and Levy (1969), Hadar and Russell (1969), and Rothschild and Stiglitz (1970). Leshno and Levy (2002) extend it to the theory of almost stochastic dominance (ASD) for *most* decision makers. Tzeng et al. (2012) show that the second-degree ASD (ASSD) introduced by Leshno and Levy (2002) does not possess the property of expected-utility maximization. They modify the ASSD definition to acquire this property. Nonetheless, Guo, et al. (2013) have constructed some examples to show that the ASD definition modified by Tzeng et al. (2012) does not possess any hierarchy property.

In this paper, we extend the work of ASD by first developing some equivalency properties for different orders of ASD. Using these results, we first prove formally that the ASD definition modified by Tzeng et al. (2012) does not possess any hierarchy property. Thereafter, we establish the relationships between different orders of ASDs and the moments of the prospects being compared. These findings lead us conclude that when the first three orders of ASD are used in the prospect comparison, decision makers prefer the one with positive gain, smaller variance, and positive skewness. This information, in turn, enables academics

and practitioners to determine the ASD relationship among prospects when they know the moments of the prospects. At last, we discuss the necessary and sufficient conditions for the ASD and the moments of the prospects.

2 Notations and Definitions

In order to develop some relationships for the ASD concepts proposed by Leshno and Levy (2002) and modified by Tzeng et al. (2012), we first state the definitions and notations being used in this paper. Suppose that random variables X and Y defined on the support $\Omega = [a, b]$ with means μ_X and μ_Y and standard deviations σ_X and σ_Y have the corresponding distribution functions F and G , respectively. The following notations will be used throughout this paper:

$$\begin{aligned} H^{(1)} &= H \quad \text{and} \quad H^{(n)}(x) = \int_a^x H^{(n-1)}(t)dt \quad \text{for} \quad H = F, G \quad \text{and} \quad n = 2, 3 ; \\ ||F^{(n)} - G^{(n)}|| &= \int_a^b |F^{(n)}(x) - G^{(n)}(x)|dx \quad \text{for} \quad n = 1, 2, 3 ; \\ S_n &\equiv S_n(F, G) = \{x \in [a, b] : G^{(n)}(x) < F^{(n)}(x)\} \quad \text{for} \quad n = 1, 2, 3 . \end{aligned} \quad (1)$$

An individual chooses between X and Y with distribution functions F and G , respectively, in accordance with a consistent set of preferences satisfying the von Neumann-Morgenstern (1944) consistency properties. Accordingly, X is preferred to Y if $E[u(X)] - E[u(Y)] \geq 0$ in which $E[u(X)] \equiv \int_a^b u(x)dF(x)$ and $E[u(Y)] \equiv \int_a^b u(x)dG(x)$. We first rewrite the definition of ASD introduced by Leshno and Levy (2002) and modified by Tzeng et al. (2012) as follows:

Definition 1 *Let F and G be the corresponding distribution functions of X and Y . For $0 < \epsilon < 1/2$,*

ϵ -AFSD: X is said to dominate Y by ϵ -AFSD, denoted by $X \succ_1^{\text{almost}(\epsilon)} Y$, if and only if

$$\int_{S_1} [F(x) - G(x)]dx \leq \epsilon ||F - G||;$$

ϵ -ASSD: X is said to dominate Y by ϵ -ASSD, denoted by $X \succ_2^{\text{almost}(\epsilon)} Y$, if and only if

$$\int_{S_2} [F^{(2)}(x) - G^{(2)}(x)]dx \leq \epsilon ||F^{(2)} - G^{(2)}|| \quad \text{and} \quad \mu_X \geq \mu_Y ;$$

ϵ -ATSD: X is said to dominate Y by ϵ -ATSD, denoted by $X \succ_3^{almost(\epsilon)} Y$, if and only if

$$\int_{S_3} [F^{(3)}(x) - G^{(3)}(x)] dx \leq \epsilon \|F^{(3)} - G^{(3)}\| \quad \text{and} \quad G^{(n)}(b) \geq F^{(n)}(b) \quad \text{for } n = 2, 3 .$$

In addition, we define the following utility functions:

Definition 2 For $n = 1, 2$, and 3 ,

$$U_n = \{u : (-1)^i u^{(i)} \leq 0, i = 1, \dots, n\},$$

$$U_n^*(\epsilon) = \{u \in U_n : (-1)^{n+1} u^{(n)}(x) \leq \inf\{(-1)^{n+1} u^{(n)}(x)\} [1/\epsilon - 1] \forall x\},$$

in which ϵ is in the range of $(0, 1/2)$.¹

3 The Theory

We first rewrite the main results in Tzeng et al. (2012) that ASD possesses the utility maximization property as stated in the following theorem.

