Higher-order volatility: dynamics and sensitivities

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HIGHER-ORDER VOLATILITY: DYNAMICS AND SENSITIVITIES

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In this addendum to Carey (2005), we draw several more analogies with the Black-Scholes model. We derive the characteristic function of the underlying log process as a function of the volatilities of all orders. Option prices are shown to satisfy an infinite-order version of the Black-Scholes partial differential equation. We find that in the same way that the option sensitivity to the cost of carry is related to delta and vega to gamma in the Black-Scholes model, the option sensitivity to j-th order volatility is related to the j-th order sensitivity to the underlying. Finally, we argue that third-order volatility provides a possible basis for the introduction of a "skew swap" product.

We begin by recalling some definitions and notation. Let (X_t) be a positive-valued adapted stochastic process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$, $t \geq 0$. Here (X_t) can be interpreted as a financial variable, $(\mathcal{F}_t)$ as the market information structure, Q as an equivalent martingale measure. Let $\delta t > 0$ denote a finite period of time, and define $\delta X_t = X_{t+\delta t} - X_t$, so that the relative change of the process over the interval t to $t + \delta t$ reads $\delta X_t \big/ X_t$. $E_t \cdot$ denotes expectation conditional on $\mathcal{F}_t$, and j is a generic positive integer.

We define j*-th order finite-period volatility* $\Sigma_{j,t}$ via the identity:

$$\Sigma_{j,t}^j \equiv E_t \left(\delta X_t / X_t \right)^j \Big/ \delta t \,,$$

with the convention that $\Sigma_{j,t}$ equals the nonnegative root when j is even, and we let $\sigma_{j,t} = \lim_{\delta t \searrow 0} \Sigma_{j,t}$ denote j*-th order instantaneous volatility*. The rationale for using j as an exponent can be found in Carey (2005), along with snapshot implied values for the S&P 500 options market.

1. CHARACTERISTIC FUNCTION

Let $\phi(v)$ denote the characteristic function at time t of $\ln X_{t+\tau}$, $\tau \geq 0$, that is:

$$\phi(v) = E_t \exp[iv(\ln X_{t+\tau})] = E_t X_{t+\tau}^{iv},$$

$i^2 \equiv -1$. Now in the general case the volatilities we have defined are stochastic. In appendix 1 we show that when the finite-period volatilities of all orders exist and are deterministic, $\phi(v)$ can be obtained as:

$$\phi(v) = X_t^{iv} \prod_u \left[1 + \sum_{j=1}^{\infty} \binom{iv}{j} \Sigma_{j,u}^j \delta t\right], \tag{1}$$

where τ is a positive multiple of δt, the product is over $u = t, t+\delta t, \ldots, t+\tau-\delta t$, and:

$$\binom{\bullet}{j} = \tfrac{1}{j!}\bullet(\bullet-1)(\bullet-2)\ldots(\bullet-j+1)$$

is a generalised binomial coefficient. If we further suppose that there exists a $\Delta > 0$ such that this assumption holds for every $\delta t \leq \Delta$, then clearly the instantaneous volatilities, when they exist, are also deterministic. Then, fixing $\tau > 0$ and taking the limit $\delta t \searrow 0$ in (1) yields:

$$\phi(v) = X_t^{iv} \exp\left[\tau \sum_{j=1}^{\infty} \binom{iv}{j} \sigma_j^j\right], \tag{2}$$

where the σ_j, which are assumed to exist, are defined via:

$$\sigma_j^j \equiv \frac{1}{\tau} \int_t^{t+\tau} \sigma_{j,u}^j du,$$

again with the convention that σ_j is nonnegative when j is even, and will be referred to as *average volatilities*.

