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Networks of Collaboration in Multi-market Oligopolies*

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Abstract

The result that firms competing in a Cournot oligopoly with pairwise collaboration form a complete network under zero or negligible link formation costs provided by Goyal and Joshi (2003) no longer hold in multi-market oligopolies. Link formation in one market affects a firm's profitability in another market in a possibly negative way resulting in the fact that it is no longer always profitable in an unambiguous manner. With non-negative link formation costs, the stable networks have a dominant group architecture and efficient networks are charecterized by at most one non-singleton component with a geodesic distance between players that is less than three.

Key words: networks, collaboration, R & D, Nested Split Graphs

JEL code: C70, L13, L20, D85

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1 Introduction

In two landmark papers, Goyal and Joshi (2003, 2006) set forth the issue of using the emerging network formation literature to discuss collaboration in R&D among a set of oligopolistic firms. The first paper characterizes stable and efficient networks. The second paper show that the model is an example of a more general category of models called *playing the field* games. In this paper, we extend the analysis to oligopolistic firms competing in more than one market.

Goyal and Joshi (2003) put forward the proposition that firms competing in a homogeneous Cournot oligopoly with constant returns to scale cost functions and forming collaborative links among themselves will form a complete network under negligible link formation costs. The rationale is straightforward. Links lower marginal costs of both players involved in forming a link. The firm gains in terms of gross profits (or profits not including link formation costs) by the lowering of its marginal cost. It loses by the lowering of its partner's marginal costs. The gain outweighs the loss and hence link formation is profitable. Similar results will follow if link formation increases demand (for instance, by increasing the demand intercept) of both firms forming a collaborative link. Such increases may be the outcome of quality enhancing collaborations.

Now, consider the case where firms compete in more than one market. Then, the mechanics of the effects associated with link formation are much more complicated. Bulow et al. (1985) investigate some of these effects in a general strategic setting. Suppose there are joint diseconomies across markets in the sense that higher quantity produced in one market reduces marginal profitability associated with an unit of production in the other. Furthermore, the market structure is such that goods produced by competing firms are strategic substitutes. Then any strategic action (such as collaborative link formation) designed to increase demand and reduce costs inevitably increases the quantity produced in one market. This will (because of joint diseconomies) reduce the marginal profitability and quantity produced in the other market. Because of strategic substitutability, rival firms increase quantities produced and this induces certainly a loss in the second market and possibly an overall loss for the firm.

We investigate the stable and efficient networks that may form in the setting of a multi-market oligopoly with non-negative link formation costs. We assume a heterogeneous product market, linear demand curves and quadratic cost functions. In a departure from Goyal and Joshi (2003), we look at quality-enchancing collaborations rather than cost-reducing collaborations. Since multiple quality levels are incompatible with the notion of a homogeneous product, we look at a market with differentiated products. The quadratic cost functions are introduced in order to make sure that the assumption of joint diseconomies (defined below) is valid. This play a key role in the inter-market effects. If we use linear cost functions, the inter-market effects disappear and we expect too see results that are similar to Goyal and Joshi (2003). A link between two firms with shift the demand curves of both firms to right as a result of quality improvements. Firms compete in two separate markets but for purposes of simplicity, quality enhancing collaborations are restricted only to one market. The cost function is a quadratic function of quantities produced in both markets.

It turns out that stable networks have, what Goyal and Joshi (2003) refer to as the dominant group architecture. Namely, the firms can be partitioned into two groups. In the first group, all firms are linked to each other. In the second group, the firms have no links whatsoever. This is a consequence of increasing returns to link formation. Namely, the more links a firm has, the greater the benefit of forming an additional link. With regard to efficient networks, we cannot arrive at a precise characterization of the networks that will result though we can derive some interesting properties of such networks and restrict the set of networks that are efficient into a small class. For four firms or more, efficient networks have only one component and the geodesic distance between two connected players cannot exceed two. In other words, dominant group architectures are possible candidates for efficient networks but we show using examples, that stable and efficient networks need not coincide.

The rest of the paper proceeds as follows. Section 2 introduces the model and discusses the notation and terminology. Section 3 discusses the inter-market effects a la´ Bulow et. al. (1985). Section 4 discusses stable networks. Section 5 discusses efficient networks. Section 7 concludes. The paper has a lot of tedious algebra most of which has been relegated to the Appendix.

2 Preliminaries

2.1 The Multi-market Cournot Model

Suppose there are n firms indexed $i = 1, 2, \ldots, n$ (where $n \geqslant 2$) that compete a la´ Cournot in two inter-related markets A and B. Demand in A for firm i is given by

$$p_i = \alpha_i - q_i - \sum_{j \neq i} q_j. \tag{1}$$

Demand in B for firm i is given by

$$P_i = \beta_i - Q_i - \sum_{j \neq i} Q_j. \tag{2}$$

The cost function of the firm i is given by

$$C_i(q_i, Q_i) = \frac{1}{2}(q_i + Q_i)^2 \tag{3}$$

and profit of firm i is given by

$$\pi_i = p_i \cdot q_i + P_i \cdot Q_i - C_i(q_i, Q_i). \tag{4}$$

We begin by giving a rationale of the demand function employed here. Products here are near substitutes but vertically differentiated. Differential quality levels allow firms to charge different prices creating a sub-market within the larger market. This demand function was introduced by Bowley (1924)[1] and used by Spence (1976) and Dixit (1979). More recently, such demand functions have been employed for instance by Chakrabarti and Haller (2007) in the context of targeted advertising.

The assumption of *joint diseconomies* is equivalent to $\dfrac{\partial^2 \pi_i}{\partial q_i \partial Q_i} < 0$. In this model, it holds because $\dfrac{\partial^2 \pi_i}{\partial q_i \partial Q_i} = -1$. The assumption of *strategic substitutes* is equivalent to $\dfrac{\partial^2 \pi_i}{\partial Q_i \partial Q_j} < 0$. In this model, it holds because $\dfrac{\partial^2 \pi_i}{\partial Q_i \partial Q_j} = -1$.

First consider market 1 in isolation by assuming a priori that $Q_i = 0$ for all i. Let q_i^*, Q_i^* and π_i^* denote equilibrium quantities and profits in the second stage. Then, it

[1] Usually, a more general formulation, $p_i = \alpha_i - q_i - \theta \sum_{j \neq i} q_j$ where $0 \leqslant \theta \leqslant 1$ is employed. The current formulation simplifies the exposition without changing the results.

is straight forward to show that

$$\pi_i^* = \frac{3}{8}\left(\frac{(n+1)\alpha_i - \sum\limits_{j\neq i}\alpha_j}{n+2}\right)^2 .$$

Let us consider now the full fledged model. Upon solving the second stage of the model, we get

$$\begin{aligned} q_i^* &= \frac{1}{3(3+4n+n^2)}\left[(6+8n+2n^2)\alpha_i - (3+4n+n^2)\beta_i\right] \\ &+\frac{1}{3(3+4n+n^2)}\left[-(5+2n)\overline{\alpha} + (4+n)\overline{\beta}\right]; \\ Q_i^* &= \frac{1}{3(3+4n+n^2)}\left[(6+8n+2n^2)\beta_i - (3+4n+n^2)\alpha_i\right] \\ &+\frac{1}{3(3+4n+n^2)}\left[-(5+2n)\overline{\beta} + (4+n)\overline{\alpha}\right]. \end{aligned}$$

where $\overline{\alpha} = \sum\limits_{i=1}^{n}\alpha_i$ and $\overline{\beta} = \sum\limits_{i=1}^{n}\beta_i$. The calculations are in the appendix. The expression for profit is complicated and given in the appendix.

2.2 Networks

Let the set of players be denoted by $N = \{1, 2, \ldots, n\}$. A network g is a list of pairs of players who are *linked* to each other. For simplicity, we denote the link between i and j (where $i \neq j$) by ij, so $ij \in g$ indicates i and j are linked in the network g. The links are undirected in the sense that we do not distinguish between ij and ji.

Let g^N be the set of all subsets of N of size 2. The network g^N is referred to as the complete network. The set $G = \{g \subset g^N\}$ denotes the set of all possible networks on N. A network in which there are no links is called an empty network and is denoted by g^0.

We let $g + ij$ denote the network formed by adding the link ij to the network g. $g - ij$ denotes the network formed by deleting the link ij from the network g. A network payoff function $u_i : G \to \mathbb{R}_+$ assigns an utility to player i by virtue of being part of a network. Let $u = (u_1, u_2, \ldots, u_n)$ denote the vector of utility functions. Then u combined with N defines a *network game*.[2]

[2] Originally, the term network game was used to denote a transferable utility version of the game by Jackson (2005).