Theorem 1 Let F and G be the corresponding distribution functions of X and Y and u is an utility function. For $n = 1, 2$, and 3 ,

$$X \succ_n^{almost(\epsilon)} Y \quad \text{if and only if} \quad E[u(X)] > E[u(Y)] \quad \text{for any } u \in U_n^*(\epsilon).$$

Since it is very difficult, if not impossible, to make comparison for utility maximization of any pair of prospects, say, X and Y , based on the results from Theorem 1 academics and practitioners could turn to compare the ϵ -ASD ranking of the prospects which could then draw the utility maximization preference of the prospects for investors in $U_n^*(\epsilon)$.

In this paper we establish some equivalent conditions for different orders of ASD. We first present in the following theorem for the first-order ASD:

Theorem 2 For any pair of random variables X and Y defined on $[a, b]$ with means μ_X and μ_Y and distribution functions F and G , respectively, the following statements are equivalent:

- a. X dominates Y by ϵ -AFSD,

¹We note that the theory can be extended to satisfy utilities defined to be non-differentiable and/or non-expected utility functions, readers may refer to Wong and Ma (2008) and the references therein for more information.

- b. $\mu_X > \mu_Y$, and
- c. $G^{(2)}(b) > F^{(2)}(b)$.

We then present the following theorem for the second-order ASD:

Theorem 3 *For any pair of random variables X and Y stated in Theorem 2, the following statements are equivalent:*

- a. X dominates Y by ϵ -ASSD,
- b. $\mu_X \geq \mu_Y$ and $2b(\mu_X - \mu_Y) > E(X^2) - E(Y^2)$, and
- c. $G^{(3)}(b) > F^{(3)}(b)$ and $G^{(2)}(b) \geq F^{(2)}(b)$.

Thereafter, we establish the following theorem for the result of the third-order ASD:

Theorem 4 *For any pair of random variables X and Y stated in Theorem 2, the following statements are equivalent:*

- a. X dominates Y by ϵ -ATSD,
- b. $\mu_X \geq \mu_Y$, $2b(\mu_X - \mu_Y) \geq E(X^2) - E(Y^2)$, and $E(X^3) - E(Y^3) > 3b(E(X^2) - E(Y^2)) - 3b^2(\mu_X - \mu_Y)$, and
- c. $G^{(4)}(b) > F^{(4)}(b)$ and $G^{(n)}(b) \geq F^{(n)}(b)$ for $n = 2, 3$.

Guo, et al. (2013) have constructed some examples to show that the ASD definition modified by Tzeng et al. (2012) does not possess any hierarchy property. In this paper, we prove this property formally by using the results of Theorems 2 to 4 as shown in the following theorem:

Theorem 5 *The almost stochastic dominance defined in Definition 1 does not possess any hierarchy property.*

In addition, the results from Theorems 2 to 4 could be used to determine the relationships between different orders of the ASD relationship for any two prospects and the moments of the prospects. We first state the relationship of the ASD relationship for any two prospects and the first moments of the prospects in the following corollary:

Corollary 6 For any pair of random variables X and Y with means μ_X and μ_Y , respectively, if $\mu_X \neq \mu_Y$, there is an ASD relationship between X and Y . In particular,

- a. if $\mu_X > \mu_Y$, then $X \succ_1^{\text{almost}(\epsilon)} Y$, and
- b. if $\mu_Y > \mu_X$, then $Y \succ_1^{\text{almost}(\epsilon)} X$.

From Corollary 6, it is clear that if the means of the prospects are different, even it is very small, one will prefer the one with larger mean by using ϵ -AFSD. It is well known that there is the hierarchy property in SD such that the first-order SD implies the second-order SD which, in turn, implies the third-order SD, and thus, practitioners could stop for any higher-order SD investigation when they find any lower-order SD relationship between the prospects. It will be good if the ASD could possess the hierarchy property. However, in this paper we formally prove in Theorem 5 that the ASD definition modified by Tzeng et al. (2012) does not possess any hierarchy property. Nonetheless, in this paper, we still recommend practitioners investigate higher-order ASD only when they do not find any lower-order ASD. Since Corollary 6 tells that there is first-order ASD relationship between two prospects if their means are different, we will examine whether there is any second-order ASD relationship between the prospects only when their means are the same. Under this condition and using the result in Theorem 3, we establish the following corollary to determine the second-order ASD relationship and the second moments of the prospects:

Corollary 7 For any pair of random variables X and Y with means μ_X and μ_Y , respectively, if $\mu_X = \mu_Y$, then

$$X \succ_2^{\text{almost}(\epsilon)} Y \iff \text{var}(X) < \text{var}(Y).$$

It is well known (Levy, 1998) that in the traditional SD theory, for any pair of prospects X and Y , if $\mu_X = \mu_Y$, then $\text{var}(X) < \text{var}(Y)$ is only a necessary condition for the second order SD of X over Y . However, one could easily show that this is not a sufficient condition. Nevertheless, the result from Corollary 7 implies that for ϵ -ASSD, under the condition of $\mu_X = \mu_Y$, the inequality $\text{var}(X) < \text{var}(Y)$ is not only the necessary condition but also the sufficient condition for the dominance of X over Y in the sense of ϵ -ASSD.

