An alternative to portfolio selection problem beyond Markowitz's: Log Optimal Growth Portfolio

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An alternative to portfolio selection problem beyond Markowitz's: Log Optimal Growth Portfolio.

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Abstract

This paper constructs an alternative investment strategy to portfolio optimization model in the framework of the Mean–Variance portfolio selection model. To differentiate it from the ubiquitously applied Mean–Variance model, which is constructed on an assumption that returns are normally distributed, our model makes two assumptions: Firstly, that asset prices follow a Geometric Brownian Motion and that secondly asset prices are Log-normally distributed meaning that continuously compounded returns are normally distributed. The traditional Mean–Variance optimization approach has only one objective, which fails to capture the stochastic nature of asset returns and their correlations. This paper presents an alternative approach to the portfolio selection problem. The proposed optimization model which is an optimal portfolio strategy is produced for investors of various risk tolerance, taking into account the stochastic nature of the returns. Detailed analysis based on log–optimal growth optimization and the application of the model are provided and compared to the standard Mean–Variance approach.

1 Introduction to Portfolio Optimization

In this research paper, we construct the growth optimal portfolio (GOP) which is a strategic asset allocation process more suited for those investors with a long term investment view and wish to maximize their expected utility of terminal wealth. Growth optimal portfolio arise from the notion of computing the investment internal rate of return which in essence is bent on constructing those portfolio that have maximal growth. In principle we build a portfolio of risky asset that maximizes the geometric mean.

The paper has primarily been inspired and written in the framework of Modern Portfolio Theory (MPT). Portfolio optimization in the context of portfolio theory is a classical problem in mathematical finance which has spawned a great amount of important academic work. In particular it is one of the well studied classical problems. Central to (MPT) is the Mean–Variance optimization theory(MVO), an important model which was a major breakthrough developed in the 50s and 60s by Markowitz (1952, 1959), his paper opened a new era in the theory of portfolio selection. It plays a important and critical role in determining passive portfolio investment strategies for rational investors and quantifies precisely the relationship between risk and return. His theory set out a way of diversifying investment portfolios so that for any degree of risk, the investor got the best return possible, or alternatively, for any risk, the investor bore the lowest risk. Tobin (1958) built on Markowitz work with the formulation of the "Capital market line" portfolio. Following Markowitz and Tobin, the theory generated a lot of interest and the general equilibrium model capital asset pricing model "CAPM" was independently developed by Sharpe (1964), Lintner (1965), Mossin (1966) .

Markowitz theories pervade the finance industry and are well-known to almost everyone vested in portfolio management. Nonetheless, the Mean–Variance theory suffers from certain well-known drawbacks. The most notable one is that, it is a static optimization problem and is only concerned with

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solving single period investment strategy hence it can not optimally be applied to any rebalancing period. Unlike the GOP, it is a strategy that can be applied in multi–period framework and can also be used for any rebalancing period such as a year, a month, a week or a day. However, in our case we investigate the application in a continuous time period. Portfolio rebalancing is also what is commonly known as "volatility pumping". When volatility is high and a portfolio is rebalanced, an investor would normally yield higher returns. In fact while volatility(risk) is considered bad in static portfolios, it is a positive for growth optimal portfolio.

In order to understand growth portfolios approach, we begin with understanding assets dynamics and how assets prices evolve over time. We characterize that asset behavior follow a stochastic process and can adequately be modeled by stochastic differential equation the "Geometric Brownian motion(GBM)". Bachelier (1900) was the first to discover that stock prices changes are random and unpredictable. The differential equation has been used in the field of asset pricing see Black & Scholes (1973) because of its unique positivity property. This property of positivity is precisely the nature exhibited in a stock price since stock prices can not be negative but returns can, hence the logarithmic or geometric growth.

The Mean–Variance model requires two model input parameters, thus, the first and second moments and the model is built on the assumption that asset returns are normally distributed $N\left[\mu, \sigma^2\right]$ and therefore that the portfolio optimal solution is quantified on the premise that we know collectively the mean vector and variance-covariance matrix. The growth model makes the assumption that continuously compounded asset returns are normally distributed and that price relatives are log–normally distributed a concept which was studied and introduced in financial economics in the late 1950s by Osborne (1959).

Campbell, Lo & MacKinlay (1997) outline two remarkable drawbacks of assets returns being independent and identically normally distributed. Firstly, most financial assets show limited liability meaning that the maximum loss for an investment is equal to the total investment and no more. This implies that the minimum return achievable is -100% [1] However, normal distribution is defined over the range $[-\infty, \infty]$ hence, the assumption of normality clearly violets this lower bound of -100%.

Secondly, if single period returns are assumed to be normal, then multi-period returns cannot be normal since they are simply products of single period returns. In probability theory, the sum of normal single period returns are of course normal, but these sums have no economically meaningful interpretation. When analyzing optimal portfolios over longer time periods or on multiple time periods, the normality hypothesis of returns leads to problems. This is because long-term returns are far from being normally distributed. Undeniably, even over a single year, it can be shown that Log-normal distribution, while still not perfect, is a much better approximation to the distribution of the observed historical returns for common financial assets like stocks and bonds see Norstad (2005).

As discussed above, it is quite intuitive to see that asset prices cannot be negative but can only be infinitely positive. It then follows logically that, the investment gross return at any point in time t, $1 + R_t = (\frac{S_t}{S_{t-1}})$ is bounded below. This fact is the basis for suggesting that asset price returns should be modelled as continuously compounded $r_t = \ln(\frac{S_t}{S_{t-1}})$.[2]. The Log-normal distribution often shows a better fit to historical asset returns as compared to the traditional normal distribution when observed over a longer time horizon. This view is also shared by Palczewski (2005), that asset returns are far from being normal and deviate from the independent identically distribution (i.i.d) assumptions. This formulation is extremely important given both the multitude of areas within economics where portfolio models have found applications and the increasing acceptance of the Log-normality of price relatives in the economic literature.

Any work such as this builds on the advances and ideas of so much knowledge already ascertained in the field and that is this paper is based on the breakthroughs and developments of countless researchers in the fields of finance and statistics and most of all mathematics. Although they are too numerous to name in this space, I acknowledge the foundation that they have built in the field. The remainder of this paper will be elucidated in the next four sections which are described in detail as follows:

[1] It is easy to see that, since asset prices cannot be negative, the price relative $\frac{S_t}{S_{t-1}} > 0$. Then by definition the smallest value that R_t can have, given that the gross return is $1 + R_t = \frac{S_t}{S_{t-1}}$ is -100%.