We make three comments on these results. First, as in Carey (2005) we point out that we have not shown that deterministic instantaneous volatilities imply (2) in and of themselves. Expression (2) is perhaps best viewed as an approximation to (1) for the case when δt is arbitrarily small. Second, if $\sigma_j = 0$ for every $j > 2$ then the characteristic function (2) can be rewritten as:

$$\phi(v) = \exp\left[iv\left(\ln X_t + \left(\sigma_1 - \tfrac{1}{2}\sigma_2^2\right)\tau\right) - \tfrac{1}{2}v^2\sigma_2^2\tau\right],$$

implying that $\ln X_{t+\tau}$ is normally distributed (with mean $\ln X_t + \left(\sigma_1 - \frac{1}{2}\sigma_2^2\right)\tau$ and variance $\sigma_2^2\tau$). Thus, nonzero values for higher-order average volatilities indicate and quantify deviations from the lognormal distribution. Third, we point out that the sequence of volatilities may not be truncated at any index greater than two. Indeed, this would imply that the characteristic function has the form $\exp P(v)$, with $P(v)$ a polynomial of order greater than two, contradicting the theorem of Marcinkiewicz (see Stuart and Ord (1994), §4.8). This does not however rule out such a strategy by way of approximation.

2. DYNAMICS AND SENSITIVITIES

We now consider the dynamics of a European-exercise option contract with maturity T. We choose the money-market account as numeraire, and assume deterministic interest rates, with R_t denoting the riskfree rate per unit time for the period t to $t + \delta t$, and with $r_t = \lim_{\delta t \searrow 0} R_t$ denoting the short rate.

We next assume that the volatilities of all orders are deterministic as per the previous section. By (2), the characteristic function of $\ln X_T$ depends only on the level of the underlying variable, the average volatilities, and time. Thus, given a set of volatilities, a payoff function and a rate of interest, the price of the contract is some function of the underlying variable and time, say $C(X,t)$, which we will assume to be infinitely differentiable in X, and once differentiable in t.

Now since the discounted price of the option is a martingale under Q, we have:

$$(1 + R_t \delta t) C(X_t, t) = E_t C(X_t + \delta X_t, t + \delta t). \tag{3}$$

for every $t < T - \delta t$. Letting $\partial_X^j C$ denote the j-th order partial derivative of C with respect to X, expand $C(X_t + \delta X_t, t + \delta t)$ in the Taylor series:

$$C(X_t + \delta X_t, t + \delta t) = C(X_t, t + \delta t) + \sum_{j=1}^{\infty} \tfrac{1}{j!} (\delta X_t)^j \partial_X^j C(X_t, t + \delta t). \tag{4}$$

As shown in appendix 2, substituting (4) into (3) and taking the limit $\delta t \searrow 0$ yields the dynamics of the option price as:

$$\partial_t C(X,t) + \sum_{j=1}^{\infty} \tfrac{1}{j!} \sigma_{j,t}^j X^j \partial_X^j C(X,t) = r_t C(X,t), \tag{5}$$

for almost every $(X,t) \in (0,\infty) \times [0,T]$, where $\partial_t C$ is the derivative of C with respect to t.[1]

As an example, consider the case where the underlying is a traded asset paying no dividends, implying $\sigma_{1,t} = r_t$. Assume further that the instantaneous rate of interest is a constant $r_t = r$, second-order instantaneous volatility is a constant $\sigma_{2,t} = \sigma$, and all higher-order volatilities are zero. Then (5) reduces to the Black-Scholes equation:

$$\partial_t C(X,t) + rX \partial_X C(X,t) + \tfrac{1}{2!} \sigma^2 X^2 \partial_X^2 C(X,t) = rC(X,t). \tag{6}$$

[1] Equation (5) is closely related to the backward Kramers-Moyal expansion, which describes the evolution of the probability density of a Markov process when the initial time and level are varied, and which is to (5) what the backward Kolmogorov equation is to (6) — see Gillespie (1992) for a detailed introduction. In fact, heuristically terminating a (forward) Kramers-Moyal expansion at the second term (yielding a forward Kolmogorov, or Fokker-Planck, equation) was the approach used by Albert Einstein in his original characterisation of Brownian motion (see Gardiner (1985), §7.2.2). However, as directly implied by Pawula's theorem, the expansion cannot be terminated at any higher order (see Van Kampen (1992), page 269). This mirrors our earlier finding that the sequence of volatilities cannot terminate at any order greater than two.

