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Many carriers, such as airlines and ocean carriers, collaborate through the formation of alliances. The detailed alliance design is clearly important for both the stability of the alliance and profitability of the alliance members. This work is motivated by a real-life liner shipping "resource exchange alliance" agreement design. We provide an economic motivation for interest in resource exchange alliances and propose a model and method to design a resource exchange alliance. The model takes into account how the alliance members compete after a resource exchange by selling substitutable products and thus enables us to obtain insight into the effect of capacity and the intensity of competition on the extent to which an alliance can provide greater profit than when in the setting without an alliance. The problem of determining the optimal amounts of resources to exchange is formulated as a stochastic mathematical program with equilibrium constraints (SMPECs). We show how to determine whether there exists a unique equilibrium after resource exchange, how to compute the equilibrium, and how to compute the optimal resource exchange. SMPEC problem, which is generally very difficult to solve, is well-posed in the paper, and robust results can be obtained with a reasonable amount of computational effort.

*Key words*: alliance, resource exchange, pricing, revenue management, stochastic mathematical programming with equilibrium constraints, non-cooperative game

## 1. Introduction

Alliances are collaborative agreements made between two or more parties in order to achieve common goals and to improve competitiveness. Increasingly, alliances are being recognized as a key component of business strategies and can be found easily in many industries. For example, in the liner shipping industry, ocean container carriers often make an alliance agreement to provide joint services. A "service is a cycle of successive port visits that repeats according to a regular schedule, typically with weekly departures at each port in the cycle. To maintain a schedule of weekly departures at each port, the headway between successive ships traversing the cycle must

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be one week. Thus, if it takes a ship $n$ weeks to complete one cycle, then $n$ ships are needed to offer the service with weekly departures at each port in the cycle. For many services between Asia and North America and services between Asia and Europe, it takes a ship between 5 and 10 weeks to complete one cycle, and thus several ships are needed to offer the service. Taking into account that a large container ship can cost a hundred million US dollars[1], it becomes clear that for even the large carriers it would require an enormous investment to introduce a new service. For this reason, ocean carriers form an alliance. Another example of a widely used alliance can be found in the airline industry. Airlines often sell tickets on each others' flights through code share agreements; an alliance member (the marketing member) can sell tickets for flights operated by another alliance member (the operating member) and the marketing member can put its own code on the flight. Code sharing dramatically increases the number of itinerary products that each airline can sell. Vacation packages provide another example of seller alliances enabling the sale of products combined from the resources/products of several sellers. For example, a vacation package may consist of airline tickets for 2 people, a hotel room for 4 nights, and car rental for 5 days. Computers and peripherals provide another example of products combined from the resources of several sellers.

Seller alliances can be structured in many different ways. The rules of an alliance are clearly critical for both the stability of the alliance and the well-being of the alliance members. The major distinguishing factor between different alliance structures involves the control of the resources involved. In a so-called "free-sale" or "soft-block" agreement, each alliance member controls the availability of the resources (such as seat space on flights or container slots on voyages) that it contributed and other alliance members can buy the resources from the owner and include it in the products that they sell as long as the owner makes the resources available. Under a "resource-exchange" or "hard-block" agreement, alliance members exchange resources, and thereafter each alliance member controls the resources allocated to them as though they were the owners of the resources.

In the liner shipping industry, the most widely used form of the alliances is the "resource exchange" type (also called a slot exchange alliance or a cross slot charter alliance). Under the resource exchange alliance, carriers agree to provide each other a specified amount of capacity and possibly a specified amount of money in advance of the sales season. For example, carrier one provides carrier two $x_1$ slots on each of its voyages on its service and carrier two provides carrier one $x_2$ slots on each of its voyages on its service. Thereafter each carrier can make bookings for itineraries that include voyages operated by both carriers, thereby dramatically increasing the

[1] e.g.,`http://www.emma-maersk.com/specification/`

number of itinerary products that each carrier can sell. Suppose the carrier one operates a service with the cycle of port visits $A$,$B$,$C$,$A$, and carrier two operates a service with the cycle of port visits $A$,$D$,$E$,$A$. If a customer wants to send a container from port $C$ to port $D$ via port $A$ and there is no alliance between the carriers, then the customer should make a booking with carrier one to send the container from $C$ to $A$ and a separate booking with carrier two to send the container from $A$ to $D$. However, after the slot exchange agreement, each carrier can offer the customer a complete booking from $D$ via $A$ to $G$ and controls the pricing for such products as they control the pricing for all their other products. A real-life example of such an alliance can be found in the Hanjin-Evergreen Agreement[2], in which Hanjin operated a service that visits New York, Norfolk, Savannah, Rio Grande, Itajai, and Santos, and Evergreen operated a service that visits Shanghai, Ningbo, Kaohsiung, Hong Kong, Yantian, Singapore, Durban, Cape Town, Montevideo, Buenos Aires, and Santos. Evergreen provided Hanjin with 100 Twenty-foot Equivalent Unit (TEU) slots on each voyage of its service and Hanjin provided Evergreen with 137 TEU slots on each voyage of its service. Thus, if a customer wants to send a container from Cape Town to Rio Grande, then with the alliance the customer can book a single itinerary from Cape Town to Rio Grande via Santos with either carrier, whereas without an alliance the customer would have to make two separate bookings. In fact, this type of alliances are very common in practice and other similar examples of such alliance agreements can be found easily[3].

The ocean container carriers also enter into a resource exchange alliance when they introduce new joint services. As described earlier, services repeat according to a regular schedule and in order to offer weekly departures at each port in the cycle, the number of ships needed to provide the service is equal to the number of weeks that it takes a ship to complete the cycle. Under a resource exchange alliance, each carrier in the alliance provides one or more ships to be used for the service, and the capacity on each ship is partitioned among the alliance members. An example of such an alliance is the China Shipping Container Lines (CSCL) / United Arab Shipping Company (UASC) Agreement[4]. The new joint service consists of the cycle of port visits of Xiamen, Hong Kong, Yantian, Shanghai, New York, Norfolk, Savannah, Xiamen. The cycle takes 9 weeks to complete, with weekly departures at each port, and thus deploys 9 vessels, 6 contributed by CSCL and 3 contributed by UASC. The alliance contract enables each carrier in the alliance to offer weekly departures at each port in the service without having to commit 9 ships to the service.

Several advantages of alliances can be easily identified. By entering carrier alliances, the seller can offer a larger number of itinerary products to a much wider choice of destinations, leading

[2] Details of the contract can be seen at `http://www2.fmc.gov/agreement_lib/011968-000.pdf`

[3] Please see similar contracts on `http://www2.fmc.gov/agreements/type_npage.aspx`

[4] The contract detail can be found on `http://www2.fmc.gov/agreement_lib/012168-000.pdf`

to enhanced marketing opportunities. It is especially apparent when carriers introduce new joint services – without an alliance, each carrier may not have sufficient resources to offer such a global service. Alliances also offer a consumer "seamless" worldwide travel/shippment experiences. Again, suppose the carrier one operates a service with the cycle of port visits $A$,$B$,$C$,$A$, and carrier two operates a service with the cycle of port visits $A$,$D$,$E$,$A$. Under no alliance, customers who want to send a container from port $C$ to port $D$ via port $A$ should make separate bookings with multiple carriers. Even worse, if something goes wrong with the acquisition of one of the components in the itinerary, it is difficult for the customer to get the other seller to make appropriate adjustments. For example, if carrier one delays the arrival of the container at port $A$, then the customer has to convince carrier two to change the booking from port $A$ without a penalty, whereas such adjustment should be easier if the customer made the booking with a single carrier. There also exists an economic motivation. As will be shown in the paper, if separate sellers sell a partial product in the itinerary, then each seller will have an incentive to charge a higher price for its component in order to (at least implicitly) extract as much of the total revenue as it can. This leads to system-wide inefficiencies and thus loss in total profit; in the example above, under no alliance, carrier one would charge more for its own itinerary product, shipment of a container from port $C$ to port $A$ and carrier two would do the same. Therefore, total prices for the complete shipment (port $C$ to port $D$ via port $A$) would be higher, resulting in lower demand and profit. In fact, when the sellers form a "well-designed contract" where they can sell a complete product to customers, profits may increase.

However, along with its obvious benefits, an alliance could pose a new problem. After formation of an alliance, alliance members can sell substitute products (in the previous example, both carriers now offer a service of shipment from port $C$ to port $D$) and thus compete with each other for the same demand. That is, alliances increase not only the sellers' product portfolios but also competition. It is easy to see that a poorly designed alliance may be detrimental to both sellers and therefore, the questions remains – are sellers better off with an alliance or not?

In this paper, our study focuses on resource exchange alliances. Since alliance members compete by selling substitute products after an alliance, we propose an alliance design model which explicitly takes into account how the exchange of resources affects the competition among the alliance members, unlike several other papers on resource exchange alliances. Our model is presented as a stochastic optimization model with equilibrium constraints (SMPECs), and we determine the optimal resource exchange amount which maximizes the sum of the alliance members' total profits after the exchange, considering the resulting competition. For each resource exchange, the competition among alliance members is modeled as a noncooperative game in which each alliance member chooses the prices for its own products, subject to its own capacity constraints (which depend on the resource exchange), to maximize its own profit.

The paper is organized as follows. We review the relevant literature in the next section and then provide an important economic motivation for interest in resource exchange alliances in Section 3. As mentioned above, if customers want to buy a product that consists of components provided by different sellers, then in an attempt to maximize their own profits, the sellers tend to choose prices for their components that are too high, which leads to loss in total profit. Intuitively this happens because each seller is implicitly attempting to gather a larger share of the total revenue without an alliance. The idea is illustrated with a specific model in Section 3. It is shown that the equilibrium prices without an alliance are higher than the prices under perfect coordination and the equilibrium quantities without an alliance are lower than the quantities under perfect coordination. We also show that the total profit of a resource exchange alliance with well-chosen exchange quantities is greater than the total profit without an alliance. In addition, we show that the equilibrium prices, quantities, and profits are equal for a resource exchange alliance with exchange quantities chosen to maximize the total profit and for perfect coordination. The major purpose of the model is to demonstrate that if customers want to buy a product that consists of components provided by different sellers, then sellers who attempt to maximize their own profits will tend to choose prices that are too high. A secondary purpose of the model is to demonstrate how a well designed resource exchange alliance may increase the total profit and the amount of increase depends upon the intensity of competition and the resource capacity.

In Section 4, we consider more general models of no alliance, perfect coordination, and a resource exchange alliance. For resource exchange alliances, we formulate an optimization model to determine the amount of each resource to be exchanged, taking into account the consequences of the exchange on subsequent competition among alliance members. If one assumes that after a resource exchange, each alliance member chooses the prices of its products to maximize its own profit, and that this behavior of the alliance members leads to an equilibrium, then the problem can be formulated as a mathematical program with equilibrium constraints. An important question is whether, for each resource exchange, there exists an equilibrium and, if so, whether it is unique.

In Section 5 we show how to determine whether a unique equilibrium exists, and how to compute it. We solve examples of the mathematical program with equilibrium constraints (which in general is a hard, poorly behaved problem), and in Section 6 we compare the results for the cases with no alliance, perfect coordination, and a resource exchange alliance.

## 2. Related Literature

This paper is motivated by the design of resource exchange alliances in the liner shipping industry and we formulate the alliance design problem as a Stochastic Mathematical Program with Equilibrium Constraints (SMPEC). Therefore, our paper is related to two streams of research: studies of liner shipping alliances and Mathematical Programs with Equilibrium Constraints (MPECs).

When studying liner shipping and airline alliances, it is useful to note that in both industries the term "alliance" is used for two different scales of agreement. The larger scale agreements usually involve more than two carriers and can be described as agreements of large scope but limited specificity with the idea that alliance members will enter into more specific collaborative agreements with each other. Current examples are the Grand Alliance, the New World Alliance, and the CKYH Alliance in the liner shipping industry and the Star Alliance, the Sky Team Alliance, and the OneWorld Alliance in the airline industry. Most smaller scale agreements are between two carriers, although sometimes more than two carriers are involved. These agreements address specific operational collaboration rules, such as resource exchange, or free sale of each other's capacity, or sharing of facilities, such as port terminals or airport gates. These smaller scale agreements are also called alliances. The agreements described in the introduction are examples of such alliances.

There is a substantial amount of literature on alliances which is mainly focused on strategic level problems such as tactical motivations, the main driving factors for alliances, and the potential outcomes of alliances. Slack et al. (2002) empirically examine the changes in services made by container shipping lines in response to the formation of alliances and Yang et al. (2011) consider the increase of ship size and new strategy of alliance and study the economic performance and stability of liner shipping alliances. Panayides and Wiedmer (2011) also describe the structure and dynamics of strategic alliances in container liner shipping and analyze the operational and strategic changes within the last ten years.

Of particular interest to us is the part of this literature that highlights the competition among alliance members as an important factor affecting the stability of alliances. For example, Midoro and Pitto (2000) identify the existence of intra-alliance competition as a key force driving alliance instability in the liner shipping industry. Lu et al. (2006) conducted a survey among members of the CKYH Alliance to determine key strategic reasons for liner shipping alliances, disadvantages of alliances, and reasons for success of alliances. They also recognize the competition between alliance members as a primary disadvantage of strategic alliances in liner shipping. In addition, they demonstrate that the two most important key strategic reasons for liner shipping alliances were to expand service coverage and to increase service frequency, which are the two motivations that we illustrated in the introduction. In this paper, we realize the importance of intra-competitions among alliance members and take those competitions into consideration in designing the optimal alliance contract.

On the other hand, literature on alliance design is very sparse. Song and Panayides (2002) analyze operations of liner shipping alliances involving two small examples in liner shipping. They propose to use cooperative game theory to analyze strategic alliance but they do not address alliance design decisions (such as resource exchange quantities or free sale prices). Agarwal and Ergun (2010) address a large scale service network design in which ocean carriers integrate the networks and share capacity on their ships in a liner shipping alliance. They propose a heuristic for choosing the services to operate under the alliance and a heuristic for assigning the vessels of each alliance carrier to the chosen services. They also propose the use of side payments made by each carrier who sends a shipment along an edge in the network to all the alliance carriers in proportion to the capacity contributed to the edge by the carriers, and then they model each carrier's decision problem as a linear program. Side payments that lead to the central optimal flows can then be obtained by solving the resulting inverse problem. They mention the importance of competitive prices, but their model assumes that the prices charged by the carriers for origin-destination shipments are fixed exogenously and are not affected by the side payments that the carriers have to make. Their model also assumes that the demands for each carrier are fixed exogenously and thus there is no competition among carriers in their model, despite every alliance carrier's ability to serve the same origin-destination shipments using the same services. In contrast, in our model prices are determined endogenously taking competition into account and these prices depend on the alliance design. Our paper is most closely related to the one written by Lu et al. (2010). They consider a slot exchange contract between two ocean carriers and propose two optimization models to determine slot exchange amounts and the number of containers flowing between different origin-destination port pairs. The optimization model in their paper is formulated from the point of view of one of the two carriers whereas our model is formulated as a mathematical program with equilibrium constraints with the upper level determining the slot exchange amounts maximizing the total profit of the alliance and the lower level representing the competition between the alliance members after slot exchange. Also, the model in Lu et al. (2010) takes prices as input whereas prices are decision variables in the lower level of our model.

In the context of airline alliance collaborations and operations, there exist several interesting studies which address questions such as the choice of flights to include in code-share agreements, the choice of transfer price or proration rates in free-sale alliances, and the effect of airline alliances on booking limits. For example, Sivakumar (2003) presents Code Share Optimizer, a tool built by United Airlines that considers the interaction between proration agreements, demand, fares, and market shares and O'Neal et al. (2007) built a code-share flight profitability tool to automate the code-share flight selection process at Delta airlines. Abdelghany et al. (2009) also present a model for airlines to determine a set of flights for a code-share agreement. On the other hand, Netessine and Shumsky (2005) examine how horizontal and vertical competition affect airline seat inventory decisions and how airlines in an alliance may coordinate these decisions by agreements similar to revenue-sharing contracts. Hu et al. (2013) extend the analysis in Netessine and Shumsky (2005) and study a model of a free-sale airline alliance. Similar to our model, their model is a two-stage model with the alliance design decision in the first stage and operational selling decisions of individual airlines in the second stage, formulated as a Nash equilibrium problem. In addition to static proration scheme, Wright et al. (2010) formulate a Markov-game dynamic model of two airlines and Wright (2011) extends the dynamic bid-price sharing scheme under incomplete information.

In the present paper, we are concerned with a particular formulation of stochastic mathematical programs that involve equilibrium constraints. MPEC plays a very important role in many fields such as operations research, economics, and engineering design and it has been receiving more and more attention in the optimization field. However, at the same time, it is well known that MPECs are still one of the most challenging problems in optimization, and there is a strong need to improve and develop the advance theoretical and numerical tools to be applied in practical problems. MPECs are constrained optimization problems with constraints resulting from an equilibrium problem. They are a generalization of a bilevel problem, where instead of a lower-level optimization problem, there is a lower-level equilibrium problem, possibly defined by a parametric variational inequality. MPECs are difficult, non-convex and non-smooth optimization problem and thus the well developed nonlinear programming theory cannot be applied to MPECs directly. We may refer to, e.g., Giallombardo and Ralph (2008), DeMiguel et al. (2005), Facchinei and Pang (2003), Luo et al. (1996), and references therein for a further discussion. Since MPECs are already very difficult to handle, obviously, stochastic MPECs are extremely difficult to deal with due to the additional efforts required to account for the uncertainty involved in the problem (e.g., Birbil et al. 2006, Shapiro and Xu 2008, Xu and Ye 2010, and Shanbhag et al. 2011). Although MPECs and stochastic MPECs are important modeling tools for numerous applications, there is only little literature that demonstrates successful formulation and provides an efficient solution approach for nontrivial examples. Kachani et al. (2008) address the dynamic pricing problem with learning of the parameters in the price-demand relationship. They show that the problem can be formulated as a MPEC problem, propose a solution method, and discuss various insights through a computational study. Also, Côté et al. (2003) propose a new modeling approach for pricing and fare optimization in the airline industry based on a bilevel mathematical programming and demonstrate that the use of this modeling paradaim allows a company to maximize revenue while taking into account the behavior of the passengers as well as the complex topology of airline networks in a detailed fashion.

## 3. An Economic Rationale for Alliances

In this section we present an economic rationale for alliances in a setting where customers want a product that is composed of products (here called resources) sold by two sellers. For example, a customer wants to send a container from $A$ to $C$ and to accomplish that in the absence of an alliance the customer buys transportation service from $A$ to $B$ provided by one seller and transportation service from $B$ to $C$ provided by another seller. The intuitive idea behind the economic rationale is that each seller chooses its price to extract the maximum profit for the seller and in the absence of an alliance, the resulting equilibrium prices are too high in the sense that lower prices maximize the total profit of the two sellers. We use a simple model to illustrate the idea. We also use the model to introduce resource exchange alliances in a simple setting before proceeding to the more general setting. For the model, it also turns out that a well designed alliance attains the maximum total profit which could be attained under the perfect coordination. We will discuss the general setting in detail later but the core insight will remain same.

We consider 2 sellers, indexed by $i = \pm 1$. Each seller produces one resource. Seller $i$ produces resource $i$ and a maximum quantity $b_i$ of resource $i$ can be consumed. Seller $i$ has a constant marginal cost of $c_i$ per unit of resource $i$ consumed and seller $i$ chooses the price $\tilde{y}_i + c_i$ per unit of resource $i$, that is, $\tilde{y}_i$ denotes the price in excess of the marginal cost $c_i$ per unit of resource $i$. Customers want to consume a product that requires one unit of each resource. (In this section, there is no demand for a product that consists of only one resource. General settings will be discussed in later sections.) Thus, customers buy products consisting of one unit of each resource and pay $c_{-1} + \tilde{y}_{-1} + c_1 + \tilde{y}_1$ per unit of product. The demand $d$ for products depends on the prices as follows:

$$d \;=\; \max\{0,\, \tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1)\} \tag{1}$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are positive constants known to each seller. Assume that $\tilde{\alpha} > 0$, that is, demand is positive if each seller charges only its marginal cost. The detailed calculations for this section are given in Appendix A.

### 3.1. No Alliance

First consider the case with no alliance which is modeled as a non-cooperative game. Let $b_{\min} := \min\{b_{-1}, b_1\}$. Thus, the number of products sold is given by $\min\{b_{\min}, \max\{0,\, \tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1)\}\}$ and the profit of seller $i$ is given by

$$\tilde{g}_i(\tilde{y}_i, \tilde{y}_{-i}) \;:=\; \tilde{y}_i \min\{b_{\min},\, \max\{0,\, \tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-i} + \tilde{y}_i)\}\}$$

If $b_{\min} \geq \tilde{\alpha}/3$, then the equilibrium prices are given by

$$\tilde{y}_i^* \;=\; \frac{\tilde{\alpha}}{3\tilde{\beta}} \tag{2}$$

the equilibrium demand is equal to

$$\tilde{\alpha} - \tilde{\beta}(\tilde{y}^*_{-1} + \tilde{y}^*_1) \;=\; \frac{\tilde{\alpha}}{3} \;>\; 0 \tag{3}$$

the resulting profit of seller $i$ is equal to

$$\tilde{y}^*_i \min\{b_{\min},\, \max\{0,\, \tilde{\alpha} - \tilde{\beta}(\tilde{y}^*_{-i} + \tilde{y}^*_i)\}\} \;=\; \frac{\tilde{\alpha}^2}{9\tilde{\beta}} \tag{4}$$

and thus the total profit of both sellers together is equal to

$$\tilde{y}^*_{-1}\left[\tilde{\alpha} - \tilde{\beta}(\tilde{y}^*_{-1} + \tilde{y}^*_1)\right] + \tilde{y}^*_1\left[\tilde{\alpha} - \tilde{\beta}(\tilde{y}^*_{-1} + \tilde{y}^*_1)\right] \;=\; \frac{2\tilde{\alpha}^2}{9\tilde{\beta}} \tag{5}$$

If $b_{\min} \leq \tilde{\alpha}/3$, then all pairs of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ on the line segment between $(b_{\min}/\tilde{\beta},\, [\tilde{\alpha} - 2b_{\min}]/\tilde{\beta})$ and $([\tilde{\alpha} - 2b_{\min}]/\tilde{\beta},\, b_{\min}/\tilde{\beta})$ are equilibria. For all of these equilibrium prices the total price is equal to $(\tilde{\alpha} - b_{\min})/\tilde{\beta}$, the demand is equal to $b_{\min}$, the resulting profit of seller $i$ is equal to $\tilde{y}_i b_{\min}$, and thus the total profit of both sellers together is equal to

$$\tilde{y}_{-1} b_{\min} + \tilde{y}_1 b_{\min} \;=\; \frac{\tilde{\alpha} - b_{\min}}{\tilde{\beta}} b_{\min} \tag{6}$$

### 3.2. Perfect Coordination

Next we determine the maximum achievable total profit of the two sellers together, that is, the total profit if the sellers would perfectly coordinate pricing.

The total profit of the two sellers is given by

$$\tilde{g}(\tilde{y}_{-1}, \tilde{y}_1) \;:=\; [\tilde{y}_{-1} + \tilde{y}_1] \min\{b_{\min},\, \max\{0,\, \tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1)\}\}$$

If $b_{\min} \geq \tilde{\alpha}/2$, then the optimal total price is equal to

$$\bar{y}_{-1} + \bar{y}_1 \;=\; \frac{\tilde{\alpha}}{2\tilde{\beta}} \tag{7}$$

Note that (2) and (7) show that $\tilde{y}^*_{-1} + \tilde{y}^*_1 > \bar{y}_{-1} + \bar{y}_1$, that is, the total of the equilibrium prices under no alliance is greater than the optimal total price. (These results are reminiscent of the comparison of the cases with and without vertical integration by Spengler (1950); however, the setting here is different because one seller does not buy a product from another seller and add a mark-up before reselling it.) The corresponding demand is equal to

$$\tilde{\alpha} - \tilde{\beta}(\bar{y}_{-1} + \bar{y}_1) \;=\; \frac{\tilde{\alpha}}{2} \;>\; \frac{\tilde{\alpha}}{3} \;=\; \tilde{\alpha} - \tilde{\beta}(\tilde{y}^*_{-1} + \tilde{y}^*_1) \tag{8}$$

and the total profit of both sellers together is equal to

$$[\bar{y}_{-1} + \bar{y}_1]\left[\tilde{\alpha} - \tilde{\beta}(\bar{y}_{-1} + \bar{y}_1)\right] \;=\; \frac{\tilde{\alpha}^2}{4\tilde{\beta}} \tag{9}$$

If $b_{\min} \le \tilde{\alpha}/2$, then the optimal total price is given by $\bar{y}_{-1} + \bar{y}_1 = (\tilde{\alpha} - b_{\min})/\tilde{\beta}$, with corresponding demand equal to $b_{\min}$. The total profit of both sellers together is equal to $(\bar{y}_{-1} + \bar{y}_1)\, b_{\min} = (\tilde{\alpha} - b_{\min})\, b_{\min}/\tilde{\beta}$,

Note that when capacity is small, $b_{\min} \le \tilde{\alpha}/3$, the total profit of the setting with no alliance cannot be increased by coordination. When capacity is large, $b_{\min} \ge \tilde{\alpha}/2$, the relative amount by which the total profit can be increased is given by

$$\frac{\frac{\tilde{\alpha}^2}{4\tilde{\beta}} - \frac{2\tilde{\alpha}^2}{9\tilde{\beta}}}{\frac{2\tilde{\alpha}^2}{9\tilde{\beta}}} \;=\; \frac{1}{8}$$

When capacity is intermediate, $\tilde{\alpha}/3 \le b_{\min} \le \tilde{\alpha}/2$, then the relative amount by which the total profit can be increased is bounded by

$$0 \;\le\; \frac{\frac{\tilde{\alpha} - b_{\min}}{\tilde{\beta}} b_{\min} - \frac{2\tilde{\alpha}^2}{9\tilde{\beta}}}{\frac{2\tilde{\alpha}^2}{9\tilde{\beta}}} \;\le\; \frac{1}{8}$$

This potential increase in profit is the major economic motivation for alliances. The extent to which this increase can be attained by an alliance depends on the capacity and the customer choice behavior, including the extent to which the sellers can differentiate their products. In the next section we consider a resource exchange alliance and investigate the effect of both capacity and product differentiation on the total profit with and without an alliance.

### 3.3. Resource Exchange Alliance

In this paper, we focus on a form of alliances, so called "resource exchange alliance". Under this type of alliance, alliance members exchange resources (e.g., container slots) in advance of sales. After exchange, each seller controls resources allocated to them as though they were the owners of the resources. The key questions is how much of each resource (including money) should be exchanged to maximize the total profit of alliance members. One important thing to note here is that after exchange, alliance members compete for the same demand since each seller chooses their own prices to sell products. Therefore, it is critical to take the resulting competition into account when determining the optimal resource exchange amounts. As we will show later, the competition is affected by capacity and product differentiation levels.

Consider a resource exchange alliance involving the two sellers. Let $x_i \in [0, b_i]$ denote the amount of resource $i$ that seller $i$ makes available to seller $-i$, and let $x := (x_{-1}, x_1)$. Then the number of units of the two-resource product that seller $i$ can sell is $q_i(x) := \min\{b_i - x_i,\, x_{-i}\}$. Typically, cost items that make up the largest part of the marginal cost $c_i$ of seller $i$ are passed on to the customer via the marketing seller even if the customer buys the product from seller $-i$. For example, in the case of ocean carriers, these marginal costs consist mostly of amounts paid to other parties such as the port operator for stevedoring services and storage. Hence, assume that seller $-i$ pays seller $i$ an amount $c_i$ for each unit of resource $i$ that seller $-i$ uses, so that each seller has marginal cost equal to $c_{-1}+c_1$ for the two-resource product. Specifically, a resource exchange alliance with zero exchange of resources ($x=0$) may be chosen, in which case the sellers sell only the separate resources as in the case without an alliance. Thus, in general, the total profit of an optimally designed resource exchange alliance is no less than the total profit without an alliance. Let $y_i$ denote the markup of seller $i$, that is, the difference between the price of seller $i$ and the marginal cost $c_{-1}+c_1$ for the two-resource product.