[2] If gross return is defined as $1 + R_t = (\frac{S_t}{S_{t-1}})$ then taking logarithms on both sides yields $r_t = \ln(1 + R_t) = \ln(\frac{S_t}{S_{t-1}})$. This implies that $S_t = S_{t-1}e^{r_t}$ hence the expression $\ln(\frac{S_t}{S_{t-1}})$ is a representation of a continuously compounded rate of return. Also, $R_t = e^{r_t} - 1, r_t \in [-\infty, \infty]$ unlike R_t, r_t is not bounded

Section 2 and Section 3 : Introduces literature review and the theoretical background to Log–optimal growth portfolios. The chapter begins the discussion on the central limit theorem and asset dynamics paying particular attention to the Log–normality of asset prices and how the geometric mean relates to simple average returns. It further presents the continuous time mathematics on Geometric Brownian Motion, Ito's Lemma and the quadratic form of the log–optimal optimization problems. Finally the section deals with the Lagrange multipliers and how the Markowitz quadratic optimization problem is solved using matrix algebra. We study optimization where an investor has more than one risky asset in his portfolio and when a riskless asset is added to a portfolio of risk assets. Finally, the section finishes with the capital market line and construction of tangency portfolio.

Section 4 and 5: The data requirements and test results are presented followed by the research conclusion. Five stocks randomly selected were used to illustrate how log optimal portfolio are constructed. The data used are the monthly prices downloaded from I-Net bridge, covering the period from the 30 November 1999 up to 31 October 2009. That makes up nine years of monthly data, and in total 120 monthly observations. The assets under consideration were randomly selected from the Johannesburg Securities Exchange namely, BHPBILL, WBHOVCO,NEDBANK,ABIL and SAB and these have been chosen from different economic sectors.

2 Previous work and contribution to existing literature

Over the past 50 years a large number of papers have been dedicated to the Growth Optimal Portfolio (GOP). In financial literature it has been applied in as diverse connections as portfolio theory and gambling, utility theory, information theory, game theory, theoretical and applied asset pricing, insurance, capital structure theory and event studies.

Theoretically, it can be shown that the GOP, which maximizes expected logarithmic utility from terminal wealth, is the portfolio that almost indisputably outperforms all other strictly positive portfolios after a sufficiently long time. Theory has its roots dating back to Benoulli (1954)with the introduction of "expected utility theory" and using the geometric mean as a performance measure of risky portfolios. Kelly (1956) is considered the father of the growth optimal portfolio where in his paper he applied it to the gambling setting. He proposed maximizing the expected exponential growth rate of an investment capital as an investment strategy in a gambling. It is comforting to know that there is a sound theoretical basis for advocating a growth portfolio investment strategy. The Kelly view:that maximizing investment growth of value is a self-evident superior strategy, probably resonates more with the investment sector. However as Christensen (2005) notes that, the first origins of the GOP can be attributed to William (1936). Following, William some remarkable papers on this theory can also be credited to Latane (1959), Hakansson (1971), Elton & Gruber (1974a) and Fernholz & Shay (1982).

The most recent work found on the GOP is by Estrada (2010) with a comparative analysis between the GOP and maximization of the Sharpe ratio. Hunt(2000, 2002, 2005b) attempts to test a simple, practical investment strategy based on portfolios selected to have maximum expected growth rate. Le & Platen (2006) on the other hand applies the theory studying well diversified world stock indices. Elton & Gruber (1974b) argued the practicality of the geometric mean as another alternative portfolio selection criterion in the GOP framework. The development of an algorithm to maximize the geometric mean with assumption log-normally distributed asset prices was entailed by this paper. The methodology was based on the proof that that the maximum geometric mean lies on the efficient frontier in the Mean–Variance space. However, in the conclusion, they pointed out one aspect noting that the construction of the portfolio has major implications for economic theory.

In Weide, Peterson & Maier (1977), it is noted that not much has been documented in terms of finding the optimal solution to the wealth allocation problem of GOP. An investigation into the necessary restrictions under which solutions exist for the case where the retuns distribution is discrete was done. The restrictions were formulated based on the assumption that returns r_i are discrete random variables which can assume a finite number n of combinations of values.

On a different note, in Roll (1973), it is stated that if investors wish to maximize the probability of achieving a given level of wealth within a fixed time, they should choose the GOP that is, the portfolio with highest expected rate of increase in value. The paper investigated the repercussions for observed common stock returns of all investors who choose, under utility maximization, such a portfolio. It is further pointed out that the GOP model has caused a schism in academia. In some tests it performs well while in other cases the results are rather perplexing. Again it has been shown to outperform the Mean–Variance model but the results are so close due to the operational similarity of the two models.

The effects of long-term out-performance of any strictly positive portfolio by the GOP has been studied, for instance, in Laten (1959). Latane is perceived to be the main scholar in financial economics to have introduced the geometric mean as another approach to portfolio selection criteria and since then the theory has recently received some attention in the academic circles.

Others like, Breiman (1961), Markowitz (1976) and Long (1990) also have made remarkable contributions to the theory. In principle, the GOP is the portfolio that cannot be beaten in any reasonable systematic way. Reviews of this portfolio properties can be found in Hakansson & Ziemba (1995) where in their investigation the GOP focused mainly on the relevance of the GOP for investment and gambling setting.

The most far reaching study can be linked to other researchers like Luenberger (1998) and Hakansson (1971) where the use of GOPs on the basis of investors expected utility portfolio maximization was justified. However, the paper compared the Mean–Variance approach to portfolio selection with the capital growth model. A comparison of the two models partly in terms of long-run results may accordingly seem inequitable to the Mean–Variance approach. The reason being that former is a single period model while the later is by definition a multi period model. However, both models are ultimately portfolio models which claim to offer guidance to sensible portfolio choices at any given decision point.

In Platen (2005), the various roles that the GOP plays in finance are discussed and a conclusion is made that the GOP can be interpreted as a fundamental building block in financial market modeling, portfolio optimization and risk measurement and the various ways that the GOP is the best performing portfolio are described. Elton, Gruber, Brown & Goetmann (2003) studied the GOP in the case of a continuous market. No doubt, this property has fascinated many researchers and created a huge and exciting literature on growth optimal investments, a field of study for financial economics.

