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Abstract: This paper first extends the theory of almost stochastic dominance (ASD) to the first four orders. We then establish some equivalent relationships for the first four orders of the ASD. Using these results, we prove formally that the ASD definition modified by Tzeng et al. (2012) does not possess any hierarchy property. Thereafter, we conclude that when the first four orders of ASD are used in the prospects comparison, risk-averse investors prefer the one with positive gain, smaller variance, positive skewness, and smaller kurtosis. This information, in turn, enables decision makers to determine the ASD relationship among prospects when they know the moments of the prospects. At last, we discuss the necessary and sufficient conditions for different orders the ASD and the moments of the prospects.

Keywords: stochastic dominance; almost stochastic dominance; risk aversion, mean, variance, skewness, kurtosis.

JEL Classification : C0, D81, G11.

1 Introduction

Stochastic dominance (SD) theory has been well established, see, for example, Hanoch and Levy (1969), Hadar and Russell (1969), and Rothschild and Stiglitz (1970). Leshno and Levy (2002) extend it to the theory of almost stochastic dominance (ASD) for *most* decision makers. Tzeng et al. (2012) show that the almost second-degree ASD (ASD_2) introduced by Leshno and Levy (2002) does not possess the property of expected-utility maximization. They modify the ASD_2 definition to acquire this property. Nonetheless, Guo, et al. (2013) have constructed some examples to show that the ASD definition modified by Tzeng et al. (2012) does not possess any hierarchy property.

In this paper, we first extend the theory of ASD to the first four orders. We then develop some equivalent properties for different orders of ASD. Using these results, we prove formally that the ASD definition modified by Tzeng et al. (2012) does not possess any hierarchy property. Thereafter, we establish the relationships between different orders of ASDs and the moments of the prospects being compared. These findings lead us conclude that when the first four orders of ASD are used in the prospect comparison, risk-averse decision makers prefer the one with positive gain, smaller variance, positive skewness, and smaller kurtosis. This information, in turn, enables academics and practitioners to determine the ASD relationship among prospects when they know the moments of the prospects. At last, we discuss the necessary and sufficient conditions for the ASD and the moments of the prospects.

2 Notations and Definitions

In order to develop some relationships for the ASD concepts proposed by Leshno and Levy (2002) and modified by Tzeng et al. (2012), we first state the definitions and notations being used in this paper. Suppose that random variables X and Y defined on the support $\Omega = [a, b]$ with means μ_X and μ_Y and standard deviations σ_X and σ_Y have the corresponding distribution functions F and G, respectively. The following notations will be used throughout this paper:

$$\begin{aligned} &H^{(1)} = H \quad \text{and} \quad H^{(n)}(x) = \int_a^x H^{(n-1)}(t)dt \quad \text{for} \quad H = F, G \quad \text{and} \quad n = 2, 3, 4 \; ; \\ &\big|\big|F^{(n)} - G^{(n)}\big|\big| = \int_a^b \big|F^{(n)}(x) - G^{(n)}(x)\big|dx \quad , \quad \text{and} \\ &S_n \equiv S_n(F, G) = \big\{x \in [a, b] : G^{(n)}(x) < F^{(n)}(x)\big\} \quad \text{for} \quad n = 1, \cdots, 4 \; . \end{aligned} \tag{1}$$

An individual chooses between X and Y with distribution functions F and G, respectively, in accordance with a consistent set of preferences satisfying the von Neumann-Morgenstern (1944) consistency properties. Accordingly, X is preferred to Y if $E[u(X)] - E[u(Y)] \geq 0$ in which $E[u(X)] \equiv \int_a^b u(x)dF(x)$ and $E[u(Y)] \equiv \int_a^b u(x)dG(x)$. We first rewrite the definition of ASD introduced by Leshno and Levy (2002) and modified by Tzeng et al. (2012) and extend it to the first four orders as follows:

Definition 1 *Let F and G be the corresponding distribution functions of X and Y. For $0 < \epsilon < 1/2$,*

ϵ-ASD_1: *X is said to dominate Y by ϵ-ASD_1, denoted by $X \succ_1^{almost(\epsilon)} Y$, if and only if*

$$\int_{S_1} \big[F(x) - G(x)\big]dx \leq \epsilon\big|\big|F - G\big|\big|;$$

ϵ-ASD_2: *X is said to dominate Y by ϵ-ASD_2, denoted by $X \succ_2^{almost(\epsilon)} Y$, if and only if*

$$\int_{S_2} \big[F^{(2)}(x) - G^{(2)}(x)\big]dx \le \epsilon\big|\big|F^{(2)} - G^{(2)}\big|\big| \ \ and \ \ \mu_X \ge \mu_Y \ ;$$