The demand $d_i(y_i, y_{-i})$ for the product sold by seller $i$ depends on the prices as follows:

$$d_i(y_i, y_{-i}) \quad = \quad \max\{0,\ \alpha - \beta y_i + \gamma y_{-i})\} \tag{10}$$

where $\alpha$ and $\beta$ are positive constants and $\gamma \in (0, \beta)$. That is, after the alliance, sellers sell substitute product and therefore compete with each other. Here, provision is made for brand distinction between the products sold by the sellers. The constants are known to each seller. To keep the number of parameters in this example small, the constants $\alpha$, $\beta$, and $\gamma$ are the same for both sellers.

Under this setting, the number of units of product sold by seller $i$ is given by $\min\{q_i(x), \max\{0,\ \alpha - \beta y_i + \gamma y_{-i})\}\}$, and the profit of seller $i$ is given by

$$g_i(x, y_i, y_{-i}) \quad := \quad y_i \min\{q_i(x), \max\{0,\ \alpha - \beta y_i + \gamma y_{-i}\}\}$$

Next we establish a relation between $\tilde{\alpha}$ and $\tilde{\beta}$, and $\alpha$, $\beta$ and $\gamma$, to facilitate a fair comparison among the settings with no alliance, with perfect coordination, and with an alliance. Consider prices $(\tilde{y}_{-1}, \tilde{y}_1)$ in the no-alliance setting, such that $\tilde{y}_{-1}+\tilde{y}_1 < \tilde{\alpha}/\tilde{\beta}$. Suppose that the two alliance members charge the same price $y_{-1}=y_1=\tilde{y}_{-1}+\tilde{y}_1$ for the two-resource products. Then the total demand in the no-alliance setting given by (1) is equal to $\tilde{\alpha}-\tilde{\beta}(\tilde{y}_{-1}+\tilde{y}_1) > 0$, and the total demand in the alliance setting given by (10) is equal to $2(\alpha - \beta y_1 + \gamma y_1) = 2\alpha - 2(\beta-\gamma)(\tilde{y}_{-1}+\tilde{y}_1)$. Thus the total demand in the two settings is the same if $\tilde{\alpha}=2\alpha$ and $\tilde{\beta}=2(\beta-\gamma)$. It is also shown in Appendix A.4 that a model of perfect coordination with demand given by (10) leads to the same optimal prices, demands, and profits as the model in Section 3.2 with demand given by (1) if $\tilde{\alpha}=2\alpha$ and $\tilde{\beta}=2(\beta-\gamma)$. Hence the results for the settings with no alliance, with perfect coordination, and with an alliance will be compared using $\tilde{\alpha}=2\alpha$ and $\tilde{\beta}=2(\beta-\gamma)$.

For the setting with an alliance, for any given resource exchange $x$, let $(y^*_{-1}(x), y^*_1(x))$ denote the equilibrium prices of the two sellers for the two-resource product. The alliance design problem is to choose $x \in [0, b_{-1}] \times [0, b_1]$ to maximize

$$f(x) \quad := \quad g_{-1}(x, y^*_{-1}(x), y^*_1(x)) + g_1(x, y^*_1(x), y^*_{-1}(x))$$

and $x^*$ denote an optimal resource exchange. (existence and uniqueness of the equilibrium are addressed in the detailed calculations in Appendix A.3. The resulting profit of seller $i$ is given by $g_i(x, y_i^*(x), y_{-i}^*(x))$.

A natural question here might be on how the total profit $f(x^*)$ should be partitioned among the alliance members. First, note that if money can be exchanged together with the other resources, then any partition of the total profit can be achieved. In that case, the Nash bargaining solution for a two-player cooperative game provides a compelling partition of the total profit that is also easy to characterize: each alliance member receives its profit in the setting without an alliance plus half the difference between the maximum total profit $f(x^*)$ of the alliance and the total profit without an alliance. Detailed discussion and derivation is given in Section 4.6.

Table 1 and Figure 1 summarize the results for the settings with no alliance, with perfect coordination, and with an alliance. The detailed calculations are given in Appendix A. Here we just mention that there are three cases regarding capacity: (1) Capacity $b_{\min}$ is large enough so that both sellers can be provided with sufficient product capacity $q_i(x)$ to make capacity not constraining in equilibrium ($b_{\min} \geq 2\alpha\beta/(2\beta-\gamma)$), (2) Capacity $b_{\min}$ is so small that the product capacity $q_i(x)$ of both sellers must be constraining in equilibrium ($b_{\min} \leq \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2)$), and (3) Capacity $b_{\min}$ is small enough that the product capacity $q_i(x)$ of at least one seller must be constraining in equilibrium but large enough so that one seller can be provided with sufficient product capacity $q_i(x)$ to make capacity not constraining in equilibrium ($\alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) \leq b_{\min} \leq 2\alpha\beta/(2\beta-\gamma)$).

Figure 2 shows a plot of the relative increase in total profit with an alliance over no alliance, that is, $(f(x^*) - [\tilde{g}_{-1}(\tilde{y}_{-1}^*, \tilde{y}_1^*) + \tilde{g}_1(\tilde{y}_1^*, \tilde{y}_{-1}^*)])/[\tilde{g}_{-1}(\tilde{y}_{-1}^*, \tilde{y}_1^*) + \tilde{g}_1(\tilde{y}_1^*, \tilde{y}_{-1}^*)]$, as a function of $b_{\min}/\alpha$ and $\gamma/\beta$. The figure shows that the relative increase is largest when the capacity is large ($b_{\min} \geq \alpha$). In addition, the total profit under an alliance equals the total profit under perfect coordination as shown in Table 1. One additional observation which is interesting to note here is that in addition to profit increase, the prices without an alliance are higher than the prices under an alliance except when the capacity is small (where the prices and profits are same). In summary, we could show that a well designed resource exchange alliance indeed captures the foregone profit without an alliance. For general models presented later in Section 4 and Section 6, the main insights remain same.

## 4. Model

In this section, we present a general model for a resource exchange alliance which could involve multiple resources. In addition to the alliance model, we also present models for the settings with no alliance and with perfect coordination to facilitate comparisons.

As we describe in Section 3.3, a resource exchange alliance can be designed in two stages. At the first stage, alliance members together determine the optimal resource exchange amount which

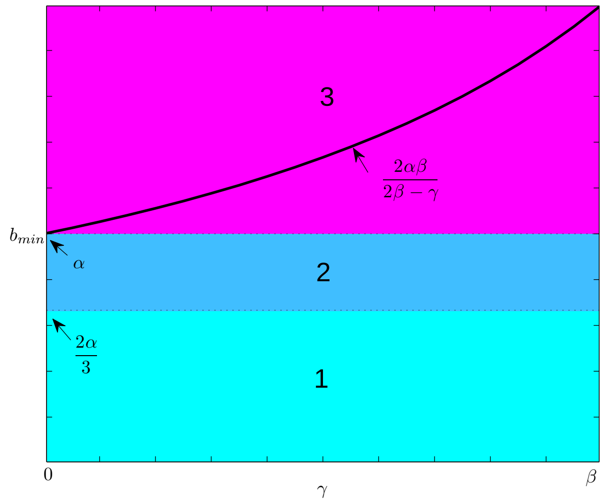


**Figure 1 The regions distinguished in Table 1**

**Table 1 Comparison of no alliance, perfect coordination, and a resource exchange alliance, in terms of price, demand, total profit, and consumer surplus, for a single product with two resources.**

| Region | Capacity | Quantity | No-Alliance | Perfect Coordination | Alliance |
|---|---|---|---|---|---|
| 1 | $0 \le b_{\min} \le \frac{2\alpha}{3}$ | Total Price | $\frac{2\alpha-b_{\min}}{2(\beta-\gamma)}$ | $\frac{2\alpha-b_{\min}}{2(\beta-\gamma)}$ | $\frac{2\alpha-b_{\min}}{2(\beta-\gamma)}$ |
| | | Total Demand | $b_{\min}$ | $b_{\min}$ | $b_{\min}$ |
| | | Total Profit | $\frac{(2\alpha-b_{\min})b_{\min}}{2(\beta-\gamma)}$ | $\frac{(2\alpha-b_{\min})b_{\min}}{2(\beta-\gamma)}$ | $\frac{(2\alpha-b_{\min})b_{\min}}{2(\beta-\gamma)}$ |
| 2 | $\frac{2\alpha}{3} \le b_{\min} \le \alpha$ | Total Price | $\frac{2\alpha}{3(\beta-\gamma)}$ | $\frac{2\alpha-b_{\min}}{2(\beta-\gamma)}$ | $\frac{2\alpha-b_{\min}}{2(\beta-\gamma)}$ |
| | | Total Demand | $\frac{2\alpha}{3}$ | $b_{\min}$ | $b_{\min}$ |
| | | Total Profit | $\frac{4\alpha^2}{9(\beta-\gamma)}$ | $\frac{(2\alpha-b_{\min})b_{\min}}{2(\beta-\gamma)}$ | $\frac{(2\alpha-b_{\min})b_{\min}}{2(\beta-\gamma)}$ |
| 3 | $\alpha \le b_{\min}$ | Total Price | $\frac{2\alpha}{3(\beta-\gamma)}$ | $\frac{\alpha}{2(\beta-\gamma)}$ | $\frac{\alpha}{2(\beta-\gamma)}$ |
| | | Total Demand | $\frac{2\alpha}{3}$ | $\alpha$ | $\alpha$ |
| | | Total Profit | $\frac{4\alpha^2}{9(\beta-\gamma)}$ | $\frac{\alpha^2}{2(\beta-\gamma)}$ | $\frac{\alpha^2}{2(\beta-\gamma)}$ |

maximizes the total profit of both sellers before the demand becomes known. At the second stage, each seller separately sets the optimal price to maximize their own profit under some constraints. As we have discussed earlier, the alliance members are now competing with each other by selling substitute products. Thus, an important aspect of this modeling is that when we make a decision at the first stage, we need to incorporate the resulting competition from the second stage. The modeling can be done with stochastic mathematical programming with equilibrium constraints. We will discuss more details in this section.

### 4.1. Multiple-Resource Network Example

We first provide an example with multiple resources to illustrate the models that will be formulated in later sections.

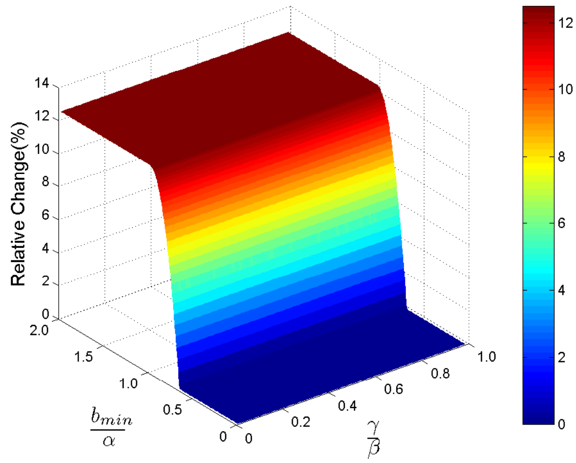


**Figure 2** **Plot of the relative increase in total profit with an alliance over no alliance, that is,** $(f(x^*) - [\tilde{g}_{-1}(\tilde{y}^*_{-1}, \tilde{y}^*_1) + \tilde{g}_1(\tilde{y}^*_1, \tilde{y}^*_{-1})])/[\tilde{g}_{-1}(\tilde{y}^*_{-1}, \tilde{y}^*_1) + \tilde{g}_1(\tilde{y}^*_1, \tilde{y}^*_{-1})]$**, as a function of** $b_{\min}/\alpha$ **and** $\gamma/\beta$**.**

Consider 2 sellers, indexed by $i = \pm 1$. (It can be easily seen from the results in Section 4.3 how to extend the model and the solution method to a setting with more than 2 sellers at the cost of more complicated notation.) Seller $i$ produces $k_i$ resource types indexed by $j = 1, \ldots, k_i$. For example, resource $j$ may denote the voyage of ocean carrier $i$ scheduled to depart from Cape Town to Santos every Monday at 8am. Initially, before any resource exchange, seller $i$ has quantity $b_{i,j}$ of resource $j$ and a constant marginal cost of $c_{i,j}$ per unit of resource $j$ consumed.

A liner shipping network is shown in Figure 3 and some voyage and port data are given in Table 2. In this network, port 1 is a connection hub for both ocean carriers. In fact, this network is similar to the one in the Hanjin-Evergreen Agreement described in the Introduction. Each carrier operates 4 voyages. For example, voyage 5, taking place from port 1 to port 4, is operated by carrier 1, and has a capacity of 3000 TEU. The set of products that can be sold by each carrier is different in the case with no alliance and the case with an alliance. Table 3 shows the products and the corresponding itineraries (here simply specified by the origin-destination pair) which could be offered by the two carriers. The column labeled "Carrier" specifies which carriers can sell each product in the case with no alliance and with an alliance. For example, in the case with no alliance, product 7 can be sold by carrier 1 only, and in the case with an alliance, product 7 can be sold by both carriers ("$A$" denotes both carriers under alliance). On the other hand, product 17, involving travel from port 3 to port 4 via port 1, can only be sold in the case with an alliance and in that case it can be sold by both carriers. However, note that there is demand for travel from port 3 to port 4 both in the case with no alliance and in the case with an alliance. In the case with no alliance, all demand for travel from port 3 to port 4 is satisfied by making two separate bookings; a booking with carrier -1 for travel from port 3 to port 1 and a booking with carrier 1 for travel

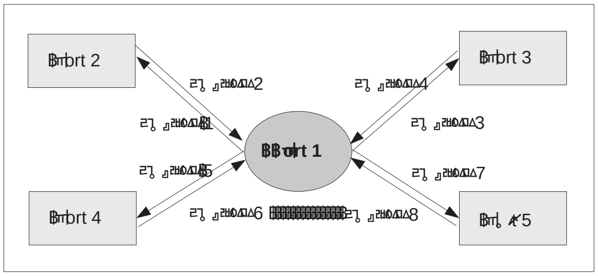

**Figure 3** **Multiple-resource network example**

| Voyage number | Carrier | Departure | Arrival | Capacity (TEU) |
|---|---|---|---|---|
| 1 | -1 | 1 | 2 | 3000 |
| 2 | -1 | 2 | 1 | 3000 |
| 3 | -1 | 1 | 3 | 3000 |
| 4 | -1 | 3 | 1 | 3000 |
| 5 | 1 | 1 | 4 | 3000 |
| 6 | 1 | 4 | 1 | 3000 |
| 7 | 1 | 1 | 5 | 3000 |
| 8 | 1 | 5 | 1 | 3000 |

**Table 2** **Voyage information**

**Table 3** **Product information for network example.**

| Product | Carrier | Origin | Destination | Product | Carrier | Origin | Destination |
|---|---|---|---|---|---|---|---|
| 1 | -1 or A | 1 | 2 | 11 | 1 or A | 4 | 5 |
| 2 | -1 or A | 2 | 1 | 12 | 1 or A | 5 | 4 |
| 3 | -1 or A | 1 | 3 | 13 | A only | 2 | 4 |
| 4 | -1 or A | 3 | 1 | 14 | A only | 4 | 2 |
| 5 | -1 or A | 2 | 3 | 15 | A only | 2 | 5 |
| 6 | -1 or A | 3 | 2 | 16 | A only | 5 | 2 |
| 7 | 1 or A | 1 | 4 | 17 | A only | 3 | 4 |
| 8 | 1 or A | 4 | 1 | 18 | A only | 4 | 3 |
| 9 | 1 or A | 1 | 5 | 19 | A only | 3 | 5 |
| 10 | 1 or A | 5 | 1 | 20 | A only | 5 | 3 |

from port 1 to port 4. In the case with an alliance, demand for travel from port 3 to port 4 can be satisfied in four different ways: (1) by making a booking with carrier -1 for travel from port 3 to port 1 and a booking with carrier 1 for travel from port 1 to port 4, or (2) by making a booking with carrier 1 for travel from port 3 to port 1 and a booking with carrier -1 for travel from port 1 to port 4, or (3) by making a booking for travel from port 3 to port 4 via port 1 with carrier -1, or (4) by making a booking for travel from port 3 to port 4 via port 1 with carrier 1. In the case with an alliance, the choices exercised by the buyers and thus the resulting aggregate demand depends on the prices of the carriers for the different products. In this paper we consider linear models of aggregate demand as specified in more detail later.

### 4.2. Resource Exchange Alliance Model

In this section we introduce a general model of a resource exchange alliance involving multiple resources. After resource exchange, seller $i$ may have some of each resource supplied by seller $-i$ as well as some of each resource supplied by itself. Index the union of the resources by $j = 1, \ldots, k$, where $k = k_{-1} + k_1$. Let $b_i = (b_{i,1}, \ldots, b_{i,k})$ denote the initial endowment of seller $i$ of each resource ($b_{i,j} = 0$ if resource $j$ is supplied by seller $-i$). Let $x_j$ denote the amount of resource $j$ that seller 1 makes available to seller $-1$. For example, $x = (-1100, -1200, -1000, -1500, 1400, 1700, 1300, 1600)$ for the network in Section 4.1 means that carrier $-1$ gives 1100 TEU on voyage 1 to carrier 1, carrier 1 gives 1400 TEU on voyage 5 to carrier $-1$, etc.

After the resource exchange, seller $i$ can sell $m_i$ products, indexed by $\ell = 1, \ldots, m_i$. In the example in Table 3, $m_i = 20$ for $i = \pm 1$. Let $y_{i,\ell}$ denote the price of seller $i$ for product $\ell$ in excess of the marginal cost of the product, and $d_{i,\ell}$ denote the demand for product $\ell$ of seller $i$. Consider the following linear demand model:

$$d_{i,\ell} \quad = \quad -\sum_{\ell'=1}^{m_i} E_{i,\ell,\ell'} y_{i,\ell'} + \sum_{\ell'=1}^{m_{-i}} B_{-i,\ell,\ell'} y_{-i,\ell'} + C_{i,\ell} \tag{11}$$

where $E_{i,\ell,\ell'}$ denotes the rate of change of the demand for product $\ell$ of seller $i$ with respect to the price of product $\ell'$ of the same seller $i$, and $B_{-i,\ell,\ell'}$ denotes the rate of change of the demand for product $\ell$ of seller $i$ with respect to the price of product $\ell'$ of the other seller $-i$. Using matrix notation, $d_i = -E_i y_i + B_{-i} y_{-i} + C_i$, where $d_i, y_i, C_i \in \mathbb{R}^{m_i}$, $E_i \in \mathbb{R}^{m_i \times m_i}$, $B_i \in \mathbb{R}^{m_{-i} \times m_i}$, and attention is restricted to values of $(y_{-1}, y_1)$ such that $d_i \geq 0$ for $i = \pm 1$. Let $A_i \in \mathbb{R}^{k \times m_i}$ be the "network matrix", i.e., $A_{i,j,\ell}$ denotes the amount of resource $j$ consumed by each unit of product $\ell$ sold by seller $i$.

Next we introduce the two-stage alliance design problem. Given a first stage resource exchange decision $x \in \mathbb{R}^k$, at the second stage each seller $i$ wants to solve the following optimization problem:

$$\begin{array}{ll} \max\limits_{y_i, d_i \in \mathbb{R}_+^{m_i}} & y_i^\top d_i \\ \text{s.t.} & A_i d_i \;\; \leq \;\; b_i - ix \\ & d_i \;\; = \;\; -E_i y_i + B_{-i} y_{-i} + C_i \;\; \geq \;\; 0 \end{array} \tag{12}$$

We are interested in the Nash equilibrium defined by the two optimization problems (12) for $i = \pm 1$. Since the demand in unknown at the first stage, we are taking the demand uncertainty into account with a stochastic version of the alliance design problem as follows. At the first stage, when $x$ is chosen, elements of matrices $E_i$ and $B_i$, and vectors $C_i$, are random. However, the network matrices $A_i$ are deterministic. Let $\xi := (E_{-1}, E_1, B_{-1}, B_1, C_{-1}, C_1)$ denote the random data vector. At the first stage, the expected value with respect to the distribution of $\xi$ of an objective (specified below) is optimized. Also, note that the Nash equilibrium associated with the second stage depends on the realization of $\xi$.

Let $Q_i := E_i + E_i^\top \in \mathbb{R}^{m_i \times m_i}$ denote the symmetric version of $E_i$. We assume that matrices $E_i$ and hence $Q_i$, are positive definite. Let $I_m$ denote the $m \times m$ identity matrix, $0_m$ denotes the zero vector in $\mathbb{R}^m$, and $0_{m,n}$ denotes the zero matrix in $\mathbb{R}^{m \times n}$. Then the optimization problem (12) can be written as follows:

$$\begin{array}{ll} \min\limits_{y_i \in \mathbb{R}_+^{m_i}} & \frac{1}{2} y_i^\top Q_i y_i - y_i^\top B_{-i} y_{-i} - C_i^\top y_i \\ \text{s.t.} & W_i \left( E_i y_i - B_{-i} y_{-i} \right) \;\; \geq \;\; \eta_i + i M_i x. \end{array} \tag{13}$$

where

$$W_i \;\; := \;\; \begin{bmatrix} A_i \\ -I_{m_i} \end{bmatrix}, \qquad \eta_i \;\; := \;\; W_i \tilde{C}_i + \begin{bmatrix} -b_i \\ 0_{m_i} \end{bmatrix}, \qquad M_i \;\; := \;\; \begin{bmatrix} I_k \\ 0_{m_i,k} \end{bmatrix}.$$

A point $(y_{-1}^*(x), y_1^*(x))$ is a solution of the equilibrium problem if $y_1^*(x)$ is an optimal solution of problem (13) for $i = 1$ when $y_{-1} = y_{-1}^*(x)$, and also $y_{-1}^*(x)$ is an optimal solution of problem (13) for $i=-1$ when $y_1=y_1^*(x)$. Note that $(y_{-1}^*(x),y_1^*(x))$ also depends on $\xi$ but the dependence is not shown in the notation. (The above problem is called a *generalized* Nash equilibrium problem since the feasible set of problem (13) depends on $y_{-i}$.) Let $V_i(x,\xi)$, $i=\pm1$, denote the optimal objective values of problem (13) at the equilibrium point given data $\xi$, i.e.,

$$V_i(x,\xi) \;:=\; \tfrac{1}{2}y_i^*(x)^\top Q_i y_i^*(x) - y_i^*(x)^\top B_{-i} y_{-i}^*(x) - C_i^\top y_i^*(x) \tag{14}$$

Note that these functions are well defined only if the equilibrium point $(y_{-1}^*(x),y_1^*(x))$ exists and is unique. We will discuss existence and uniqueness of the equilibrium point in Section 4.3.

Now, at the first stage, we consider designs of the resource exchange alliance that aim to maximize the total profit of the sellers. Let $b=b_1-b_{-1}\in\mathbb{R}^k$. Note that $b_j>0$ if resource $j$ is supplied by seller 1 and $b_j<0$ if resource $j$ is supplied by seller $-1$. Let $l_j$ and $u_j$ be lower and upper bounds, respectively, such that $b_j l_j\geq 0$ and $b_j u_j\geq 0$, that is, $l_j$, $u_j$, and $b_j$ have the same sign, and $|l_j|\leq|u_j|\leq|b_j|$. Then the first stage problem is as follows:

$$\begin{array}{ll}\max\limits_{x\in\mathbb{R}^k} & \left\{f(x) \;:=\; \mathbb{E}\left[V_{-1}(x,\xi)+V_1(x,\xi)\right]\right\}\\ \text{s.t.} & b_j x_j \;\geq\; 0 \qquad\qquad \forall\, j=1,\ldots,k\\ & |l_j| \;\leq\; |x_j| \;\leq\; |u_j| \qquad\qquad \forall\, j=1,\ldots,k\end{array} \tag{15}$$

As mentioned, the expectation in (15) is with respect to a specified probability distribution of the data vector $\xi$. In particular, if a single value for $\xi$ is considered in the first stage, then the problem (15) is deterministic and the expectation operator can be removed.

### 4.3. Existence and Uniqueness of Nash Equilibrium

Recall that the matrices $Q_i$ are positive definite and hence problem (13) is a convex quadratic programming problem. The first order (KKT) necessary and sufficient optimality conditions for problem (13) are

$$\begin{aligned} Q_i y_i - B_{-i}y_{-i} - C_i - E_i^\top W_i^\top \lambda_i &= 0\\ W_i\left(E_i y_i - B_{-i}y_{-i}\right) - \eta_i - iM_i x &\geq 0\\ \lambda_i &\geq 0\\ \lambda_i^\top\left[W_i\left(E_i y_i - B_{-i}y_{-i}\right) - \eta_i - iM_i x\right] &= 0\end{aligned} \tag{16}$$

where $\lambda_i$ denotes the vector of Lagrange multipliers associated with the inequality constraints in (13).

The optimality conditions (16) can be written as a variational inequality. A widely used "standard" approach to establish existence and uniqueness of a solution to the optimality conditions and thus existence and uniqueness of a Nash equilibrium is to exploit monotonicity of the variational inequality. However, in our case the variational inequality is not monotone and therefore a different approach is required. We explain how we could solve the MPEC problem next.

Consider the optimization problem

$$\begin{array}{llll} \min\limits_{y_{-1},y_1,\lambda_{-1},\lambda_1} & \sum\limits_{i=\pm 1} \lambda_i^{\mathsf{T}} \left[W_i \left(E_i y_i - B_{-i} y_{-i}\right) - \eta_i - i M_i x\right] \\ \text{s.t.} & Q_i y_i - B_{-i} y_{-i} - C_i - E_i^{\mathsf{T}} W_i^{\mathsf{T}} \lambda_i \;=\; 0, \quad i = \pm 1 \\ & W_i \left(E_i y_i - B_{-i} y_{-i}\right) - \eta_i - i M_i x \;\geq\; 0, \quad i = \pm 1 \\ & \lambda_i \;\geq\; 0, \quad i = \pm 1 \end{array} \tag{17}$$

Note that the objective value of problem (17) is nonnegative at all feasible points, and $(y_{-1}^*, y_1^*, \lambda_{-1}^*, \lambda_1^*)$ is a solution of the optimality conditions (16) if and only if its objective value in problem (17) is zero, in which case it is an optimal solution of problem (17). It follows from the first equation of (16) that

$$\lambda_i^{\mathsf{T}} W_i \;=\; y_i^{\mathsf{T}} Q_i E_i^{-1} - y_{-i}^{\mathsf{T}} B_{-i}^{\mathsf{T}} E_i^{-1} - C_i^{\mathsf{T}} E_i^{-1}$$

After substitution of this into the objective, problem (17) becomes

$$\begin{array}{llll} \min\limits_{y_{-1},y_1,\lambda_{-1},\lambda_1} & \sum\limits_{i=\pm 1} \left(y_i^{\mathsf{T}} Q_i E_i^{-1} - y_{-i}^{\mathsf{T}} B_{-i}^{\mathsf{T}} E_i^{-1} - C_i^{\mathsf{T}} E_i^{-1}\right) \left(E_i y_i - B_{-i} y_{-i}\right) - \lambda_i^{\mathsf{T}} \left(\eta_i + i M_i x\right) \\ \text{s.t.} & Q_i y_i - B_{-i} y_{-i} - C_i - E_i^{\mathsf{T}} W_i^{\mathsf{T}} \lambda_i \;=\; 0, \quad i = \pm 1 \\ & W_i \left(E_i y_i - B_{-i} y_{-i}\right) - \eta_i - i M_i x \;\geq\; 0, \quad i = \pm 1 \\ & \lambda_i \;\geq\; 0, \quad i = \pm 1 \end{array} \tag{18}$$

We note that the objective function of problem (18) is quadratic with its quadratic term $(y_{-1}^{\mathsf{T}}, y_1^{\mathsf{T}})\, \Psi\, (y_{-1}^{\mathsf{T}}, y_1^{\mathsf{T}})^{\mathsf{T}}$, where

$$\Psi \;:=\; \begin{bmatrix} Q_{-1} + B_{-1}^{\mathsf{T}} E_1^{-1} B_{-1} & -B_{-1} - Q_{-1} E_{-1}^{-1} B_1 \\ -B_1 - Q_1 E_1^{-1} B_{-1} & Q_1 + B_1^{\mathsf{T}} E_{-1}^{-1} B_1 \end{bmatrix} \tag{19}$$

The problem (18) is a convex quadratic program if and only if the matrix $\Psi$, or equivalently the symmetric matrix $\Psi + \Psi^{\mathsf{T}}$, is positive semidefinite.