In Samuelson (1971), evidence for the use of the geometric mean is given and argues, that the law of large numbers or of the central limit theorem when applied to logs can show that a maximum-geometric-mean selection criterion does indeed make it "virtually certain" that, in a "long" sequence one will end with a higher terminal wealth and utility, are given. The prescription to select a portfolio that maximizes an investors expected utility is hardly new. Nor are applications in the area of asset allocation. Particularly relevant in this respect is also very recent work by Cremers & Page (2005), and Tim & Kritzman (2007) in which a full-scale optimization numerical search algorithm is used to find an asset allocation that maximizes expected utility under a variety of assumptions about investor preferences. The GOP, however, is unique in that it has dominating characteristics over all other investment strategies.

The determination of the true population mean and variance-covariance matrix is indeed a significant part to the entire optimization problem. In order to estimate the expected portfolio return, the Mean–Variance methodology uses historical data on the assumption that the sample mean is a true representation of the population mean. The sample mean and sample variance are simply average values of a finite data sample, and are not entirely true representation population parameters. As a consequence of that, it is imperative that we revisit how we parameterize the Mean–Variance model. In addition, due to the amount of volatility in asset returns, *sample means* may not provide accurate estimates of the true population means. As a consequence, utilizing the sample version into a model expecting the population equivalent can produce off-the-wall results. In particular, an investor could be inclined to invest large amounts of money into securities and sectors that performed better than expected in the past.

Supporting this statement, Merton (1980) admits that there has been diminutive academic research on estimating the expected return of assets in comparison to extensive research that has been conducted since the pioneering work of Black & Scholes (1973) in option pricing where they used the second moment as the input parameter in the model. Whether this argument still stands in the 21st century is yet to determined. Perhaps, now the factor models or Bayes-Stein models address this shortcoming. Another paper reinforcing this point is by Black (1993), who also agrees that estimates of expected return based on past data are indeed inaccurate and we need theory to quantify returns.

3 Methodology: Log Optimal Portfolio Selection Criterion

As a preamble to understanding the growth optimal portfolio, we start by introducing the theory on the geometric mean followed by a background to the mathematics of continuous time processes underlying stochastic variables.

3.1 Geometric Mean and Asset Prices as Random Processes

In an investors wealth perspective, the growth of the asset over the entire period $[0, T]$ should be expressed as a geometric mean. The use of geometric mean has far better properties in terms of the interpretation of asset returns as compared to the arithmetic mean. In analyzing wealth over a longer period of time, the geometric mean conveys what the average financial rate of return would have been over the whole duration of the investment period.

Combining the results discussed in the previous section, where $R_t = \frac{S_t}{S_{t-1}} - 1$ represents price return for a single period, between dates $t-1$ and t. Then for n single periods returns, a sequence of asset returns defined as $\{R_t\}_{t=1}^n$ based on the sequence of asset prices $\{S_t\}_{t=0}^n$, the geometric mean return or the so-called "time-weighted rate of return," $r_{(0,n)}$, can be expressed as follows:

$$r_{(0,n)} = \left(\prod_{t=1}^{n} (1 + R_t) \right)^{\frac{1}{n}} - 1. \tag{1}$$

There is, however, a mathematical relationship between the geometric mean and the arithmetic mean of logarithmic returns. The geometric mean can technically be viewed as an average of logarithm values of asset returns. Thus:

$$\left(\prod_{t=1}^{n} (1 + R_t) \right)^{\frac{1}{n}} = e^{\frac{1}{n}\sum_{t=1}^{n} \ln\left(\frac{S_t}{S_{t-1}}\right)}. \tag{2}$$

However, this also implies that, $e^{\frac{1}{n}\sum_{t=1}^{n} \ln\left(\frac{S_t}{S_{t-1}}\right)} = e^{\frac{1}{n}\ln\left(\frac{S_n}{S_{n-1}}\frac{S_{n-1}}{S_{n-2}}\cdots\frac{S_1}{S_0}\right)}$ and this represents the expression $\left(\frac{S_n}{S_0}\right)^{\frac{1}{n}}$. Hence from equation [2] we have $\ln\left(\frac{S_n}{S_0}\right) = \ln\left(\prod_{t=1}^{n}(1 + R_t)\right)$. This equation implies that the geometric mean is just a function of the mean of the log-returns [3], if they are interpreted as statistics, that is:

$$r_{(0,n)} = e^{\mathbb{E}\left[\ln\left(\frac{S_t}{S_{t-1}}\right)\right]} - 1, \tag{3}$$

from the above equation, we can therefore infer that the objective of the Growth Optimal Portfolio (GOP) rests in maximizing the mean ($G_{\max}$) which is the same as maximizing the mean of the log-returns due to the monoticity of the exponential function, thus:

$$G_{\max} = \max\left\{ \mathbb{E}\left[\ln\left(\frac{S_t}{S_{t-1}} \right) \right] \right\}, \tag{4}$$

where $\mathbb{E}$ is the expectation operator.

Now let $\{R_t\}_{t=1}^n$ be a sequence of independent and identically distributed (i.i.d.) random variables with finite mean and variance, then so is the sequence:$\{Z_t | Z_t = \ln(1 + R_t) \text{ and } t \in 1, 2, \cdots, n\}$. Then it is clear that,$\ln\left(\frac{S_n}{S_0}\right) = \sum_{i=t}^{n} Z_t$. Choosing a time horizon T and letting $\triangle t = T/n$, we demonstrate the distribution dynamics of $\ln\left(\frac{S_n}{S_0}\right)$. First we look at the Central Limit Theorem (CLT) which says that:

Theorem 3.1. [Central Limit Theorem] *Let random variables $Y_1, Y_2 \cdots$, be independent and identically distributed with finite mean μ and variance σ^2 generated from any distribution. Then as the sample sizes gets large, the sampling distribution of the mean approaches a normal distribution with mean μ and variance σ^2/N.*

[3] This is stated more formally in the textbook by Luenburger, D. (1998), page 420, were the rationale for the result is as a result of the Strong Law of Large Numbers

Having the CLT in mind and that $\mathbb{E}(Z_t) = \mu\triangle t$ and $\mathbb{V}(Z_t) = \sigma^2\triangle t$, then $\ln\left(\frac{S_n}{S_0}\right)/n$ is approximately normally distributed:

$$\frac{\ln\left(\frac{S_n}{S_0}\right)}{n} \sim N\left(\mu\triangle t, \sigma^2\frac{\triangle t}{n}\right). \tag{5}$$