A firm i's neighborhood $N_i(g)$ is given by $\{j \in N\backslash\{i\}|ij \in g\}$ and its cardinality is given by $\eta_i(g) = |N_i(g)|$. $\eta_i(g)$ is called the degree of player i in network g. We also define $N(g) = \cup_{i\in N} N_i(g)$. $N(g)$ refers to the set of players that have at least one link. Let $\eta(g) = \#N(g)$ with the convention that if $N(g) = \varnothing$, we let $\eta(g) = 1$.[3]

Player i therefore is participating in the links in her *link set* $L_i(g) = \{ij \in g \mid j \in N_i(g)\} \subset g$. Let $L_i = L_i(g^N)$ denote the set of all possible links involving player i. Let $\lambda(g) = \frac{1}{2}\sum_{i\in N} \eta_i(g)$ be the total number of links in a network g.

For any $h \subset g$, let $g - h$ denotes the network formed by deleting the link set h from the network g. Similarly, for $h \subset g^N \backslash g$, $g + h$ denotes the network formed by adding the link set h from the network g.

A network g is *regular* if each player has the same number of neighbors. Namely, for all $i \neq j$, $\eta_i(g) = \eta_j(g)$.

A *path* in g connecting i and j is a set of distinct players $\{i_1, i_2, \ldots, i_p\} \subset N(g)$ with $p \geqslant 2$ such that $i_1 = i$, $i_p = j$, and $\{i_1i_2, i_2i_3, \ldots, i_{p-1}i_p\} \subset g$. We refer to the number of links on this path, here $p - 1$, as the length of the path.

We say i and j are *connected* to each other if a path exists between them and they are *disconnected* otherwise. The number of links on the shortest path between two distinct players i and j is called the geodesic distance between i and j.

The network $g' \subset g$ is a *component* of g if $\eta(g') \geqslant 2$ and for all $i \in N(g')$ and $j \in N(g')$, $i \neq j$, there exists a path in g' connecting i and j and for any $i \in N(g')$ and $j \in N(g)$, $ij \in g$ implies $ij \in g'$. In other words, a component is simply a maximally connected subnetwork of g. We denote the set of network components of the network g by $C(g)$. The set of players that are not connected in the network g are collected in the set of (fully) disconnected players in g denoted by

$$N_0(g) = N \setminus N(g) = \{i \in N \mid N_i(g) = \varnothing\}.$$

Such players are known as singletons. A component $g' \subset g$ is complete if for all distinct $i, j \in N(g')$, $ij \in g$. A component $g' \subset g$ is regular if for all distinct $i, j \in N(g')$, $\eta_i(g) = \eta_j(g)$. The *dominant group architecture* g^k is characterized by one complete non-singleton component with $k \geqslant 2$ players and $n - k$ singletons.

[3] We emphasize here that if $N(g) \neq \varnothing$, we have that $\eta(g) \geqslant 2$. Namely, in those cases the network has to consist of at least one link.

A network is a *pairwise equilibrium network* with regard to a profile of utility functions u if

(a) for all i and $h \subset L_i(g)$, $u_i(g) \geqslant u_i(g-h)$, and
(b) for all i and $ij \notin g$, if $u_i(g+ij) > u_i(g)$ then $u_j(g+ij) < u_j(g)$.

An equivalent definition can be given as follows. Consider a non-cooperative game where each player i has a strategy set $s_i = \left\{\{s_{ij}\}_{j\neq i}\right\}$ with $s_{ij} \in \{0,1\}$. $s_{ij} = 1$ means i intends to form a link with j, while $s_{ij} = 0$ means i does not intend to form such a link. A link between two players is formed if and only if $s_{ij} = s_{ji} = 1$. A strategy profile $s = \{s_1, s_2, \cdots, s_n\}$ induces a network $g(s) = \left\{\bigcup_{i\neq j, i\in N, j\in N} ij | s_{ij} = s_{ji} = 1\right\}$. We say that the network $g(s)$ is induced by the strategy profile s. A network g is a *pairwise equilibrium network* (or simply and equilibrium network) if

(a) There is a Nash equilibrium strategy profile that induces g;
(b) for all i and $ij \notin g$, if $u_i(g+ij) > u_i(g)$ then $u_j(g+ij) < u_j(g)$.

For any network g, and $h \subset g^N \backslash g$, we denote the marginal benefit of link formation by

$$\Delta u_i(g,h) = u_i(g+h) - u_i(g).$$

Obviously, for a pairwise equilibrium network, $\Delta u_i(g-h,h) \geqslant 0$ for all $h \subset L_i(g)$ and if $\Delta u_i(g,ij) > 0$, then $\Delta u_j(g,ij) < 0$.

Next, we define efficient networks. Consider a social welfare function W given by sums of payoffs of all the players. Therefore,

$$W(g) = \sum_{i=1}^{n} u_i(g).$$

A network is *efficient* it is maximizes the social welfare function. More specifically, g' is efficient if

$$W(g') \geqslant W(g)$$

for all $g \neq g'$. For any network g, and $h \subset g^N \backslash g$, we denote the marginal change in social welfare as a result of link formation by

$$\Delta W(g,h) = W(g+h) - W(g).$$

Obviously, for an efficient network, $\Delta W(g-h,h) \geqslant 0$ and $\Delta W(g,h) \leqslant 0$.

3 Inter-market Effects

We assume that firms can improve quality via collaborative links in market 1. This enhances demand. Hence, if a firm has formed k links, then

$$\alpha_i = \gamma_0 + \gamma \cdot k \tag{5}$$

It is reasonable to assume that $\gamma_0 > \gamma$. For reasons that will be clear later, we assume γ_0 is sufficiently large compared to γ, namely,

$$\gamma_0 > \left[\frac{(n-1)^2}{2}\right]\gamma. \tag{6}$$

Link formation costs are given by a real number c where $c \geqslant 0$. To keep the model tractable, assume that no link formation is possible in the second market. Hence, assume a two stage game where first stage consists of a link formation game and in the second stage, Cournot competition ensues. Define

$$\alpha_i(g) = \gamma_0 + \gamma \cdot \eta_i(g).$$

Then, the relevant network network payoff function is given by

$$u_i(g) = \pi_i^*(\alpha_i(g)) - c \cdot \eta_i(g)$$

where $\pi_i^*(\alpha_i(g))$ is derived by expressing optimal profits π_i^* in (36) as a function of $\alpha_i = \alpha_i(g)$. Let us assume for time being that $c = 0$.

First consider market 1 in isolation by assuming a priori that $Q_i = 0$ for all i. For an incomplete network g, if i forms a link with $k \neq i$ (where $ik \notin g$), its net profits increase by

$$\Delta u_i(g, ik) = \frac{3}{4}\left(\frac{n\gamma}{(n+2)^2}\right)\left[(n+1)\alpha_i(g) - \sum_{j\neq i}\alpha_j(g) + \frac{n\gamma}{2}\right].$$

Now, $(n+1)\alpha_i(g) - \sum_{j\neq i}\alpha_j(g) \geqslant (n+1)\gamma_0 - (n-1)(\gamma_0 + (n-1)\gamma) = 2\cdot\gamma_0 - (n-1)^2\cdot\gamma > 0$ from (6). Hence, each link unambiguously increases profitability and a complete network is the unique pairwise stable network. This leads to the following lemma.

Lemma 1 *Let us exogenously impose the condition that $Q_i = 0$ for all i. Then, with $c = 0$, the unique pairwise equilibrium network is given by the complete network.*

Let us consider now the full fledged model. If i forms a link with $k \neq i$, its net profits increase by

$$\begin{aligned}\Delta u_i(g, ik) &= \left[\frac{\alpha_i(g) \cdot \gamma}{18(1+n)^2(3+n)^2}\right]\left(22n^4 + 110n^3 + 90n^2 - 134n + 8\right) \\ &- \left[\frac{\left(\sum_{j\neq i}\alpha_j(g)\right) \cdot \gamma}{18(1+n)^2(3+n)^2}\right]\left(22n^3 + 102n^2 + 66n - 158\right) \\ &- \left[\frac{\beta_i \cdot \gamma}{18(1+n)^2(3+n)^2}\right]\left(14n^4 + 70n^3 + 18n^2 - 190n - 8\right) \\ &+ \left[\frac{\left(\sum_{j\neq i}\beta_j\right) \cdot \gamma}{18(1+n)^2(3+n)^2}\right]\left(14n^3 + 78n^2 + 42n - 166\right) \\ &+ \left[\frac{\gamma^2}{18(1+n)^2(3+n)^2}\right]\left(11n^4 + 44n^3 - 6n^2 - 100n + 83\right). \qquad (7)\end{aligned}$$

Clearly, $\Delta u_i(g, ik)$ is not necessary positive. For instance, consider for all i, $n = 6, \beta_i = 400, \gamma_0 = 100, \gamma = 0.05$. Consider an empty network for which $\alpha_i = \gamma_0$. In such a network, $\Delta u_i(g^0, ik) = -0.0894772$ making the empty network an equilibrium network. In a complete network, deleting a link yields a positive payoff of 0.0890931 and hence the complete network is not an equilibrium network. In other words, the mechanics driving the results of Goyal and Joshi (2003) fail to hold. In the Appendix, we derive precise conditions for the empty network to be an equilibrium network when the costs of link formation are zero.