We further investigate the comparison of prospects X and Y by the third-order ASD. Similarly, though ASD does not possess any hierarchy property, we still recommend to examine whether there is any third-order ASD only when one does not find any first two

orders of ASD between prospects X and Y . Thus, we will compare the preference of prospects X and Y in the sense of the third-order ASD only under the situation in which $\mu_X = \mu_Y$ and $var(X) = var(Y)$. In this situation, both ϵ -AFSD and ϵ -ASSD fail to distinguish which prospect is better and we can use ϵ -ATSD to draw preference between two prospects. From Theorem 4, we conclude that the one with larger third-order moment is preferred even the difference is very small. Formally, we establish the following corollary:

Corollary 8 *For any pair of random variables X and Y , if $\mu_X = \mu_Y$ and $var(X) = var(Y)$, then*

$$X \succ_3^{almost(\epsilon)} Y \iff E[(X - \mu_X)^3] > E[(Y - \mu_Y)^3].$$

The above three corollaries imply that when ϵ -AFSD, ϵ -ASSD, and ϵ -ATSD are used in the prospects comparison, investors prefer the one with positive gain, smaller variance and positive skewness. We note that there are some studies draw a similar conclusion. For example, Post and Levy (2005) suggest that a third-order polynomial utility function implies that investors care only about the first three central moments of the return distribution (mean, variance, and skewness). Post and Versijp (2007) suggest that third-order stochastic dominance (TSD) efficiency applies if and only if a portfolio is optimal for some nonsatiable, risk-averse, and skewness-loving investor.

4 Concluding Remarks and Discussions

The paper establishes some relationships of the first three orders of ASD. Using these results, we first prove formally that the ASD definition modified by Tzeng et al. (2012) does not possess any hierarchy property. Thereafter, we could conclude that when ϵ -AFSD, ϵ -ASSD, and ϵ -ATSD are used in the prospect comparison, investors prefer the one with positive gain, smaller variance and positive skewness.² This information enables academics and practitioners to determine the ASD relationship among prospects when they know the moments of the prospects. This information, in turn, enables investors to make wise decision in the investment.

We note that the preference of positive gain, smaller variance, and positive skewness is

²We note that one could easily extend our work to $n > 3$ including studying the relationship of the forth order ASD and the kurtosis. However, though some studies, see, for example, Eeckhoudt and Schlesinger (2006), Eeckhoudt, et al. (2009), and Denuit and Eeckhoudt (2010), study risk to $n > 3$, most academics and practitioners are only interested in studying the case up to $n = 3$. Thus, we stop at $n = 3$.

not only a necessary condition but also a sufficient condition for the almost ASD if ASD has hierarchy property. However, it is well known that ASD does not possess any hierarchy property, and thus, the preference of positive gain, smaller variance, and positive skewness is only a necessary condition but not a sufficient condition for the ASD. Nonetheless, if one only considers investors in $U_1^*(\epsilon)$, $U_2^{*'}(\epsilon) = U_2^*(\epsilon) \cap U_1^*(\epsilon)$ and $U_3^{*'}(\epsilon) = U_3^*(\epsilon) \cap U_2^{*'}(\epsilon)$, then the preference of positive gain, smaller variance and positive skewness is not only a necessary condition but also a sufficient condition for the ASD.

At last, academics and practitioners may not like to see the results in which if the means (variances, skewness) of the prospects are bigger (smaller, bigger), even it is very small, one will prefer the one with larger mean (smaller variance, larger skewness) by using ϵ -ASD rule. One may wish to have a way to overcome this “limitation.” The answer is very simple - to choose ϵ to be significantly smaller than 1/2. Actually, Levy, et al. (2010) have provided a good solution. They suggest two approaches. We modify their suggestion as follows:

The first approach is to check the actual area violation ϵ in Definition 1 that is significantly smaller than 1/2. The second approach is to find for a given group of subjects what is the allowed area violation by each investor and whether for all subjects belonging to this group the allowed area violation is greater than the actual area violation.

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