THEOREM 1. *Suppose that the problem* (18) *is feasible and that the matrix* $\Psi$*, defined in* (19)*, is positive definite. Then problem* (18) *has an optimal solution* $(y_{-1}^*, y_1^*, \lambda_{-1}^*, \lambda_1^*)$ *with* $(y_{-1}^*, y_1^*)$ *being unique. Moreover, if the optimal objective value of problem* (18) *is zero, then* $(y_{-1}^*, y_1^*)$ *is the unique Nash equilibrium.*

The proof is given in Appendix B.

A similar approach can be used if there are more than two sellers. In such a case more than two sets of optimality conditions of the form (16) will be involved and in the quadratic program (18) the index $i$ will take on more than two values.

Given the reformulation above, the question of existence and uniqueness of the Nash equilibrium can be answered with the following steps: (1) verify that the matrix $\Psi$ (or the symmetric matrix $\Psi + \Psi^{\mathsf{T}}$) is positive definite, (2) solve the convex quadratic program (18) if $\Psi$ is positive definite, and (3) verify that the optimal objective value is zero. Note that if $\Psi$ is positive definite, then the quadratic program (18) can be solved efficiently and hence existence and uniqueness of the equilibrium point can easily be verified numerically. Some simple necessary conditions and sufficient conditions for $\Psi$ to be positive definite can be identified. A necessary condition for $\Psi$ to be positive definite is that its block diagonal matrices $Q_{-1} + B_{-1}^{\mathsf{T}} E_1^{-1} B_{-1}$ and $Q_1 + B_1^{\mathsf{T}} E_{-1}^{-1} B_1$ be positive definite. Note that these matrices are indeed positive definite because $E_{-1}$ and $E_1$ are positive definite. Also, note that if $B_{-1}$ and $B_1$ are null matrices, then matrix $\Psi$ is the block diagonal matrix $\text{diag}(Q_{-1}, Q_1)$, and hence $\Psi$ is positive definite because $Q_{-1}$ and $Q_1$ are positive definite. More general, if matrices $E_i$ are "significantly bigger" than $B_i$, then one may expect matrix $\Psi$ to be positive definite. The intuitive explanation of this sufficient condition is that if the demand for each seller's product depends on the price of the product more than it depends on the prices of other products, then the matrix is positive definite. Another instructive example is the following.

Example 1. Suppose that the products of the two sellers are direct substitutes for each other, that is, for each product of seller $i$ there is a product of seller $-i$ that is a close substitute. This allows the possibility that seller $-i$ may not be able to sell the substitute product because it does not have the resources to do so. It seems that in the applications of interest, the set of products can always be chosen so that this property holds. Hence, the matrices $B_i$ are squared, i.e., $m_{-1} = m_1$. Suppose that the matrices $E_i$ and $B_i$, $i = \pm 1$, are diagonal. Then $Q_i = E_i$ and

$$\Psi \quad = \quad \begin{bmatrix} E_{-1} + B_{-1}^2 E_1^{-1} & -B_{-1} - B_1 \\ -B_{-1} - B_1 & E_1 + B_1^2 E_{-1}^{-1} \end{bmatrix}.$$

Since matrices $E_i$ are positive definite it follows that $E_1 + B_1^2 E_{-1}^{-1}$ is positive definite, and thus it follows by the Schur complement condition for positive definiteness that $\Psi$ is positive definite if and only if the matrix $E_{-1} + B_{-1}^2 E_1^{-1} - (B_{-1} + B_1)^2 (E_1 + B_1^2 E_{-1}^{-1})^{-1}$ is positive definite. Since matrices $E_i$ and $B_i$ are diagonal, this matrix is positive definite if and only if the matrix

$$(E_{-1} + B_{-1}^2 E_1^{-1})(E_1 + B_1^2 E_{-1}^{-1}) - (B_{-1} + B_1)^2 \quad = \quad E_{-1}E_1 + B_{-1}^2 B_1^2 E_{-1}^{-1} E_1^{-1} - 2B_{-1}B_1$$

is positive definite. In turn this matrix is positive definite if and only if the matrix

$$E_{-1}^2 E_1^2 + B_{-1}^2 B_1^2 - 2E_{-1}E_1 B_{-1} B_1 \quad = \quad (E_{-1}E_1 - B_{-1}B_1)^2$$

is positive definite. Note that the last matrix is always positive semidefinite and is positive definite if and only if matrix $E_{-1}E_1 - B_{-1}B_1$ does not have any zero diagonal elements.

### 4.4. No Alliance Model

In this section, we present a model for the setting with no alliance. This model will be used to compare the profits under no alliance, under an alliance, and under perfect coordination. First we describe the demand model for the setting with no alliance.

Under an alliance, there are a total of $m$ distinct products. Some of the products may be offered by only one seller and some of the products may be offered by both sellers. In the example in Table 3, $m = 20$ and each of the 20 products is offered by both sellers in an alliance. These $m$ products can be partitioned into three subsets: sets $L_i$, for $i = \pm 1$, of products which can be offered by seller $i$ with and without an alliance and set $L_0$ of products which could be offered only under an alliance. For the example in Table 3, $L_{-1}$ contains products 1 to 6, $L_1$ contains products 7 to 12, and $L_0$ contains products 13 to 20.

As before, let $\tilde{y}_{i,\ell}$ denote the price of seller $i$ for product $\ell \in L_i$. Suppose that the resulting demand for product $\ell \in L_i$ is given by

$$\tilde{d}_{i,\ell} \;=\; -\sum_{\ell' \in L_i} \tilde{E}_{i,\ell,\ell'} \tilde{y}_{i,\ell'} + \sum_{\ell' \in L_{-i}} \tilde{B}_{-i,\ell,\ell'} \tilde{y}_{-i,\ell'} + \tilde{C}_{i,\ell} \tag{20}$$

Using matrix notation, $\tilde{d}_i = -\tilde{E}_i \tilde{y}_i + \tilde{B}_{-i} \tilde{y}_{-i} + \tilde{C}_i$, where $\tilde{d}_i, \tilde{y}_i, \tilde{C}_i \in \mathbb{R}^{|L_i|}$, $\tilde{E}_i \in \mathbb{R}^{|L_i| \times |L_i|}$, $\tilde{B}_i \in \mathbb{R}^{|L_{-i}| \times |L_i|}$, and attention is restricted to values of $(\tilde{y}_{-1}, \tilde{y}_1)$ such that $\tilde{d}_i \geq 0$ for $i = \pm 1$. Let $\tilde{A}_{i,j,\ell}$ denote the amount of resource $j$ consumed by each unit of product $\ell \in L_i$, and let $\tilde{A}_i \in \mathbb{R}^{k_i \times |L_i|}$ denote the network matrix.

Similar to the example with two resources in Section 3, the parameters $E, B, C$ in demand model (11) and the parameters $\tilde{E}, \tilde{B}, \tilde{C}$ in demand model (20) should be related in a particular way to facilitate a fair comparison of the prices, demands, and total profits between the settings with and without an alliance. The derivation of the relation is given in Appendix C.

The setting with no alliance is formulated as a non-cooperative game in which each seller $i$ wants to solve the optimization problem

$$\begin{array}{lll} \max\limits_{\tilde{y}_i, \tilde{d}_i \in \mathbb{R}_+^{|L_i|}} & \tilde{y}_i^{\mathsf{T}} \tilde{d}_i & \\ \text{s.t.} & \tilde{A}_i \tilde{d}_i \;\leq\; b_i & \\ & \tilde{d}_i \;=\; -\tilde{E}_i \tilde{y}_i + \tilde{B}_{-i} \tilde{y}_{-i} + \tilde{C}_i \;\geq\; 0 & \end{array} \tag{21}$$

The no alliance outcome is the Nash equilibrium defined by the two optimization problems (21) for $i = \pm 1$ as long as it exists and is unique. The Nash equilibrium is computed using the same approach described in Section 4.3.

### 4.5. Perfect Coordination Model

The models with and without an alliance presented above are compared with a perfect coordination model given in this section. The perfect coordination model considers a setting in which the sellers coordinate pricing together to maximize the sum of the sellers' profits as given by the following optimization problem:

$$\begin{array}{ll} \max\limits_{(y_{-1}, y_1) \in \mathbb{R}^{m_{-1}} \times \mathbb{R}^{m_1}} & \sum\limits_{i=\pm 1} y_i^{\mathsf{T}} \left(-E_i y_i + B_{-i} y_{-i} + C_i\right) \\ \text{s.t.} & \sum\limits_{i=\pm 1} A_i \left(-E_i y_i + B_{-i} y_{-i} + C_i\right) \;\leq\; b_{-1} + b_1 \\ & -E_i y_i + B_{-i} y_{-i} + C_i \;\geq\; 0\,, \quad i = \pm 1 \end{array} \tag{22}$$

### 4.6. Alliance Profit Allocation

In a resource exchange alliance, resources including money are exchanged between the alliance members. Thus the allocation of the alliance profit is one of the alliance design decisions. In this section, we address the question of how the alliance profit should be allocated to each members.

We follow an axiomatic approach, similar to Nash (1950, 1953), Kalai and Smorodinsky (1975) and Kalai (1977). To better explain the meaning of one of the axioms, here we allow the profits of the sellers to be measured in different currency units with exchange rate coefficients $e_{-1}, e_1$ to convert the profits to a common currency unit, but in the rest of the paper, it is assumed that the costs and revenues have already been converted to the same currency unit. Let $a = (a_{-1}, a_1)$ denote the Nash equilibrium profits if the sellers do not form an alliance, that is, $a_i \in \mathbb{R}$ denotes the profit of seller $i$ if the sellers do not collaborate.

Let the maximum total profit $g^* := \sup\{e_{-1}g_{-1}(x, y_{-1}, y_1) + e_1 g_1(x, y_1, y_{-1}) : 0 \leq x_i \leq b_i, i = \pm 1,$ $(y_{-1}, y_1)$ is a Nash equilibrium after resource exchange $x\}$. The set of achievable profit pairs under a resource exchange alliances is denoted by $A := \{(b_{-1}, b_1) : a_i \leq b_i, i = \pm 1, e_{-1}b_{-1} + e_1 b_1 \leq g^*\}$. Also, let $\mathcal{A}$ denote the set of all such allocation problems characterized by $(a, A)$. An allocation solution is a function $f : \mathcal{A} \mapsto \mathbb{R}^2$ such that $f(a, A) \in A$ for all $(a, A) \in \mathcal{A}$. Given these notations, we impose the following axioms on an allocation solution:

*Pareto optimality:* For each $(a, A) \in \mathcal{A}$, it holds that $f(a, A)$ is strongly Pareto optimal on $A$, that is, for each $b \in A$ such that $b \neq f(a, A)$ it holds that $f_i(a, A) > b_i$ for at least one $i$.

*Symmetry:* For each symmetric $(a, A) \in \mathcal{A}$, that is, $a_{-1} = a_1$ and if $(b_{-1}, b_1) \in A$ then $(b_1, b_{-1}) \in A$, it holds that $f_{-1}(a, A) = f_1(a, A)$.

*Invariance under positively homogeneous affine transformations:* For each $(a, A) \in \mathcal{A}$ and coefficients $c_{-1}, c_1 > 0$, $d_{-1}, d_1 \in \mathbb{R}$, consider the allocation problem $(a', A') \in \mathcal{A}$ given by $a'_i := c_i a_i + d_i$ and $A' := \{(c_{-1}b_{-1} + d_{-1}, c_1 b_1 + d_1) \in \mathbb{R}^2 : (b_{-1}, b_1) \in A\}$. Then $f(a', A')$ satisfies $f_i(a', A') = c_i f_i(a, A) + d_i$ for $i = \pm 1$.

The last axiom means that the profit allocation to each seller is independent of the currency units in which the seller's profits are measured and independent of the inclusion of constant costs or revenues in the profit function.

Proposition 1. *The unique allocation solution that satisfies the axioms of Pareto optimality, symmetry, and invariance under positively homogeneous affine transformations is given by*

$$f_i(a, A) = \frac{g^* - e_{-i}a_{-i} + e_i a_i}{2e_i} \tag{23}$$

The proof is given in Appendix D. Note that the allocation solution satisfies

$$e_{-1}\left[f_{-1}(a, A) - a_{-1}\right] = e_1\left[f_1(a, A) - a_1\right] = \frac{g^* - e_{-1}a_{-1} - e_1 a_1}{2}$$

that is, the incremental profit $g^* - e_{-i}a_{-i} - e_i a_i$ relative to the Nash equilibrium is divided equally between the sellers.

## 5. Solution Approach

In this section, we present a solution method for the multiple-resource model described in Section 4. Recall that we first solve the problem (18) to solve the second-stage Nash equilibrium problem and that problem (18) can be solved efficiently if the matrix $\Psi$ defined in (19) is positive definite. Next consider the first stage problem (15). The expectation in (15) is taken with respect to the probability distribution of the random data vector $\xi$. We assume that we can sample from that distribution by using Monte Carlo sampling techniques and hence generate an independent and identically distributed sample $\xi^1, \ldots, \xi^N$. Next we approximate the expectation with the sample average and construct the following Sample Average Approximation (SAA) problem:

$$\begin{array}{lllll} \max\limits_{x \in \mathbb{R}^k} & \left\{ \hat{f}_N(x) \ := \ \sum_{n=1}^N \left[ V_{-1}(x, \xi^n) + V_1(x, \xi^n) \right] \right\} & & & \\ \text{s.t.} & b_j x_j \ \geq \ 0 & & \forall\, j = 1, \ldots, k & \\ & |l_j| \ \leq \ |x_j| \ \leq \ |u_j| & & & \forall\, j = 1, \ldots, k \end{array} \tag{24}$$

Theoretical properties of the SAA approach have been studied extensively (e.g., Shapiro et al. 2009). Under mild conditions, the optimal objective value and optimal solution of the SAA problem (24) converge exponentially fast to the optimal objective value and optimal solution of the problem (15) (cf., Shapiro and Xu 2008). The first-stage problem may not be convex and thus it may be hard to solve problem (24) to optimality. For that reason, we may only ensure convergence to a stationary point of the problem (15). Nevertheless, in our numerical experiments, solutions typically seem to be stable and insensitive to the choice of starting point.

In order to solve the SAA problem (24) numerically, we need to compute derivatives $\nabla_x V_i(x, \xi^n)$ of the first-stage objective functions $V_i$ at a feasible point $x$ and sample point $\xi^n$. Consider a feasible point $x$ and assume that $\Psi$ is positive definite and that the second-stage problem has an equilibrium point $(y^*_{-1}(x), y^*_1(x))$ (the equilibrium depends on $\xi^n$ as well but the dependence is not shown in the notation). Let $(y^*_{-1}(x), y^*_1(x), \lambda^*_{-1}(x), \lambda^*_1(x))$ be a solution of the system (16) of first order optimality conditions (and thus $(y^*_{-1}(x), y^*_1(x), \lambda^*_{-1}(x), \lambda^*_1(x))$ is also a solution of the quadratic programming problem (17)). Note that, since $\Psi$ is positive definite, it holds that $(y^*_{-1}(x), y^*_1(x))$ is unique and is a continuous function of $x$ (e.g., Bonnans and Shapiro 2000).

Recall that Lagrange multipliers corresponding to inactive constraints are zeros. Let

$$\mathcal{I}_i(y_i, y_{-i}, x) \ := \ \left\{ j \in \{1, \ldots, k + m_i\} \, : \, \left[ W_i \left( E_i y_i - B_{-i} y_{-i} \right) - \eta_i - i M_i x \right]_j = 0 \right\}$$

denote the index set of active constraints of the problem (13). It is said that the strict complementarity condition holds at an equilibrium point $(y^*_{-1}(x), y^*_1(x))$ if among the corresponding Lagrange multiplier vectors $\lambda_i$, there exists at least one such that $[\lambda_i]_j > 0$ for all $j \in \mathcal{I}_i(y_i^*(x), y_{-i}^*(x), x)$, for $i = \pm 1$, i.e., there are Lagrange multipliers corresponding to the active constraints that are positive.

Now, suppose that the strict complementarity condition holds at $(y_{-1}^*(x), y_1^*(x))$, with $[\lambda_i^*(x)]_j > 0$ for all $j \in \mathcal{I}_i(y_i^*(x), y_{-i}^*(x), x)$, for $i = \pm 1$. Then for small perturbations $dx$ of $x$, the active constraints remain active and the inactive constraints remain inactive. Therefore, by linearizing the optimality conditions (16) at $(y_{-1}^*(x), y_1^*(x), \lambda_{-1}^*(x), \lambda_1^*(x))$, the following system of $m_{-1} + m_1 + 2k$ linear equations in $m_{-1} + m_1 + 2k$ unknowns $(dy_{-1}, dy_1, d\lambda_{-1}, d\lambda_1)$ is obtained:

$$\begin{aligned} Q_i dy_i - B_{-i} dy_{-i} - E_i^\mathsf{T} W_i^\mathsf{T} d\lambda_i &= 0, && i = \pm 1 \\ [W_i (E_i dy_i - B_{-i} dy_{-i}) - i M_i dx]_j &= 0, \ j \in \mathcal{I}_i(y_i^*(x), y_{-i}^*(x), x), && i = \pm 1 \\ [d\lambda_i]_j &= 0, \ j \notin \mathcal{I}_i(y_i^*(x), y_{-i}^*(x), x), && i = \pm 1 \end{aligned} \tag{25}$$

Suppose that the linear system (25) is nonsingular. Then for any $dx$ sufficiently small, the system (25) has a unique solution and by the Implicit Function Theorem, the solution of (25) gives the differential of $(y_{-1}^*(x), y_1^*(x), \lambda_{-1}^*(x), \lambda_1^*(x))$ at $x$. More specifically, the system (25) can be written in the form $S(dy_{-1}, dy_1, d\lambda_{-1}, d\lambda_1) = T\,dx$, where $S \in \mathbb{R}^{(m_{-1}+m_1+2k)\times(m_{-1}+m_1+2k)}$ and $T \in \mathbb{R}^{(m_{-1}+m_1+2k)\times k}$. If $S$ is nonsingular, then $(dy_{-1}, dy_1, d\lambda_{-1}, d\lambda_1) = S^{-1}T\,dx$, and thus $\nabla(y_{-1}^*(x), y_1^*(x), \lambda_{-1}^*(x), \lambda_1^*(x)) = S^{-1}T$. It follows from (14) that

$$\nabla_x V_i(x,\xi) = \nabla y_i^*(x)^\mathsf{T} Q_i y_i^*(x) - \nabla y_i^*(x)^\mathsf{T} B_{-i} y_{-i}^*(x) - \nabla y_{-i}^*(x)^\mathsf{T} B_{-i}^\mathsf{T} y_i^*(x) - \nabla y_i^*(x)^\mathsf{T} C_i \tag{26}$$

$$\nabla_{xx}^2 V_i(x,\xi) = \nabla y_i^*(x)^\mathsf{T} Q_i \nabla y_i^*(x) - \nabla y_i^*(x)^\mathsf{T} B_{-i} \nabla y_{-i}^*(x) - \nabla y_{-i}^*(x)^\mathsf{T} B_{-i}^\mathsf{T} \nabla y_i^*(x) \tag{27}$$

can be calculated easily.

The analysis above shows that sufficient conditions for differentiability of $V_i$ with respect to $x$ at $(x, \xi)$ are the strict complementarity condition and nondegeneracy of the system (25). These conditions are not necessary — for example, if $M_i = 0$ for $i = \pm 1$, then $V_i(x, \xi)$ is constant and hence differentiable with respect to $x$. Also, the expectation operator often smooths nondifferentiable functions. For example, if $\nabla_x V_i(x, \xi)$ exists for almost every $\xi$ and a mild boundedness condition holds, then $\mathbb{E}[V_i(x, \xi)]$ is differentiable at $x$ and $\nabla_x \mathbb{E}[V_i(x,\xi)] = \mathbb{E}[\nabla_x V_i(x,\xi)]$ (e.g., Shapiro et al. 2009, Theorem 7.44). The derivatives in (26) and (27) are used to solve SAA problems (24) with a trust-region method. Numerical results are given in Section 6.

## 6. Numerical Examples

In this section, we present numerical results to compare profits in settings with an alliance, no alliance, and perfect coordination. We present results for the network example given in Section 4.1. We first provide the results for the deterministic case with known demand functions in Section 6.1, and then present results for the stochastic case with random demand functions in Section 6.2.

### 6.1. Deterministic Examples

We first describe how the input data $E_i$, $B_i$, and $C_i$ for the numerical examples were chosen. For the example network, $m_{-1} = m_1 = 20$, and thus $E_i, B_i \in \mathbb{R}^{20\times 20}$ and $C_i \in \mathbb{R}^{20}$ for $i = \pm 1$. For each instance, a specific ratio $r_1 \in [0, 1)$ is chosen such that $|B_{-i,\ell,\ell'}| = r_1 |E_{i,\ell,\ell'}|$. Thus, $r_1$ is similar to the ratio $\gamma/\beta$ of the two-resource example in Section 3.3 and represents the level of differentiation between the sellers' products (smaller ratios mean higher levels of differentiation). For all instances, it was verified that the resulting matrix $\Psi$ defined in (19) was positive definite.

For the no alliance setting, we used the transformations in Appendix C to obtain $\tilde{E}_i$, $\tilde{B}_i$, and $\tilde{C}_i$. In addition, we investigated the effect of a difference in product attractiveness between the settings with and without an alliance. As mentioned, in a setting without an alliance, a buyer may have to buy products from multiple sellers and combine them to obtain the product desired by the buyer. Under an alliance a seller may offer the combined product to the buyer, making it more convenient for the buyer to obtain the product ("one-stop shopping"). There may be additional ways in which an alliance increases demand. For example, with a liner shipping alliance, the coordination of connecting voyage schedules to reduce lay-over time or missed connections and rebooking cost may further enhance the combined product under an alliance. This might increase the potential demand level under an alliance compared to that under no alliance. Motivated by these observations, we solved some instances in which the demands under no alliance is obtained using the transformations in Appendix C, but with a reduction in the demand for products assembled from more than one seller by a factor of $r_2 \in (0, 1]$ (in the notation of that section, the part of the demand for products in $L_i$ derived from the demand for products in $L_{0,-1} \cup L_{0,1}$ was reduced by a factor of $r_2$).

The two-stage alliance design problem (15) was solved using a trust region algorithm. At each iteration, given the current value of the resource exchange vector $x$, the convex quadratic program (17) was solved. It was verified that the optimal objective value of (17) was zero, that is, the solution of (17) gave a solution of the second stage equilibrium problem (12) for $i = \pm 1$. It was also verified that the strict complimentary condition held and that the system (25) was nonsingular. Next the derivatives of the objective function of (15) with respect to $x$ could be computed, and the trust region algorithm could execute the next iteration. As mentioned, the objective function of (15) may not be convex. To address the concern of potential multiple local optima, we used 50 different starting points $x_0$ for the first iteration for each instance. In each scenario, all 50 starting points led to similar final solutions and final objective values.

For the no alliance model, the second-stage equilibrium problem had to be solved only once for each instance. For the perfect coordination model, the convex quadratic optimization problem (22) also had to be solved only once for each instance.

**Table 4 Comparison of total profit for a resource exchange alliance, no alliance, and perfect coordination, for different levels of product differentiation.**

| Deterministic Model ($r_2 = 1$) | $r_1 = 0.2$ | | $r_1 = 0.5$ | | $r_1 = 0.8$ | |
|---|---|---|---|---|---|---|
| | Total Revenue | Relative increase (%) | Total Revenue | Relative increase (%) | Total Revenue | Relative increase (%) |
| No alliance | 3180600 | | 3227900 | | 3269800 | |
| Perfect Coordination | 3434300 | 7.98 | 3433400 | 6.37 | 3433000 | 4.99 |
| Alliance | 3432355 | 7.92 | 3416153 | 5.83 | 3363869 | 2.88 |

**Table 5 Comparison of maximum achievable total revenue under different convenience level**

| Deterministic Model ($r_1 = 0.5$) | $r_2 = 0.2$ (High) | | $r_2 = 0.6$ | | $r_2 = 1$ (No Difference) | |
|---|---|---|---|---|---|---|
| | Total Revenue | Relative increase (%) | Total Revenue | Relative increase (%) | Total Revenue | Relative increase (%) |
| No alliance | 3115900 | | 3184500 | | 3227900 | |
| Perfect Coordination | 3433400 | 10.19 | 3433400 | 7.82 | 3433400 | 6.37 |
| Alliance | 3416153 | 9.64 | 3416173 | 7.27 | 3416153 | 5.83 |

Table 4 presents the total profits under different levels of product differentiation represented by different values of $r_1$ given $r_2 = 1$ and with diagonal matrices $E_i$ and $B_i$. The largest increase in profits relative to the no alliance setting was obtained under high levels of product differentiation. For example, when $r_1 = 0.2$, an alliance increases the profit of the no alliance setting by 7.92%, and perfect coordination increases the profit by 7.98%. Even under a low level of product differentiation ($r_1 = 0.8$), an alliance still increases the profit by 2.88%, and perfect coordination increases the profit by 4.99%. Similar results were obtained with non-diagonal matrices.

In order to further investigate the impact of capacity and product differentiation level on the competition and thus the potential profit improvement, we performed the experiments given different capacity levels and $r_1$ values. In order to compare the results easily, we generated similar plots to the ones given in Section 3.3 for the two-resource model. Here, there exists demand for a product that consists of only one resource thus it represents more general cases. Figure 4 shows a plot of the relative increase in total profit with an alliance over no alliance as a function of the capacity $b_{min}/\alpha$ and the product differentiation level $\gamma/\beta$. The general insights remain same; the relative increase is largest when the capacity is large and the product differentiation level is high. Interestingly, in addition to the profit increase, we also found that the prices under no alliance is generally higher than the price under an alliance as shown in Figure 5. These are consistent with what we have observed in Section 3.3.

We also compared profits for different values of $r_2$. Table 5 compares the total profits under different levels of convenience represented by different values of $r_2$ for $r_1 = 0.5$ and with diagonal matrices $E_i$ and $B_i$. As expected, the relative increase in profit is larger for smaller values of $r_2$.