We give some intuition behind these results. It follows from strategic complementarity and joint economies analyzed by Bulow et al. (1985). Note that

$$\begin{aligned}\frac{d\pi_i^*}{d\alpha_i} &= \left(\frac{\partial \pi_i^*}{\partial q_i^*}\right)\left(\frac{dq_i^*}{d\alpha_i}\right) + \sum_{j\neq i}\left(\frac{\partial \pi_i^*}{\partial q_j^*}\right)\left(\frac{dq_j^*}{d\alpha_i}\right) + \left(\frac{\partial \pi_i^*}{\partial Q_i^*}\right)\left(\frac{dQ_i^*}{d\alpha_i}\right) \\ &+ \sum_{j\neq i}\left(\frac{\partial \pi_i^*}{\partial Q_j^*}\right)\left(\frac{dQ_j^*}{d\alpha_i}\right) + \left(\frac{\partial \pi_i^*}{\partial \alpha_i}\right). \qquad (8)\end{aligned}$$

Also,

$$\pi_i^* = \left(\alpha_i - \sum_{k=1}^{n} q_k^*\right) q_i^* + \left(\beta_i - \sum_{k=1}^{n} Q_k^*\right) Q_i^* - \frac{1}{2}(q_i^* + Q_i^*)^2.$$

Hence, for $j \neq i$,

$$\frac{\partial \pi_i^*}{\partial q_j^*} = -q_i^* < 0; \tag{9}$$

$$\frac{\partial \pi_i^*}{\partial \alpha_i} = q_i^* > 0; \tag{10}$$

$$\frac{\partial \pi_i^*}{\partial Q_j^*} = -Q_i^* < 0. \tag{11}$$

Also, from first order conditions of profit maximization,

$$\frac{\partial \pi_i^*}{\partial q_i^*} = 0; \tag{12}$$

$$\frac{\partial \pi_i^*}{\partial Q_i^*} = 0. \tag{13}$$

resulting in two terms of (8) dropping out. In Cournot competition, quantity produced by a firm in equilibrium is decreasing in the demand intercepts of it's rivals. Hence,

$$\left(\frac{dq_j^*}{d\alpha_i}\right) < 0. \tag{14}$$

Further, Bulow et al. (1985) show that

$$sign\left(\frac{dQ_j^*}{d\alpha_i}\right) = sign\left[\left(\frac{\partial^2 \pi_i}{\partial q_i \partial Q_i}\right) \cdot \left(\frac{\partial^2 \pi_j}{\partial Q_i \partial Q_j}\right)\right].$$

Given $\left(\frac{\partial^2 \pi_i}{\partial q_i \partial Q_i}\right) = -1$ and $\left(\frac{\partial^2 \pi_j}{\partial Q_i \partial Q_j}\right) = -1$ in this multi-market model,[4]

$$\left(\frac{dQ_j^*}{d\alpha_i}\right) > 0. \tag{15}$$

Using inequalities (15), (14), (9), (10) and (11), we can sign each term to get the following:

$$\frac{d\pi_i^*}{d\alpha_i} = \sum_{j \neq i} \underbrace{\left(\frac{\partial \pi_i^*}{\partial q_j^*}\right)}_{<0} \underbrace{\left(\frac{dq_j^*}{d\alpha_i}\right)}_{<0} + \sum_{j \neq i} \underbrace{\left(\frac{\partial \pi_i^*}{\partial Q_j^*}\right)}_{<0} \underbrace{\left(\frac{dQ_j^*}{d\alpha_i}\right)}_{>0} + \underbrace{\left(\frac{\partial \pi_i^*}{\partial \alpha_i}\right)}_{>0}. \tag{16}$$

[4]Note that inequalities (15) and (14) can be verified from (34) and (35).

So, the *aberrant sign* is introduced by the presence of $\left(\frac{\partial Q_j^*}{\partial \alpha_i}\right)$ being positive. If it were negative, then any increase in α_i would only boost profits and a complete network would result in equilibrium. In fact, the precise expression for $\frac{d\pi_i^*}{d\alpha_i}$ is given by

$$\frac{d\pi_i^*}{d\alpha_i} = q_i^* \left(1 + \frac{(5+2n)(n-1)}{3(3+4n+n^2)}\right) - Q_i^* \left(\frac{(4+n)(n-1)}{3(3+4n+n^2)}\right). \tag{17}$$

Given that an increase in demand in one market no longer unambiguously increases profit, the result follows.

4 Configuration of Equilibrium Networks

While the payoff functions are quite complicated, this game has features that were analyzed by Goyal and Joshi (2006). We will devote some space to reproducing their definitions and terminology. Let us assume $c \geqslant 0$.

Suppose from the network g, we remove player i and all his links, and call the resulting network g_{-i}. Namely, $g_{-i} = g - L_i(g)$. Now, the total number of links in this network g_{-i} is given by $\frac{1}{2}\sum_{j\neq i}\eta_j\left(g_{-i}\right) = \lambda\left(g_{-i}\right)$.

Definition 1 *A network game is called playing the field game if the payoff function of player i is a function of her degree $\eta_i(g)$ and $\lambda(g_{-i})$, namely,*

$$u_i(g) = \Phi\left(\eta_i(g), \lambda(g_{-i})\right) - c \cdot \eta_i(g).$$

Definition 2 *The payoff function Φ is convex in its own links if the marginal returns $\Phi(k+1,l) - \Phi(k,l)$ is strictly increasing in k.*

Definition 3 *Suppose $l' > l$. The payoff function Φ satisfies the strategic substitutes property if $\Phi(k+1,l') - \Phi(k,l') < \Phi(k+1,l) - \Phi(k,l)$.*

The next lemma is a reproduction of Proposition 3.1 of Goyal and Joshi (2006).

Lemma 2 *For a playing the field game, if the payoff function satisfies convexity in own links and the strategic substitutes property, then a pairwise equilibrium network always exists. Furthermore, if the payoff function satisfies convexity in own links, the pairwise equilibrium network is either complete or empty or has the dominant group architecture.*

In the appendix we show that the network games qualifies as playing the field game. Furthermore, the payoff function satisfies convexity in own links as well as the strategic substitutes property. In fact, if we define

$$\Delta(k,l) = \Phi\left(k+1,l\right) - \Phi\left(k,l\right),$$

then we show in the appendix that

$$\frac{\partial\Delta}{\partial k} = \frac{2\gamma^2\left(11n^4 + 44n^3 - 6n^2 - 100n + 83\right)}{18\left(1+n\right)^2\left(3+n\right)^2} > 0$$

and

$$\frac{\partial\Delta}{\partial l} = -\frac{4\gamma^2\left(22n^3 + 102n^2 + 66n - 158\right)}{18\left(1+n\right)^2\left(3+n\right)^2} < 0.$$

Therefore, applying Lemma 2, we get the following corollary.

Corollary 1 *The pairwise equilibrium network exists and is either complete or empty or has a dominant group architecture.*

We note that in the one-market Cournot model, the dominant group architecture emerges. To see this, one can verify that the one-market Cournot game is also an example of *playing the field* game and then apply Lemma 2.

5 Configuration of Efficient Networks

We shall distinguish between three kinds of efficiency. First the efficient networks for firms is one that maximizes the joint profits of firms. This corresponds to the usual notion of efficiency as defined by Jackson and Wolinsky (1996) because firms are involved in the link formation process. However, one can define two other kinds of efficiency. The efficient networks for consumers are ones that maximize the overall consumer surplus. Overall efficiency refers in our case to networks maximizing the sum of joint profits and overall consumer surplus. If we just use the words, efficient networks, we are referring to efficient networks for firms.

5.1 Efficient Networks for Firms

In this section, we shall discuss efficient networks where the social welfare function is defined by the sum of profits of all the firms. While we do not obtain an exact characterization of efficient networks, we can identify certain properties of such networks. Let $\Pi : G \to \mathbb{R}_+$ denote the joint profit of firms as a function of the network. In other words,

$$\Pi(g) = \sum_{i=1}^{n} u_i(g).$$

Consider the effect of link formation between two arbitrary firms i and k in a network g. In the Appendix we show that such link formation alters the joint profit of all firms by

$$\Delta\Pi(g, ik) = \kappa' \left[(\alpha_i + \alpha_k) - \frac{\tau'}{n} \left(\sum_{l \neq i,k} \alpha_l \right) + \Lambda' \right] - 2c \tag{18}$$

where $\tau' > 0, \kappa' > 0$ and Λ' are constants independent of network structure. τ' has an upper bound less than 11 (at $n = 2$, its value is 10.791) and is strictly decreasing in n. It has a lower bound of 2 and converges asymptotically to 2. It is important to note that at $n = 4$, $\tau' = 3.5$. For the discussion that follows, let us assume $n \geqslant 3$.