Hence, $\ln\left(\frac{S_n}{S_0}\right) \sim N\left(\mu T, \sigma^2 T\right)$. Another way of expressing this is in the form $\ln\left(\frac{S_n}{S_0}\right) = \mu T + \sigma\sqrt{T}\epsilon$. where ϵ has a standard normal distribution. In particular, it is not an unreasonable assumption to assume that the Z_t above are normally distributed i.i.d. and then we have that (by definition),

$$\ln\left(\frac{S_t}{S_{t-1}}\right) = Z_t = \mu\triangle t + \sigma\sqrt{\triangle t}\epsilon_t, \tag{6}$$

which is by definition just the discrete random walk, ϵ_t being just a sequence of standard normal variables. Note that by the following theorem on random variable transformation;

Theorem 3.2. [Transformation of random variables] *Let X be a continuous random variables having probability density function f_X. Suppose $g(x)$ is strictly monotone, differentiable function of x. The random variable Y defined by $Y = g(X)$ has probability density function given by*

$$f_Y(y) = \begin{cases} f_X\left[g^{-1}(y)\right]\frac{d}{dy}g^{-1}(y) & \text{if } y = g(x) \text{ for some } x, \\ 0 & \text{if } y \neq g(x) \text{ for all } x, \end{cases}$$

we have that, $\frac{S_n}{S_0} \sim$ Log-normal $\left(\mu T, \sigma^2 T\right)$ which is an important result showing that price relatives are log-normally distributed, given the assumptions above.

3.2 Geometric Wiener Process

We have so far given a thorough introduction to asset dynamics and specifically the stochastic process of how asset prices change over time in discrete terms. Having said that, this section provides a more rigorous approach to asset behavior in continuous time mathematics. We now formalize the stochastic growth model for the optimal portfolio selection optimization problem. In continuous time mathematics, the log normal random walk model in equation for non dividend paying asset is usually formulated in terms of the following stochastic differential equation.

$$d\ln S(t) = \mu dt + \sigma\sqrt{dt}\epsilon_t. \tag{7}$$

Where the parameters μ and σ represent percentage drift rate or *instantaneous rate of return* and the percentage rate of volatility respectively. The two parameters could be constant or could depend on the stock price and/or time. The term $\sqrt{dt}\epsilon_t$ is the standard wiener process where $\epsilon_t \sim N(0,1)$, the process represents something like the infinitesimal stochastic change in $\ln S(t)$ over an infinitesimal instant of time. It then follows that, $\mu dt + \sigma dw(t) \sim N\left(\mu dt, \sigma^2 dt\right)$, hence $\frac{dS(t)}{S(t)} \sim N\left(\mu dt, \sigma^2 dt\right)$. Equation [7] cannot be interpreted as an ordinary differential equation, since the Brownian paths $\sqrt{dt}\epsilon_t$ are not differentiable with respect to time. It was precisely for the purpose of dealing with differential equations incorporating stochastic differentials that Itō developed what is now called the Itō calculus. The left hand side represents the percentage change in asset value and it equivalent to stating it as $d\ln S(t) = \frac{dS(t)}{S(t)}$[4], substituting the expression into equation [7], we obtain the following expression,

$$\begin{aligned} d\ln S(t) = \frac{dS(t)}{S(t)} &= \mu dt + \sigma\sqrt{dt}\epsilon_t. && (8) \\ \Rightarrow dS(t) &= \mu S(t)dt + \sigma S(t)\sqrt{dt}\epsilon_t. && (9) \end{aligned}$$

[4]If $y = \ln f(x)$, then the derivative of y is given by $d\ln S(t) = \frac{dS(t)}{S(t)}$

which is the Itōs diffusion process, now let $\sqrt{dt}\epsilon_t = dw$, Itōs lemma states that for the process above and a function $F(S,t)$ which is twice continuously differentiable in both S and t then, the change in $F(S(t),t)$, $dF(S(t),t)$ is also an Itō process, subsequently,

$$dF(S,t) = \left(\frac{\partial F}{\partial S}\mu S + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial F}{\partial S}\sigma S dw. \tag{10}$$

We know that $S(t)$ follows the process defined by equation [9]. Then, the process followed by $d\ln S(t)$ can easily be solved by defining a function $F(S(t)) = \ln(S(t))$ and since this is a function of only S and not t, then the Itō lemma takes the form $dF = \left(\mu SF' + \frac{1}{2}\sigma^2 S^2 F''\right)dt + \sigma SF'dw$. Applying the Itō lemma, we derive the governing process outlined by $F(S(t)) = \ln S(t)$. Then, the process followed by dF is

$$dF = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dw_t \quad \text{where} \quad dF \quad \text{is representing} \quad d\ln S \quad \text{and} \quad dw_t is \quad \sqrt{dt}\epsilon_t \tag{11}$$

Equation [11] concludes that $d\ln S$ follows the Wiener process. [5]

Integrating and set the initial value condition $F(0) = \ln S(0)$ yields;

$$\begin{aligned} F(t) &= F(0) + \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right)ds + \int_0^t \sigma dw \\ &= F(0) + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma w(t). \end{aligned}$$

We know that $F(s) = \ln S$ then,

$$\begin{aligned} \ln S(t) - \ln S(0) &= \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma w(t) \\ \ln\left(\frac{S(t)}{S(0)}\right) &= \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma w(t) \\ S(t) &= S(0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma w(t)}, \end{aligned} \tag{12}$$

discretized, this can be written as:

$$S_t = S_{t-1}e^{\left(\mu - \frac{1}{2}\sigma^2\right)\triangle t + \sigma\epsilon\sqrt{\triangle t}}. \tag{13}$$

In essence we are stating that stock prices follow a Geometric Wiener process and that over any time increment, $\triangle t$, the distribution of logarithmic returns is normally distributed with mean $\alpha\triangle t$, where $\alpha = \left(\mu - \frac{1}{2}\sigma^2\right)$, proportional to the time increment and the volatility, $\sigma\sqrt{\triangle t}$ is proportional to the square root of time increment.