A further analogy with the Black-Scholes model can be drawn with respect to the sensitivities of the option price. We note that in the Black-Scholes model, the price sensitivity of a plain vanilla option to a change in the cost of carry b is related to the option delta via:

$$\frac{\partial C}{\partial b} = \tau X \frac{\partial C}{\partial X},$$

and vega is related to gamma via:

$$\frac{\partial C}{\partial \sigma} = \tau \sigma X^2 \frac{\partial^2 C}{\partial X^2}$$

(see Haug (1997), §1.3). As shown in appendix 3, under (2) these results generalise to:[2]

$$\frac{\partial C}{\partial \sigma_j} = \tau \frac{\sigma_j^{j-1}}{(j-1)!} X^j \frac{\partial^j C}{\partial X^j}, \tag{7}$$

for a generic European-exercise contract. Thus, the option price sensitivity to j-th order volatility is directly proportional to the j-th order sensitivity to the underlying, by a factor which increases linearly with time to maturity.

3. SOME APPLICATIONS

In Carey (2005) we suggest investigating the extent to which derivatives pricing and risk management can be carried out based on a finite set of leading-order volatilities, say up to order four. This essentially involves identifying a pricing formula or procedure which both dispenses the user from (exogenously) specifying an infinite set of volatilities, and prices the product(s) at hand with acceptable precision. The generalised Edgeworth expansion formula of Jarrow and Rudd (1982) and other moment-based formulae are obvious candidates, but other approaches are possible. Regardless of the implementation, the simple relationship between sensitivities to the underlying and sensitivities to the volatilities promises computational savings as well as conceptual clarity in risk management.

Here we suggest another possible application, making use of the intuitive nature of these new risk metrics. As shown in Carey (2005), the volatilities can be interpreted, via a simple thought experiment, as break-even relative changes. Indeed, as illustrated by a sample of implied values for S&P 500 options, the volatilities of order two, three and four (and those beyond, although they are not reported) have the same order of magnitude, each falling between 5% and 25% in absolute value.[3] Thus, the intuitive nature of ordinary volatility carries over to the new quantities both in concept and in practice.

While this quality in a risk metric holds little interest from an analytical or computational perspective, it is fundamentally important whenever the metric is traded, be it in isolation (as in a variance swap) or packaged with other exposures (as in a generic derivatives book). Indeed, it is the vehicle for the opinions of the trader with respect to a wide range of future scenarios, often synthesised in a nonanalytical way, and as such must provide an intuitive handle on market conditions.

There is a degree of interest among investors in “trading the skew”, which at this time must be done via unwieldy and expensive options trades. While the variance can be

[2] Here C is as defined in appendix 3.

[3] We recall that in this example, first-order volatility equals the cost of carry.

traded directly via variance swaps, there is currently no such product for skewness. This could arguably be ascribed in part to the absence of a sufficiently compelling quantity from a trading perspective, and in this regard it is worth noting that variance swaps are quoted in terms of the corresponding volatilities, not as raw variances. Whereas traditional measures of skewness almost certainly do not possess the right qualities, we suggest third-order volatility could help spark interest in a “skew swap” product.