Lemma 3 *(i) For any network g and player i such that $ik, im \notin g$ and $\eta_m(g) > \eta_k(g)$, $\Delta\Pi(g + ik, im) > \Delta\Pi(g, ik)$.*
(ii) If $\eta_m(g) = \eta_k(g)$ but $n \geqslant 4$, $\Delta\Pi(g + ik, im) > \Delta\Pi(g, ik)$ as well.

Proof. (i) Starting from an arbitrary network g with $ik, im \notin g$ suppose two players i and k form a link. This implies from (18), the increase in social welfare is proportional to

$$\frac{\Delta\Pi(g, ik)}{\kappa'} = 2 \cdot \gamma_0 + \gamma \cdot [\eta_i(g) + \eta_k(g)] - \frac{\tau'}{n} \left((n-2)\gamma_0 + \gamma \sum_{l \neq i,k,m} \eta_l(g) + \gamma \cdot \eta_m(g) \right) + \Lambda' - 2 \left(\frac{c}{\kappa'} \right)$$

Then, for forming yet another link say im, the the increase in social welfare is proportional to

$$\begin{aligned}
\frac{\Delta\Pi(g+ik,im)}{\kappa'} &= 2\cdot\gamma_0+\gamma\cdot[\eta_i(g)+\eta_m(g)+1] \\
&\quad -\frac{\tau'}{n}\left((n-2)\gamma_0+\gamma\sum_{l\neq i,k,m}\eta_l(g)+\gamma\cdot(\eta_k(g)+1)\right)+\Lambda'-2\left(\frac{c}{\kappa'}\right) \\
&= \frac{\Delta\Pi(g,ik)}{\kappa'}+\gamma\left[\left(1-\frac{\tau'}{n}\right)+\left(1+\frac{\tau'}{n}\right)(\eta_m(g)-\eta_k(g))\right] \\
&= \frac{\Delta\Pi(g,ik)}{\kappa'}+\gamma\left[2+\left(1+\frac{\tau'}{n}\right)(\eta_m(g)-\eta_k(g)-1)\right]
\end{aligned}$$

Now, if $\eta_m(g) > \eta_k(g)$, it implies $\eta_m(g) \geqslant \eta_k(g)+1$. Hence, $\frac{\Delta\Pi(g+ik,im)}{\kappa'} > \frac{\Delta\Pi(g,ik)}{\kappa'}$ completing the proof.

(ii) Now,

$$\frac{\Delta\Pi(g+ik,im)}{\kappa'} = \frac{\Delta\Pi(g,ik)}{\kappa'}+\gamma\left[\left(1-\frac{\tau'}{n}\right)\right].$$

Since $\tau' = 3.5$ for $n=4$, $\frac{\tau'}{n} < 1$. Furthermore, τ' and hence $\frac{\tau'}{n}$ is strictly decreasing in n, therefore $\frac{\tau'}{n} < 1$ for all $n \geqslant 4$. Hence, it follows that in all cases,

$$\frac{\Delta\Pi(g+ik,im)}{\kappa'} > \frac{\Delta\Pi(g,ik)}{\kappa'}.$$

■

The following lemma plays a key role in the results that follow.

Lemma 4 *For any efficient network for firms g, if $ij \in g$ and $ik \notin g$, then $\eta_j(g) \geqslant \eta_k(g)$. If $n \geqslant 4$, $\eta_j(g) > \eta_k(g)$.*

Proof. Suppose there exists an efficient network g and $ij \in g$ and $ik \notin g$. Then, $\Delta\Pi(g-ij,ij) \geqslant 0$. Suppose, towards a contradiction, $\eta_k(g) > \eta_j(g)$. This implies by Lemma 3 that $\Delta\Pi(g,ik) > 0$ contradicting that g is efficient. Hence, $\eta_k(g) \leqslant \eta_j(g)$.

Next let $\eta_j(g) = \eta_k(g)$ and $n \geqslant 4$. Again, $\Delta\Pi(g,ik) > \Delta\Pi(g-ij,ij) \geqslant 0$ which contradicts that g is efficient. Therefore, $\eta_k(g) < \eta_j(g)$. ■

The proposition below sets forth properties that characterize efficient networks for firms.

Proposition 1 *If $n \geqslant 4$: (i) The efficient network for firms cannot consist of more than one component.*

(ii) The geodesic distance between any two connected players in an efficient network for firms is less than or equal to 2.

Proof. (i) Suppose $h_1, h_2 \in C(g)$ where g is an efficient network and $ij \in h_1$ and $kl \in h_2$. Now, i is linked to j and not to k which implies using Lemma 4, $\eta_j(g) > \eta_k(g)$. But l is linked to k but not to j which implies $\eta_k(g) > \eta_j(g)$. Hence, we arrive at a contradiction.

(ii) Take two players i and j such that i and j belong to $N(h)$ where $h \in C(g)$. Hence, a path exists between i and j. Suppose the shortest path is $\{i_1i_2, i_2i_3, \ldots, i_{p-1}i_p\}$ where $i_1 = i$ and $i_p = j$ and $p \geqslant 4$. i is linked to i_2 but i is not linked to i_3. Hence, from Lemma 4, we get

$$\eta_{i_2}(g) > \eta_{i_3}(g). \tag{19}$$

Now, i_4 is linked to i_3 but not linked to i_2. Hence,

$$\eta_{i_3}(g) > \eta_{i_2}(g). \tag{20}$$

But (20) contradicts (19). ■

5.2 Efficient Networks for Consumers

In this section, we shall discuss efficient networks with regard to consumers. Namely, these are networks that maximize the consumer surplus. Consider the total consumer surplus of agents in both markets. It is given by

$$CS = \sum_i (\alpha_i - p_i^*)\, q_i^* + \sum_i (\beta_i - P_i^*)\, Q_i^*$$

where $p_i^* = \alpha_i - \overline{q}^*$ and $P_i^* = \beta_i - \overline{Q}^*$ denotes prices in both markets at equilibrium. Let $\overline{Q} = \sum_{i=1}^{n} Q_i$ and $\overline{q} = \sum_{i=1}^{n} q_i$. Hence,

$$\begin{aligned} CS &= \sum_i (\alpha_i - p_i^*)\, q_i^* + \sum_i (\beta_i - P_i^*)\, Q_i^* \\ &= \sum_i (\overline{q}^*)\, q_i^* + \sum_i \left(\overline{Q}^*\right) Q_i^* \\ &= (\overline{q}^*)^2 + \left(\overline{Q}^*\right)^2 \\ &= \left[\frac{(n+2)\overline{\alpha} - \overline{\beta}}{3+4n+n^2}\right]^2 + \left[\frac{(n+2)\overline{\beta} - \overline{\alpha}}{3+4n+n^2}\right]^2 \\ &= \frac{(n^2+4n+5)\left(\overline{\alpha}^2 + \overline{\beta}^2\right) - 4(n+2)\overline{\alpha}\overline{\beta}}{(3+4n+n^2)^2}. \end{aligned}$$

We can express consumer surplus CS as a function of the network. To this end, let $CS : G \to \mathbb{R}_+$ denote the overall consumer surplus as a function of the network. Now, suppose two players i and k form a link in a network g where initially $ik \notin g$. Then, $\overline{\alpha}$ increases by 2γ and hence,

$$\Delta CS(g, ik) = \frac{(n^2+4n+5)\,(2\gamma)\,(2\overline{\alpha} + 2\gamma) - 4(n+2)\overline{\beta}\,(2\gamma)}{(3+4n+n^2)^2} - 2c.$$

Proposition 2 *The efficient network with regard to consumers is either complete or empty.*

Proof. Consider any two arbitrary links ij and kl where neither link belongs to the network. Now,

$$\Delta CS(g+ij, kl) - \Delta CS(g, ij) = \frac{8\gamma^2(n^2+4n+5)}{(3+4n+n^2)^2} > 0.$$

Hence, if $\Delta CS(g, ij) > 0$, then $\Delta CS(g+ij, kl) > 0$ as well. Hence, starting from any arbitrary network, if forming one link increases consumer surplus, then forming all subsequent links enhances welfare as well. Hence, we end up in the complete network. If on the other hand, link formation costs are sufficiently high, the empty network is efficient. ■

5.3 Overall Efficiency

Overall efficient networks have similar properties to that of networks for firms. Let us define

$$W(g) = CS(g) + \Pi(g) \tag{21}$$

for all $g \in G$. Suppose two players i and k form a link in a network g where initially $ik \notin g$. Then, let

$$\Delta W(g, ik) = \Delta CS(g, ik) + \Delta\Pi(g, ik). \tag{22}$$

Lemma 5 *(i) For any network g and player i such that $ik, im \notin g$ and $\eta_m(g) > \eta_k(g)$, $\Delta W(g + ik, im) > \Delta W(g, ik)$.*
(ii) If $\eta_m(g) = \eta_k(g)$ but $n \geqslant 4$, $\Delta W(g + ik, im) > \Delta W(g, ik)$ as well.