The probability distribution function of $X = \ln\left(\frac{S(t)}{S(0)}\right) \sim \mathcal{N}\left(\left(\mu - \frac{1}{2}\sigma^2\right)t, \sigma^2 t\right)$ is;

$$f(X) = \frac{1}{\sqrt{2\pi\sigma^2 t}}e^{-\frac{1}{2}\left[\frac{X - \left(\mu - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right]^2} \tag{14}$$

and through Jacobian variable transformation $Y = e^X = e^{\ln\frac{S_t}{S_0}} = e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma t}$ is lognormally distributed with parameters $\left(\mu - \frac{1}{2}\sigma^2\right)t$ and $\sigma^2 t$. $\mathbb{E}\left(\ln\left(\frac{S_t}{S_0}\right)\right) = \left(\mu - \frac{1}{2}\sigma^2\right)t$ and the variance is, $\mathbb{V}\left(\ln\left(\frac{S_t}{S_0}\right)\right) = \sigma^2 t$ This statement stated differently is saying that the the change in asset prices when modelled using the Geometric Brownian motion are log-normally distributed. Applying theorem 3.2, the probability density function of Y is as follows;

[5] notice that the left hand side is a expression of the form $a + bx$ and if $x \sim N(0,1)$ then $y = a + bx \sim \mathcal{N}\left(a, b^2\right)$. Hence $d\ln S \sim \mathcal{N}\left(\left(\mu - \frac{1}{2}\sigma^2\right)dt, \sigma\epsilon\sqrt{dt}\right)$

$$f(Y) = \frac{1}{y\sqrt{2\pi\sigma^2 t}} e^{-\frac{1}{2}\left[\frac{\ln y - \left(\mu - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right]^2} \tag{15}$$

Putting all the pieces of information discussed so far together, we conclude that the Geometric Brownian motion is lognomally distributed thus,

$$\frac{S_t}{S_0} \sim LN\left(\left(\mu - \frac{1}{2}\right)t, \sigma^2 t\right) \text{ and}$$

The expected value $\mathbb{E}\left(\frac{S_t}{S_0}\right) = \mathbb{E}\left(e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \frac{1}{2}\sigma^2 t}\right) = e^{\mu t}$

If we put all the pieces of information discussed so far, then, by employing the GBM to estimate the model input parameters, we significantly change the whole Markowitz's setting and in particular we construct a portfolio which maximizes the expected growth rate hence the name Log-Optimal growth portfolio. The methodology of how this is achieved is the topic of our discussion the last chapter.

3.3 Log Optimal Portfolio Selection Criterion

We also focus on the key analytic tools employed in portfolio optimization methods where we introduce the elements of stochastic calculus as an important tool in modeling of financial processes, see Wilmott (2001) for a detailed introduction on the subject of SDE[6]. We further review the standard analytic approach based on mean-variance assumptions and then describe a more general procedure that assumes investors seek to maximize expected utility. We will show, later, that mean-variance procedures are special cases of the more general expected utility formulations. We term the more general approaches Expected Utility Optimization and the traditional methods Mean-Variance optimization.

The problem dealt in this paper realistically deals also with the problem of rebalancing the portfolio in any time horizon. In small time horizons, the portfolio rebalancing a log optimal growth portfolio is constructed as a as continuous-time stochastic process.

Suppose an investor has a choice of investing in a finite number of correlated risky assets represented by an $n \times 1$ vector $\mathcal{S} = [S_1, S_2, \cdots S_n]$ where, without loss of generality, it can be assumed that each S_i represents the asset and its price. Then, for each asset $\{S_i\}_{i=1}^n$, the price is assumed to be governed by a Geometric Brownian Motion:

$$\frac{dS_i}{S_i} = \mu_i dt + \sigma_i dw_t, \text{where } i \in \{1, 2, \ldots, n\}. \tag{16}$$

By our earlier assumption, the assets prices are correlated through the Wiener process $dw_t = \epsilon_i\sqrt{dt}$ with a probability density function $f_{dw_t}(x) = \frac{1}{\sqrt{2\pi dt}} e^{-\frac{x^2}{2dt}}$ and the covariance, $\mathrm{Cov}[dw_i, dw_j] = \sigma_{ij} dt$ where σ_{ij} can be understood as the correlation between assets i and j. In the Markowitz framework the Wiener vector, $\mathbf{dw} = [dw_1, dw_2, \ldots, dw_n]$ has a multivariate Gaussian distribution hence,

$$f_{\mathbf{dw}}(x_1, \ldots, x_n) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} e^{\left[-\frac{1}{2}(\mathbf{x}-\mu)' \Sigma^{-1} (\mathbf{x}-\mu)\right]}. \tag{17}$$

The vector μ is just the zero vector since $\epsilon_i\sqrt{dt} \sim N(0, dt)$, $|\Sigma|$ is the determinant of the covariance matrix Σ, and the $n \times n$ covariance matrix is defined having diagonal elements $\sigma_i^2 dt$, for $i = j$ and $\sigma_{ij} dt$ for $i \neq j$. The symbol e is an exponential function representing the number approximately 2.7182818

In the multivariate framework we have the log price change for time $(t - t_0)$ from the univariate solution in equation [12] as normally distributed,

$$\ln S_{i,t} \sim N\left[\ln S_{i,t_0} + \left(\mu_i - \frac{\sigma_i^2}{2}\right)(t - t_0), \sigma_i^2 (t - t_0)\right]. \tag{18}$$

[6]Stands for stochastic differential Equation

The asset prices at some future deterministic date (t), given that we know the asset price at time (t_0) is lognormally distributed and the standard deviation is proportional to the square root of the time interval of how far ahead we are looking. From Equation [18] we have:

$$\mathbb{E}\left[\ln\left(\frac{S_{(i,t)}}{S_{(i,t_0)}}\right)\right] = \left(\mu_i - \frac{1}{2}\sigma_i^2\right)(t-t_0) \quad \text{And the variance is } \mathbb{V}\left[\ln\left(\frac{S_{(i,t)}}{S_{(i,t_0)}}\right)\right] = \sigma_i^2(t-t_0).$$

Let the sequence of portfolio weights be denoted $\{\omega_i\}_{i=1}^n$, such that $\sum_{i=1}^n \omega_i = 1$ represent the budget constraint. The overall portfolio value P can be formulated in terms of the aforementioned stochastic processes as follows:

$$\begin{aligned}\frac{dP}{P} &= \omega_1\frac{dS_1}{S_1} + \omega_2\frac{dS_2}{S_2} + \cdots + \omega_n\frac{dS_n}{S_n} = \sum_{i=1}^n \omega_i\frac{dS_i}{S_i} \\ &= \sum_{i=1}^n \omega_i\left(\mu_i dt + \sigma_i dw_i\right). \end{aligned} \tag{19}$$

One thing to note is that the portfolio is a weighted sum of the assets.