### 6.2. Stochastic Examples

In this section, we present results for the stochastic model (that is, the first stage problem (15) with expectation in the objective) presented in Section 4. The random data $E_i$, $B_i$, and $C_i$ followed

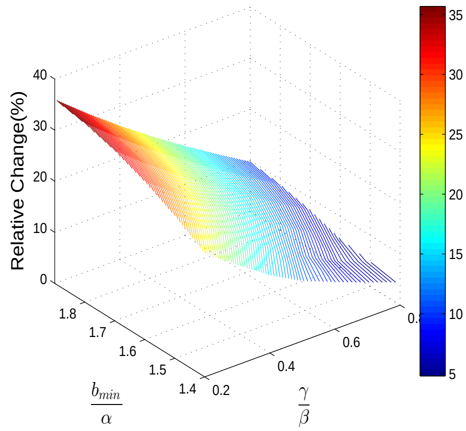


**Figure 4 Plot of the relative increase in profit with an alliance over no alliance.**

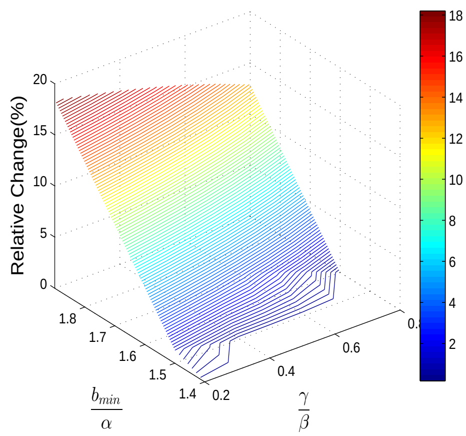


**Figure 5 Plot of the relative decrease in price with an alliance over no alliance.**

a multivariate normal distribution with means as described in Section 6.1, standard deviations proportional to the means, and correlation coefficients of 0.6.

We generated and solved SAA problems with different sample sizes $N = 20, 40, \ldots, 500$. At each iteration of the first-stage problem, the second-stage problem was solved for each of the $N$ sample points $\xi^n$. Then, for each of the $N$ sample points $\xi^n$, the derivatives of $V_i(x, \xi^n)$ were computed as given in (26) and (27). The averages of these derivatives over the $N$ sample points then gave the derivatives of the first-stage objective of the SAA problem (24).

Finally, after a resource exchange $x$ was chosen by solving a SAA problem, we evaluated the total profits in the alliance, no alliance, and perfect coordination settings with an independent and identically distributed sample of 1000 sample points, independent of the samples used in the SAA problem. Table 6 reports the number of iterations of the trust region algorithm until termination, the resource exchange solution $x_{opt}$ at termination, the objective value ($obj_{opt}$) of the SAA problem at $x_{opt}$, and the gradient norm ($\|g\|$) of the SAA objective function at $x_{opt}$, for different sample

**Table 6 Optimal solution under different sample sizes for the stochastic case**

| $n$ | $iter$ | $obj_{opt}$ | $\|g\|$ | $x_{opt}$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 20 | 41 | -3409501 | 1.08E-04 | 1444 | 1550 | 1395 | 1480 | -1501 | -1586 | -1393 | -1523 |
| 100 | 39 | -3408869 | 3.53E-05 | 1443 | 1549 | 1394 | 1479 | -1503 | -1585 | -1393 | -1525 |
| 300 | 43 | -3409336 | 3.25E-05 | 1447 | 1553 | 1398 | 1483 | -1499 | -1582 | -1388 | -1521 |
| 500 | 41 | -3413295 | 8.62E-05 | 1446 | 1553 | 1397 | 1482 | -1500 | -1582 | -1389 | -1522 |

[a] $n$: sample size
[b] $iter$: number of iterations when algorithm stopped
[c] $obj_{opt}$: objective function value at the optimal solution
[d] $\|g\|$: gradient norm at the optimal solution
[e] $x_{opt}$: optimal solution

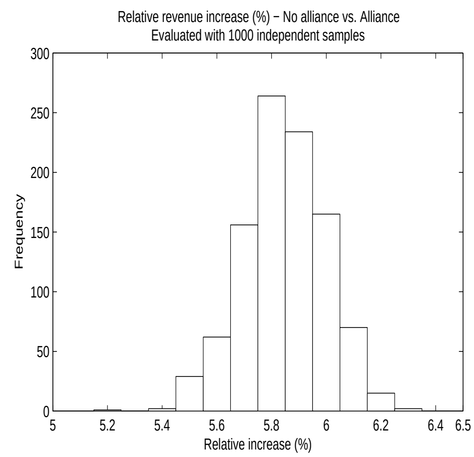


**Figure 6 Histogram of the pairwise difference in total profit between an alliance and no alliance, using a sample of 1000 sample points.**

sizes $N$, for the network example in Section 4.1. As far as we know, these are the first stochastic mathematical programs with equilibrium constraints motivated by an application that has been solved reasonably.

Figure 6 presents a histogram of the pairwise difference in total profit between an alliance and no alliance, using a sample of 1000 sample points, independent of the samples used in the SAA problem. The total profit under an alliance was larger for *all* 1000 sample points, with the percentage increase varying from 5.24% to 6.31%.

## 7. Conclusion

In this paper we presented an economic motivation for interest in alliances by showing that without an alliance sellers will tend to price their products too high and sell too little, thereby foregoing potential profit. We showed that under a resource exchange alliance, some of the foregone profit can be captured. In fact, the relative increase is largest when the capacity is large and the product differentiation level is high.

We formulated the problem of determining the optimal amounts of resources to exchange as a mathematical program with equilibrium constraints, taking the competition that results from alliance members selling similar products into account. In general, Mathematical Programs with Equilibrium Constraints (MPEC) are extremely badly behaved, often because the lower level equilibrium, if it exists, is discontinuous in the upper level decision. As a result MPECs are notoriously intractable and stochastic MPECs are even more intractable. Our solution approach utilizeing a trust region algorithm provides a method to search for an optimal exchange and the stochastic MPECs for the alliance design problems could be solved reasonably well.

Many research questions regarding alliances remain. In this paper we consider one type of alliances, namely resource exchange alliances. Such alliances are attractive because they do not require complicated coordination after the resource exchange has taken place and do not raise anti-trust concerns. since they enhance competition instead of reducing competition. However, there are many other potential alliance structures of interest that remain to be analyzed and compared in greater detail. In addition, the problem of optimal revenue management under an alliance is very challenging and has not received much attention in the literature. Thus the problem of optimal revenue management under an alliance remains to be addressed.

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## Appendix A: Derivation of Results for Two-Resource Model

### Appendix A.1: No Alliance

First consider the case in which $b_{\min} \geq \tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1) > 0$ (it is shown later for which input parameter values this condition holds). In this case the profit function of seller $i$ is given by

$$\tilde{g}_i(\tilde{y}_i, \tilde{y}_{-i}) = \tilde{y}_i \left[\tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-i} + \tilde{y}_i)\right]$$

Then the best response function of seller $i$ is given by

$$B_i(\tilde{y}_{-i}) = \frac{\tilde{\alpha} - \tilde{\beta}\tilde{y}_{-i}}{2\tilde{\beta}}$$

Solving the system

$$\tilde{y}_i = \frac{\tilde{\alpha} - \tilde{\beta}\tilde{y}_{-i}}{2\tilde{\beta}}$$

for $i = \pm 1$, the equilibrium $(\tilde{y}_{-1}^*, \tilde{y}_1^*)$ is obtained, where

$$\tilde{y}_i^* = \frac{\tilde{\alpha}}{3\tilde{\beta}} > 0$$

The demand at the equilibrium prices $(\tilde{y}_{-1}^*, \tilde{y}_1^*)$ is equal to

$$\tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1}^* + \tilde{y}_1^*) = \frac{\tilde{\alpha}}{3} > 0 \tag{28}$$

Therefore, if $b_{\min} \geq \tilde{\alpha}/3$, then the equilibrium prices are given by (2), the equilibrium demand is given by (3), the resulting profit of seller $i$ is given by (4), and thus the total profit of both sellers together is given by (5).

Next, consider the case in which $b_{\min} \leq \tilde{\alpha}/3$. Note that in this case $\tilde{\alpha} \geq 3b_{\min} > b_{\min}$.

Case (1): First, consider any pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ such that $\tilde{y}_{-1} + \tilde{y}_1 < (\tilde{\alpha} - b_{\min})/\tilde{\beta}$. In Figure 7, this corresponds to (a). Then $\tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1) > b_{\min} > 0$, and thus the profit of seller $i$ is given by

$$\tilde{g}_i(\tilde{y}_i, \tilde{y}_{-i}) = \tilde{y}_i b_{\min}$$

Thus, if $\tilde{y}_{-1} + \tilde{y}_1 < (\tilde{\alpha} - b_{\min})/\tilde{\beta}$, then the profit of seller $i$ is increasing in $\tilde{y}_i$, and hence such a pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ cannot be an equilibrium.

Case (2): Next, consider any pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ such that $\tilde{y}_{-1} + \tilde{y}_1 \geq \tilde{\alpha}/\tilde{\beta}$. In Figure 7, this corresponds to (b). Then the demand and profit of each seller is zero.

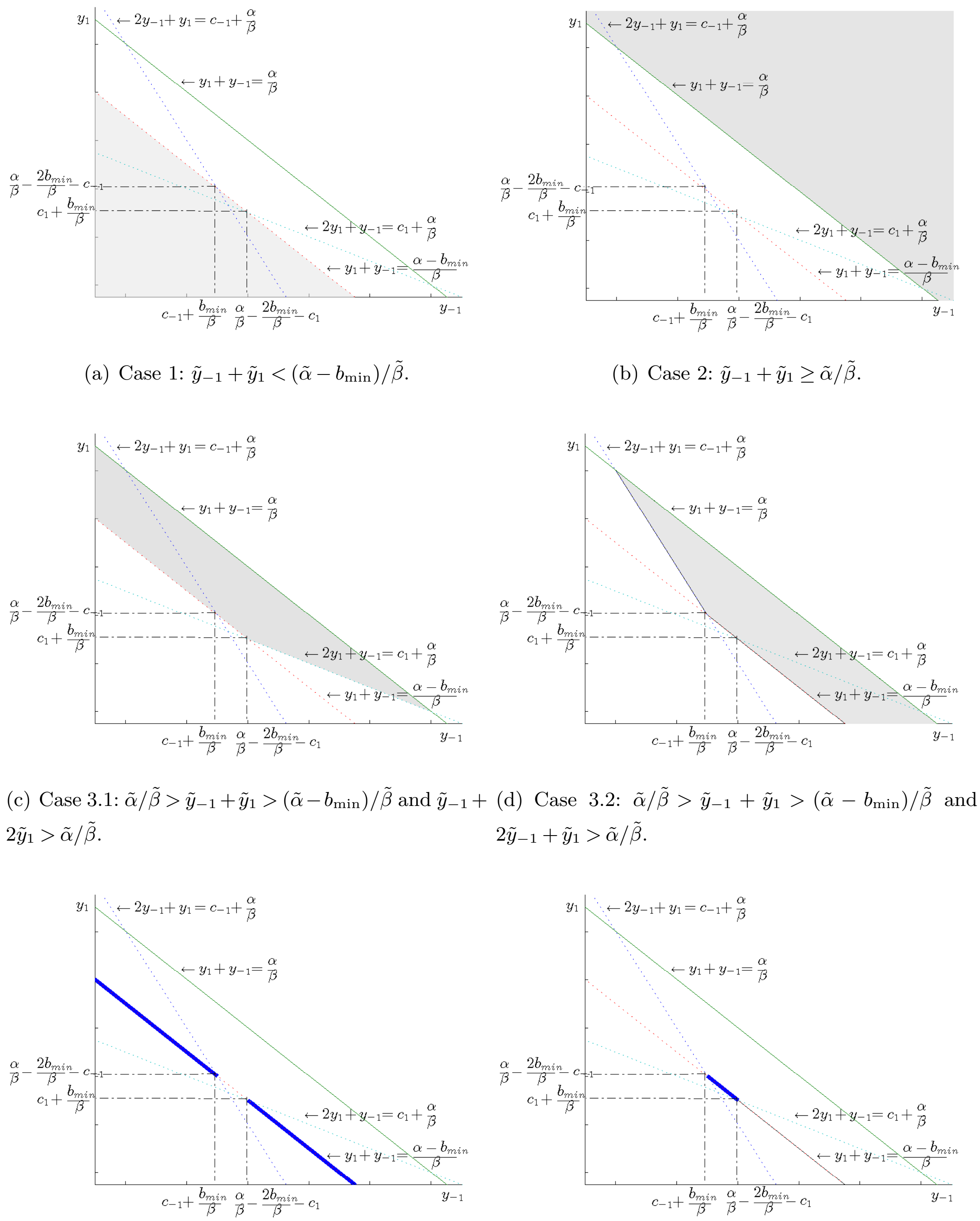
Case (3.1): Next, consider any pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ such that $\tilde{\alpha}/\tilde{\beta} > \tilde{y}_{-1} + \tilde{y}_1 > (\tilde{\alpha} - b_{\min})/\tilde{\beta}$ and $\tilde{y}_{-1} + 2\tilde{y}_1 > \tilde{\alpha}/\tilde{\beta}$. In Figure 7, this corresponds to (c). Then $0 < \tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1) < b_{\min}$, and thus the profit of seller $i$ is given by

$$\tilde{g}_i(\tilde{y}_i, \tilde{y}_{-i}) = \tilde{y}_i \left[\tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-i} + \tilde{y}_i)\right]$$

Note that

$$\partial\tilde{g}_1(\tilde{y}_1, \tilde{y}_{-1})/\partial\tilde{y}_1 = \tilde{\alpha} - \tilde{\beta}\tilde{y}_{-1} - 2\tilde{\beta}\tilde{y}_1 < 0$$

Thus, if $\tilde{\alpha}/\tilde{\beta} > \tilde{y}_{-1} + \tilde{y}_1 > (\tilde{\alpha} - b_{\min})/\tilde{\beta}$ and $\tilde{y}_{-1} + 2\tilde{y}_1 > \tilde{\alpha}/\tilde{\beta}$, then the profit of seller 1 is decreasing in $\tilde{y}_1$, and hence such a pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ cannot be an equilibrium.




(a) Case 1: $\tilde{y}_{-1} + \tilde{y}_1 < (\tilde{\alpha} - b_{\min})/\tilde{\beta}$.

(b) Case 2: $\tilde{y}_{-1} + \tilde{y}_1 \geq \tilde{\alpha}/\tilde{\beta}$.

(c) Case 3.1: $\tilde{\alpha}/\tilde{\beta} > \tilde{y}_{-1} + \tilde{y}_1 > (\tilde{\alpha} - b_{\min})/\tilde{\beta}$ and $\tilde{y}_{-1} + 2\tilde{y}_1 > \tilde{\alpha}/\tilde{\beta}$.

(d) Case 3.2: $\tilde{\alpha}/\tilde{\beta} > \tilde{y}_{-1} + \tilde{y}_1 > (\tilde{\alpha} - b_{\min})/\tilde{\beta}$ and $2\tilde{y}_{-1} + \tilde{y}_1 > \tilde{\alpha}/\tilde{\beta}$.

(e) Case 4: $\tilde{y}_{-1} + \tilde{y}_1 = (\tilde{\alpha} - b_{\min})/\tilde{\beta}$ and ($\tilde{y}_{-1} < b_{\min}/\tilde{\beta}$ or $\tilde{y}_1 < b_{\min}/\tilde{\beta}$).

(f) Case 5: The line segment between $(b_{\min}/\tilde{\beta}, \tilde{\alpha}/\tilde{\beta} - 2b_{\min}/\tilde{\beta})$ and $(\tilde{\alpha}/\tilde{\beta} - 2b_{\min}/\tilde{\beta}, b_{\min}/\tilde{\beta})$.

**Figure 7 Different regions of the pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ corresponding to different cases.**

Case (3.2): Next, consider any pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ such that $\tilde{\alpha}/\tilde{\beta} > \tilde{y}_{-1} + \tilde{y}_1 > (\tilde{\alpha} - b_{\min})/\tilde{\beta}$ and $2\tilde{y}_{-1} + \tilde{y}_1 > \tilde{\alpha}/\tilde{\beta}$. In Figure 7, this corresponds to (d). It follows similarly to Case 3.1 that the profit of seller $-1$ is decreasing in $\tilde{y}_{-1}$, and hence such a pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ cannot be an equilibrium.

Case (4.1): Next, consider any pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ such that $\tilde{y}_{-1} + \tilde{y}_1 = (\tilde{\alpha} - b_{\min})/\tilde{\beta}$ and $0 \leq \tilde{y}_{-1} < b_{\min}/\tilde{\beta}$. Note that $\tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1) = b_{\min}$, and thus the corresponding profit of seller $-1$ is given by

$$\tilde{g}_{-1}(\tilde{y}_{-1}, \tilde{y}_1) \quad = \quad \tilde{y}_{-1} b_{\min}$$

Next, consider $\hat{y}_{-1} := \left(\tilde{\alpha}/\tilde{\beta} - \tilde{y}_1\right)/2$. First, note that

$$\begin{aligned} \tilde{y}_1 \quad \leq \quad \tilde{y}_{-1} + \tilde{y}_1 \quad = \quad \frac{\tilde{\alpha} - b_{\min}}{\tilde{\beta}} &< \frac{\tilde{\alpha}}{\tilde{\beta}} \\ \Rightarrow \quad \frac{\tilde{\alpha} - \tilde{\beta}\tilde{y}_1}{2} &> 0 \\ \Leftrightarrow \quad \tilde{\alpha} - \tilde{\beta}\left(\frac{\tilde{\alpha}/\tilde{\beta} - \tilde{y}_1}{2} + \tilde{y}_1\right) &> 0 \\ \Leftrightarrow \quad \tilde{\alpha} - \tilde{\beta}\left(\hat{y}_{-1} + \tilde{y}_1\right) &> 0 \end{aligned}$$

Also, note that

$$\begin{aligned} \tilde{y}_{-1} &< b_{\min}/\tilde{\beta} \\ \Leftrightarrow \quad \tilde{y}_{-1} + (\tilde{\alpha} - b_{\min})/\tilde{\beta} &< \tilde{\alpha}/\tilde{\beta} \\ \Leftrightarrow \quad 2\tilde{y}_{-1} + \tilde{y}_1 &< \tilde{\alpha}/\tilde{\beta} \\ \Leftrightarrow \quad \tilde{y}_{-1} &< \frac{\tilde{\alpha}/\tilde{\beta} - \tilde{y}_1}{2} \quad = \quad \hat{y}_{-1} \end{aligned}$$

and thus $\tilde{\alpha} - \tilde{\beta}\left(\hat{y}_{-1} + \tilde{y}_1\right) < \tilde{\alpha} - \tilde{\beta}\left(\tilde{y}_{-1} + \tilde{y}_1\right) = b_{\min}$. Thus the corresponding profit of seller $-1$ is given by

$$\tilde{g}_{-1}(\hat{y}_{-1}, \tilde{y}_1) \quad = \quad \hat{y}_{-1}\left[\tilde{\alpha} - \tilde{\beta}\left(\hat{y}_{-1} + \tilde{y}_1\right)\right]$$

Next, note that

$$\begin{aligned} & \tilde{y}_{-1} \quad < \quad b_{\min}/\tilde{\beta} \\ \Rightarrow \quad & \left(b_{\min} - \tilde{\beta}\tilde{y}_{-1}\right)^2 \quad > \quad 0 \\ \Leftrightarrow \quad & b_{\min}^2 + 2b_{\min}\tilde{\beta}\tilde{y}_{-1} + \tilde{\beta}^2\tilde{y}_{-1}^2 \quad > \quad 4b_{\min}\tilde{\beta}\tilde{y}_{-1} \\ \Leftrightarrow \quad & \left(b_{\min} + \tilde{\beta}\tilde{y}_{-1}\right)^2 \quad > \quad 4\tilde{\beta}\tilde{y}_{-1}b_{\min} \\ \Leftrightarrow \quad & \left(\frac{b_{\min}/\tilde{\beta} + \tilde{y}_{-1}}{2}\right)\left(\frac{b_{\min} + \tilde{\beta}\tilde{y}_{-1}}{2}\right) \quad > \quad \tilde{y}_{-1}b_{\min} \\ \Leftrightarrow \quad & \left(\frac{\tilde{\alpha}/\tilde{\beta} - \left(\tilde{\alpha}/\tilde{\beta} - b_{\min}/\tilde{\beta} - \tilde{y}_{-1}\right)}{2}\right)\left(\frac{\tilde{\alpha} - \tilde{\beta}\left(\tilde{\alpha}/\tilde{\beta} - b_{\min}/\tilde{\beta} - \tilde{y}_{-1}\right)}{2}\right) \quad > \quad \tilde{y}_{-1}b_{\min} \\ \Leftrightarrow \quad & \left(\frac{\tilde{\alpha}/\tilde{\beta} - \tilde{y}_1}{2}\right)\left(\frac{\tilde{\alpha} - \tilde{\beta}\tilde{y}_1}{2}\right) \quad > \quad \tilde{y}_{-1}b_{\min} \\ \Leftrightarrow \quad & \left(\frac{\tilde{\alpha}/\tilde{\beta} - \tilde{y}_1}{2}\right)\left(\tilde{\alpha} - \frac{\tilde{\beta}\left(\tilde{\alpha}/\tilde{\beta} - \tilde{y}_1\right)}{2} - \tilde{\beta}\tilde{y}_1\right) \quad > \quad \tilde{y}_{-1}b_{\min} \\ \Leftrightarrow \quad & \hat{y}_{-1}\left(\tilde{\alpha} - \tilde{\beta}\hat{y}_{-1} - \tilde{\beta}\tilde{y}_1\right) \quad > \quad \tilde{y}_{-1}b_{\min} \\ \Leftrightarrow \quad & \tilde{g}_{-1}(\hat{y}_{-1}, \tilde{y}_1) \quad > \quad \tilde{g}_{-1}(\tilde{y}_{-1}, \tilde{y}_1) \end{aligned}$$

Thus such a pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ cannot be an equilibrium.

Case (4.2): Next, consider any pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ such that $\tilde{y}_{-1} + \tilde{y}_1 = (\tilde{\alpha} - b_{\min})/\tilde{\beta}$ and $0 \leq \tilde{y}_1 < b_{\min}/\tilde{\beta}$. Consider $\hat{y}_1 := \left(\tilde{\alpha}/\tilde{\beta} - \tilde{y}_{-1}\right)/2$. It follows similarly to Case 4.1 that $\tilde{g}_1(\hat{y}_1, \tilde{y}_{-1}) > \tilde{g}_1(\tilde{y}_1, \tilde{y}_{-1})$ and thus such a pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ cannot be an equilibrium. In Figure 7, Case (4.1) and Case (4.2) correspond to (e).

Case (5): The only remaining pairs of prices to check are pairs $(\tilde{y}_{-1}, \tilde{y}_1)$ on the line segment between $(b_{\min}/\tilde{\beta},\, \tilde{\alpha}/\tilde{\beta} - 2b_{\min}/\tilde{\beta})$ and $(\tilde{\alpha}/\tilde{\beta} - 2b_{\min}/\tilde{\beta},\, b_{\min}/\tilde{\beta})$. In Figure 7, this corresponds to the line segment on (f). Consider any pair of prices $(\tilde{y}_{-1}, \tilde{y}_1) = (1-\gamma)(b_{\min}/\tilde{\beta},\, \tilde{\alpha}/\tilde{\beta} - 2b_{\min}/\tilde{\beta}) + \gamma(\tilde{\alpha}/\tilde{\beta} - 2b_{\min}/\tilde{\beta},\, b_{\min}/\tilde{\beta})$ for $\gamma \in [0,1]$. It follows from $b_{\min} \leq \tilde{\alpha}/3$ that $0 < b_{\min}/\tilde{\beta} \leq \tilde{\alpha}/\tilde{\beta} - 2b_{\min}/\tilde{\beta}$, and thus $\tilde{y}_i > 0$. Note that $\tilde{y}_{-1} + \tilde{y}_1 = (1-\gamma)(\tilde{\alpha}/\tilde{\beta} - b_{\min}/\tilde{\beta}) + \gamma(\tilde{\alpha}/\tilde{\beta} - b_{\min}/\tilde{\beta}) = (\tilde{\alpha} - b_{\min})/\tilde{\beta}$, that $\tilde{y}_{-1} + 2\tilde{y}_1 = (1-\gamma)(2\tilde{\alpha}/\tilde{\beta} - 3b_{\min}/\tilde{\beta}) + \gamma\tilde{\alpha}/\tilde{\beta} \geq \tilde{\alpha}/\tilde{\beta}$, where the inequality follows from $b_{\min} \leq \tilde{\alpha}/3$, and similarly $2\tilde{y}_{-1} + \tilde{y}_1 \geq \tilde{\alpha}/\tilde{\beta}$. Then, for any $\hat{y}_1 < \tilde{y}_1$, it holds that $\tilde{y}_{-1} + \hat{y}_1 < (\tilde{\alpha} - b_{\min})/\tilde{\beta}$, and thus it follows from Case (a) that $\tilde{g}_1(\hat{y}_1, \tilde{y}_{-1}) < \tilde{g}_1(\tilde{y}_1, \tilde{y}_{-1})$. Also, for any $\hat{y}_1 > \tilde{y}_1$, it holds that $\tilde{y}_{-1} + \hat{y}_1 > (\tilde{\alpha} - b_{\min})/\tilde{\beta}$ and $\tilde{y}_{-1} + 2\hat{y}_1 > \tilde{\alpha}/\tilde{\beta}$, and thus it follows from Case (c) that $\tilde{g}_1(\hat{y}_1, \tilde{y}_{-1}) < \tilde{g}_1(\tilde{y}_1, \tilde{y}_{-1})$. Hence, given $\tilde{y}_{-1}$, $\tilde{y}_1$ is the best response for seller 1. Similarly, given $\tilde{y}_1$, $\tilde{y}_{-1}$ is the best response for seller $-1$.

Therefore, if $b_{\min} \leq \tilde{\alpha}/3$, then all pairs of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ on the line segment between $(b_{\min}/\tilde{\beta},\, \tilde{\alpha}/\tilde{\beta} - 2b_{\min}/\tilde{\beta})$ and $(\tilde{\alpha}/\tilde{\beta} - 2b_{\min}/\tilde{\beta},\, b_{\min}/\tilde{\beta})$ are equilibria. For all of these equilibrium prices total price is equal to $(\tilde{\alpha} - b_{\min})/\tilde{\beta}$, the demand is equal to $b_{\min}$, the resulting profit of seller $i$ is equal to $\tilde{y}_i b_{\min}$, and thus the total profit of both sellers together is given by (6).

### Appendix A.2: Perfect Coordination

In this section, we determine the maximum achievable total profit of the two sellers together, that is, the total profit if the sellers would perfectly coordinate pricing.

The total profit of the two sellers is given by

$$\tilde{g}(\tilde{y}_{-1}, \tilde{y}_1) \;:=\; (\tilde{y}_{-1} + \tilde{y}_1)\min\{b_{\min},\, \max\{0,\, \tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1)\}\}$$

First consider the case in which $b_{\min} \geq \tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1) > 0$. In this case the total profit of the two sellers is given by

$$\tilde{g}(\tilde{y}_{-1}, \tilde{y}_1) \;:=\; (\tilde{y}_{-1} + \tilde{y}_1)\left[\tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1)\right]$$

The optimal total price $\bar{y}_{-1} + \bar{y}_1$ that maximizes the total profit is given by

$$\bar{y}_{-1} + \bar{y}_1 \;=\; \frac{\tilde{\alpha}}{2\tilde{\beta}} \;>\; 0$$

The demand at the optimal total price $\bar{y}_{-1} + \bar{y}_1$ is equal to

$$\tilde{\alpha} - \tilde{\beta}(\bar{y}_{-1} + \bar{y}_1) \;=\; \frac{\tilde{\alpha}}{2} \;>\; \frac{\tilde{\alpha}}{3} \;=\; \tilde{\alpha} - \tilde{\beta}(\tilde{y}^*_{-1} + \tilde{y}^*_1) \tag{29}$$

Therefore, if $b_{\min} \geq \tilde{\alpha}/2$, then the optimal total price is given by (7), the corresponding demand is given by (8), the total profit of both sellers together is given by (9), and the consumer surplus is given by (**??**).