Proof. (i) From Lemma 3, $\Delta\Pi(g + ik, im) > \Delta\Pi(g, ik)$. From Lemma 2, $\Delta CS(g + ik, im) > \Delta CS(g, ik)$. Hence, applying (22), the result follows.

(ii) The result is similar to (i). ■

The result leads to Lemma 6 which is the analog of Lemma 4.

Lemma 6 *For any overall efficient network for firms g, if $ij \in g$ and $ik \notin g$, then $\eta_j(g) \geqslant \eta_k(g)$. If $n \geqslant 4$, $\eta_j(g) > \eta_k(g)$.*

The proof follows from Lemma 5 in an analogous manner to that of the proof of Lemma 4 so we skip it to avoid repetition. Now, Lemma 6 directly leads to Proposition 3 and the proof is identical to the proof of Proposition 1, and so we skip it to avoid repetition.

Proposition 3 *If $n \geqslant 4$: (i) The overall efficient network for firms cannot consist of more than one component.*

(ii) The geodesic distance between any two connected players in an overall efficient network for firms is less than or equal to 2.

The networks defined above belong to a special category of networks called Nested Split Graphs (NSG) for $n \geq 4$. Belhaj, Bervoets, and Deroïan (2013, p.9) define a NSG as a graph which satisfies the following condition: if $ij \in g$ and $\eta_k(g) \geq \eta_j(g)$, then $ik \in g$. Clearly, this is equivalent to the condition in Lemma 6.

6 Some Examples

Next, for the purposes for illustration, let us consider some examples. By efficiency, we are referring to the traditional notion, namely the efficiency of firms. Such examples besides being interesting in themselves also help us verify the above theorems. They can give us additional insights that are not included in the above proven theorems.[5]

Example 1 *Let* $n = 3$. *Then there are eight possible networks, namely complete, empty,* $\{12\}$, $\{13\}$, $\{23\}$, $\{12, 13\}$, $\{12, 23\}$, $\{13, 23\}$. *Let* $\beta_i = 400, \gamma_0 = 100, \gamma = 0.05, c = 0$. *From our above lemmas the candidates for stability are complete, empty,* $\{12\}$, $\{13\}$, $\{23\}$, $\{12, 13, 23\}$ *while the candidates for efficiency include all networks. The payoffs are summarized in the table below.*

Network	u_1	u_2	u_3	Π
g^0	9756.94	9756.94	9756.94	29270.83
$\{12\}$	9756.92	9756.92	9757.89	29271.70
$\{13\}$	9756.92	9757.89	9756.92	29271.70
$\{23\}$	9756.92	9756.92	9756.92	29271.70
$\{12, 13\}$	9756.90	9757.87	9757.87	29272.60
$\{12, 23\}$	9757.87	9756.90	9757.87	29272.60
$\{13, 23\}$	9757.87	9757.87	9756.90	29272.60
g^N	9757.85	9757.85	9757.85	29273.60

The unique stable network is the empty network and the unique efficient network is the complete network.

[5] All computations were done in *Wolfram Mathematica 6.0* and available upon request.

This example confirms that the sets of stable and efficient networks need not coincide. Now, let us in Example 1 increase γ from 0.05 to 5. The complete network remains the efficient network, but now the stable networks are given by $\{ij\}$, $i, j \in \{1, 2, 3\}$ and the complete network. We show this in Example 2.

Example 2 *Let* $n = 3$. *Let* $\beta_i = 400, \gamma_0 = 100, \gamma = 5, c = 0$. *The payoffs are summarized in the table below.*

Network	u_1	u_2	u_3	Π
g^0	9756.94	9756.94	9756.94	29270.83
$\{12\}$	9758.99	9758.99	9855.29	29373.26
$\{13\}$	9758.99	9855.29	9758.99	29373.26
$\{23\}$	9855.29	9758.99	9758.99	29373.26
$\{12, 13\}$	9769.75	9849.85	9849.85	29469.44
$\{12, 23\}$	9849.85	9769.75	9849.85	29469.44
$\{13, 23\}$	9849.85	9849.85	9769.75	29469.44
g^N	9853.13	9853.13	9853.13	29559.38

Now, let us in Example 1 further increase γ from 5 to 45. We find that the unique efficient network is the complete network and the unique stable network is $\{ij\}$, $i, j \in \{1, 2, 3\}$. We show this in Example 3 below.

Example 3 *Let* $n = 3$. *Let* $\beta_i = 400, \gamma_0 = 100, \gamma = 45$. *The payoffs are summarized in the table below.*

Network	u_1	u_2	u_3	Π
g^0	9756.94	9756.94	9756.94	29270.83
$\{12\}$	10089.24	10089.24	10889.24	31067.71
$\{13\}$	10089.24	10889.24	10089.24	31067.71
$\{23\}$	10889.24	10089.24	10089.24	31067.71
$\{12, 13\}$	11127.78	10615.28	10615.28	32358.33
$\{12, 23\}$	10615.28	11127.78	10615.28	32358.33
$\{13, 23\}$	10615.28	10615.28	11127.78	32358.33
g^N	11047.57	11047.57	11047.57	33142.71

The last example (Example 4) shows that the possibility exists that stable and efficient networks might coincide.

Example 4 *Let* $n = 3$. *Let* $\beta_i = 50, \gamma_0 = 100, \gamma = 40$. *The payoffs are summarized in the table below.*

Network	u_1	u_2	u_3	Π
g^0	703.13	703.13	703.13	2109.38
$\{12\}$	1676.58	1676.58	450.66	3803.82
$\{13\}$	1676.58	450.66	1676.58	3803.82
$\{23\}$	450.66	1676.58	1676.58	3803.82
$\{12, 13\}$	3208.06	945.10	945.10	5098.26
$\{12, 23\}$	945.10	3208.06	945.10	5098.26
$\{13, 23\}$	945.10	945.10	3208.06	5098.26
g^N	1997.57	1997.57	1997.57	5992.71

The unique stable and efficient network is the complete network.

Now, let us focus on networks with four players. From the previous examples, it should be abundantly clear that the network payoff function satisfies a condition known as *anonymity*, which means that the payoff of a player in a network depends on its position in the network rather than its identity. Hence, we shall use a generic notation. The property also helps us reduce a total of 64 networks into 11. Let us also look at consumer surplus depicted in the table below by CS and overall welfare depicted in the table below by W. We shall employ exactly the same parameters as above.

Example 5 *Let* $n = 4$. *Let* $\beta_i = 400, \gamma_0 = 100, \gamma = 0.05, c = 0$. *The payoffs are summarized in the table below.*

Network	u_i	u_j	u_k	u_l	Π	CS	W
g^0	6902.04	6902.04	6902.04	6902.04	27608.16	69616.33	97224.49
$\{ij\}$	6901.94	6901.94	6902.56	6902.56	27609.00	69615.60	97224.60
$\{ij, jk\}$	6902.46	6901.84	6902.46	6903.08	27609.84	69614.90	97224.70
$\{ij, kl\}$	6902.46	6902.46	6902.46	6902.46	27609.84	69614.90	97224.70
$\{ij, jk, kl\}$	6902.98	6902.36	6902.36	6902.98	27610.68	69614.20	97224.70
$\{ij, jk, ik\}$	6902.36	6902.36	6902.36	6903.60	27610.68	69614.20	97224.70
$\{ij, ik, il\}$	6901.75	6902.98	6902.98	6902.98	27610.68	69614.20	97224.70
$\{ij, jk, kl, il\}$	6902.88	6902.88	6902.88	6902.88	27611.52	69613.50	97225.00
$\{ij, jk, ik, jl\}$	6902.88	6902.27	6902.88	6903.50	27611.52	69613.50	97225.00
$\{ij, jk, ik, jl, il\}$	6903.40	6902.78	6902.78	6903.40	27612.40	69612.70	97225.10
g^N	6903.30	6903.30	6903.30	6903.30	27613.20	69612.00	97225.20

The empty network is stable and efficient for consumers. The complete network is efficient for firms and overall efficient.