APPENDIX 1

To establish (1), let z denote an arbitrary complex number, and note that:

$$\left(X_{t+\tau}/X_t\right)^z = \left(\prod_u X_{u+\delta t}/X_u\right)^z = \prod_u \left(1+\delta X_u/X_u\right)^z . \qquad \text{(A1.1)}$$

Next recall the binomial identity:

$$(1+\beta)^z = \sum_{j=0}^{\infty} \binom{z}{j} \beta^j , \qquad |\beta| < 1 .$$

We may then write:

$$\left(1+\delta X_u/X_u\right)^z = \sum_{j=0}^{\infty} \binom{z}{j} \left(\delta X_u/X_u\right)^j , \qquad \text{(A1.2)}$$

assuming that $\left|\delta X_u/X_u\right| < 1$ almost surely for every u. This assumption is safe for typical financial variables, provided that the time step δt is sufficiently small. Suppose further that:

$$E_u\left[\sum_{j=0}^{\infty} \binom{z}{j} \left(\delta X_u/X_u\right)^j\right] = \sum_{j=0}^{\infty} \binom{z}{j} E_u\left(\delta X_u/X_u\right)^j \qquad \text{(A1.3)}$$

for every u. Now $E_t \cdot = \left(\prod_u E_u\right) \cdot$, so that taking expectations in (A1.1) and using (A1.2) and (A1.3) iteratively, along with the fact that the $\Sigma^j_{j,u}$ are deterministic, yields:

$$E_t\left(X_{t+\tau}/X_t\right)^z = \prod_u \left[1+\sum_{j=1}^{\infty} \binom{z}{j} \Sigma^j_{j,u} \delta t\right]. \qquad \text{(A1.4)}$$

Setting $z = iv$, $v \in \mathbb{R}$, yields (1), while if $z = n$, $n = 1,2,\ldots$, then this reduces to the discrete-period moment formula in Carey (2005).

To derive (2), note that:

$$1+\sum_{j=1}^{\infty} \binom{z}{j} \Sigma^j_{j,u} \delta t = \exp\left[o(\delta t)+\sum_{j=1}^{\infty} \binom{z}{j} \Sigma^j_{j,u} \delta t\right],$$

where $o(\delta t)$ represents terms which vanish with δt faster than δt (that is, $o(\delta t)/\delta t \to 0$ as $\delta t \searrow 0$). Replacing in (A1.4), rearranging and simplifying yields:

$$E_t\left(X_{t+\tau}/X_t\right)^z = \exp\left[\frac{o(\delta t)}{\delta t}\tau + \sum_{j=1}^{\infty} \sum_u \binom{z}{j} \Sigma^j_{j,u} \delta t\right]. \qquad \text{(A1.5)}$$

Now by definition $\Sigma_{j,u}^{j} \to \sigma_{j,u}^{j}$ as $\delta t \searrow 0$, hence $\sum_{u} \Sigma_{j,u}^{j} \delta t \to \sigma_{j}^{j} \tau$. Assuming that:

$$\lim_{\delta t \searrow 0} \sum_{j=1}^{\infty} \sum_{u} \binom{z}{j} \Sigma_{j,u}^{j} \delta t = \sum_{j=1}^{\infty} \lim_{\delta t \searrow 0} \sum_{u} \binom{z}{j} \Sigma_{j,u}^{j} \delta t \,,$$

taking limits in (A1.5) yields:

$$E_t \left(X_{t+\tau} / X_t \right)^z = \exp\left[\tau \sum_{j=1}^{\infty} \binom{z}{j} \sigma_j^j \right].$$

As above, setting $z = iv$ yields (2), while $z = n$ yields the average-volatilities moment formula in Carey (2005).

APPENDIX 2

We begin with:

$$(1 + R_t\delta t)C(X_t, t) = E_t C(X_t + \delta X_t, t + \delta t), \tag{A2.1}$$

and expand $C(X_t + \delta X_t, t + \delta t)$ as:

$$C(X_t + \delta X_t, t + \delta t) = C(X_t, t + \delta t) + \sum_{j=1}^{\infty} \tfrac{1}{j!}(\delta X_t)^j \partial_X^j C(X_t, t + \delta t). \tag{A2.2}$$