Next, consider the case in which $b_{\min} \leq \tilde{\alpha}/2$. In this case the optimal total price is given by

$$\tilde{y}_{-1} + \tilde{y}_1 \;=\; \frac{\tilde{\alpha} - b_{\min}}{\tilde{\beta}} \;>\; 0$$

with corresponding demand equal to $b_{\min}$. The total profit of both sellers together is equal to

$$(\tilde{y}_{-1}+\tilde{y}_1)\,b_{\min} \quad = \quad \frac{\tilde{\alpha}-b_{\min}}{\tilde{\beta}}b_{\min}$$

and the consumer surplus is equal to

$$\frac{1}{2}\left[\frac{\tilde{\alpha}}{\tilde{\beta}}-\frac{\tilde{\alpha}-b_{\min}}{\tilde{\beta}}\right]b_{\min} \quad = \quad \frac{b_{\min}^2}{2\tilde{\beta}}$$

**Appendix A.3: Resource Exchange Alliance**

For given values of $b_{-1}$ and $b_1$, the feasible set $S_1$ of two-resource products that can be sold by the two sellers is given by $S_1 := \{(q_{-1}(x), q_1(x)) : x_i \in [0, b_i],\, i = \pm 1\}$. Next we show that this set $S_1$ is equal to $S_2 := \{(q_{-1}, q_1) \in [0, b_{\min}]^2 : q_{-1} + q_1 \leq b_{\min}\}$. First, consider any $(q_{-1}(x), q_1(x)) \in S_1$ with corresponding $(x_{-1}, x_1) \in [0, b_{-1}] \times [0, b_1]$. Without loss of generality, suppose that $b_{-1} = b_{\min}$. Then $q_{-1}(x) + q_1(x) = \min\{b_{-1} - x_{-1},\, x_1\} + \min\{b_1 - x_1,\, x_{-1}\} \leq b_{-1} - x_{-1} + x_{-1} = b_{-1} = b_{\min}$, and thus $(q_{-1}(x), q_1(x)) \in S_2$. Next, consider any $(q_{-1}, q_1) \in S_2$. Choose $x_i = q_{-i}$ for $i = \pm 1$. Note that $x_i \in [0, b_i]$ since $q_{-i} \in [0, b_{\min}]$. Also, $x_i = q_{-i} \leq b_{\min} - q_i = b_{\min} - x_{-i} \leq b_{-i} - x_{-i}$, and thus $q_{-i}(x) = \min\{b_{-i} - x_{-i},\, x_i\} = x_i = q_{-i}$. Thus $(q_{-1}, q_1) \in S_1$, and hence $S_1 = S_2$. Hence, the first-stage decision variables may be considered to be the resource exchange quantities $x = (x_{-1}, x_1) \in [0, b_{-1}] \times [0, b_1]$, or equivalently the capacities $q = (q_{-1}, q_1) \in S_2$ of two-resource products after exchange.

*Case 1.* First consider the case in which $q_i > \alpha - \beta y_i + \gamma y_{-i} > 0$ for $i = \pm 1$ (it is considered later for which input parameter values and values of $q$ and $y$ this condition holds). In this case the profit function of each seller $i$ is given by

$$g_i(y_i, y_{-i}) \quad = \quad y_i\,[\alpha - \beta y_i + \gamma y_{-i}]$$

Then the best response function of each seller $i$ is given by

$$B_i(y_{-i}) \quad = \quad \frac{\alpha + \gamma y_{-i}}{2\beta}$$

Solving the system

$$y_i \quad = \quad \frac{\alpha + \gamma y_{-i}}{2\beta}$$

for $i = \pm 1$, the equilibrium $(y_{-1}^*, y_1^*)$ is obtained, where

$$y_i^* \quad = \quad \frac{\alpha}{2\beta - \gamma} \quad > \quad 0 \tag{30}$$

Note that the equilibrium prices are greater than the marginal cost $c_{-1} + c_1$ of the two-resource product. The demand at the equilibrium prices $(y_{-1}^*, y_1^*)$ is equal to

$$\alpha - \beta y_i^* + \gamma y_{-i}^* \quad = \quad \frac{\alpha\beta}{2\beta - \gamma} \quad > \quad 0 \tag{31}$$

The resulting profit of each seller is equal to

$$y_i^* \min\{q_i,\, \max\{0,\, \alpha - \beta y_i^* + \gamma y_{-i}^*\}\} \quad = \quad \frac{\alpha^2\beta}{(2\beta - \gamma)^2} \tag{32}$$

and thus the total profit of both sellers together is equal to

$$2\frac{\alpha^2\beta}{(2\beta - \gamma)^2} \tag{33}$$

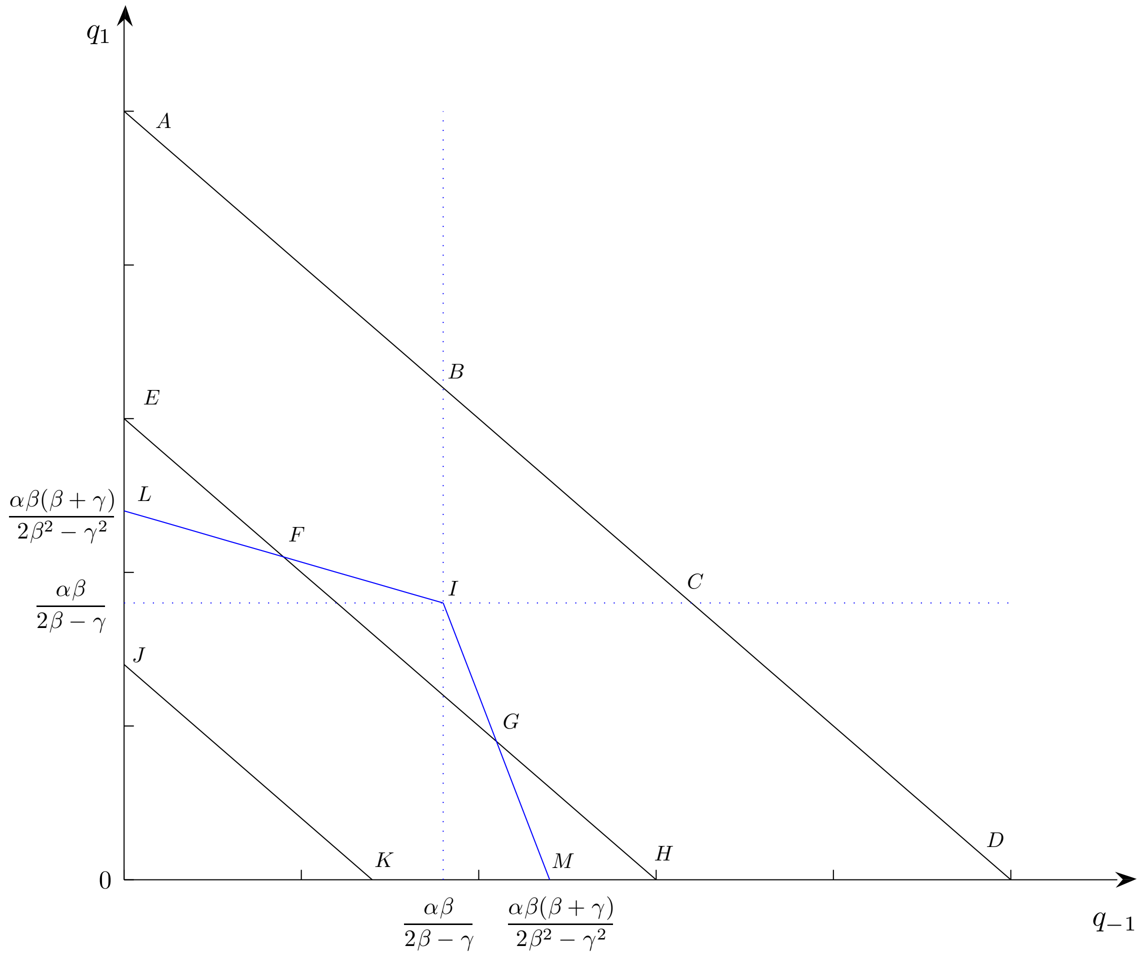



**Figure 8 Different cases of capacity $b_{\min}$ for a resource exchange alliance.**

Therefore, if $q_i \geq \alpha\beta/(2\beta-\gamma)$ for $i=\pm 1$, then the equilibrium prices are given by (30), the equilibrium demand is given by (31), the resulting profit of each seller is given by (32), and thus the total profit of both sellers together is given by (33). Note that $q_i \geq \alpha\beta/(2\beta-\gamma)$ for $i=\pm 1$ requires that $b_{\min} \geq 2\alpha\beta/(2\beta-\gamma)$. Thus the results above hold if $b_{\min} \geq 2\alpha\beta/(2\beta-\gamma)$ and the resource exchange $x$ is chosen such that $q_i \geq \alpha\beta/(2\beta-\gamma)$ for $i=\pm 1$. In Figure 8, the line $ABCD$ shows pairs $(q_{-1}, q_1)$ such that $q_{-1}+q_1 = b_{\min} > 2\alpha\beta/(2\beta-\gamma)$, obtained with resource exchange $x=(x_{-1},x_1)$ such that $x_i = q_{-i} = b_{\min} - q_i = b_{\min} - x_{-i} \leq b_{-i} - x_{-i}$. Thus, for the given value of $b_{\min} > 2\alpha\beta/(2\beta-\gamma)$, the set of points $(q_{-1}, q_1)$ such that $q_i \geq \alpha\beta/(2\beta-\gamma)$ for $i=\pm 1$ and $q_{-1}+q_1 \leq b_{\min}$ corresponds to triangle $BCI$. All corresponding resource exchanges $x$ lead to sales of two-resource products of $\alpha\beta/(2\beta-\gamma)$ by each seller, corresponding to point $I$, and provide total profit of $2\alpha^2\beta/(2\beta-\gamma)^2$.

*Case 2.* Next, consider the case in which $0 \leq q_{-i} \leq \alpha - \beta y_{-i} + \gamma y_i$ and $q_i > \alpha - \beta y_i + \gamma y_{-i} > 0$ (as before, it is considered later for which input parameter values and values of $q$ and $y$ this condition holds). In this case the profit function of seller $-i$ is given by

$$g_{-i}(y_{-i}, y_i) = y_{-i} q_{-i}$$

and the profit function of seller $i$ is given by

$$g_i(y_i, y_{-i}) = y_i \left[\alpha - \beta y_i + \gamma y_{-i}\right]$$

Then the best response function of seller $-i$ is given by

$$B_{-i}(y_i) = \max\{y_{-i} : q_{-i} \leq \alpha - \beta y_{-i} + \gamma y_i\} = \frac{\alpha + \gamma y_i - q_{-i}}{\beta}$$

and the best response function of seller $i$ is given by

$$B_i(y_{-i}) = \frac{\alpha + \gamma y_{-i}}{2\beta}$$

Solving the system

$$\begin{aligned} y_{-i} &= \frac{\alpha + \gamma y_i - q_{-i}}{\beta} \\ y_i &= \frac{\alpha + \gamma y_{-i}}{2\beta} \end{aligned}$$

the solution $(y^*_{-1}, y^*_1)$ is obtained, where

$$\begin{aligned} y^*_{-i} &= \frac{2\alpha\beta + \alpha\gamma - 2\beta q_{-i}}{2\beta^2 - \gamma^2} \\ y^*_i &= \frac{\alpha\beta + \alpha\gamma - \gamma q_{-i}}{2\beta^2 - \gamma^2} \end{aligned} \quad (34)$$

(It is checked later under what conditions $y^*_{-i}, y^*_i > 0$ and $(y^*_{-i}, y^*_i)$ is the unique equilibrium.) The demands at the prices $(y^*_{-i}, y^*_i)$ are equal to

$$d_{-i}(y^*_{-i}, y^*_i) = \alpha - \beta y^*_{-i} + \gamma y^*_i = q_{-i} \quad (35)$$

$$d_i(y^*_i, y^*_{-i}) = \alpha - \beta y^*_i + \gamma y^*_{-i} = \frac{\alpha\beta(\beta + \gamma) - \beta\gamma q_{-i}}{2\beta^2 - \gamma^2} \quad (36)$$

Recall that we are considering the case in which $q_{-i} \leq \alpha - \beta y_{-i} + \gamma y_i$ and $q_i > \alpha - \beta y_i + \gamma y_{-i}$. Note that $q_{-i} = \alpha - \beta y^*_{-i} + \gamma y^*_i$. Also note that $q_i > \alpha - \beta y^*_i + \gamma y^*_{-i}$ if and only if $q_i > \alpha\beta(\beta+\gamma)/(2\beta^2 - \gamma^2) - \beta\gamma q_{-i}/(2\beta^2 - \gamma^2)$. Examples of the line $q_i = \alpha\beta(\beta+\gamma)/(2\beta^2 - \gamma^2) - \beta\gamma q_{-i}/(2\beta^2 - \gamma^2)$ are given in Figure 8 by line $LFI$ for $i = 1$ and by line $MGI$ for $i = -1$. It can be verified that the intercept satisfies $\alpha\beta(\beta+\gamma)/(2\beta^2 - \gamma^2) \in (0, 2\alpha\beta/(2\beta - \gamma))$. The slope of the lines are negative if $\gamma > 0$ and positive if $\gamma < 0$. Note that if $q_{-i} = \alpha\beta/(2\beta - \gamma)$, then $\alpha\beta(\beta+\gamma)/(2\beta^2 - \gamma^2) - \beta\gamma q_{-i}/(2\beta^2 - \gamma^2) = \alpha\beta/(2\beta - \gamma)$, and thus in all cases the lines go through $I = (\alpha\beta/(2\beta - \gamma), \alpha\beta/(2\beta - \gamma))$. In Figure 8, if $b_{\min} > 2\alpha\beta/(2\beta - \gamma)$, such as in the case in which the line $ABCD$ shows pairs $(q_{-1}, q_1)$ such that $q_{-1} + q_1 = b_{\min}$, then the set of points $(q_{-1}, q_1)$ such that $0 \leq q_{-1} \leq \alpha - \beta y^*_{-1} + \gamma y^*_1$, $q_1 > \alpha - \beta y^*_1 + \gamma y^*_{-1}$, and $q_{-1} + q_1 \leq b_{\min}$, corresponds to quadrilateral $ABIL$. (Note that $q_{-1} \leq \alpha\beta/(2\beta - \gamma)$, since it has already been shown that $q_{-1} > \alpha - \beta y^*_{-1} + \gamma y^*_1$ in triangle $BCI$.) Similarly, the set of points $(q_{-1}, q_1)$ such that $0 \leq q_1 \leq \alpha - \beta y^*_1 + \gamma y^*_{-1}$, $q_{-1} > \alpha - \beta y^*_{-1} + \gamma y^*_1$, and $q_{-1} + q_1 \leq b_{\min}$, corresponds to quadrilateral $DCIM$ (note that $q_1 \leq \alpha\beta/(2\beta - \gamma)$). If $\alpha\beta(\beta+\gamma)/(2\beta^2 - \gamma^2) < b_{\min} \leq 2\alpha\beta/(2\beta - \gamma)$, such as in the case in which the line $EFGH$ shows pairs $(q_{-1}, q_1)$ such that $q_{-1} + q_1 = b_{\min}$, then the set of points $(q_{-1}, q_1)$ such that $0 \le q_{-1} \le \alpha - \beta y^*_{-1} + \gamma y^*_1$, $q_1 > \alpha - \beta y^*_1 + \gamma y^*_{-1}$, and $q_{-1} + q_1 \le b_{\min}$, corresponds to triangle $EFL$, and the set of points $(q_{-1}, q_1)$ such that $0 \le q_1 \le \alpha - \beta y^*_1 + \gamma y^*_{-1}$, $q_{-1} > \alpha - \beta y^*_{-1} + \gamma y^*_1$, and $q_{-1} + q_1 \le b_{\min}$, corresponds to triangle $HGM$. It is verified in Case 3 that, if $b_{\min} \le \alpha\beta(\beta+\gamma)/(2\beta^2 - \gamma^2)$, then $q_i \le \alpha - \beta y^*_i + \gamma y^*_{-i}$ for $i = \pm 1$.

Next we verify that, if $q_{-i} \le \alpha\beta/(2\beta - \gamma)$, then the prices $y^*_{-i}, y^*_i$ given in (34) satisfy $y^*_{-i}, y^*_i > 0$, that is, the prices are greater than the marginal cost $c_{-1} + c_1$ of the two-resource product. First note that the denominator in the expressions for $y^*_{-i}$ and $y^*_i$ is positive. Next consider the numerator in the expression for $y^*_{-i}$. Note that

$$\begin{aligned} & 2\beta^2 < 4\beta^2 - \gamma^2 \;=\; (2\beta+\gamma)(2\beta-\gamma) \\ \Leftrightarrow \quad & \frac{\alpha\beta}{2\beta-\gamma} < \frac{2\alpha\beta+\alpha\gamma}{2\beta} \end{aligned}$$

Thus, if $q_{-i} \le \alpha\beta/(2\beta - \gamma)$, then

$$\begin{aligned} & q_{-i} < \frac{2\alpha\beta+\alpha\gamma}{2\beta} \\ \Leftrightarrow \quad & 0 < 2\alpha\beta + \alpha\gamma - 2\beta q_{-i} \\ \Leftrightarrow \quad & 0 < \frac{2\alpha\beta+\alpha\gamma-2\beta q_{-i}}{2\beta^2-\gamma^2} \;=\; y^*_{-i} \end{aligned}$$

Next consider the numerator in the expression for $y^*_i$. If $\gamma \le 0$, then $\alpha(\beta+\gamma) - \gamma q_{-i} > 0$ (recall that $\gamma \in (-\beta, \beta)$), and thus

$$y^*_i \;=\; \frac{\alpha\beta+\alpha\gamma-\gamma q_{-i}}{2\beta^2-\gamma^2} \;>\; 0$$

Next, consider the case with $\gamma > 0$. Note that

$$\frac{\alpha\beta}{2\beta-\gamma} \;<\; \frac{\alpha\beta}{\gamma} \;<\; \frac{\alpha\beta+\alpha\gamma}{\gamma}$$

Thus, if $q_{-i} \le \alpha\beta/(2\beta - \gamma)$, then

$$\begin{aligned} & q_{-i} < \frac{\alpha\beta+\alpha\gamma}{\gamma} \\ \Leftrightarrow \quad & 0 < \alpha\beta + \alpha\gamma - \gamma q_{-i} \\ \Leftrightarrow \quad & 0 < \frac{\alpha\beta+\alpha\gamma-\gamma q_{-i}}{2\beta^2-\gamma^2} \;=\; y^*_i \end{aligned}$$

Next we verify that, if $q_{-i} \le \alpha\beta/(2\beta - \gamma)$ and $q_i > \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-i}/(2\beta^2-\gamma^2)$, then $(y^*_{-i}, y^*_i)$ given in (34) is the unique equilibrium. First, recall that $B_i(y_{-i}) = (\alpha + \gamma y_{-i})/(2\beta)$ is the unique best response for seller $i$ if the capacity $q_i$ of seller $i$ is not constraining. Note that if seller $-i$ chooses price $y^*_{-i}$ and $q_i > \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-i}/(2\beta^2-\gamma^2)$, then the capacity $q_i$ of seller $i$ is not constraining, and thus $y^*_i$ given in (34) is the unique best response for seller $i$ to $y^*_{-i}$. Next we verify that $y^*_{-i}$ given in (34) is the unique best response for seller $-i$ to $y^*_i$. Given $y^*_i$, the profit of seller $-i$ is given by

$$\begin{aligned} g_{-i}(y_{-i}, y^*_i) &= y_{-i} \min\{q_{-i}, \max\{0,\, \alpha - \beta y_{-i} + \gamma y^*_i\}\} \\ &= \begin{cases} y_{-i} q_{-i} & \text{if} \quad y_{-i} \le \frac{\alpha+\gamma y^*_i - q_{-i}}{\beta} \\ y_{-i}\left(\alpha - \beta y_{-i} + \gamma y^*_i\right) & \text{if} \quad \frac{\alpha+\gamma y^*_i - q_{-i}}{\beta} \le y_{-i} \le \frac{\alpha+\gamma y^*_i}{\beta} \\ 0 & \text{if} \quad y_{-i} \ge \frac{\alpha+\gamma y^*_i}{\beta} \end{cases} \end{aligned}$$

Thus $g_{-i}(y_{-i}, y_i^*)$ is a nondecreasing linear function of $y_{-i}$ if $y_{-i} \leq (\alpha + \gamma y_i^* - q_{-i})/\beta$. If $(\alpha + \gamma y_i^* - q_{-i})/\beta < y_{-i} < (\alpha + \gamma y_i^*)/\beta$, then $g_{-i}(y_{-i}, y_i^*)$ is a concave quadratic function of $y_{-i}$, with

$$\begin{aligned} g'_{-i}(y_{-i}, y_i^*) &= -2\beta y_{-i} + \alpha + \gamma y_i^* \\ &< -2\left(\alpha + \gamma y_i^* - q_{-i}\right) + \alpha + \gamma y_i^* \\ &= -\alpha - \gamma y_i^* + 2q_{-i} \\ &= -\alpha - \gamma \frac{\alpha\beta + \alpha\gamma - \gamma q_{-i}}{2\beta^2 - \gamma^2} + 2q_{-i} \\ &= \frac{-2\alpha\beta^2 - \alpha\beta\gamma + (4\beta^2 - \gamma^2)q_{-i}}{2\beta^2 - \gamma^2} \end{aligned}$$

Note that

$$\begin{aligned} \frac{-2\alpha\beta^2 - \alpha\beta\gamma + (4\beta^2 - \gamma^2)q_{-i}}{2\beta^2 - \gamma^2} &\leq 0 \\ \Leftrightarrow \quad -2\alpha\beta^2 - \alpha\beta\gamma + (4\beta^2 - \gamma^2)q_{-i} &\leq 0 \\ \Leftrightarrow \quad -\alpha\beta(2\beta + \gamma) + (2\beta - \gamma)(2\beta + \gamma)q_{-i} &\leq 0 \\ \Leftrightarrow \quad -\alpha\beta + (2\beta - \gamma)q_{-i} &\leq 0 \\ \Leftrightarrow \quad q_{-i} &\leq \frac{\alpha\beta}{2\beta - \gamma} \end{aligned}$$

Hence, if $q_{-i} \leq \alpha\beta/(2\beta - \gamma)$, then $g'_{-i}(y_{-i}, y_i^*) < 0$ for all $y_{-i} \in ((\alpha + \gamma y_i^* - q_{-i})/\beta, (\alpha + \gamma y_i^*)/\beta)$. Hence, the unique best response for seller $-i$ to $y_i^*$ is $B_{-i}(y_i^*) = (\alpha + \gamma y_i^* - q_{-i})/\beta$. Therefore, if $q_{-i} \leq \alpha\beta/(2\beta - \gamma)$ and $q_i > \alpha\beta(\beta + \gamma)/(2\beta^2 - \gamma^2) - \beta\gamma q_{-i}/(2\beta^2 - \gamma^2)$, then $(y_{-i}^*, y_i^*)$ given in (34) is the unique equilibrium.

The resulting profit of each seller is equal to

$$\begin{aligned} g_{-i}(y_{-i}^*, y_i^*) &= y_{-i}^* q_{-i} \\ &= \frac{\alpha\left(2\beta + \gamma\right) q_{-i} - 2\beta q_{-i}^2}{2\beta^2 - \gamma^2} \\ g_i(y_i^*, y_{-i}^*) &= y_i^* \left(\alpha - \beta y_i^* + \gamma y_{-i}^*\right) \\ &= \left(\frac{\alpha\beta + \alpha\gamma - \gamma q_{-i}}{2\beta^2 - \gamma^2}\right)\left(\frac{\alpha\beta\left(\beta + \gamma\right) - \beta\gamma q_{-i}}{2\beta^2 - \gamma^2}\right) \\ &= \frac{\alpha^2\beta\left(\beta + \gamma\right)^2 - 2\alpha\beta\gamma\left(\beta + \gamma\right) q_{-i} + \beta\gamma^2 q_{-i}^2}{\left(2\beta^2 - \gamma^2\right)^2} \end{aligned} \tag{37}$$

and thus the total profit of both sellers together is equal to

$$\begin{aligned} G(q_{-i}) &= \frac{\alpha\left(2\beta + \gamma\right) q_{-i} - 2\beta q_{-i}^2}{2\beta^2 - \gamma^2} + \frac{\alpha^2\beta\left(\beta + \gamma\right)^2 - 2\alpha\beta\gamma\left(\beta + \gamma\right) q_{-i} + \beta\gamma^2 q_{-i}^2}{\left(2\beta^2 - \gamma^2\right)^2} \\ &= \frac{\alpha\left(2\beta + \gamma\right)\left(2\beta^2 - \gamma^2\right) q_{-i} - 2\beta\left(2\beta^2 - \gamma^2\right) q_{-i}^2 + \alpha^2\beta\left(\beta + \gamma\right)^2 - 2\alpha\beta\gamma\left(\beta + \gamma\right) q_{-i} + \beta\gamma^2 q_{-i}^2}{\left(2\beta^2 - \gamma^2\right)^2} \\ &= \frac{\alpha^2\beta\left(\beta + \gamma\right)^2 + \alpha\left(4\beta^3 - 4\beta\gamma^2 - \gamma^3\right) q_{-i} - \beta\left(4\beta^2 - 3\gamma^2\right) q_{-i}^2}{\left(2\beta^2 - \gamma^2\right)^2} \end{aligned} \tag{38}$$

Therefore, if $q_{-i} \leq \alpha\beta/(2\beta - \gamma)$ and $q_i > \alpha\beta(\beta + \gamma)/(2\beta^2 - \gamma^2) - \beta\gamma q_{-i}/(2\beta^2 - \gamma^2)$, then the equilibrium prices are given by (34), the equilibrium demand is given by (36), the resulting profit of each seller is given by (37), and thus the total profit of both sellers together is given by (38).