In this particular case where link formation costs are low, the formation of a link seem to unambiguously lower the profits of the two players forming the link and increase the profits of the other players. Total profits however seem to be increasing in the number of links. As a consequence, the unique stable network is empty and the unique efficient network is complete.

Now, let us look at consumer surplus and overall welfare. Consumer surplus is monotonically decreasing in the number of links. As as result, the efficient network for consumers is actually the empty network. Inspite of that, the complete network remains the overall efficient network, because the gains for producers outweighs the losses for consumers.

Let us now increase γ from 0.05 to 5. The results are summarized below.

Example 6 *Let $n = 4$. Let $\beta_i = 400, \gamma_0 = 100, \gamma = 5, c = 0$. The payoffs are summarized in the table below.*

Network	u_i	u_j	u_k	u_l	Π	CS	W
g^0	6902.04	6902.04	6902.04	6902.04	27608.16	69616.33	97224.49
$\{ij\}$	6898.03	6898.03	6956.24	6956.24	27609.00	69547.51	97256.04
$\{ij, jk\}$	6945.09	6905.85	6945.09	7014.89	27810.92	69484.73	97295.66
$\{ij, kl\}$	6945.09	6945.09	6945.09	6945.09	27780.37	69484.73	97265.10
$\{ij, jk, kl\}$	6996.61	6945.78	6945.78	6996.61	27610.68	69428.00	97312.78
$\{ij, jk, ik\}$	6945.78	6945.78	6945.78	7078.00	27915.33	69428.00	97343.33
$\{ij, ik, il\}$	6925.50	6996.61	6996.61	6996.61	27610.68	69428.00	97343.33
$\{ij, jk, kl, il\}$	6990.16	6990.16	6990.16	6990.16	27960.65	69377.31	97337.96
$\{ij, jk, ik, jl\}$	6990.16	6958.30	6990.16	7052.58	27611.52	69377.31	97368.51
$\{ij, jk, ik, jl, il\}$	6995.55	6995.55	7039.00	7039.00	28069.10	69332.65	97401.76
g^N	7037.26	7037.26	7037.26	7037.26	28149.02	69294.04	97443.06

The empty network is stable, but so is the partial circle $\{ij, jk, ik\}$. *The complete network is overall efficient and efficient for firms. The empty network is efficient for consumers.*

Finally, let us increase γ from 5 to 45. The results are summarized below.

Example 7 *Let* $n = 4$. *Let* $\beta_i = 400, \gamma_0 = 100, \gamma = 45, c = 0$. *The payoffs are summarized in the table below.*

Network	u_i	u_j	u_k	u_l	Π	CS	W
g^0	6902.04	6902.04	6902.04	6902.04	27608.16	69616.33	97224.49
$\{ij\}$	7291.95	7291.95	7550.16	7550.16	29684.22	69214.45	98898.67
$\{ij, jk\}$	7362.24	8640.45	7362.24	8559.02	31923.94	69301.88	101225.80
$\{ij, kl\}$	7362.24	7362.24	7362.24	7362.24	27780.37	69301.88	98750.82
$\{ij, jk, kl\}$	7793.26	8132.90	8132.90	7793.26	31852.31	69878.61	101730.90
$\{ij, jk, ik\}$	8132.90	8132.90	8132.90	9928.61	34327.31	69878.61	104205.90
$\{ij, ik, il\}$	10947.54	7793.26	7793.26	7793.26	34327.31	69878.61	104205.90
$\{ij, jk, kl, il\}$	7986.08	7986.08	7986.08	7986.08	31944.33	70944.65	102889.00
$\{ij, jk, ik, jl\}$	7986.08	9862.15	7986.08	8585.01	34419.33	70944.65	105364.00
$\{ij, jk, ik, jl, il\}$	9137.50	9137.50	8200.00	8200.00	34765.00	72500.00	107175.00
g^N	8773.58	8773.58	8773.58	8773.58	35094.33	74544.65	109639.00

The stable network is $\{ij\}$, *but so is the partial circle* $\{ij, jk, ik\}$. *The complete network is overall efficient and efficient for firms and consumers as well.*

7 Conclusion

The dynamics of multi-market oligopolies first discussed in Bulow et. al. (1985) can upset many results which would hold in isolated oligopoly markets. Here we take the situation of collaborative link formation among Cournot oligopolists with zero link formation costs. The results that a complete network materializes in equilibrium no longer holds one we introduces participation of the same set of firms in another not completely unrelated market. A variety of networks including the empty network can materialize in equilibrium.

With positive link formation costs, stable networks have a dominant group architecture. Efficient networks have the interesting feature that they consist of only one non-empty component and in that component, the geodesic distance between any two players is two or less. An exact characterization of efficient networks in this example, or more broadly, in *playing the field* games in general is an open question, and is reserved as a future endeavour.

8 Appendix

8.1 Derivation of the Multi-market equilibrium

First substituting (1)-(3) in (4), we get an expression for profits namely,

$$\pi_i = \left(\alpha_i - q_i - \sum_{j\neq i} q_j\right) \cdot q_i + \left(\beta_i - Q_i - \sum_{j\neq i} Q_j\right) \cdot Q_i - \frac{1}{2}(q_i + Q_i)^2. \tag{23}$$

Differentiating (23) with respect to q_i and Q_i, we get the first order conditions:

$$\begin{aligned} \frac{\partial \pi_i}{\partial q_i} &= 0; \\ \frac{\partial \pi_i}{\partial Q_i} &= 0. \end{aligned}$$

These result in the following two equations:

$$\alpha_i - 2q_i - \sum_{j\neq i} q_j - q_i - Q_i = 0; \tag{24}$$

$$\beta_i - 2Q_i - \sum_{j\neq i} Q_j - Q_i - q_i = 0. \tag{25}$$

Let $\overline{Q} = \sum_{i=1}^{n} Q_i$ and $\overline{q} = \sum_{i=1}^{n} q_i$. Then, (24) and (25) can be rewritten as

$$\alpha_i - 2q_i - \overline{q} - Q_i = 0; \tag{26}$$

$$\beta_i - 2Q_i - \overline{Q} - q_i = 0. \tag{27}$$

Summing up (26) over all i, we get

$$\sum_{i=1}^{n} \alpha_i - 2 \cdot \overline{q} - n \cdot \overline{q} - \overline{Q} = 0.$$

Summing up (27) over all i, we get

$$\sum_{i=1}^{n} \beta_i - 2 \cdot \overline{Q} - n \cdot \overline{Q} - \overline{q} = 0.$$

Denoting $\overline{\alpha} = \sum_{i=1}^{n} \alpha_i$ and $\overline{\beta} = \sum_{i=1}^{n} \beta_i$, the above two equations can be re-written as:

$$\overline{\alpha} - (n+2)\overline{q} - \overline{Q} = 0; \tag{28}$$
$$\overline{\beta} - (n+2)\overline{Q} - \overline{q} = 0. \tag{29}$$

(28) and (29) constitute a simultaneous equation system of two equations in two unknowns which can be solved to yield

$$\overline{q}^* = \frac{(n+2)\overline{\alpha} - \overline{\beta}}{3 + 4n + n^2}; \tag{30}$$
$$\overline{Q}^* = \frac{(n+2)\overline{\beta} - \overline{\alpha}}{3 + 4n + n^2}. \tag{31}$$

Substituting (30) and (31) in (26) and (27), we again get a linear system of two equations in two unknowns, namely,

$$\alpha_i - 2q_i - \frac{(n+2)\overline{\alpha} - \overline{\beta}}{3 + 4n + n^2} - Q_i = 0; \tag{32}$$
$$\beta_i - 2Q_i - \frac{(n+2)\overline{\beta} - \overline{\alpha}}{3 + 4n + n^2} - q_i = 0. \tag{33}$$

Solving (32) and (33), we get

$$\begin{aligned} q_i^* &= \frac{1}{3(3+4n+n^2)}\left[(6+8n+2n^2)\alpha_i - (3+4n+n^2)\beta_i\right] \\ &\quad + \frac{1}{3(3+4n+n^2)}\left[-(5+2n)\overline{\alpha} + (4+n)\overline{\beta}\right]; \end{aligned} \tag{34}$$
$$\begin{aligned} Q_i^* &= \frac{1}{3(3+4n+n^2)}\left[(6+8n+2n^2)\beta_i - (3+4n+n^2)\alpha_i\right] \\ &\quad + \frac{1}{3(3+4n+n^2)}\left[-(5+2n)\overline{\beta} + (4+n)\overline{\alpha}\right]. \end{aligned} \tag{35}$$