In continuous time, the percentage change in the value of our portfolio, is normally distributed with mean:

$$\mathbb{E}\left[\ln\left(\frac{P_t}{P_{t_0}}\right)\right] = \sum_{i=1}^n \omega_i\mu_i(t-t_0) - \frac{1}{2}\sum_{i,j=1}^n \omega_i\sigma_{i,j}\omega_j(t-t_0); \tag{20}$$

and the portfolio variance is:

$$\mathbb{V}\left[\ln\left(\frac{P_t}{P_{t_0}}\right)\right] = \sum_{i,j=1}^n \omega_i\sigma_{i,j}\omega_j(t-t_0). \tag{21}$$

3.4 Deriving the Weights that Maximize Portfolio Growth Rates with Short Selling Allowed

For n assets with linearly independent growth returns, a portfolio of risky assets modelled in continuous time has a growth rate $\nu = \frac{1}{(t-t_0)}\mathbb{E}\left[\ln\left(\frac{P_t}{P_{t_0}}\right)\right] = \omega'\mu - \frac{1}{2}\omega'\Sigma\omega$. Denote $\mu = \nu + \frac{1}{2}\omega'\Sigma\omega$ as a column vector representing the expected growth rates, let Σ denote the variance-covariance matrix of the growth rates and ω represent a column vector of portfolio weights which are chosen to maximize the growth rate G. Let $\theta = 1$ be the risk aversion parameter this equal to a unity since it does not affect our finale solution. In the Markowitz framework the portfolio optimization problem entails minimizing the portfolio variance for some specified portfolio mean. However the duplex to the problem, is the maximization of the portfolio growth for some specified portfolio mean- the GOP problem. Symbolically the the two problems are formulated as follows;

$$\underset{\omega}{\text{argmin}}\left\{\frac{\theta}{2}\omega'\Sigma\omega \quad |\omega'\mu = \mathbb{E}\left(R_p\right) \quad \omega' e = 1\right\} \tag{22}$$

$$\underset{\omega}{\text{argmax}}\left\{\omega'\mu - \frac{\theta}{2}\omega'\Sigma\omega \quad |\omega'\mu = \mathbb{E}\left(R_p\right) \quad \omega' e = 1\right\}. \tag{23}$$

where R_p in the portfolio return defined in the following way: $\mathbb{E}(R_p) = \sum_{i=1}^n \omega_i\mathbb{E}(R_i)$ and R_i is the i-th asset return, and e is a column vector of ones. Notice that the risk aversion parameter θ

3.4.1 Minimizing the Portfolio variance ($\sigma_p^2 = \omega'\Sigma\omega$) for a Specified Mean $\mathbb{E}(R_p)$

The solution to the problem in [22] in the Mean–Variance framework can be easily be derived as follows:

Definition 3.1. *A portfolio, P, is the minimum variance portfolio of all portfolios with mean return μ_p if its portfolio vector of weights ω is the solution to the following unconstrained optimization problem,*

$$\underset{\omega}{\operatorname{argmin}} \left\{ \frac{}{2}\omega'\Sigma\omega \quad |\omega'\mu = \mathbb{E}\left(R_p\right) \quad \omega'e = 1 \right\}. \tag{24}$$

The Lagrangian function with λ and γ as multipliers is constructed for the optimization problem above, hence:

$$\mathcal{L} = \frac{\omega'\Sigma\omega}{2} - \lambda\left[\omega'\mu - \mu_p\right] - \gamma\left[\omega'e - 1\right]. \tag{25}$$

We optimize the Langrangian by differentiating with respect to ω, λ and γ and set the derivative equal to zero to yield these three sets of matrix equations:

$$\frac{dL}{dw} = \Sigma w - \lambda\mu - \gamma e = 0, \tag{26}$$
$$\frac{dL}{d\lambda} = w'\mu - \mu_p = 0, \tag{27}$$
$$\frac{dL}{d\gamma} = w'e - 1 = 0. \tag{28}$$

Solving for ω, notice that we can rewrite the first equation as follows,

$$\omega = \Sigma^{-1}\left[\lambda\mu + \gamma e\right]. \tag{29}$$

Recall from the equations in 26 that $\frac{dL}{d\gamma} = \omega'e - 1 = 0$, then by substitution, we have an equation of the form,

$$\lambda\mu'\Sigma^{-1}e + \gamma e'\Sigma^{-1}e = 1. \tag{30}$$

Similarly, we know that $\mu_p = \omega'\mu$, hence

$$\mu_p = \lambda\mu'\Sigma^{-1}\mu + \gamma e'\Sigma^{-1}\mu. \tag{31}$$

In order to get the closed form solution for λ and γ we solve a system of two simultaneous equation(30 and 31),

$$\begin{bmatrix}\mu_p \\ 1\end{bmatrix} = \begin{bmatrix}\lambda\mu'\Sigma^{-1}\mu + \gamma e'\Sigma^{-1}\mu \\ \lambda\mu'\Sigma^{-1}e + \gamma e'\Sigma^{-1}e\end{bmatrix}. \tag{32}$$

The system above can be represented in a more elaborate matrix form as follows:

$$\begin{bmatrix}\mu_p \\ 1\end{bmatrix} = \begin{bmatrix}\mu'\Sigma^{-1}\mu & e'\Sigma^{-1}\mu \\ \mu'\Sigma^{-1}e & e'\Sigma^{-1}e\end{bmatrix}\begin{bmatrix}\lambda \\ \gamma\end{bmatrix}. \tag{33}$$

Notice in Equation 33 that $e'\Sigma^{-1}\mu$=$\mu'\Sigma^{-1}e$, then the expression can be simplified to:

$$\begin{bmatrix}\mu_p \\ 1\end{bmatrix} = \begin{bmatrix}A & B \\ B & C\end{bmatrix}\begin{bmatrix}\lambda \\ \gamma\end{bmatrix}, \tag{34}$$

where we define:

$$A = \mu'\Sigma^{-1}\mu, B = \mu'\Sigma^{-1}e \text{ and } C = e'\Sigma^{-1}e.$$

A closer inspection of all the entries in the matrix above reveals that they are scalars. Also notice that the equation is a linear system of the form $\Lambda\mathbf{x} = \mathbf{b}$, so solving for this system would be as $\mathbf{x} = \Lambda^{-1}\mathbf{b}$:

From the inverse of the matrix, thus solving for the two unknown multipliers γ and λ we get,

$$\begin{bmatrix}\lambda \\ \gamma\end{bmatrix} = \frac{1}{AC - BB}\begin{bmatrix}C & -B \\ -B & A\end{bmatrix}\begin{bmatrix}\mu_p \\ 1\end{bmatrix}, \tag{35}$$

Multiplying out the expression we get:

$$\lambda = \frac{C\mu_p - B}{AC - BB} \quad \text{and} \quad \gamma = \frac{-B\mu_p + A}{AC - BB}.$$

Finally putting all the pieces together we write down the solution for ω which will be a $n \times 1$ vector of portfolio weights as follows:

$$\begin{aligned} \omega^* &= \Sigma^{-1}\left[\lambda\mu + \gamma e\right] \\ &= \frac{1}{D}\Sigma^{-1}\left[(Ae - B\mu) + \mu_p\left(C\mu - Be\right)\right]. \end{aligned} \tag{36}$$

The equation of the minimum variance set of the portfolio can then be shown to be

$$\begin{aligned} \sigma_p^2 &= \omega'\Sigma\omega = \lambda\mu_p + \gamma, \\ &= \frac{C\mu_p^2 - 2B\mu_p + A}{AC - B^2}. \end{aligned} \tag{37}$$

The portfolio variance is a quadratic function of the mean portfolio return. In the (μ_p, σ_p) space this plots the parabola while in the (σ_p, μ_p) space this plots the inverted parabola. The global minimum variance portfolio (gmv) has the mean return which is found by differentiating the equation for the variance and setting the derivative equal to zero $\frac{d\left(\sigma_p^2\right)}{d\mu_p} = \frac{2C\mu_p - 2B}{AC - BB} = 0$. This yields $\mu_p^{\text{gmv}} = \frac{B}{C}$ which simplifies to the following,

$$\text{Mean of Global Minimum Variance} = \frac{\mu'\Sigma^{-1}e}{e'\Sigma^{-1}e}, \tag{38}$$

It is easy to see that the variance of the GMV portfolio $\sigma_{\text{gmv}}^2 = \frac{C\frac{B^2}{C^2} - 2B\frac{B}{C} + A}{AC - BB}$ and further algebraic manipulation this can be expressed as follows:

$$\text{Minimum Variance} = \frac{1}{e'\Sigma^{-1}e}. \tag{39}$$

The portfolio weights for the global minimum variance portfolio (gmv) can also easily be computed from Equation [36] by substituting μ_p for $\frac{B}{C}$. This gives $\omega^* = \Sigma^{-1}\left[\frac{\mu\left(C\frac{B}{C} - B\right) + e\left(A - B\frac{B}{C}\right)}{AC - BB}\right]$ and the weights are therefore equal to the following,

$$\text{Portfolio Weights of the GMV} = \frac{\Sigma^{-1}e}{e'\Sigma^{-1}e}.$$

3.4.2 Maximizing the Log-Optimal Growth portfolio for a Specified Mean $\mathbb{E}\left(R_p\right)$

In the GOP framework, we demonstrate the similarity with the weights of both the GOP and the Mean–Variance optimal weights. Solving for the optimal solution to this problem is similar to the problem above and a complete derivation is outlined in appendix B. The Lagrangian is therefore,

$$\mathcal{L} = \omega'\mu - \frac{\omega'\Sigma\omega}{2} - \lambda_1\left[\omega'\mu - \mu_p\right] - \gamma_1\left[\omega'e - 1\right] \tag{40}$$

It is easy to show that

$$\omega = \Sigma^{-1}\left[\mu - \lambda_1\mu - \gamma_1 e\right].$$

Hence, the system above can be represented in a more elaborate matrix form as follows

$$\begin{bmatrix} \mu'\Sigma^{-1}\mu - \mu_p \\ e'\Sigma^{-1}\mu - 1 \end{bmatrix} = \begin{bmatrix} \mu'\Sigma^{-1}\mu & e'\Sigma^{-1}\mu \\ \mu'\Sigma^{-1}e & e'\Sigma^{-1}e \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \gamma_1 \end{bmatrix}, \tag{41}$$

therefore the optimal weights are given as $\lambda_1 = \frac{CX-BY}{D}and\gamma_1 = \frac{AY-BX}{D}$, where $D = AC - B^2$ and $X = \mu'\Sigma^{-1}\mu - \mu_p$ and $Y = e'\Sigma^{-1}\mu - 1$. Hence, the weights are given as.

$$\omega* = \Sigma^{-1}\mu - \frac{CX - BY}{D}\Sigma^{-1}\mu - \frac{AY - BX}{D}\Sigma^{-1}e. \tag{42}$$

The results display a remarkable similarity between the log optimal and mean variance optimization problem as is evident from equation [42] which is identical to equation [36]. This can be shown as follows; To show that the two equations are identical we need break the two expression into two parts, firstly, equation [36] can be written as follows,

$$\omega_1^* = \lambda\Sigma^{-1}\mu + \gamma\Sigma^{-1}e \quad \text{and} \quad \omega_2^* = \Sigma^{-1}\mu - \lambda_1\Sigma^{-1}\mu + \gamma_1\Sigma^{-1}e,$$

With simple algebraic manipulation, we can show that: $\gamma = \gamma_1$. Comparing the two weight vectors above, we have that,

$$\begin{aligned} \gamma\Sigma^{-1}e &= \gamma_1\Sigma^{-1}e \\ (\gamma - \gamma_1)\,\Sigma^{-1}e &= 0. \end{aligned}$$

The expression $\Sigma^{-1}e$ is non zero vector, hence $\gamma - \gamma_1 = 0$ or $\gamma = \gamma_1$. The second part we need to show that $\lambda = 1 - \lambda_1$, we have that,

$$\begin{aligned} \Sigma^{-1}\mu &= \Sigma^{-1}\mu - \lambda_1\Sigma^{-1}\mu \\ (\lambda + \lambda_1)\Sigma^{-1}\mu &= \Sigma^{-1}\mu \\ (\lambda + \lambda_1)\mu &= \mu. \end{aligned}$$

This implies that $(\lambda + \lambda_1) = 1$, that is $(\lambda = 1 - \lambda_1)$ hence $\omega_1^* = \omega_2^*$.