*Case 3.* Next consider the case in which $0 \leq q_i \leq \alpha - \beta y_i + \gamma y_{-i}$ for $i = \pm 1$. (It will be shown that this case holds if and only if $0 \leq q_i \leq \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-i}/(2\beta^2-\gamma^2)$ for $i = \pm 1$. In Figure 8 this case corresponds to two-resource product capacities $(q_{-1}, q_1)$ in region $0LIM$. Thus the entire region $\{(q_{-1}, q_1) : q_i \geq 0, i = \pm 1\}$ is covered by Cases 1–3.) In this case the profit function of each seller $i$ is given by

$$g_i(y_i, y_{-i}) \quad = \quad y_i q_i$$

Then the best response function of each seller $i$ is given by

$$B_i(y_{-i}) \quad = \quad \max\{y_i : q_i \leq \alpha - \beta y_i + \gamma y_{-i}\} \quad = \quad \frac{\alpha + \gamma y_{-i} - q_i}{\beta}$$

Solving the system

$$y_i \quad = \quad \frac{\alpha + \gamma y_{-i} - q_i}{\beta}$$

for $i = \pm 1$, the equilibrium $(y^*_{-1}, y^*_1)$ is obtained, where

$$y^*_i \quad = \quad \frac{\alpha(\beta+\gamma) - \beta q_i - \gamma q_{-i}}{\beta^2 - \gamma^2} \tag{39}$$

(It is checked later under what conditions $y^*_i > 0$ and $(y^*_{-1}, y^*_1)$ is the unique equilibrium.) The demand of seller $i$ at the prices $(y^*_{-1}, y^*_1)$ is equal to

$$\alpha - \beta y^*_i + \gamma y^*_{-i} \quad = \quad q_i \quad > \quad 0 \tag{40}$$

Next we verify that, if $q_i \leq \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-i}/(2\beta^2-\gamma^2)$ for $i = \pm 1$, then the prices $y^*_i$ given in (39) satisfy $y^*_i > 0$ for $i = \pm 1$, that is, the prices are greater than the marginal cost $c_{-1} + c_1$ of the two-resource product. Note that $q_i \leq \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-i}/(2\beta^2-\gamma^2)$ for $i = \pm 1$ implies that $q_{-1} + q_1 \leq 2\alpha\beta/(2\beta-\gamma)$. For a given pair $(q_{-1}, q_1)$ such that $q_i \leq \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-i}/(2\beta^2-\gamma^2)$ for $i = \pm 1$, consider the line with slope $-1$ through the point $(q_{-1}, q_1)$. For example, in Figure 8, $EFGH$ is such a line, with points $(q_{-1}, q_1)$ on line segment $FG$ satisfying $q_i \leq \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-i}/(2\beta^2-\gamma^2)$ for $i = \pm 1$; and $JK$ is also such a line, with all points $(q_{-1}, q_1)$ on line segment $JK$ satisfying $q_i \leq \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-i}/(2\beta^2-\gamma^2)$ for $i = \pm 1$. We show that the prices $y^*_i$ given by (39) corresponding to all points $(q_{-1}, q_1)$ on line segment $FG$ satisfy $y^*_i > 0$. It follows that the prices $y^*_i$ given by (39) corresponding to all points $(q_{-1}, q_1)$ on line segment $JK$ also satisfy $y^*_i > 0$. The coordinates of point $F$ are $([(2\beta^2-\gamma^2)(q_{-1}+q_1) - \alpha\beta(\beta+\gamma)]/(2\beta^2-\beta\gamma-\gamma^2), [\alpha\beta(\beta+\gamma) - \beta\gamma(q_{-1}+q_1)]/(2\beta^2-\beta\gamma-\gamma^2))$ and the coordinates of point $G$ are $([\alpha\beta(\beta+\gamma) - \beta\gamma(q_{-1}+q_1)]/(2\beta^2-\beta\gamma-\gamma^2), [(2\beta^2-\gamma^2)(q_{-1}+q_1) - \alpha\beta(\beta+\gamma)]/(2\beta^2-\beta\gamma-\gamma^2))$. Consider the prices $y^*_i$ given in (39). Note that

$$\begin{aligned} y^*_i \quad &= \quad \frac{\alpha(\beta+\gamma) - \beta q_i - \gamma q_{-i}}{\beta^2 - \gamma^2} > 0 \\ &\Leftrightarrow \quad \alpha(\beta+\gamma) - \beta q_i - \gamma q_{-i} > 0 \\ &\Leftrightarrow \quad \beta q_i + \gamma(q_{-i} + q_i - q_i) < \alpha(\beta+\gamma) \\ &\Leftrightarrow \quad (\beta-\gamma) q_i + \gamma(q_{-i} + q_i) < \alpha(\beta+\gamma) \end{aligned} \tag{41}$$

If $(q_{-1}, q_1)$ is on line segment $FG$, then

$$
\begin{aligned}
q_i &\le \frac{\alpha\beta(\beta+\gamma) - \beta\gamma(q_{-1}+q_1)}{2\beta^2-\beta\gamma-\gamma^2} \\
\Leftrightarrow \quad (\beta-\gamma)q_i + \gamma(q_{-i}+q_i) &\le (\beta-\gamma)\frac{\alpha\beta(\beta+\gamma) - \beta\gamma(q_{-1}+q_1)}{2\beta^2-\beta\gamma-\gamma^2} + \gamma(q_{-i}+q_i) \\
&= \frac{\alpha\beta^3 - \alpha\beta\gamma^2 + \beta^2\gamma(q_{-1}+q_1) - \gamma^3(q_{-i}+q_i)}{2\beta^2-\beta\gamma-\gamma^2} \\
&= \frac{\alpha\beta(\beta^2-\gamma^2) + (\beta^2-\gamma^2)\gamma(q_{-1}+q_1)}{2\beta^2-\beta\gamma-\gamma^2} \\
&= \frac{(\beta-\gamma)(\beta+\gamma)[\alpha\beta+\gamma(q_{-1}+q_1)]}{(\beta-\gamma)(2\beta+\gamma)} \\
&= \frac{(\beta+\gamma)[\alpha\beta+\gamma(q_{-1}+q_1)]}{2\beta+\gamma} \qquad (42)
\end{aligned}
$$

Next, by separately considering the cases $\gamma \le 0$ and $\gamma \ge 0$, we show that $[\alpha\beta+\gamma(q_{-1}+q_1)]/(2\beta+\gamma) < \alpha$, then it follows from (42) that $(\beta-\gamma)q_i + \gamma(q_{-i}+q_i) < \alpha(\beta+\gamma)$, and hence it follows from (41) that $y_i^* > 0$. First, suppose that $\gamma \le 0$. Note that

$$
\begin{aligned}
-\gamma &< \beta \\
\Leftrightarrow \quad \beta &< 2\beta+\gamma \\
\Leftrightarrow \quad \frac{\alpha\beta}{2\beta+\gamma} &< \alpha \\
\Rightarrow \quad \frac{\alpha\beta+\gamma(q_{-1}+q_1)}{2\beta+\gamma} &< \alpha \qquad (43)
\end{aligned}
$$

The last step follows since $\gamma \le 0$ and $q_{-1}+q_1 \ge 0$. It follows from (41), (42) and (43) that, if $\gamma \le 0$, then $y_i^* > 0$.

Next, suppose that $\gamma \ge 0$. Note that

$$
\begin{aligned}
\gamma &< \beta \\
\Leftrightarrow \quad \beta &< 2\beta-\gamma \\
\Leftrightarrow \quad \frac{\alpha\beta(2\beta-\gamma+2\gamma)}{(2\beta-\gamma)(2\beta+\gamma)} &< \alpha \\
\Leftrightarrow \quad \frac{\alpha\beta+\frac{2\alpha\beta\gamma}{2\beta-\gamma}}{2\beta+\gamma} &< \alpha \\
\Rightarrow \quad \frac{\alpha\beta+\gamma(q_{-1}+q_1)}{2\beta+\gamma} &< \alpha \qquad (44)
\end{aligned}
$$

The last step follows since $\gamma \ge 0$ and $q_{-1}+q_1 \le 2\alpha\beta/(2\beta-\gamma)$. It follows from (41), (42) and (44) that, if $\gamma \ge 0$, then $y_i^* > 0$.

Next we verify that, if $q_i \le \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-i}/(2\beta^2-\gamma^2)$ for $i = \pm 1$, then $(y_{-1}^*, y_1^*)$ given in (39) is the unique equilibrium. We verify that $y_i^*$ given in (39) is the unique best response for seller $i$ to $y_{-i}^*$. Given $y_{-i}^*$, the profit of seller $i$ is given by

$$
\begin{aligned}
g_i(y_i, y_{-i}^*) &= y_i \min\left\{q_i, \max\{0,\, \alpha-\beta y_i+\gamma y_{-i}^*\}\right\} \\
&= \begin{cases} y_i q_i & \text{if} \quad y_i \le \frac{\alpha+\gamma y_{-i}^*-q_i}{\beta} \\ y_i\left(\alpha-\beta y_i+\gamma y_{-i}^*\right) & \text{if} \quad \frac{\alpha+\gamma y_{-i}^*-q_i}{\beta} \le y_i \le \frac{\alpha+\gamma y_{-i}^*}{\beta} \\ 0 & \text{if} \quad y_i \ge \frac{\alpha+\gamma y_{-i}^*}{\beta} \end{cases}
\end{aligned}
$$

Thus $g_i(y_i, y^*_{-i})$ is a nondecreasing linear function of $y_i$ if $y_i \le (\alpha + \gamma y^*_{-i} - q_i)/\beta$. If $(\alpha + \gamma y^*_{-i} - q_i)/\beta < y_i < (\alpha + \gamma y^*_{-i})/\beta$, then $g_i(y_i, y^*_{-i})$ is a concave quadratic function of $y_i$, with

$$\begin{aligned}
g'_i(y_i, y^*_{-i}) &= -2\beta y_i + \alpha + \gamma y^*_{-i} \\
&< -2\left(\alpha + \gamma y^*_{-i} - q_i\right) + \alpha + \gamma y^*_{-i} \\
&= -\alpha - \gamma y^*_{-i} + 2q_i \\
&= -\alpha - \gamma \frac{\alpha(\beta+\gamma) - \beta q_{-i} - \gamma q_i}{\beta^2 - \gamma^2} + 2q_i \\
&= \frac{-\alpha\beta^2 - \alpha\beta\gamma + \beta\gamma q_{-i} + (2\beta^2 - \gamma^2)q_i}{\beta^2 - \gamma^2}
\end{aligned}$$

If $(q_{-1}, q_1)$ is on line segment $FG$, then

$$\begin{aligned}
q_i &\le \frac{\alpha\beta(\beta+\gamma) - \beta\gamma(q_{-i} + q_i)}{2\beta^2 - \beta\gamma - \gamma^2} \\
\Leftrightarrow \quad 0 &\ge -\alpha\beta^2 - \alpha\beta\gamma + \beta\gamma(q_{-i} + q_i) + (2\beta^2 - \beta\gamma - \gamma^2)q_i \\
&= -\alpha\beta^2 - \alpha\beta\gamma + \beta\gamma q_{-i} + (2\beta^2 - \gamma^2)q_i \\
\Leftrightarrow \quad 0 &\ge \frac{-\alpha\beta^2 - \alpha\beta\gamma + \beta\gamma q_{-i} + (2\beta^2 - \gamma^2)q_i}{\beta^2 - \gamma^2} \\
\Leftrightarrow \quad g'_i(y_i, y^*_{-i}) &< 0
\end{aligned}$$

Hence, if $(q_{-1}, q_1)$ is on line segment $FG$, then $g'_i(y_i, y^*_{-i}) < 0$ for all $y_i \in ((\alpha + \gamma y^*_{-i} - q_i)/\beta,\, (\alpha + \gamma y^*_{-i})/\beta)$. Hence, the unique best response for seller $i$ to $y^*_{-i}$ is $B_i(y^*_{-i}) = (\alpha + \gamma y^*_{-i} - q_i)/\beta$. It follows in the same way that if $(q_{-1}, q_1)$ is on line segment $JK$, then the unique best response for seller $i$ to $y^*_{-i}$ is $B_i(y^*_{-i}) = (\alpha + \gamma y^*_{-i} - q_i)/\beta$. Therefore, if $q_i \le \alpha\beta(\beta+\gamma)/(2\beta^2 - \gamma^2) - \beta\gamma q_{-i}/(2\beta^2 - \gamma^2)$ for $i = \pm 1$, then $(y^*_{-1}, y^*_1)$ given in (39) is the unique equilibrium.

The resulting profit of each seller $i$ is equal to

$$y^*_i \min\{q_i,\, \max\{0,\, \alpha - \beta y^*_i + \gamma y^*_{-i}\}\} \;=\; \frac{\alpha(\beta+\gamma)q_i - \beta q_i^2 - \gamma q_{-i} q_i}{\beta^2 - \gamma^2} \tag{45}$$

and thus the total profit of both sellers together is equal to

$$\frac{\alpha(\beta+\gamma)(q_{-1} + q_1) - \beta(q_{-1}^2 + q_1^2) - 2\gamma q_{-1} q_1}{\beta^2 - \gamma^2} \tag{46}$$

Therefore, if $q_i \le \alpha\beta(\beta+\gamma)/(2\beta^2 - \gamma^2) - \beta\gamma q_{-i}/(2\beta^2 - \gamma^2)$ for $i = \pm 1$, then the equilibrium prices are given by (39), the equilibrium demand is given by (40), the resulting profit of each seller is given by (45), and thus the total profit of both sellers together is given by (46).

Next we determine the value of $(q_{-1}, q_1)$ that maximizes the total profit of both sellers together under Case 3. First we fix the value of $q_{-1} + q_1$ at some value $q \le b_{\min}$, and choose $q_1$ to maximize the total profit subject to $q_{-1} + q_1 = q$. Thereafter we choose $q$ to maximize the total profit subject to $q \le b_{\min}$. It follows from (46) that the total profit is equal to

$$\begin{aligned}
\frac{\alpha(\beta+\gamma)(q_{-1} + q_1) - \beta(q_{-1}^2 + q_1^2) - 2\gamma q_{-1} q_1}{\beta^2 - \gamma^2} &= \frac{\alpha(\beta+\gamma)(q_{-1} + q_1) - \beta(q_{-1}^2 + 2q_{-1}q_1 + q_1^2) + 2\beta q_{-1}q_1 - 2\gamma q_{-1} q_1}{\beta^2 - \gamma^2} \\
&= \frac{\alpha(\beta+\gamma)q - \beta q^2 + 2(\beta-\gamma)(q - q_1)q_1}{\beta^2 - \gamma^2} \\
&= \frac{\alpha(\beta+\gamma)q - \beta q^2 + 2(\beta-\gamma)q q_1 - 2(\beta-\gamma)q_1^2}{\beta^2 - \gamma^2}
\end{aligned}$$

Let

$$H_1(q_1) \ := \ \frac{\alpha(\beta+\gamma)q - \beta q^2 + 2(\beta-\gamma)qq_1 - 2(\beta-\gamma)q_1^2}{\beta^2-\gamma^2}$$

Note that $H_1$ is a concave quadratic function that is maximized at $q_1^* = q/2$, and the corresponding value of $q_{-1}$ is also $q_{-1}^* = q/2$. Recall that (46) applies if $q_i \le \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-i}/(2\beta^2-\gamma^2)$ for $i = \pm 1$. Note that

$$\begin{aligned} q_i^* &\le \frac{\alpha\beta(\beta+\gamma)}{2\beta^2-\gamma^2} - \frac{\beta\gamma}{2\beta^2-\gamma^2} q_{-i}^* \qquad \text{for } i = \pm 1 \\ \Leftrightarrow \quad \frac{q}{2} &\le \frac{\alpha\beta(\beta+\gamma)}{2\beta^2-\gamma^2} - \frac{\beta\gamma}{2\beta^2-\gamma^2}\frac{q}{2} \\ \Leftrightarrow \quad q &\le \frac{2\alpha\beta}{2\beta-\gamma} \end{aligned}$$

Next we choose $q$ to maximize the total profit subject to $q \le b_{\min}$ and $q \le 2\alpha\beta/(2\beta-\gamma)$. Let

$$\begin{aligned} H_2(q) &:= H_1(q/2) \\ &= \frac{\alpha(\beta+\gamma)q - \beta q^2 + 2(\beta-\gamma)q^2/2 - 2(\beta-\gamma)q^2/4}{\beta^2-\gamma^2} \\ &= \frac{2\alpha(\beta+\gamma)q - (\beta+\gamma)q^2}{2(\beta-\gamma)(\beta+\gamma)} \\ &= \frac{2\alpha q - q^2}{2(\beta-\gamma)} \end{aligned}$$

Note that $H_2$ is a concave quadratic function and $H_2'(q^*) = 0 \Leftrightarrow q^* = \alpha$. Also note that $q^* = \alpha \le 2\alpha\beta/(2\beta-\gamma)$ if and only if $\gamma \ge 0$. Let $a_{\min} := \min\{\alpha, b_{\min}, 2\alpha\beta/(2\beta-\gamma)\}$. Then the value of $(q_{-1}, q_1)$ that maximizes the total profit and that satisfies $q_i \le \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-i}/(2\beta^2-\gamma^2)$ for $i = \pm 1$, is $q_{-1}^* = q_1^* = a_{\min}/2$. The corresponding total profit is $H_2(a_{\min}) = (2\alpha - a_{\min})a_{\min}/[2(\beta-\gamma)]$. This concludes Case 3.

*Optimal exchange.* Next, we compare the profits under Cases 1, 2, and 3, and determine the value of $(q_{-1}, q_1)$, that is, the value of the exchange $x = (x_{-1}, x_1)$, that maximizes the total profit of both sellers together. Different cases hold, depending on the capacity ratio $b_{\min}/\alpha$ and the price coefficient ratio $\gamma/\beta$ (recall that $\gamma/\beta \in (-1, 1)$). The different cases are depicted in Figure 9.

*Case A (small capacity).* $b_{\min}/\alpha \le [1+\gamma/\beta]/[2-(\gamma/\beta)^2]$, that is, $b_{\min} \le \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2)$:

In Figure 8, line $JK$ shows an example of pairs $(q_{-1}, q_1)$ such that $q_{-1} + q_1 = b_{\min}$ for a given value of $b_{\min} < \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2)$, and triangle $0JK$ shows pairs $(q_{-1}, q_1) \ge 0$ such that $q_{-1} + q_1 \le b_{\min}$. In this case, the capacity $b_{\min}$ is so small that all feasible values of $(q_{-1}, q_1)$ correspond to Case 3. Recall that $\alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) \in (0, 2\alpha\beta/(2\beta-\gamma))$.

*Case A1.* $\gamma/\beta \le 0$ and $b_{\min}/\alpha \le [1+\gamma/\beta]/[2-(\gamma/\beta)^2]$, that is, $\gamma \le 0$ and $b_{\min} \le \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2)$:

Recall that $2\alpha\beta/(2\beta-\gamma) \le \alpha$ if and only if $\gamma \le 0$. Since $b_{\min} \le \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) < 2\alpha\beta/(2\beta-\gamma) \le \alpha$, it follows that $b_{\min} = \min\{\alpha, b_{\min}, 2\alpha\beta/(2\beta-\gamma)\}$, and thus the value of $(q_{-1}, q_1)$ that maximizes the total profit is $q_{-1}^* = q_1^* = b_{\min}/2$, and the maximum total profit is $(2\alpha - b_{\min})b_{\min}/[2(\beta-\gamma)]$. The resulting equilibrium price of each seller, given by (39), is $y_i^* = (2\alpha - b_{\min})/[2(\beta-\gamma)]$, and the resulting equilibrium demand of each seller, given by (40), is equal to $q_i^* = b_{\min}/2$.

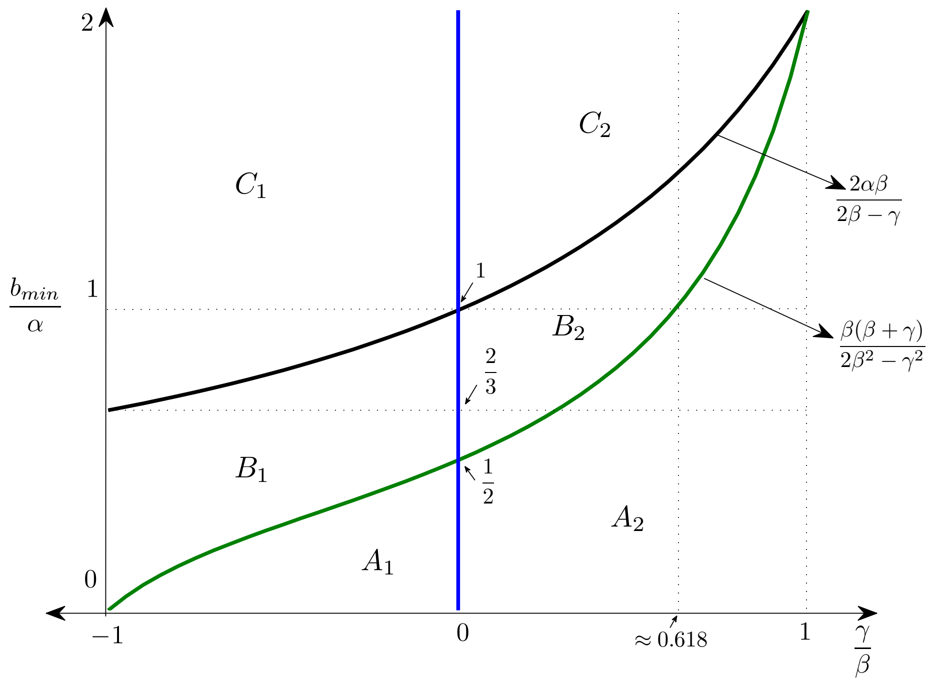



**Figure 9** **Different cases of the capacity ratio $b_{\min}/\alpha$ and the price coefficient ratio $\gamma/\beta$.**

*Case A2.* $\gamma/\beta \geq 0$ and $b_{\min}/\alpha \leq [1+\gamma/\beta]/[2-(\gamma/\beta)^2]$, that is, $\gamma \geq 0$ and $b_{\min} \leq \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2)$:

In this case, $b_{\min} \leq \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) < 2\alpha\beta/(2\beta-\gamma)$ and $\alpha \leq 2\alpha\beta/(2\beta-\gamma)$. If $\alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) \leq \alpha$, then $b_{\min} \leq \alpha$ and thus $b_{\min} = \min\{\alpha, b_{\min}, 2\alpha\beta/(2\beta-\gamma)\}$, the value of $(q_{-1}, q_1)$ that maximizes the total profit is $q_{-1}^* = q_1^* = b_{\min}/2$, and the maximum total profit is $(2\alpha - b_{\min})b_{\min}/[2(\beta-\gamma)]$. The resulting equilibrium price of each seller, given by (39), is $y_i^* = (2\alpha - b_{\min})/[2(\beta-\gamma)]$, and the resulting equilibrium demand of each seller, given by (40), is equal to $q_i^* = b_{\min}/2$. Note that $\alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) \leq \alpha$ if and only if $\gamma/\beta \leq (\sqrt{5}-1)/2 = 1/\varphi = \varphi - 1 \approx 0.618$, where $\varphi$ denotes the golden ratio. If $\gamma/\beta > (\sqrt{5}-1)/2$ (and thus $\alpha < \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2)$), then there are two possibilities. If $b_{\min} \leq \alpha$, then as before, $q_{-1}^* = q_1^* = b_{\min}/2$, the equilibrium price of each seller is $y_i^* = (2\alpha - b_{\min})/[2(\beta-\gamma)]$, the equilibrium demand of each seller is equal to $q_i^* = b_{\min}/2$, and the maximum total profit is $(2\alpha - b_{\min})b_{\min}/[2(\beta-\gamma)]$. Otherwise, if $\alpha < b_{\min}$, then $q_{-1}^* = q_1^* = \alpha/2$, the resulting equilibrium price of each seller, given by (39), is $y_i^* = \alpha/[2(\beta-\gamma)]$, the resulting equilibrium demand of each seller, given by (40), is equal to $q_i^* = \alpha/2$, and the maximum total profit is $(2\alpha-\alpha)\alpha/[2(\beta-\gamma)] = \alpha^2/[2(\beta-\gamma)]$. Note that in this case the optimal resource exchange $x^*$ is such that $q_{-1}^* + q_1^* = \alpha < b_{\min}$, that is, some capacity is not used.

*Case B (intermediate capacity).* $[1+\gamma/\beta]/[2-(\gamma/\beta)^2] \leq b_{\min}/\alpha \leq 2/(2-\gamma/\beta)$, that is, $\alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) \leq b_{\min} \leq 2\alpha\beta/(2\beta-\gamma)$:

In Figure 8, line $EFGH$ shows an example of pairs $(q_{-1}, q_1)$ such that $q_{-1} + q_1 = b_{\min}$ for a given value of $b_{\min} \in (\alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2),\, 2\alpha\beta/(2\beta-\gamma))$, and triangle $0EH$ shows pairs $(q_{-1}, q_1) \geq 0$ such that $q_{-1} + q_1 \leq b_{\min}$. In this case with intermediate capacity $b_{\min}$, there are feasible values of $(q_{-1}, q_1)$ corresponding to

Case 3, for example in pentagon $0LFGM$ in Figure 8, and there are feasible values of $(q_{-1}, q_1)$ corresponding to Case 2, for example in triangles $EFL$ and $GHM$ in Figure 8.

Consider any two pairs $(q_{-1}, q_1)$ and $(q'_{-1}, q'_1)$ in triangle $EFL$ such that $q_{-1} = q'_{-1}$. It follows from (34), (36), (37), and (38) that the equilibrium prices, the equilibrium demand, the profit of each seller, and thus the total profit of both sellers together are the same for $(q_{-1}, q_1)$ and $(q'_{-1}, q'_1)$. Therefore, for any point $(q_{-1}, q_1)$ in triangle $EFL$, there is a point $(q_{-1},\, \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-1}/(2\beta^2-\gamma^2))$ on the boundary $LF$ between triangle $EFL$ and pentagon $0LFGM$ with the same total profit as at point $(q_{-1}, q_1)$. Next, we show that the total profit as a function of $(q_{-1}, q_1)$ is continuous on the boundary between triangle $EFL$ and pentagon $0LFGM$. Recall from (46) that the total profit at a point $(q_{-1}, q_1)$ in pentagon $0LFGM$ is equal to

$$\frac{\alpha(\beta+\gamma)\left(q_{-1}+q_1\right) - \beta\left(q_{-1}^2+q_1^2\right) - 2\gamma q_{-1}q_1}{\beta^2-\gamma^2}$$

Specifically, at the boundary point $(q_{-1},\, \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-1}/(2\beta^2-\gamma^2))$ the total profit is equal to

$$\begin{aligned}
&\frac{\alpha(\beta+\gamma)\left(q_{-1}+\frac{\alpha\beta(\beta+\gamma)-\beta\gamma q_{-1}}{2\beta^2-\gamma^2}\right) - \beta\left(q_{-1}^2+\left[\frac{\alpha\beta(\beta+\gamma)-\beta\gamma q_{-1}}{2\beta^2-\gamma^2}\right]^2\right) - 2\gamma q_{-1}\frac{\alpha\beta(\beta+\gamma)-\beta\gamma q_{-1}}{2\beta^2-\gamma^2}}{\beta^2-\gamma^2} \\
&= \frac{\left\{\begin{array}{c}\left[\alpha^2\beta(\beta+\gamma)^2\left(2\beta^2-\gamma^2\right) - \alpha^2\beta^3(\beta+\gamma)^2\right] \\ +\left[\alpha(\beta+\gamma)\left(2\beta^2-\gamma^2\right)^2 - \alpha\beta\gamma(\beta+\gamma)\left(2\beta^2-\gamma^2\right) + 2\alpha\beta^3\gamma(\beta+\gamma) - 2\alpha\beta\gamma(\beta+\gamma)\left(2\beta^2-\gamma^2\right)\right] q_{-1} \\ +\left[-\beta\left(2\beta^2-\gamma^2\right)^2 - \beta^3\gamma^2 + 2\beta\gamma^2\left(2\beta^2-\gamma^2\right)\right] q_{-1}^2\end{array}\right\}}{\left(2\beta^2-\gamma^2\right)^2\left(\beta^2-\gamma^2\right)} \\
&= \frac{\left\{\begin{array}{c}\alpha^2\beta\left(2\beta^2-\gamma^2-\beta^2\right)(\beta+\gamma)^2 \\ +\alpha\left(4\beta^4-4\beta^2\gamma^2+\gamma^4-2\beta^3\gamma+\beta\gamma^3+2\beta^3\gamma-4\beta^3\gamma+2\beta\gamma^3\right)(\beta+\gamma)q_{-1} \\ -\beta\left(4\beta^4-4\beta^2\gamma^2+\gamma^4+\beta^2\gamma^2-4\beta^2\gamma^2+2\gamma^4\right)q_{-1}^2\end{array}\right\}}{\left(2\beta^2-\gamma^2\right)^2\left(\beta^2-\gamma^2\right)} \\
&= \frac{\left\{\begin{array}{c}\alpha^2\beta\left(\beta^2-\gamma^2\right)(\beta+\gamma)^2 \\ +\alpha\left(4\beta^4-4\beta^3\gamma-4\beta^2\gamma^2+3\beta\gamma^3+\gamma^4\right)(\beta+\gamma)q_{-1} \\ -\beta\left(4\beta^4-7\beta^2\gamma^2+3\gamma^4\right)q_{-1}^2\end{array}\right\}}{\left(2\beta^2-\gamma^2\right)^2\left(\beta^2-\gamma^2\right)} \\
&= \frac{\left\{\begin{array}{c}\alpha^2\beta(\beta-\gamma)(\beta+\gamma)^3 \\ +\alpha\left(4\beta^3-4\beta\gamma^2-\gamma^3\right)(\beta-\gamma)(\beta+\gamma)q_{-1} \\ -\beta\left(4\beta^2-3\gamma^2\right)(\beta-\gamma)(\beta+\gamma)q_{-1}^2\end{array}\right\}}{\left(2\beta^2-\gamma^2\right)^2(\beta-\gamma)(\beta+\gamma)} \\
&= \frac{\alpha^2\beta(\beta+\gamma)^2 + \alpha\left(4\beta^3-4\beta\gamma^2-\gamma^3\right)q_{-1} - \beta\left(4\beta^2-3\gamma^2\right)q_{-1}^2}{\left(2\beta^2-\gamma^2\right)^2}
\end{aligned}$$

which is the same as the total profit given by (38) for point $(q_{-1},\, \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-1}/(2\beta^2-\gamma^2))$ in triangle $EFL$. Thus the total profit as a function of $(q_{-1}, q_1)$ is continuous on the boundary between triangle $EFL$ and pentagon $0LFGM$. The same observation applies to the total profit as a function of $(q_{-1}, q_1)$ in triangle $GHM$. Hence, in Case B with intermediate capacity, it is sufficient to optimize $(q_{-1}, q_1)$ over pentagon $0LFGM$ only, that is, it is sufficient to restrict attention to feasible values of $(q_{-1}, q_1)$ corresponding to Case 3. The rest of Case B follows in the same way as for Case A with small capacity.