ϵ-ASD_3: *X is said to dominate Y by ϵ-ASD_3, denoted by $X \succ_3^{almost(\epsilon)} Y$, if and only if*

$$\int_{S_3} \big[F^{(3)}(x) - G^{(3)}(x)\big]dx \le \epsilon\big|\big|F^{(3)} - G^{(3)}\big|\big| \ \ and \ \ G^{(n)}(b) \ge F^{(n)}(b) \ for \ \ n = 2, 3 \ .$$

ϵ-ASD_4: *X is said to dominate Y by ϵ-ASD_4, denoted by $X \succ_4^{almost(\epsilon)} Y$, if and only if*

$$\int_{S_4} \big[F^{(4)}(x) - G^{(4)}(x)\big]dx \le \epsilon\big|\big|F^{(4)} - G^{(4)}\big|\big| \ \ and \ \ G^{(n)}(b) \ge F^{(n)}(b) \ for \ \ n = 2, 3, 4 \ ,$$

where ϵ-ASD_n is the n-order ASD for $n = 1, \cdots, 4$.

In addition, we define the following utility functions:

Definition 2 *For $n = 1, \cdots, 4$,*

$$U_n = \big\{u : (-1)^i u^{(i)} \le \ 0 \ , \ i = 1, \cdots, n\big\} \ ,$$
$$U_n^*(\epsilon) = \big\{u \in U_n : (-1)^{n+1}u^{(n)}(x) \le \inf\{(-1)^{n+1}u^{(n)}(x)\}[1/\epsilon - 1] \ \forall x\big\} \ ,$$

in which ϵ is in the range of $(0, 1/2)$.[1]

We call investors the n-order risk averters if their utility functions $u \in U_n$ and the n-order ϵ-risk averters if their utility functions $u \in U_n^*(\epsilon)$. Without loss of generality, we call them risk averters or risk-averse investors.

3 The Theory

We first rewrite the main results in Tzeng et al. (2012) and extend it to the first four orders that ASD possesses the utility maximization property as stated in the following theorem:

Theorem 1 *Let F and G be the corresponding distribution functions of X and Y and u is an utility function. For $n = 1, \cdots, 4$,*

$X \succ_n^{almost(\epsilon)} Y$ if and only if $E[u(X)] > E[u(Y)]$ for any $u \in U_n^*(\epsilon)$.

[1]We note that the theory can be extended to satisfy utilities defined to be non-differentiable and/or non-expected utility functions, readers may refer to Wong and Ma (2008) and the references therein for more information.

Since it is very difficult, if not impossible, to make comparison for utility maximization of any pair of prospects, say, X and Y, based on the results from Theorem 1 academics and practitioners could turn to compare the ϵ-ASD ranking of the prospects which could then be able to draw the utility maximization preference of the prospects for investors in $U_n^*(\epsilon)$.

In this paper we establish some equivalent conditions for different orders of ASD. We first present the following theorem for the first-order ASD:

Theorem 2 *For any pair of random variables X and Y defined on $[a, b]$ with means μ_X and μ_Y and distribution functions F and G, respectively, the following statements are equivalent:*

a. *X dominates Y by ϵ-ASD_1,*

b. *$\mu_X > \mu_Y$, and*

c. *$G^{(2)}(b) > F^{(2)}(b)$.*

We then present the following theorem for the second-order ASD:

Theorem 3 *For any pair of random variables X and Y stated in Theorem 2, the following statements are equivalent:*

a. *X dominates Y by ϵ-ASD_2,*

b. *$\mu_X \geq \mu_Y$ and $2b(\mu_X - \mu_Y) > E(X^2) - E(Y^2)$, and*

c. *$G^{(3)}(b) > F^{(3)}(b)$ and $G^{(2)}(b) \geq F^{(2)}(b)$.*

Thereafter, we establish the following theorem for the third-order ASD:

Theorem 4 *For any pair of random variables X and Y stated in Theorem 2, the following statements are equivalent:*

a. *X dominates Y by ϵ-ASD_3,*

b. *$\mu_X \geq \mu_Y$, $2b(\mu_X - \mu_Y) \geq E(X^2) - E(Y^2)$, and $E(X^3) - E(Y^3) > 3b(E(X^2) - E(Y^2)) - 3b^2(\mu_X - \mu_Y)$, and*

c. *$G^{(4)}(b) > F^{(4)}(b)$ and $G^{(n)}(b) \geq F^{(n)}(b)$ for $n = 2, 3$.*

Finally, we establish the following theorem for the fourth-order ASD:

Theorem 5 *For any pair of random variables X and Y stated in Theorem 2, the following statements are equivalent:*

a. *X dominates Y by ϵ-ASD_4,*

b. $\mu_X \geq \mu_Y$, $2b(\mu_X - \mu_Y) \geq E(X^2) - E(Y^2)$, $E(X^3) - E(Y^3) \geq 3b(E(X^2) - E(Y^2)) - 3b^2(\mu_X - \mu_Y)$ *and* $4b^3\big(\mu_X - \mu_Y\big) - 12b^2\big[G^{(3)}(b) - F^{(3)}(b)\big] + 24b\big[G^{(4)}(b) - F^{(4)}(b)\big] > (E(X^4) - E(Y^4))$ *with* $2\big[G^{(3)}(b) - F^{(3)}(b)\big] = 2b\big(\mu_X - \mu_Y\big) - \big[E(X^2) - E(Y^2)\big], 6\big[G^{(4)}(b) - F^{(4)}(b)\big] = E(X^3) - E(Y^3) - 3b(E(X^2) - E(Y^2)) + 3b^2(\mu_X - \mu_Y)$, *and*

c. $G^{(5)}(b) > F^{(5)}(b)$ *and* $G^{(n)}(b) \geq F^{(n)}(b)$ *for* $n = 2, 3, 4$.

Guo, et al. (2013) have constructed some examples to show that the ASD definition modified by Tzeng et al. (2012) does not possess any hierarchy property. In this paper, we prove this property formally by using the results of Theorems 2 to 5 as shown in the following theorem:

Theorem 6 *The almost stochastic dominance defined in Definition 1 does not possess any hierarchy property.*

In addition, the results from Theorems 2 to 5 could be used to determine the relationships between different orders of the ASD and the moments of the prospects. We first state the relationship between the first-order ASD and the first moments of the prospects as shown in the following corollary:

Corollary 7 *For any pair of random variables X and Y with means μ_X and μ_Y, respectively, $\mu_X \neq \mu_Y$ if and only if there is a first-order ASD relationship between X and Y. In particular, $X \succ_1^{almost(\epsilon)} Y \iff \mu_X > \mu_Y$.*

From Corollary 7, it is clear that if the means of the prospects are different, even it is very small, one will prefer the one with larger mean by using ϵ-ASD_1. It is well known that SD possesses the hierarchy property such that the first-order SD implies the second-order SD which, in turn, implies the third-order SD, and so on, and thus, practitioners could stop for any higher-order SD investigation when they find any lower-order SD relationship between the prospects. It will be good if the ASD could possess the hierarchy property. However, in this paper we formally prove in Theorem 6 that the ASD definition modified by

Tzeng et al. (2012) does not possess any hierarchy property. Nonetheless, in this paper, we still recommend practitioners investigate higher-order ASD only when they do not find any lower-order ASD. Since Corollary 7 tells that there is first-order ASD relationship between two prospects if their means are different, we will examine whether there is any second-order ASD relationship between the prospects only when their means are the same. Under this condition and using the result in Theorem 3, we establish the following corollary to determine the relationship between the second-order ASD and the second moments of the prospects:

Corollary 8 *For any pair of random variables X and Y with means μ_X and μ_Y, respectively, if $\mu_X = \mu_Y$, then*

$$X \succ_2^{almost(\epsilon)} Y \Longleftrightarrow var(X) < var(Y).$$

It is well known (Levy, 1998) that in the traditional SD theory, for any pair of prospects X and Y, if $\mu_X = \mu_Y$, then $var(X) < var(Y)$ is only a necessary condition but not a sufficient condition for the second order SD of X over Y. Nevertheless, the result from Corollary 8 implies that for ϵ-ASD_2, under the condition of $\mu_X = \mu_Y$, the inequality $var(X) < var(Y)$ is not only the necessary condition but also the sufficient condition for the dominance of X over Y in the sense of ϵ-ASD_2.