Substituting (30), (31), (34), and (35) in (23), we get an expression for profits, namely,

$$
\begin{aligned}
\pi_i^* \quad = \quad & \left[\frac{\alpha_i^2}{18(1+n)^2(3+n)^2}\right]\left(8+24n+107n^2+66n^3+11n^4\right) \\
& + \left[\frac{\alpha_i \cdot \beta_i}{18(1+n)^2(3+n)^2}\right]\left(16+48n-110n^2-84n^3-14n^4\right) \\
& + \left[\frac{\beta_i^2}{18(1+n)^2(3+n)^2}\right]\left(8+24n+107n^2+66n^3+11n^4\right) \\
& + \left[\frac{\alpha_i \cdot \left(\sum\limits_{j \neq i} \alpha_j\right)}{18(1+n)^2(3+n)^2}\right]\left(-8-182n-124n^2-22n^3\right) \\
& + \left[\frac{\beta_i \cdot \left(\sum\limits_{j \neq i} \alpha_j\right)}{18(1+n)^2(3+n)^2}\right]\left(-8+142n+92n^2+14n^3\right) \\
& + \left[\frac{\left(\sum\limits_{j \neq i} \alpha_j\right)^2}{18(1+n)^2(3+n)^2}\right]\left(83+58n+11n^2\right) \\
& + \left[\frac{\alpha_i \cdot \left(\sum\limits_{j \neq i} \beta_j\right)}{18(1+n)^2(3+n)^2}\right]\left(-8+142n+92n^2+14n^3\right) \\
& + \left[\frac{\beta_i \cdot \left(\sum\limits_{j \neq i} \beta_j\right)}{18(1+n)^2(3+n)^2}\right]\left(-8-182n-124n^2-22n^3\right) \\
& + \left[\frac{\left(\sum\limits_{j \neq i} \beta_j\right)^2}{18(1+n)^2(3+n)^2}\right]\left(83+58n+11n^2\right) \\
& + \left[\frac{\left(\sum\limits_{j \neq i} \alpha_j\right)\left(\sum\limits_{j \neq i} \beta_j\right)}{18(1+n)^2(3+n)^2}\right]\left(-158-100n-14n^2\right) \qquad (36)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
u_i(g) \quad = \quad & \left[\frac{\alpha_i(g)^2}{18(1+n)^2(3+n)^2}\right]\left(8+24n+107n^2+66n^3+11n^4\right) \\
& + \left[\frac{\alpha_i(g)\cdot\beta_i}{18(1+n)^2(3+n)^2}\right]\left(16+48n-110n^2-84n^3-14n^4\right) \\
& + \left[\frac{\beta_i^2}{18(1+n)^2(3+n)^2}\right]\left(8+24n+107n^2+66n^3+11n^4\right) \\
& + \left[\frac{\alpha_i(g)\cdot\left(\sum_{j\neq i}\alpha_j(g)\right)}{18(1+n)^2(3+n)^2}\right]\left(-8-182n-124n^2-22n^3\right) \\
& + \left[\frac{\beta_i\cdot\left(\sum_{j\neq i}\alpha_j(g)\right)}{18(1+n)^2(3+n)^2}\right]\left(-8+142n+92n^2+14n^3\right) \\
& + \left[\frac{\left(\sum_{j\neq i}\alpha_j(g)\right)^2}{18(1+n)^2(3+n)^2}\right]\left(83+58n+11n^2\right) \\
& + \left[\frac{\alpha_i(g)\cdot\left(\sum_{j\neq i}\beta_j\right)}{18(1+n)^2(3+n)^2}\right]\left(-8+142n+92n^2+14n^3\right) \\
& + \left[\frac{\beta_i\cdot\left(\sum_{j\neq i}\beta_j\right)}{18(1+n)^2(3+n)^2}\right]\left(-8-182n-124n^2-22n^3\right) \\
& + \left[\frac{\left(\sum_{j\neq i}\beta_j\right)^2}{18(1+n)^2(3+n)^2}\right]\left(83+58n+11n^2\right) \\
& + \left[\frac{\left(\sum_{j\neq i}\alpha_j(g)\right)\left(\sum_{j\neq i}\beta_j\right)}{18(1+n)^2(3+n)^2}\right]\left(-158-100n-14n^2\right) \\
& -c\cdot\eta_i(g).
\end{aligned}
\tag{37}
$$

8.2 Pairwise Equilibrium Networks

First, we will show that the network game is playing the field game. There are two arguments in the payoff function. The first one is $\alpha_i\,(g)$ and the second one is

$\sum_{j \neq i} \alpha_j(g)$. First,

$$\alpha_i(g) = \gamma_0 + \gamma \cdot \eta_i(g).$$

Second,

$$\sum_{j \neq i} \alpha_j(g) = (n-1)\gamma_0 + \gamma \sum_{j \neq i} \eta_j(g).$$

Consider the links of player $j \neq i$ in network g given by $N_j(g)$. One can divide this set into two subsets. First, the links with player i, given by say $N_j^i(g)$ which is either ij or $\varnothing$. The second is the links with players other than i, given by $N_j^{-i}(g)$. Let the respective cardinalities be given by $\eta_j^i(g)$ and $\eta_j^{-i}(g)$. Therefore,

$$\begin{aligned} \sum_{j \neq i} \eta_j(g) &= \sum_{j \neq i} \eta_j^i(g) + \sum_{j \neq i} \eta_j^{-i}(g) \\ &= \eta_i(g) + 2 \cdot \lambda(g_{-i}). \end{aligned}$$

Hence,

$$u_i(g) = \Phi(\eta_i(g), \lambda(g_{-i})) - c \cdot \eta_i(g).$$

The rest of the derivation is an exercise in tedious algebra. Let us define a set of positive parameters.

$$\begin{aligned} &\nu = 18(1+n)^2(3+n)^2; \\ &\rho_1 = (8 + 24n + 107n^2 + 66n^3 + 11n^4); \\ &\rho_2 = (-16 - 48n + 110n^2 + 84n^3 + 14n^4); \\ &\rho_3 = (8 + 182n + 124n^2 + 22n^3); \\ &\rho_4 = (-8 + 142n + 92n^2 + 14n^3); \\ &\rho_5 = (83 + 58n + 11n^2); \\ &\rho_6 = (158 + 100n + 14n^2). \end{aligned}$$

Then,

$$\begin{aligned}
\Phi(k,l) &= \frac{1}{\nu}\left[\rho_1(\gamma_0+\gamma\cdot k)^2-\rho_2\cdot\beta_i(\gamma_0+\gamma\cdot k)\right] \\
&\quad -\frac{1}{\nu}\left[\rho_3(\gamma_0+\gamma\cdot k)((n-1)\gamma_0+\gamma\cdot k+2\cdot\gamma\cdot l)\right] \\
&\quad +\frac{1}{\nu}\left[\beta_i\cdot\rho_4((n-1)\gamma_0+\gamma\cdot k+2\cdot\gamma\cdot l)\right] \\
&\quad +\frac{1}{\nu}\left[\rho_5((n-1)\gamma_0+\gamma\cdot k+2\cdot\gamma\cdot l)^2\right] \\
&\quad +\frac{1}{\nu}\left[\rho_4\left(\sum_{j\neq i}\beta_j\right)(\gamma_0+\gamma\cdot k)\right] \\
&\quad -\frac{1}{\nu}\left[\rho_6\left(\sum_{j\neq i}\beta_j\right)((n-1)\gamma_0+\gamma\cdot k+2\cdot\gamma\cdot l)\right] \\
&\quad +C
\end{aligned} \tag{38}$$

where C is a collection of term unrelated to k or l and hence can be treated as a constant.