*Case B1.* $\gamma/\beta \leq 0$ and $[1+\gamma/\beta]/[2-(\gamma/\beta)^2] \leq b_{\min}/\alpha \leq 2/(2-\gamma/\beta)$, that is, $\gamma \leq 0$ and $\alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) \leq b_{\min} \leq 2\alpha\beta/(2\beta-\gamma)$:

Consider the optimal value of $(q_{-1}, q_1)$ in pentagon $0LFGM$. Since $b_{\min} \leq 2\alpha\beta/(2\beta-\gamma) \leq \alpha$, it follows that $b_{\min} = \min\{\alpha, b_{\min}, 2\alpha\beta/(2\beta-\gamma)\}$, and thus the value of $(q_{-1}, q_1)$ in pentagon $0LFGM$ that maximizes the total profit is $q_{-1}^* = q_1^* = b_{\min}/2$, and the maximum total profit is $(2\alpha - b_{\min})b_{\min}/[2(\beta-\gamma)]$. The resulting equilibrium price of each seller is $y_i^* = (2\alpha - b_{\min})/[2(\beta-\gamma)]$, and the resulting equilibrium demand of each seller is equal to $q_i^* = b_{\min}/2$.

*Case B2.* $\gamma/\beta \geq 0$ and $[1+\gamma/\beta]/[2-(\gamma/\beta)^2] \leq b_{\min}/\alpha \leq 2/(2-\gamma/\beta)$, that is, $\gamma \geq 0$ and $\alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) \leq b_{\min} \leq 2\alpha\beta/(2\beta-\gamma)$:

If $\gamma/\beta \geq (\sqrt{5}-1)/2$ (and thus $\alpha \leq \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2)$), then $\alpha = \min\{\alpha, b_{\min}, 2\alpha\beta/(2\beta-\gamma)\}$, the value of $(q_{-1}, q_1)$ that maximizes the total profit is $q_{-1}^* = q_1^* = \alpha/2$, and the maximum total profit is $(2\alpha-\alpha)\alpha/[2(\beta-\gamma)] = \alpha^2/[2(\beta-\gamma)]$. The resulting equilibrium price of each seller, given by (39), is $y_i^* = \alpha/[2(\beta-\gamma)]$, and the resulting equilibrium demand of each seller, given by (40), is equal to $q_i^* = \alpha/2$. In this case the optimal resource exchange $x^*$ is such that $q_{-1}^* + q_1^* = \alpha \leq b_{\min}$, that is, some capacity is not used. If $\gamma/\beta < (\sqrt{5}-1)/2$ (and thus $\alpha > \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2)$), then there are two possibilities. If $\alpha \leq b_{\min}$, then as before, $q_{-1}^* = q_1^* = \alpha/2$, the equilibrium price of each seller is $y_i^* = \alpha/[2(\beta-\gamma)]$, the equilibrium demand of each seller is equal to $q_i^* = \alpha/2$, and the maximum total profit is $\alpha^2/[2(\beta-\gamma)]$. Otherwise, if $b_{\min} \leq \alpha$, then $q_{-1}^* = q_1^* = b_{\min}/2$, the equilibrium price of each seller is $y_i^* = (2\alpha - b_{\min})/[2(\beta-\gamma)]$, the equilibrium demand of each seller is equal to $q_i^* = b_{\min}/2$, and the maximum total profit is $(2\alpha - b_{\min})b_{\min}/[2(\beta-\gamma)]$.

*Case C (large capacity).* $b_{\min}/\alpha \geq 2/(2-\gamma/\beta)$, that is, $b_{\min} \geq 2\alpha\beta/(2\beta-\gamma)$:

In Figure 8, line $ABCD$ shows an example of pairs $(q_{-1}, q_1)$ such that $q_{-1} + q_1 = b_{\min}$ for a given value of $b_{\min} \geq 2\alpha\beta/(2\beta-\gamma)$, and triangle $0AD$ shows pairs $(q_{-1}, q_1) \geq 0$ such that $q_{-1} + q_1 \leq b_{\min}$. In this case with large capacity $b_{\min}$, there are feasible values of $(q_{-1}, q_1)$ in quadrilateral $0LIM$ in Figure 8 corresponding to Case 3, there are feasible values of $(q_{-1}, q_1)$ corresponding to Case 2, for example in quadrilaterals $ABIL$ and $DCIM$ in Figure 8, and there are feasible values of $(q_{-1}, q_1)$ corresponding to Case 1, for example in triangle $BCI$ in Figure 8.

For any point $(q_{-1}, q_1)$ in $ABIL$, there is a point $(q_{-1}, \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-1}/(2\beta^2-\gamma^2))$ on the boundary $IL$ between $ABIL$ and $0LIM$ with the same total profit as at point $(q_{-1}, q_1)$. It was shown under Case B that the total profit as a function of $(q_{-1}, q_1)$ is continuous on the boundary. The same observation applies to the total profit as a function of $(q_{-1}, q_1)$ in $DCIM$. Hence, in Case C with large capacity, it is sufficient to optimize $(q_{-1}, q_1)$ over quadrilateral $0LIM$ and triangle $BCI$ only, that is, it is sufficient to restrict attention to feasible values of $(q_{-1}, q_1)$ corresponding to Case 3 and Case 1.

*Case C1.* $\gamma/\beta \leq 0$ and $b_{\min}/\alpha \geq 2/(2-\gamma/\beta)$, that is, $\gamma \leq 0$ and $b_{\min} \geq 2\alpha\beta/(2\beta-\gamma)$:

Since $2\alpha\beta/(2\beta-\gamma) \leq \alpha$ and $b_{\min} \geq 2\alpha\beta/(2\beta-\gamma)$, it follows that $2\alpha\beta/(2\beta-\gamma) = \min\{\alpha, b_{\min}, 2\alpha\beta/(2\beta-\gamma)\}$, and thus the value of $(q_{-1}, q_1)$ that maximizes the total profit over $0LIM$ is given by $q_{-1}^* = q_1^* = \alpha\beta/(2\beta-\gamma)$ represented by point $I$, and the corresponding total profit is $(2\alpha - 2\alpha\beta/(2\beta-\gamma))2\alpha\beta/(2\beta-\gamma)/[2(\beta-\gamma)] = 2\alpha^2\beta/(2\beta-\gamma)^2$. Also, as shown in Case 1, all values of $(q_{-1}, q_1)$ in triangle $BCI$ have the same total profit of $2\alpha^2\beta/(2\beta-\gamma)^2$. Thus, any point $(q_{-1}, q_1)$ in triangle $BCI$ represents an optimal resource exchange for Case C1. For all such optimal resource exchanges, the resulting equilibrium price of each seller, given by both (30) and (39), is $y_i^* = \alpha/(2\beta-\gamma)$, and the resulting equilibrium demand of each seller, given by both (31) and (40), is equal to $\alpha\beta/(2\beta-\gamma)$.

*Case C2.* $\gamma/\beta \geq 0$ and $b_{\min}/\alpha \geq 2/(2-\gamma/\beta)$, that is, $\gamma \geq 0$ and $b_{\min} \geq 2\alpha\beta/(2\beta-\gamma)$:

Since $b_{\min} \geq 2\alpha\beta/(2\beta-\gamma) \geq \alpha$, it follows that $\alpha = \min\{\alpha, b_{\min}, 2\alpha\beta/(2\beta-\gamma)\}$, and thus the value of $(q_{-1}, q_1)$ that maximizes the total profit over $0LIM$ is $q_{-1}^* = q_1^* = \alpha/2$, and the corresponding total profit is $(2\alpha-\alpha)\alpha/[2(\beta-\gamma)] = \alpha^2/[2(\beta-\gamma)]$. Also, all values of $(q_{-1}, q_1)$ in triangle $BCI$ have the same total profit of $2\alpha^2\beta/(2\beta-\gamma)^2$. Note that

$$\begin{aligned} 4\beta^2 - 4\beta\gamma + \gamma^2 &\geq 4\beta^2 - 4\beta\gamma \\ \Rightarrow \quad (2\beta-\gamma)^2 &\geq 4\beta(\beta-\gamma) \\ \Rightarrow \quad \frac{\alpha^2}{2(\beta-\gamma)} &\geq \frac{2\alpha^2\beta}{(2\beta-\gamma)^2} \end{aligned}$$

Thus the optimal point for Case C2 is $q_{-1}^* = q_1^* = \alpha/2$, and the maximum total profit is $\alpha^2/[2(\beta-\gamma)]$. The resulting equilibrium price of each seller, given by (39), is $y_i^* = \alpha/[2(\beta-\gamma)]$, and the resulting equilibrium demand of each seller, given by (40), is equal to $q_i^* = \alpha/2$.

Inspection of the results above for the settings with no alliance, perfect coordination, and a resource exchange alliance reveal that the results can be summarized by 5 cases, as in Table 1.

*Consumer surplus.* To calculate the consumer surplus associated with demand model (10), it is instructive to start with a utility model that leads to demand model (10). Consider a representative consumer who consumes $z_{-1}$ units of the product sold by seller $-1$ and $z_1$ units of the product sold by seller 1. Suppose that the resulting utility is given by $U(z_{-1}, z_1) := a_{-1}z_{-1} + a_1 z_1 - b_{-1}z_{-1}^2/2 - b_1 z_1^2/2 - cz_{-1}z_1$ with $b_{-1}, b_1, b_{-1}b_1 - c^2 > 0$. Given a price $p_i$ for the product sold by each seller $i$, the consumer chooses quantities $(z_{-1}, z_1)$ to maximize the consumer surplus $U(z_{-1}, z_1) - p_{-1}z_{-1} - p_1 z_1$. It follows that the chosen quantities satisfy

$$z_i = \frac{a_i b_{-i} - a_{-i}c}{b_{-1}b_1 - c^2} - \frac{b_{-i}}{b_{-1}b_1 - c^2}p_i + \frac{c}{b_{-1}b_1 - c^2}p_{-i}$$

This utility model leads to the demand model (10) if $\alpha = (a_i b_{-i} - a_{-i}c)/(b_{-1}b_1 - c^2)$, $\beta = b_i/(b_{-1}b_1 - c^2)$, and $\gamma = c/(b_{-1}b_1 - c^2)$ for $i = \pm 1$, that is, if $a_i = \alpha/(\beta-\gamma)$, $b_i = \beta/(\beta^2-\gamma^2)$, and $c = \gamma/(\beta^2-\gamma^2)$ for $i = \pm 1$.

In regions 1 and 2 in Table 1, the resulting consumer surplus is given by

$$U(b_{\min}/2, b_{\min}/2) - \frac{2\alpha - b_{\min}}{2(\beta-\gamma)}\frac{b_{\min}}{2} - \frac{2\alpha - b_{\min}}{2(\beta-\gamma)}\frac{b_{\min}}{2} = \frac{b_{\min}^2}{4(\beta-\gamma)}$$

In regions 3 and 4, the resulting consumer surplus is given by

$$U(\alpha\beta/(2\beta-\gamma), \alpha\beta/(2\beta-\gamma)) - \frac{\alpha}{2\beta-\gamma}\frac{\alpha\beta}{2\beta-\gamma} - \frac{\alpha}{2\beta-\gamma}\frac{\alpha\beta}{2\beta-\gamma} = \frac{\alpha^2\beta^2}{(\beta-\gamma)(2\beta-\gamma)^2}$$

In region 5, the resulting consumer surplus is given by

$$U(\alpha/2, \alpha/2) - \frac{\alpha}{2(\beta-\gamma)}\frac{\alpha}{2} - \frac{\alpha}{2(\beta-\gamma)}\frac{\alpha}{2} = \frac{\alpha^2}{4(\beta-\gamma)}$$

Thus, in region 1 all three settings have the same consumer surplus. In region 2, the consumer surplus under perfect coordination and under the alliance are the same, and as shown in Section 3.2, both are larger than the consumer surplus under no alliance. To compare the consumer surplus under the alliance and under no alliance in regions 3 and 4, note that

$$\begin{aligned} \frac{\alpha^2}{9(\beta-\gamma)} &\leq \frac{\alpha^2\beta^2}{(\beta-\gamma)(2\beta-\gamma)^2} \\ \Leftrightarrow \quad -4\beta\gamma+\gamma^2 &\leq 5\beta^2 \end{aligned}$$

which holds since $\gamma \in (-\beta, \beta)$, and thus in regions 3 and 4 the consumer surplus under the alliance is greater than the consumer surplus under no alliance. To compare the consumer surplus under the alliance and under perfect coordination in region 3, note that

$$\begin{aligned} \frac{b_{\min}^2}{4(\beta-\gamma)} &\geq \frac{\alpha^2\beta^2}{(\beta-\gamma)(2\beta-\gamma)^2} \\ \Leftrightarrow \quad b_{\min} &\geq \frac{2\alpha\beta}{2\beta-\gamma} \end{aligned}$$

and thus in region 3 the consumer surplus under perfect coordination is greater than the consumer surplus under the alliance. To compare the consumer surplus under the alliance and under perfect coordination in region 4, note that

$$\begin{aligned} \frac{\alpha^2}{4(\beta-\gamma)} &\geq \frac{\alpha^2\beta^2}{(\beta-\gamma)(2\beta-\gamma)^2} \\ \Leftrightarrow \quad (2\beta-\gamma)^2 &\geq 4\beta^2 \end{aligned}$$

which holds since $\gamma \leq 0$ in region 4, and thus in region 4 the consumer surplus under perfect coordination is greater than the consumer surplus under the alliance. Finally, in region 5 the consumer surplus under perfect coordination and under the alliance are the same, and both are larger than the consumer surplus under no alliance by a factor of 9/4. Note that, similar to total profit, the consumer surplus under perfect coordination and under the alliance are the same except when capacity is large ($b_{\min} \geq 2\alpha\beta/(2\beta-\gamma)$) and the sellers' products are complements ($\gamma \leq 0$).

**Appendix A.4: Perfect Coordination with Product Differentiation**

The model of perfect coordination introduced in Section 3.2 (with details given in Section 7) was based on a model of demand $d$ for the two-resource product given by $d = \max\{0,\ \tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1)\}$, and the model of an alliance introduced in Section 3.3 (with details given in Section 7) was based on a model of demand $d_i(y_i, y_{-i})$ for the two-resource product of seller $i$ given by $d_i(y_i, y_{-i}) = \max\{0,\ \alpha - \beta y_i + \gamma y_{-i}\}$, where $\tilde{\alpha} = 2\alpha + 2(\beta-\gamma)(c_{-1}+c_1)$ and $\tilde{\beta} = 2(\beta-\gamma)$. Thus, the model of perfect coordination in Section 3.2 does not make provision for different brands of the two-resource product, but the model of an alliance in Section 3.3 makes provision for different brands of the two-resource product. In this section, we consider a model of perfect coordination that makes provision for different brands of the two-resource product.

The demand $d_i(y_i, y_{-i})$ for the brand $i$ product sold is given as follows:

$$d_i(y_i, y_{-i}) \;=\; \alpha - \beta y_i + \gamma y_{-i}$$

where as before $y_i$ denotes the excess of the price of the brand $i$ product over the marginal cost $c_{-1} + c_1$, and we consider only values of $(y_{-1}, y_1)$ such that $\alpha - \beta y_i + \gamma y_{-i} \geq 0$ for $i = \pm 1$.

First consider the case in which the capacity is not constraining (it is determined later what amount of capacity is sufficient for this condition to hold). In this case, the total profit is given by

$$g(y_{-1}, y_1) \;:=\; y_{-1} d_{-1}(y_{-1}, y_1) + y_1 d_1(y_1, y_{-1}) \;=\; \alpha(y_{-1} + y_1) - \beta(y_{-1}^2 + y_1^2) + 2\gamma y_{-1} y_1$$

Note that

$$\begin{aligned} \nabla g(y_{-1}, y_1) &= \begin{bmatrix} \alpha - 2\beta y_{-1} + 2\gamma y_1 \\ \alpha - 2\beta y_1 + 2\gamma y_{-1} \end{bmatrix} \\ \nabla^2 g(y_{-1}, y_1) &= \begin{bmatrix} -2\beta & 2\gamma \\ 2\gamma & -2\beta \end{bmatrix} \end{aligned}$$

and thus $\nabla^2 g(y_{-1}, y_1)$ is negative definite ($\beta > 0$, $\beta^2 - \gamma^2 > 0$), and hence $g$ is a concave quadratic function. Therefore, the prices that maximize the total profit are given by

$$y_{-1}^* \;=\; y_1^* \;=\; \frac{\alpha}{2(\beta - \gamma)}, \tag{47}$$

and the corresponding total demand at the optimal prices is equal to $\alpha$. Thus, if $b_{\min} \geq \alpha$, then the total profit of the two sellers under perfect coordination is given by $\frac{\alpha^2}{2(\beta-\gamma)}$. Note that the optimal prices, demand, profit, and consumer surplus are the same as for perfect coordination in Section 3.2 when $b_{\min} \geq \alpha$.

Next consider the case in which $b_{\min} < \alpha$. First we consider price points $(y_{-1}, y_1)$ such that $d_{-1}(y_{-1}, y_1) + d_1(y_1, y_{-1}) \leq b_{\min}$, and then we consider price points $(y_{-1}, y_1)$ such that $d_{-1}(y_{-1}, y_1) + d_1(y_1, y_{-1}) \geq b_{\min}$. It follows from the results above for $g$ that the point $(\check{y}_{-1}, \check{y}_1)$ that maximizes $g$ subject to the constraint $d_{-1}(y_{-1}, y_1) + d_1(y_1, y_{-1}) \leq b_{\min}$ satisfies $d_{-1}(\check{y}_{-1}, \check{y}_1) + d_1(\check{y}_1, \check{y}_{-1}) = b_{\min}$, that is, $2\alpha - (\beta - \gamma)(\check{y}_{-1} + \check{y}_1) = b_{\min}$. Let

$$\begin{aligned} g_1(y_1) &:= g\left([2\alpha - b_{\min}]/[\beta - \gamma] - y_1, y_1\right) \\ &= \alpha \frac{2\alpha - b_{\min}}{\beta - \gamma} - \beta \frac{(2\alpha - b_{\min})^2}{(\beta - \gamma)^2} + 2(\beta + \gamma)\left(\frac{2\alpha - b_{\min}}{\beta - \gamma} - y_1\right) y_1 \end{aligned}$$

Note that $g_1$ is a concave quadratic function with maximum at $\check{y}_1 = (2\alpha - b_{\min})/[2(\beta - \gamma)]$ (and thus $\check{y}_{-1} = \check{y}_1 = (2\alpha - b_{\min})/[2(\beta - \gamma)]$).

Next consider price points $(y_{-1}, y_1)$ such that $d_{-1}(y_{-1}, y_1) + d_1(y_1, y_{-1}) \geq b_{\min}$, that is, $2\alpha - (\beta - \gamma)(y_{-1} + y_1) \geq b_{\min}$. The model should specify how capacity $b_{\min}$ is to be allocated between the two brands if $d_{-1}(y_{-1}, y_1) + d_1(y_1, y_{-1}) > b_{\min}$. There are various ways to allocate constrained capacity. Here we present one such way, the equal rationing rule, in detail, and then we point out other ways that lead to the same results. Under the equal rationing rule, if $d_{-1}(y_{-1}, y_1) + d_1(y_1, y_{-1}) > b_{\min}$, then the same fraction $\lambda$ of the demands $d_i(y_i, y_{-i})$ for the different brands is satisfied, where

$$\lambda \;=\; \frac{b_{\min}}{d_{-1}(y_{-1}, y_1) + d_1(y_1, y_{-1})} \;=\; \frac{b_{\min}}{2\alpha - (\beta - \gamma)(y_{-1} + y_1)}$$

Then, the total profit is given by

$$\begin{aligned} g_2(y_{-1}, y_1) &= \lambda y_{-1}(\alpha - \beta y_{-1} + \gamma y_1) + \lambda y_1(\alpha - \beta y_1 + \gamma y_{-1}) \\ &= b_{\min} \frac{\alpha(y_{-1} + y_1) - \beta(y_{-1} + y_1)^2 + 2(\beta + \gamma) y_{-1} y_1}{2\alpha - (\beta - \gamma)(y_{-1} + y_1)} \end{aligned}$$

Let $y := y_{-1} + y_1$, and let

$$\begin{aligned} g_3(y, y_1) &:= g_2(y - y_1, y_1) \\ &= b_{\min} \frac{\alpha y - \beta y^2 + 2(\beta+\gamma) y y_1 - 2(\beta+\gamma) y_1^2}{2\alpha - (\beta-\gamma) y} \end{aligned}$$

Recall that, in this case, $2\alpha - (\beta-\gamma)(y_{-1} + y_1) \geq b_{\min}$, and thus $y \leq (2\alpha - b_{\min})/(\beta - \gamma)$. First, consider any fixed value of $y \in [0, (2\alpha - b_{\min})/(\beta-\gamma)]$, and maximize $g_3(y, \cdot)$ with respect to $y_1$. Note that $g_3(y, \cdot)$ is a concave quadratic function with maximum at $\hat{y}_1 = y/2$ (and thus $\hat{y}_{-1} = \hat{y}_1 = y/2$). Next, let

$$\begin{aligned} g_4(y) &:= g_2(y/2, y/2) \\ &= \frac{b_{\min}}{2} \frac{2\alpha y + \gamma y^2 - \beta y^2}{2\alpha - (\beta-\gamma) y} \\ &= \frac{b_{\min}}{2} y \end{aligned}$$

Note that the maximum of $g_4$ over $y \in [0, (2\alpha - b_{\min})/(\beta-\gamma)]$ is attained at $y = (2\alpha - b_{\min})/(\beta-\gamma)$, and thus $\hat{y}_{-1} = \hat{y}_1 = (2\alpha - b_{\min})/[2(\beta-\gamma)]$. Therefore, if $b_{\min} < \alpha$, then the optimal prices are

$$y^*_{-1} = y^*_1 = \check{y}_{-1} = \check{y}_1 = \hat{y}_{-1} = \hat{y}_1 = \frac{2\alpha - b_{\min}}{2(\beta-\gamma)} \tag{48}$$

with corresponding total demand equal to $b_{\min}$. Thus, the total profit under perfect coordination is equal to $(2\alpha - b_{\min}) b_{\min}/[2(\beta-\gamma)]$. Note that the optimal prices, demand, profit and consumer surplus are also the same as for perfect coordination in Section 3.2 when $b_{\min} \leq \alpha$.

Other rationing rules also lead to the same results. For example, suppose that the demand for brand $-1$ is satisfied first and then the remaining capacity, if any, is used for brand 1. In this case, the total profit is given by

$$g_5(y_{-1}, y_1) = y_{-1} \min\{b_{\min}, \alpha - \beta y_{-1} + \gamma y_1\} + y_1 \min\{\max\{0, b_{\min} - (\alpha - \beta y_{-1} + \gamma y_1)\}, \alpha - \beta y_1 + \gamma y_{-1}\}$$

For this rationing rule the optimal prices are same as in (48).

## Appendix B: Proof of Theorem 1

In the problem (18), the objective value is bounded below by zero. It is known that a quadratic program with a bounded objective value has an optimal solution. To establish uniqueness, consider the problem

$$\min_{(x,y)\in\mathcal{X}} \left\{ f(x,y) := x^{\mathsf{T}} Q x + a^{\mathsf{T}} x + b^{\mathsf{T}} y \right\} \tag{49}$$

where $\mathcal{X} \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is a convex set and $Q$ is an $n_1 \times n_1$ positive definite matrix. Let $(x_1^*, y_1^*)$ and $(x_2^*, y_2^*)$ be two optimal solutions of (49). Consider the function $\phi(t) := f(t x_1^* + (1-t) x_2^*, t y_1^* + (1-t) y_2^*)$. Note that $\phi$ is a quadratic function, $\phi(t) = \alpha t^2 + \beta t + \gamma$, where $\alpha = (x_1^* - x_2^*)^{\mathsf{T}} Q (x_1^* - x_2^*)$. Note that $\alpha \geq 0$ since $Q$ is positive definite, and thus $\phi$ is convex. Convexity of $\mathcal{X}$ and optimality of $(x_1^*, y_1^*)$ and $(x_2^*, y_2^*)$ implies that $\phi(t) \geq \phi(0) = \phi(1)$ for all $t \in [0,1]$. Moreover, convexity of $\phi$ implies that $\phi(t) \leq \phi(0) = \phi(1)$ for all $t \in [0,1]$. Hence $\phi(t) = \phi(0) = \phi(1)$ for all $t \in [0,1]$, and thus $\alpha = 0$. Since $Q$ is positive definite it follows that $x_1^* = x_2^*$. Finally, if the optimal objective value of problem (18), and hence of problem (17), is zero, then $(y^*_{-1}, y^*_1, \lambda^*_{-1}, \lambda^*_1)$ satisfies the necessary and sufficient optimality conditions (16), and thus $(y^*_{-1}, y^*_1)$ is the Nash equilibrium. ∎

## Appendix C: Details of Demand Transformation for No Alliance Model

The parameters $E, B, C$ in demand model (11) and the parameters $\tilde{E}, \tilde{B}, \tilde{C}$ in demand model (20) should be related in a particular way to facilitate a fair comparison of the prices, demands, total profit, and consumer surplus between the settings with and without an alliance. In this section we derive the relation.

The relation between the demand models with and without an alliance is based on the assumption that the overall demand level for each product is the same with and without an alliance. Recall that $L_i$ denotes the set of products which can be offered by seller $i$ with and without an alliance, for $i = \pm 1$, and $L_0$ denotes the set of products which could be offered only under an alliance. In addition, let $L_{0,i} \subset L_0$ denote the set of products in $L_0$ that can be offered by seller $i$ under an alliance, and let $L_{i,-i} \subset L_i$ denote the set of products in $L_i$ that can be offered by seller $-i$ under an alliance, but not without an alliance. Thus, for the setting with an alliance the number of demand equations (and prices) for each seller $i$ is $m_i = |L_i| + |L_{0,i}| + |L_{-i,i}|$, and for the setting without an alliance the number of demand equations (and prices) for each seller $i$ is only $|L_i|$.