We further investigate the comparison of prospects X and Y by the third-order ASD. Similarly, though ASD does not possess any hierarchy property, we still recommend to examine whether there is any third-order ASD only when one does not find any first two orders of ASD between prospects X and Y. Thus, we will compare the preference of prospects X and Y in the sense of the third-order ASD only under the situation in which $\mu_X = \mu_Y$ and $var(X) = var(Y)$. In this situation, both ϵ-ASD_1 and ϵ-ASD_2 fail to distinguish which prospect is better and we can use ϵ-ASD_3 to draw preference between two prospects. From Theorem 4, we conclude that the one with larger third moment is preferred even the difference is very small. Formally, we establish the following corollary:

Corollary 9 *For any pair of random variables X and Y, if $\mu_X = \mu_Y$ and $var(X) = var(Y)$, then*

$$X \succ_3^{almost(\epsilon)} Y \Longleftrightarrow E[(X - \mu_X)^3] > E[(Y - \mu_Y)^3].$$

Similarly, we discuss the the fourth-order ASD when the first three moments are equal. The following corollary leads us to conclude that in this situation, the prospect with smaller kurtosis is preferred.

Corollary 10 *For any pair of random variables X and Y, if $\mu_X = \mu_Y$, $var(X) = var(Y)$ and $E[(X-\mu_X)^3] = E[(Y-\mu_Y)^3]$, then*

$$X \succ_4^{almost(\epsilon)} Y \Longleftrightarrow E[(X-\mu_X)^4] < E[(Y-\mu_Y)^4].$$

The above four corollaries imply that when ϵ-ASD_n are used in the prospects comparison for $n = 1, \cdots, 4$, risk-averse investors prefer the one with positive gain, smaller variance, positive skewness, and smaller kurtosis. We note that there are some studies draw a similar conclusion for the first three orders. For example, Post and Levy (2005) suggest that a third-order polynomial utility function implies that investors care only about the first three central moments of the return distribution (mean, variance, and skewness). Post and Versijp (2007) suggest that third-order stochastic dominance (TSD) efficiency applies if and only if a portfolio is optimal for some nonsatiable, risk-averse, and skewness-loving investor.

4 Concluding Remarks and Discussions

In this paper we extend the theory of ASD to the first four orders and develop some equivalent relationships of the first four orders of ASD. Using these results, we first prove formally that the ASD definition modified by Tzeng et al. (2012) does not possess any hierarchy property. Thereafter, we conclude that when ϵ-ASD_n are used in the prospect comparison for $n = 1, \cdots, 4$, risk-averse investors prefer the one with higher mean, smaller variance, higher skewness, and smaller kurtosis.[2] This information enables academics and practitioners to determine the ASD relationship among prospects when they know the moments of the prospects. This information, in turn, enables investors to make wiser decision in their investment.

We note that the preference of higher mean, smaller variance, higher skewness, and smaller kurtosis is not only a necessary condition but also a sufficient condition for the ASD if ASD has hierarchy property. However, it is well known that ASD does not possess any hierarchy property, and thus, the preference of higher mean, smaller variance, higher skewness, and smaller kurtosis is only a necessary condition but not a sufficient condition

[2]We note that one could easily extend our work to $n > 4$. However, though some studies, see, for example, Eeckhoudt and Schlesinger (2006), Eeckhoudt, et al. (2009), and Denuit and Eeckhoudt (2010), study risk to $n > 4$, most academics and practitioners are only interested in studying the case up to $n = 4$. Thus, we stop at $n = 4$.

for the ASD. Nonetheless, if one only considers investors in $U_1^*(\epsilon)$, $U_2^{*'}(\epsilon) = U_2^*(\epsilon) \cap U_1^*(\epsilon)$, $U_3^{*'}(\epsilon) = U_3^*(\epsilon) \cap U_2^{*'}(\epsilon)$, and $U_4^{*'}(\epsilon) = U_4^*(\epsilon) \cap U_3^{*'}(\epsilon)$, then the preference of higher mean, smaller variance, higher skewness, and smaller kurtosis is not only a necessary condition but also a sufficient condition for the ASD.

At last, academics and practitioners may not like to see the results in which if the means (variances, skewness, kurtosis) of the prospects are bigger (smaller, bigger, smaller), even it is very small, one will prefer the one with larger mean (smaller variance, larger skewness, smaller kurtosis) by using the ϵ-ASD rule. One may wish to find a way to overcome this "limitation." The answer is very simple - to choose ϵ to be significantly smaller than 1/2. Actually, Levy, et al. (2010) have provided a good solution. They suggest two approaches. We modify their suggestion as follows:

The first approach is to check the actual area violation ϵ in Definition 1 that is (significantly) smaller than 1/2*. The second approach is to find for a given group of subjects what is the allowed area violation by each investor and whether for all subjects belonging to this group the allowed area violation is greater than the actual area violation.*

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