Therefore,

$$\begin{aligned}
\Delta(k,l) &= \Phi(k+1,l)-\Phi(k,l) \\
&= \frac{1}{\nu}\left[\rho_1\left(2\gamma_0\cdot\gamma+\gamma^2(2k+1)\right)\right] \\
&\quad -\left(\frac{1}{\nu}\right)\gamma\cdot\rho_2\cdot\beta_i+\left(\frac{1}{\nu}\right)\gamma\cdot\rho_4\cdot\beta_i+\left(\frac{1}{\nu}\right)\gamma\cdot\rho_4\left(\sum_{j\neq i}\beta_j\right) \\
&\quad +\frac{1}{\nu}\left[\rho_5\cdot\gamma(2(n-1)\gamma_0+\gamma(2k+1)+4\cdot\gamma\cdot l)\right] \\
&\quad -\frac{1}{\nu}\left[\rho_3\left(\gamma_0\cdot\gamma\cdot n+\gamma^2(2k+2l+1)\right)\right] \\
&\quad -\frac{1}{\nu}\left[\left(\sum_{j\neq i}\beta_j\right)\rho_6\cdot\gamma\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\partial\Delta}{\partial k} &= \left(\frac{2\gamma^2}{\nu}\right)(\rho_1-\rho_3+\rho_5) \\
&= \left(\frac{2\gamma^2}{\nu}\right)\left(11n^4+44n^3-6n^2-100n+83\right)>0.
\end{aligned}$$

Finally,

$$\begin{aligned}\frac{\partial \Delta}{\partial l} &= 2\left(\frac{\gamma^2}{\nu}\right)(2\rho_5 - \rho_3) \\ &= -2\left(\frac{\gamma^2}{\nu}\right)\big(22n^3 + 102n^2 + 66n - 158\big) < 0.\end{aligned}$$

8.3 Efficient Networks for Firms

Consider the effect of link formation between two arbitrary firms i and k on firm $l \neq k, i$. The change in profits is given by

$$\begin{aligned}\Delta\pi_l^* &= -\left[\frac{\alpha_l \cdot \gamma}{18(1+n)^2(3+n)^2}\right]\big(44n^3 + 248n^2 + 364n + 16\big) \\ &\quad + \left[\frac{\left(\sum_{m \neq l} \alpha_m\right) \cdot \gamma}{18(1+n)^2(3+n)^2}\right]\big(44n^2 + 232n + 332\big) \\ &\quad + \left[\frac{\beta_l \cdot \gamma}{18(1+n)^2(3+n)^2}\right]\big(28n^3 + 184n^2 + 284n - 16\big) \\ &\quad - \left[\frac{\left(\sum_{m \neq l} \beta_m\right) \cdot \gamma}{18(1+n)^2(3+n)^2}\right]\big(28n^2 + 200n + 316\big) \\ &\quad + \left[\frac{\gamma^2}{18(1+n)^2(3+n)^2}\right]\big(44n^2 + 232n + 332\big)\,. \end{aligned} \tag{39}$$

Further, from (7), the change in profits of k gross of link formation costs are given by:

$$\begin{aligned}
\Delta\pi_k^* &= \left[\frac{\alpha_k \cdot \gamma}{18(1+n)^2(3+n)^2}\right]\left(22n^4 + 110n^3 + 90n^2 - 134n + 8\right) \\
&\quad -\left[\frac{\left(\sum_{j\neq k}\alpha_j\right)\cdot\gamma}{18(1+n)^2(3+n)^2}\right]\left(22n^3 + 102n^2 + 66n - 158\right) \\
&\quad -\left[\frac{\beta_k \cdot \gamma}{18(1+n)^2(3+n)^2}\right]\left(14n^4 + 70n^3 + 18n^2 - 190n - 8\right) \\
&\quad +\left[\frac{\left(\sum_{j\neq k}\beta_j\right)\cdot\gamma}{18(1+n)^2(3+n)^2}\right]\left(14n^3 + 78n^2 + 42n - 166\right) \\
&\quad +\left[\frac{\gamma^2}{18(1+n)^2(3+n)^2}\right]\left(11n^4 + 44n^3 - 6n^2 - 100n + 83\right). \qquad (40)
\end{aligned}$$

Hence,

$$\begin{aligned}
\Delta\Pi(g, ik) &= \Delta\pi_i^* + \Delta\pi_k^* + \sum_{l\neq i,k}\Delta\pi_l^* - 2c \\
&= \left[\frac{(\alpha_i + \alpha_k)\cdot\gamma}{18(1+n)^2(3+n)^2}\right]\left(22n^4 + 132n^3 + 132n^2 - 332n - 498\right) \\
&\quad -\left[\frac{\left(\sum_{l\neq i,k}\alpha_l\right)\cdot\gamma}{18(1+n)^2(3+n)^2}\right]\left(44n^3 + 352n^2 + 860n + 696\right) \\
&\quad -\left[\frac{(\beta_i + \beta_k)\cdot\gamma}{18(1+n)^2(3+n)^2}\right]\left(14n^4 + 84n^3 + 84n^2 - 316n - 474\right) \\
&\quad +\left[\frac{\left(\sum_{l\neq i,k}\beta_l\right)\cdot\gamma}{18(1+n)^2(3+n)^2}\right]\left(28n^3 + 224n^2 + 652n + 600\right) \\
&\quad +\left[\frac{\gamma^2}{18(1+n)^2(3+n)^2}\right]\left(22n^4 + 132n^3 + 132n^2 - 332n - 498\right) \\
&\quad -2c \qquad (41)
\end{aligned}$$

Note that we can simplify (41) into

$$\Delta\Pi(g, ik) = \kappa' \left[(\alpha_i + \alpha_k) - \frac{1}{n} \left(\sum_{l \neq i,k} \alpha_l \right) \frac{(44n^3 + 352n^2 + 860n + 696)}{\left(22n^3 + 132n^2 + 132n - 332 - \frac{498}{n}\right)} + \Lambda' \right] - 2c$$
$$= \kappa' \left[(\alpha_i + \alpha_k) - \frac{\tau'}{n} \left(\sum_{l \neq i,k} \alpha_l \right) + \Lambda' \right] - 2c \quad (42)$$

where $\tau' > 0, \kappa' > 0$ and Λ' are constants independent of network structure. τ' has an upper bound less than 11 (at $n = 2$, its value is 10.791) and is strictly decreasing in n. It has a lower bound of 2 and converges asymptotically to 2. It is important to note that at $n = 4$, $\tau' = 3.5$.

8.4 Conditions for the Empty Network to be an Equilibrium Network with Zero Link Formation Costs

Let c be zero. Then, substituting $k = 0$ and $l = 0$ in (22), we get

$$\Delta(0,0) = \frac{1}{\nu} \left[\rho_1 \left(2\gamma_0 \cdot \gamma + \gamma^2 \right) + \rho_5 \cdot \gamma \left(2(n-1)\gamma_0 + \gamma \right) - \rho_3 \left(\gamma_0 \cdot \gamma \cdot n + \gamma^2 \right) \right]$$
$$- \left(\frac{1}{\nu} \right) \gamma \beta_i (\rho_2 - \rho_4)$$
$$+ \left(\frac{1}{\nu} \right) \gamma \left(\sum_{j \neq i} \beta_j \right) (\rho_4 - \rho_6).$$

Let $\beta_1 < \beta_2 < \cdots < \beta_n$. Then, if

$$\beta_2 > \left[\frac{\rho_1 (2\gamma_0 + \gamma) + \rho_5 (2(n-1)\gamma_0 + \gamma) - \rho_3 (\gamma_0 \cdot n + \gamma) + \left(\sum_{j \neq 2} \beta_j \right) (\rho_4 - \rho_6)}{\rho_2 - \rho_4} \right],$$

then $\Delta(0,0) < 0$ and hence the empty network is an equilibrium network.

References

[1] Belhaj, M., Bervoets, S. and F. Deroïan, (2013), "Network Design under Local Complementarities", Working Papers halshs-00796487.

[2] Bernheim, B.D. and M.D. Whinston, "Multimarket Contact and Collusive Behavior", *Rand Journal of Economics*, Vol. 21, 1-26.

[3] Billand, P., C. Bravard, S. Chakrabarti and S. Sarangi, (2008) "Corporate Espionage", Working Paper.

[4] Bowley, A.L., (1924) *The Mathematical Groundwork of Economics*, Oxford: Oxford University Press.

[5] Bulow, J.I., J.D. Geanakopolos and Paul D. Klemperer, (1985) "Multimarket Oligopoly: Strategic Substitutes and Complements", *The Journal of Political Economy*, Vol. 93, 488-511.

[6] Dixit, A. (1979), "A Model of Duopoly Suggesting a Theory of Entry Barriers", *Bell Journal of Economics*, Vol. 10, 20-32.

[7] Chakrabarti, S. and H. Haller, (2011) "An Analysis of Advertising Wars", *Manchester School*, Vol. 79, 100-124.

[8] Goyal, G. and S. Joshi, (2003) "Networks of Collaboration in Oligopoly", *Games and Economic Behavior*, Vol. 43, 57-85.

[9] Goyal, G. and S. Joshi (2006) "Unequal Connections", *International Journal of Game Theory*, Vol. 34, 319-349.

[10] Jackson, M. and A. Wolinsky, (1996) "A Strategic Model of Social and Economic Networks", *Journal of Economic Theory*, Vol. 71, 44-74.

[11] Jackson, M., (2005) "Allocation Rules for Network Games", *Games and Economic Behavior*, Vol. 51, 128-154.

[12] Martin, S., (1993) *Advanced Industrial Economics*, Oxford: Blackwell.

[13] Spence, M., (1976) "Product Differentiation and Welfare", *American Economic Review*, Vol. 66, 407-414.

[14] Tirole, J., (1988) *The Theory of Industrial Organization,* Cambridge, MA, and London: The MIT Press.

[15] Vives, X., (1999) *Oligopoly Pricing: Old Ideas and New Tools*, Cambridge, MA, and London: The MIT Press.