The following example is used to explain the derivation of the relation between the demand models. Seller $-1$ produces resource $A$, and seller 1 produces resources $B$ and $C$. With an alliance, the following products are offered by each seller: Product $A$ using 1 unit of resource $A$ each, product $B$ using 1 unit of resource $B$ each, product $C$ using 1 unit of resource $C$ each, product $BC$ using 1 unit of resource $B$ and 1 unit of resource $C$ each, and product $A^2BC$ using 2 units of resource $A$, 1 unit of resource $B$, and 1 unit of resource $C$ each. Without an alliance, product $A$ is offered by seller $-1$ only and seller $-1$ captures all the demand for product $A$, and products $B$, $C$, and $BC$ are offered by seller 1 only and seller 1 captures all the demand for products $B$, $C$, and $BC$. Product $A^2BC$ is not offered by either seller, but there still is the same demand for product $A^2BC$; buyers buy each unit of product $A^2BC$ by buying 2 units of product $A$ from seller $-1$, and 1 unit of product $BC$ from seller 1. As shown later, the demands for products $A$ and $BC$ derived from the demand for product $A^2BC$ is added to the respective demands for products $A$ and $BC$ by themselves. Note that this derivation assumes that buyers buy each unit of product $A^2BC$ by buying 1 unit of product $BC$ from seller 1 instead of buying 1 unit of product $B$ and 1 unit of product $C$ separately from the same seller. This assumption may be questionable if the price of buying products $B$ and $C$ separately is less than the price of product $BC$. In the numerical work, we verified that the prices of multiple resource products offered by a seller were less than the sum of the prices of any products that could be bought separately to make up the multiple resource product. Thus, in this example, $L_{-1} = \{A\}$, $L_1 = \{B, C, BC\}$, $L_{0,-1} = \{A^2BC\}$, $L_{0,1} = \{A^2BC\}$, $L_{-1,1} = \{A\}$, and $L_{1,-1} = \{B, C, BC\}$. With an alliance, the demand for each product is given by (11):

$$
\begin{aligned}
d_{i,A} &= -E_{i,A,A}y_{i,A} - E_{i,A,B}y_{i,B} - E_{i,A,C}y_{i,C} - E_{i,A,BC}y_{i,BC} - E_{i,A,A^2BC}y_{i,A^2BC} \\
&\quad + B_{-i,A,A}y_{-i,A} + B_{-i,A,B}y_{-i,B} + B_{-i,A,C}y_{-i,C} + B_{-i,A,BC}y_{-i,BC} + B_{-i,A,A^2BC}y_{-i,A^2BC} + C_{i,A} \\
d_{i,B} &= -E_{i,B,A}y_{i,A} - E_{i,B,B}y_{i,B} - E_{i,B,C}y_{i,C} - E_{i,B,BC}y_{i,BC} - E_{i,B,A^2BC}y_{i,A^2BC} \\
&\quad + B_{-i,B,A}y_{-i,A} + B_{-i,B,B}y_{-i,B} + B_{-i,B,C}y_{-i,C} + B_{-i,B,BC}y_{-i,BC} + B_{-i,B,A^2BC}y_{-i,A^2BC} + C_{i,B} \\
d_{i,C} &= -E_{i,C,A}y_{i,A} - E_{i,C,B}y_{i,B} - E_{i,C,C}y_{i,C} - E_{i,C,BC}y_{i,BC} - E_{i,C,A^2BC}y_{i,A^2BC}
\end{aligned}
$$

$$
\begin{aligned}
&+B_{-i,C,A}y_{-i,A}+B_{-i,C,B}y_{-i,B}+B_{-i,C,C}y_{-i,C}+B_{-i,C,BC}y_{-i,BC}+B_{-i,C,A^2BC}y_{-i,A^2BC}+C_{i,C}\\
d_{i,BC} &= -E_{i,BC,A}y_{i,A}-E_{i,BC,B}y_{i,B}-E_{i,BC,C}y_{i,C}-E_{i,BC,BC}y_{i,BC}-E_{i,BC,A^2BC}y_{i,A^2BC}\\
&+B_{-i,BC,A}y_{-i,A}+B_{-i,BC,B}y_{-i,B}+B_{-i,BC,C}y_{-i,C}+B_{-i,BC,BC}y_{-i,BC}\\
&+B_{-i,BC,A^2BC}y_{-i,A^2BC}+C_{i,BC}\\
d_{i,A^2BC} &= -E_{i,A^2BC,A}y_{i,A}-E_{i,A^2BC,B}y_{i,B}-E_{i,A^2BC,C}y_{i,C}-E_{i,A^2BC,BC}y_{i,BC}-E_{i,A^2BC,A^2BC}y_{i,A^2BC}\\
&+B_{-i,A^2BC,A}y_{-i,A}+B_{-i,A^2BC,B}y_{-i,B}+B_{-i,A^2BC,C}y_{-i,C}+B_{-i,A^2BC,BC}y_{-i,BC}\\
&+B_{-i,A^2BC,A^2BC}y_{-i,A^2BC}+C_{i,A^2BC}
\end{aligned}
$$

To use these observations and the demand functions given by (11) for the alliance setting to derive the demand functions for the products with no alliance, first note that the demands in (11) depend on $|L_{0,-1}|+|L_{0,1}|+|L_{-1}|+|L_1|+|L_{-1,1}|+|L_{1,-1}|$ prices $y_{i,\ell}$, but the demands in (20) depend on only $|L_{-1}|+|L_1|$ prices. Thus, to derive the demands of the products with no alliance (as a function of the $|L_{-1}|+|L_1|$ prices $\tilde{y}$ with no alliance), it remains to determine appropriate values to substitute into (11) for the $|L_{0,-1}|+|L_{0,1}|+|L_{-1}|+|L_1|+|L_{-1,1}|+|L_{1,-1}|$ prices $y$ given the prices $\tilde{y}$. First, consider the easy case: if a product $\ell$ is offered by the same seller $i$ in both the setting with an alliance and the setting without an alliance, that is, $\ell \in L_i$, then simply substitute price $\tilde{y}_{i,\ell}$ for $y_{i,\ell}$ in the demand model (11). Thus, in the example above, $\tilde{y}_{-1,A}$, $\tilde{y}_{1,B}$, $\tilde{y}_{1,C}$, and $\tilde{y}_{1,BC}$ are substituted for $y_{-1,A}$, $y_{1,B}$, $y_{1,C}$, and $y_{1,BC}$ respectively. Next, if a product $\ell$ offered by a seller $i$ in the alliance setting is not offered by any seller in the no alliance setting, that is, $\ell \in L_{0,i}$, but it can be assembled in the no alliance setting by buying $a_{-1}$ units of product $\ell_{-1}$ from seller $-1$ and $a_1$ units of product $\ell_1$ from seller 1, then substitute price $a_{-1}\tilde{y}_{-1,\ell_{-1}}+a_1\tilde{y}_{1,\ell_1}$ for $y_{i,\ell}$ in the demand model (11). Thus, in the example above, $2\tilde{y}_{-1,A}+\tilde{y}_{1,BC}$ is substituted for $y_{-1,A^2BC}$ and $y_{1,A^2BC}$. Next, if a product $\ell$ offered by a seller $i$ in the alliance setting is not offered by seller $i$ in the no alliance setting, but it is offered by seller $-i$ in the no alliance setting, that is, $\ell \in L_{-i,i}$), then we choose the price $y_{i,\ell}$ in the demand model (11) so that together with the other prices $y_{i',\ell'}$, $i'=\pm 1$, $\ell' \in L_{i'} \cup L_{0,i'}$, already determined as described above, will equate $d_{i,\ell}$ to zero. Note that if there are $n$ such products, then $n$ linear equations are obtained by equating the $n$ linear expressions for $d_{i,\ell}$ to zero, and under reasonable conditions these equations can be solved for the $n$ desired values of $y_{i,\ell}$. Thus, for the example above, the system of equations

$$
\begin{aligned}
-E_{1,A,A}y_{1,A}-E_{1,A,B}\tilde{y}_{1,B}-E_{1,A,C}\tilde{y}_{1,C}-E_{1,A,BC}\tilde{y}_{1,BC}-E_{1,A,A^2BC}(2\tilde{y}_{-1,A}+\tilde{y}_{1,BC})&\\
+B_{-1,A,A}\tilde{y}_{-1,A}+B_{-1,A,B}y_{-1,B}+B_{-1,A,C}y_{-1,C}+B_{-1,A,BC}y_{-1,BC}+B_{-1,A,A^2BC}(2\tilde{y}_{-1,A}+\tilde{y}_{1,BC})+C_{1,A}&\\
&= 0\\
-E_{-1,B,A}\tilde{y}_{-1,A}-E_{-1,B,B}y_{-1,B}-E_{-1,B,C}y_{-1,C}-E_{-1,B,BC}y_{-1,BC}-E_{-1,B,A^2BC}(2\tilde{y}_{-1,A}+\tilde{y}_{1,BC})&\\
+B_{1,B,A}y_{1,A}+B_{1,B,B}\tilde{y}_{1,B}+B_{1,B,C}\tilde{y}_{1,C}+B_{1,B,BC}\tilde{y}_{1,BC}+B_{1,B,A^2BC}(2\tilde{y}_{-1,A}+\tilde{y}_{1,BC})+C_{-1,B}&\\
&= 0\\
-E_{-1,C,A}\tilde{y}_{-1,A}-E_{-1,C,B}y_{-1,B}-E_{-1,C,C}y_{-1,C}-E_{-1,C,BC}y_{-1,BC}-E_{-1,C,A^2BC}(2\tilde{y}_{-1,A}+\tilde{y}_{1,BC})&\\
+B_{1,C,A}y_{1,A}+B_{1,C,B}\tilde{y}_{1,B}+B_{1,C,C}\tilde{y}_{1,C}+B_{1,C,BC}\tilde{y}_{1,BC}+B_{1,C,A^2BC}(2\tilde{y}_{-1,A}+\tilde{y}_{1,BC})+C_{-1,C}&\\
&= 0
\end{aligned}
$$

$$\begin{aligned} -E_{-1,BC,A}\tilde{y}_{-1,A} - E_{-1,BC,B}y_{-1,B} - E_{-1,BC,C}y_{-1,C} - E_{-1,BC,BC}y_{-1,BC} - E_{-1,BC,A^2BC}(2\tilde{y}_{-1,A} + \tilde{y}_{1,BC}) \\ +B_{1,BC,A}y_{1,A} + B_{1,BC,B}\tilde{y}_{1,B} + B_{1,BC,C}\tilde{y}_{1,C} + B_{1,BC,BC}\tilde{y}_{1,BC} + B_{1,BC,A^2BC}(2\tilde{y}_{-1,A} + \tilde{y}_{1,BC}) + C_{-1,BC} \\ = \; 0 \end{aligned}$$

is solved for $y_{1,A}$, $y_{-1,B}$, $y_{-1,C}$, and $y_{-1,BC}$ as linear functions of $\tilde{y}_{-1,A}$, $\tilde{y}_{1,B}$, $\tilde{y}_{1,C}$, and $\tilde{y}_{1,BC}$.

To state the general relation between parameters $E, B, C$ in demand model (11) and the parameters $\tilde{E}, \tilde{B}, \tilde{C}$ in demand model (20) in general, we first develop the notation needed for a concise representation. Let the rows and columns of matrix $E_i$ be grouped so that the first group of rows and columns correspond to products in $L_i$, the second group of rows and columns correspond to products in $L_{0,i}$, and the third group of rows and columns correspond to products in $L_{-i,i}$. Hence $E_i$ can be partitioned into submatrices as follows:

$$E_i = \begin{array}{c} \begin{matrix} L_i & L_{0,i} & L_{-i,i} \end{matrix} \\ \begin{bmatrix} E_{i,i} & E_{i,0,i} & E_{i,-i,i} \\ E_{0,i,i} & E_{0,i,0,i} & E_{0,i,-i,i} \\ E_{-i,i,i} & E_{-i,i,0,i} & E_{-i,i,-i,i} \end{bmatrix} \end{array} \begin{matrix} L_i \\ L_{0,i} \\ L_{-i,i} \end{matrix}$$

This grouping of the rows and columns of $E_i$ implies that the rows and columns of $d_i$, $y_i$, $B_i$, and $C_i$ are similarly grouped:

$$B_{-i} = \begin{array}{c} \begin{matrix} L_{-i} & L_{0,-i} & L_{i,-i} \end{matrix} \\ \begin{bmatrix} B_{i,-i} & B_{i,0,-i} & B_{i,i,-i} \\ B_{0,i,-i} & B_{0,i,0,-i} & B_{0,i,i,-i} \\ B_{-i,i,-i} & B_{-i,i,0,-i} & B_{-i,i,i,-i} \end{bmatrix} \end{array} \begin{matrix} L_i \\ L_{0,i} \\ L_{-i,i} \end{matrix}, \quad y_i = \begin{bmatrix} y_{i,i} \\ y_{i,0,i} \\ y_{i,-i,i} \end{bmatrix}, \quad C_i = \begin{bmatrix} C_{i,i} \\ C_{i,0,i} \\ C_{i,-i,i} \end{bmatrix}, \quad d_i = \begin{bmatrix} d_{i,i} \\ d_{i,0,i} \\ d_{i,-i,i} \end{bmatrix}$$

Note that given the prices $\tilde{y}$ in the no alliance setting, the prices for the same products in the alliance setting are $y_{i,i} = \tilde{y}_i \in \mathbb{R}^{|L_i|}$. Let $R_{i,i',\ell,\ell'}$ denote the number of units of product $\ell' \in L_{i'}$ used to assemble one unit of product $\ell \in L_{0,i}$. Then, given the prices $\tilde{y}$ in the no alliance setting, the price paid to assemble one unit of product $\ell \in L_{0,i}$ in the no alliance setting is

$$\sum_{i'=\pm 1} \sum_{\ell' \in L_{i'}} R_{i,i',\ell,\ell'} \tilde{y}_{i',\ell'}$$

Let $R_{i,i'} \in \mathbb{R}^{|L_{0,i}| \times |L_{i'}|}$ denote the matrix with entry $R_{i,i',\ell,\ell'}$ in the row corresponding to $\ell \in L_{0,i}$ and the column corresponding to $\ell' \in L_{i'}$. Then, given the prices $\tilde{y}$ in the no alliance setting, the prices paid to assemble each unit of product in $L_{0,i}$ is given by

$$y_{i,0,i} \;=\; \sum_{i'=\pm 1} R_{i,i'} \tilde{y}_{i'}$$

Next, consider the demand for products in $L_{-i,i}$.

$$\begin{aligned} d_{i,-i,i} &= -E_{-i,i,i}y_{i,i} - E_{-i,i,0,i}y_{i,0,i} - E_{-i,i,-i,i}y_{i,-i,i} + B_{-i,i,-i}y_{-i,-i} + B_{-i,i,0,-i}y_{-i,0,-i} + B_{-i,i,i,-i}y_{-i,i,-i} + C_{i,-i,i} \\ &= -E_{-i,i,i}\tilde{y}_i - E_{-i,i,0,i} \sum_{i'=\pm 1} R_{i,i'}\tilde{y}_{i'} - E_{-i,i,-i,i}y_{i,-i,i} \\ &\quad + B_{-i,i,-i}\tilde{y}_{-i} + B_{-i,i,0,-i} \sum_{i'=\pm 1} R_{-i,i'}\tilde{y}_{i'} + B_{-i,i,i,-i}y_{-i,i,-i} + C_{i,-i,i} \end{aligned}$$

Then, given the prices $\tilde{y}$ in the no alliance setting, the value of $(y_{-1,1,-1}, y_{1,-1,1})$ is chosen to set $(d_{-1,1,-1}, d_{1,-1,1}) = 0$. The system of equations $(d_{-1,1,-1}, d_{1,-1,1}) = 0$ can be written as $-Dy_- + F\tilde{y} + C_- = 0$, where

$$y_- := \begin{bmatrix} y_{-1,1,-1} \\ y_{1,-1,1} \end{bmatrix}, \quad \tilde{y} := \begin{bmatrix} \tilde{y}_{-1} \\ \tilde{y}_1 \end{bmatrix}, \quad C_- := \begin{bmatrix} C_{-1,1,-1} \\ C_{1,-1,1} \end{bmatrix}, \quad D := \begin{bmatrix} E_{1,-1,1,-1} & -B_{1,-1,-1,1} \\ -B_{-1,1,1,-1} & E_{-1,1,-1,1} \end{bmatrix}$$
$$F := \begin{bmatrix} -E_{1,-1,-1} - E_{1,-1,0,-1}R_{-1,-1} + B_{1,-1,0,1}R_{1,-1} & -E_{1,-1,0,-1}R_{-1,1} + B_{1,-1,1} + B_{1,-1,0,1}R_{1,1} \\ -E_{-1,1,0,1}R_{1,-1} + B_{-1,1,-1} + B_{-1,1,0,-1}R_{-1,-1} & -E_{-1,1,1} - E_{-1,1,0,1}R_{1,1} + B_{-1,1,0,-1}R_{-1,1} \end{bmatrix}$$

Under reasonable conditions $D$ is nonsingular (more specifically, positive definite), and then the unique solution is $y_- = D^{-1}F\tilde{y} + D^{-1}C_-$. Let

$$D^{-1} = \begin{array}{c} \\ \\ \end{array}\overset{\quad L_{1,-1} \qquad L_{-1,1}}{\begin{bmatrix} D^{-1}_{-1,-1} & D^{-1}_{-1,1} \\ D^{-1}_{1,-1} & D^{-1}_{1,1} \end{bmatrix}} \begin{array}{l} L_{1,-1} \\ L_{-1,1} \end{array}, \qquad F = \overset{\quad L_{-1} \qquad L_{1}}{\begin{bmatrix} F_{-1,-1} & F_{-1,1} \\ F_{1,-1} & F_{1,1} \end{bmatrix}} \begin{array}{l} L_{1,-1} \\ L_{-1,1} \end{array}$$

Then

$$\begin{aligned} y_{i,-i,i} &= (D^{-1}_{i,-i}F_{-i,i} + D^{-1}_{i,i}F_{i,i})\tilde{y}_i + (D^{-1}_{i,-i}F_{-i,-i} + D^{-1}_{i,i}F_{i,-i})\tilde{y}_{-i} + (D^{-1}_{i,-i}C_{-i,i,-i} + D^{-1}_{i,i}C_{i,-i,i}) \\ &= \sum_{i'=\pm 1} \left( \sum_{i''=\pm 1} D^{-1}_{i,i''}F_{i'',i'}\tilde{y}_{i'} + D^{-1}_{i,i'}C_{i',-i',i'} \right) \end{aligned}$$

Next, the demand model (11) is used to derive the demand for each product $\ell \in L_i$ that is offered in the no alliance setting:

$$\begin{aligned} d_{i,\ell} = &\left[ -\sum_{\ell' \in L_i} E_{i,\ell,\ell'} y_{i,i,\ell'} - \sum_{\ell' \in L_{0,i}} E_{i,\ell,\ell'} y_{i,0,i,\ell'} - \sum_{\ell' \in L_{-i,i}} E_{i,\ell,\ell'} y_{i,-i,i,\ell'} \right. \\ &\left. + \sum_{\ell' \in L_{-i}} B_{-i,\ell,\ell'} y_{-i,-i,\ell'} + \sum_{\ell' \in L_{0,-i}} B_{-i,\ell,\ell'} y_{-i,0,-i,\ell'} + \sum_{\ell' \in L_{i,-i}} B_{-i,\ell,\ell'} y_{-i,i,-i,\ell'} + C_{i,\ell} \right] \\ &+ \sum_{i'=\pm 1} \left[ \sum_{\ell' \in L_{0,i'}} R_{i',i,\ell',\ell} \left( -\sum_{\ell'' \in L_{i'}} E_{i',\ell',\ell''} y_{i',i',\ell''} - \sum_{\ell'' \in L_{0,i'}} E_{i',\ell',\ell''} y_{i',0,i',\ell''} - \sum_{\ell'' \in L_{-i',i'}} E_{i',\ell',\ell''} y_{i',-i',i',\ell''} \right. \right. \\ &\left. \left. + \sum_{\ell'' \in L_{-i'}} B_{-i',\ell',\ell''} y_{-i',-i',\ell''} + \sum_{\ell'' \in L_{0,-i'}} B_{-i',\ell',\ell''} y_{-i',0,-i',\ell''} + \sum_{\ell'' \in L_{i',-i'}} B_{-i',\ell',\ell''} y_{-i',i',-i',\ell''} + C_{i',\ell'} \right) \right] \end{aligned}$$

The first term in brackets above corresponds to the demand for product $\ell \in L_i$ by itself, and the second term in brackets corresponds to the demand for product $\ell$ to assemble products $\ell' \in L_{0,i'}$, $i' = \pm 1$. In terms of matrix notation, the demands for the products in $L_i$ that are offered in the no alliance setting is given by

$$\begin{aligned} d_{i,i} = &[-E_{i,i}y_{i,i} - E_{i,0,i}y_{i,0,i} - E_{i,-i,i}y_{i,-i,i} + B_{i,-i}y_{-i,-i} + B_{i,0,-i}y_{-i,0,-i} + B_{i,i,-i}y_{-i,i,-i} + C_{i,i}] \\ &+ \sum_{i'=\pm 1} [R^{\mathsf{T}}_{i',i} (-E_{0,i',i'}y_{i',i'} - E_{0,i',0,i'}y_{i',0,i'} - E_{0,i',-i',i'}y_{i',-i',i'} \\ &\qquad + B_{0,i',-i'}y_{-i',-i'} + B_{0,i',0,-i'}y_{-i',0,-i'} + B_{0,i',i',-i'}y_{-i',i',-i'} + C_{i',0,i'})] \end{aligned}$$

Next, replace $y_{i,i}$, $y_{i,0,i}$, and $y_{i,-i,i}$ with the expressions in terms of $\tilde{y}$ derived above. Then the demands $\tilde{d}_i$ for the products in $L_i$ in the no alliance setting as a function of the prices $\tilde{y}$ in the no alliance setting are obtained, as follows:

$$\tilde{d}_i = \left[ -E_{i,i}\tilde{y}_i - E_{i,0,i} \sum_{i'=\pm 1} R_{i,i'}\tilde{y}_{i'} - E_{i,-i,i} \sum_{i'=\pm 1} \left( \sum_{i''=\pm 1} D^{-1}_{i,i''}F_{i'',i'}\tilde{y}_{i'} + D^{-1}_{i,i'}C_{i',-i',i'} \right) \right.$$

$$
\begin{aligned}
&+B_{i,-i}\tilde{y}_{-i}+B_{i,0,-i}\sum_{i'=\pm1}R_{-i,i'}\tilde{y}_{i'}+B_{i,i,-i}\sum_{i'=\pm1}\left(\sum_{i''=\pm1}D^{-1}_{-i,i''}F_{i'',i'}\tilde{y}_{i'}+D^{-1}_{-i,i'}C_{i',-i',i'}\right)+C_{i,i}\Bigg]\\
&+\sum_{i'=\pm1}\Bigg[R^{\mathsf{T}}_{i',i}\Bigg(-E_{0,i',i'}\tilde{y}_{i'}-E_{0,i',0,i'}\sum_{i''=\pm1}R_{i',i''}\tilde{y}_{i''}-E_{0,i',-i',i'}\sum_{i''=\pm1}\left(\sum_{i'''=\pm1}D^{-1}_{i',i'''}F_{i''',i''}\tilde{y}_{i''}+D^{-1}_{i',i''}C_{i'',-i'',i''}\right)\\
&\qquad+B_{0,i',-i'}\tilde{y}_{-i'}+B_{0,i',0,-i'}\sum_{i''=\pm1}R_{-i',i''}\tilde{y}_{i''}\\
&\qquad+B_{0,i',i',-i'}\sum_{i''=\pm1}\left(\sum_{i'''=\pm1}D^{-1}_{-i',i'''}F_{i''',i''}\tilde{y}_{i''}+D^{-1}_{-i',i''}C_{i'',-i'',i''}\right)+C_{i',0,i'}\Bigg)\Bigg]
\end{aligned}
$$

Note that the demands $\tilde{d}_i$ above are consistent with the demand model (20), for the following parameter values:

$$
\begin{aligned}
\tilde{E}_i &= E_{i,i}+E_{i,0,i}R_{i,i}+E_{i,-i,i}\sum_{i'=\pm1}D^{-1}_{i,i'}F_{i',i}-B_{i,0,-i}R_{-i,i}-B_{i,i,-i}\sum_{i'=\pm1}D^{-1}_{-i,i'}F_{i',i}\\
&\quad+R^{\mathsf{T}}_{i,i}E_{0,i,i}-R^{\mathsf{T}}_{-i,i}B_{0,-i,i}\\
&\quad+\sum_{i'=\pm1}R^{\mathsf{T}}_{i',i}\left(E_{0,i',0,i'}R_{i',i}+E_{0,i',-i',i'}\sum_{i''=\pm1}D^{-1}_{i',i''}F_{i'',i}-B_{0,i',0,-i'}R_{-i',i}-B_{0,i',i',-i'}\sum_{i''=\pm1}D^{-1}_{-i',i''}F_{i'',i}\right)\\
\tilde{B}_{-i} &= -E_{i,0,i}R_{i,-i}-E_{i,-i,i}\sum_{i'=\pm1}D^{-1}_{i,i'}F_{i',-i}+B_{i,-i}+B_{i,0,-i}R_{-i,-i}+B_{i,i,-i}\sum_{i'=\pm1}D^{-1}_{-i,i'}F_{i',-i}\\
&\quad-R^{\mathsf{T}}_{-i,i}E_{0,-i,-i}+R^{\mathsf{T}}_{i,i}B_{0,i,-i}\\
&\quad+\sum_{i'=\pm1}R^{\mathsf{T}}_{i',i}\left(-E_{0,i',0,i'}R_{i',-i}-E_{0,i',-i',i'}\sum_{i''=\pm1}D^{-1}_{i',i''}F_{i'',-i}+B_{0,i',0,-i'}R_{-i',-i}+B_{0,i',i',-i'}\sum_{i''=\pm1}D^{-1}_{-i',i''}F_{i'',-i}\right)\\
\tilde{C}_i &= -E_{i,-i,i}\sum_{i'=\pm1}D^{-1}_{i,i'}C_{i',-i',i'}+B_{i,i,-i}\sum_{i'=\pm1}D^{-1}_{-i,i'}C_{i',-i',i'}+C_{i,i}\\
&\quad+\sum_{i'=\pm1}R^{\mathsf{T}}_{i',i}\left(-E_{0,i',-i',i'}\sum_{i''=\pm1}D^{-1}_{i',i''}C_{i'',-i'',i''}+B_{0,i',i',-i'}\sum_{i''=\pm1}D^{-1}_{-i',i''}C_{i'',-i'',i''}+C_{i',0,i'}\right)
\end{aligned}
$$

## Appendix D: Proof of Proposition 1

Consider any $(a,A)\in\mathcal{A}$. Let $c_i=e_i>0$, $d_i=-e_ia_i$.

Consider the allocation problem $(a',A')\in\mathcal{A}$ given by $a'_i:=c_ia_i+d_i=0$ and $A':=\{(c_{-1}b_{-1}+d_{-1},c_1b_1+d_1)\in\mathbb{R}^2:(b_{-1},b_1)\in A\}=\{(b'_{-1},b'_1):0=a'_i\le b'_i,i=\pm1,b'_{-1}+b'_1\le g^*-e_{-1}a_{-1}-e_1a_1\}$. Note that $(a',A')$ is symmetric. It follows from the axioms of Pareto optimality and symmetry that $f_{-1}(a',A')=f_1(a',A')=(g^*-e_{-1}a_{-1}-e_1a_1)/2$. Next it follows from the axiom of invariance under positively homogeneous affine transformations that $c_if_i(a,A)+d_i=f_i(a',A')$, that is, $e_i[f_i(a,A)-a_i]=(g^*-e_{-1}a_{-1}-e_1a_1)/2$. ∎