Assuming that:

$$E_t \sum_{j=1}^{\infty} \tfrac{1}{j!}(\delta X_t)^j \partial_X^j C(X_t, t + \delta t) = \sum_{j=1}^{\infty} E_t \left[\tfrac{1}{j!}(\delta X_t)^j \partial_X^j C(X_t, t + \delta t)\right],$$

substituting (A2.2) into (A2.1) yields:

$$(1 + R_t\delta t)C(X_t, t) = C(X_t, t + \delta t) + \sum_{j=1}^{\infty} \tfrac{1}{j!} E_t\left[(\delta X_t)^j\right] \partial_X^j C(X_t, t + \delta t).$$

Noting that $E_t\left[(\delta X_t)^j\right] = \Sigma_{j,t}^j X_t^j \delta t$ and rearranging gives:

$$\frac{C(X_t, t + \delta t) - C(X_t, t)}{\delta t} + \sum_{j=1}^{\infty} \tfrac{1}{j!} \Sigma_{j,t}^j X_t^j \partial_X^j C(X_t, t + \delta t) = R_t C(X_t, t)$$

almost surely. Taking the limit $\delta t \searrow 0$ yields (5).

We note here that equation (5) can be liberalised by the change of variable $Y = \ln X$. A straightforward induction shows that $\tfrac{1}{j!} X^j \partial_X^j \cdot = \begin{pmatrix} \partial_Y \\ j \end{pmatrix} \cdot$,[4] so that:

$$\partial_t C(Y, t) + \sum_{j=1}^{\infty} \sigma_{j,t}^j \begin{pmatrix} \partial_Y \\ j \end{pmatrix} C(Y, t) = r_t C(Y, t),$$

highlighting the beginnings of a correspondence with formula (2). A solution to this equation can in principle be sought using the standard Fourier transformation technique, although there are potential technical complications attached to the infinite series. In any event, this achieves nothing more than confirm (2).

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[4] Use $\partial_X^j\,(X\partial_X)\cdot = \left(j\partial_X^j + X\partial_X^{j+1}\right)\cdot$.

APPENDIX 3

Under (2), the density of the underlying at expiration, given $X_t = X$, is:

$$q = q(X') = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-iv\ln X'\right) X^{iv} \exp\left[\tau\sum_{j=1}^{\infty}\binom{iv}{j}\sigma_j^j\right]dv\,,$$

and an option with payoff $f(X_T)$ therefore has price:

$$C = \exp\left(-\int_t^T r_u du\right)\int_0^{\infty} f(X')q(X')dX'\,.$$

Now:[5]

$$\begin{aligned}\frac{\partial^k q}{\partial X^k} &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-iv\ln X'\right)\frac{\partial^k}{\partial X^k}\left[X^{iv}\right]\exp\left[\tau\sum_{j=1}^{\infty}\binom{iv}{j}\sigma_j^j\right]dv \\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-iv\ln X'\right)k!\binom{iv}{k}X^{iv-k}\exp\left[\tau\sum_{j=1}^{\infty}\binom{iv}{j}\sigma_j^j\right]dv\end{aligned}$$

and:

$$\begin{aligned}\frac{\partial q}{\partial \sigma_k^k} &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-iv\ln X'\right)X^{iv}\frac{\partial}{\partial \sigma_k^k}\exp\left[\tau\sum_{j=1}^{\infty}\binom{iv}{j}\sigma_j^j\right]dv \\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-iv\ln X'\right)\tau\binom{iv}{k}X^{iv}\exp\left[\tau\sum_{j=1}^{\infty}\binom{iv}{j}\sigma_j^j\right]dv\end{aligned}$$

from which:

$$\frac{\partial q}{\partial \sigma_k^k} = \tfrac{1}{k!}\tau X^k\frac{\partial^k q}{\partial X^k}\,.$$

Thus:

$$\frac{\partial C}{\partial \sigma_k^k} = \tfrac{1}{k!}\tau X^k\frac{\partial^k C}{\partial X^k}\,,$$

and (7) follows.

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[5] Here we assume that the differentiations and integrations can be performed indifferently in either order.

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