3.4.3 Maximizing the portfolio growth regardless of the mean

Next we show that maximizing the portfolio growth rate without indicating the mean implies the optimal mean growth for the portfolio, for an investor with risk aversion parameter θ. In matrix algebra notation, the problem of maximizing the portfolio growth rate in continuous time is formulated as follows:

$$\underset{\omega}{\operatorname{argmax}} \left\{ \omega'\mu - \frac{\theta}{2}\omega'\Sigma\omega \quad |\omega'e = 1 \right\}. \tag{43}$$

Note that μ and Σ are time dependent and the subscripts denoting that are dropped for ease of analysis.

We now use the method of Lagrangian multipliers to solve the problem analytically. However we arrive at the same solution using block matrix decomposition. Thus the Lagrangian expression is given by:

$$\mathcal{L}(\omega, \lambda) = \omega'\mu - \frac{\theta}{2}\omega'\Sigma\omega - \lambda\left[\omega e' - 1\right], \tag{44}$$

To find the critical points of the Lagrangian, we determine the first order equations:

$$\frac{d\mathcal{L}}{d\omega} = \mu - \theta\Sigma\omega - \lambda e = 0 \tag{45}$$

$$\frac{d\mathcal{L}}{d\lambda} = 1 - \omega' e = 0, \tag{46}$$

Notice that $\omega'e = e'\omega$ and that $\Sigma^{-1}\Sigma$ exists which is just the identity matrix, then we have,

$$1 = e'\Sigma^{-1}\Sigma\omega. \tag{47}$$

Right multiplying $e'\Sigma^{-1}$ into equation [45] and solve for λ, we have,

$$\lambda = \frac{e'\Sigma^{-1}\mu - \theta}{e'\Sigma^{-1}e}. \tag{48}$$

The optimal solution- the portfolio vector of weights (ω) that maximize the portfolio growth rate is:

$$\begin{aligned}\omega^* &= \frac{1}{\theta}\Sigma^{-1}\left[\mu - \lambda e\right] \\ &= \frac{1}{\theta}\Sigma^{-1}\left[\mu - \left(\frac{e'\Sigma^{-1}\mu - \theta}{e'\Sigma^{-1}e}\right)e\right] \\ &= \left[\Sigma^{-1}\mu - \left(\frac{e'\Sigma^{-1}\mu - 1}{e'\Sigma^{-1}e}\right)\Sigma^{-1}e\right] \quad \text{for} \quad \theta = 1. \end{aligned} \tag{49}$$

The portfolio mean $\mathbb{E}\left(R_p\right)$ and variance $\mathbb{V}\left(R_p\right)$ are calculated as follows:

$$\mu'\omega = \theta^{-1}\mu'\Sigma^{-1}\mu - \theta^{-1}\left[\frac{e'\Sigma^{-1}\mu}{e'\Sigma^{-1}e} - \frac{\theta}{e'\Sigma^{-1}e}\right]\mu'\Sigma^{-1}e, \tag{50}$$

now let $\mathcal{A} = e'\Sigma^{-1}\mu$, $\mathcal{B} = \mu'\Sigma^{-1}\mu$, $\mathcal{C} = e'\Sigma^{-1}e$ and $\mathcal{D} = \mathcal{CB} - \mathcal{A}^2$. Then the expression simplifies to $\frac{\mathcal{D}}{\theta\mathcal{C}} + \frac{\mathcal{A}}{\mathcal{C}}$ and the variance is simply $\mathbb{V}\left(R_p\right) = \omega'\Sigma\omega$ which is $\frac{\mathcal{D}}{\theta^2\mathcal{C}} + \frac{1}{\mathcal{C}}$.

It is easy to see that $\lim_{\theta\to\infty}\mathbb{E}\left(R_p\right) \to \frac{e'\Sigma^{-1}\mu}{e'\Sigma^{-1}e} = \frac{\mathcal{A}}{\mathcal{C}}$ and $\lim_{\theta\to\infty}\mathbb{V}\left(R_p\right) \to \frac{1}{e'\Sigma^{-1}e} = \frac{1}{\mathcal{C}}$, so the Mean–Variance portfolio mean and variance are asymptotes to the GOP ones hence the more risk averse the investor is the more he/she is likely to miss out on the benefits of portfolio growth, namely that the more inefficient the strategy is. This we know since the Markowitz strategy is less efficient than the GOP/Log-optimal one. On the contrary, when $\lim_{\theta\to 0}$, both $\mathbb{E}\left(R_p\right)$ and $\mathbb{V}\left(R_p\right) \to +\infty$ showing that risk-seeking investors benefit in the mean.

3.4.4 Growth Portfolio with a Riskless Asset

In the previous section, we discussed the construction of an efficient portfolio with risky assets. However the same reasoning can be expanded with the inclusion of a riskless asset. The process followed by a riskless asset is therefore $\frac{dS}{S} = \mu_f dt$ where μ_f representing the mean return of the riskless asset and that the riskless asset is deterministic, since on assumption there is no risk, in other words the Wiener term is zero. Now let P represent the vector of n risky assets, μ_f a unit vector of a riskless asset and e a column vector of ones of dimension $n+1$. Then the budget constraint is therefore, noting the appropriate notation for the weights:

$$\omega'e + \omega_f = 1 \quad \Leftrightarrow \omega_f = 1 - \omega'e \quad \text{so that this can be written as} \quad \omega'e + (1 - \omega'e) = 1.$$

The portfolio expected return is therefore stated as:

$$\mathbb{E}\left[R_p\right] = \omega'\mu + (1 - \omega'e)\,\mu_f,$$

where μ_f represents a $(n+1)\times 1$ vector which is zero everywhere except for the position $(n+1, 1)$ and μ the vector of means with the same dimension as the former but zero where the former is non-zero. The log optimal maximization problem is formulated as follows;

$$\underset{\omega}{\operatorname{argmax}}\left\{(1 - \omega')\,\mu_f + \omega'\mu - \frac{1}{2}\omega'\Sigma\omega\right\}. \tag{51}$$

Forming the Lagrange is: $\mathcal{L} = (1 - \omega')\,\mu_f + \omega'\mu - \frac{1}{2}\omega'\Sigma\omega$, then the maximum (it can be shown that $\frac{d^2\mathcal{L}}{d\omega^2} < 0$) of this function occurs at $\frac{d\mathcal{L}}{dw} = -\mu_f + \mu - \Sigma\omega = 0$. which happens when,

$$\omega^* = \Sigma^{-1}\left(\mu - \mu_f\right). \tag{52}$$

However a more general formulation is to restate the problem as follows, where the mean is specified:

$$\underset{\omega}{\operatorname{argmin}}\left\{\frac{1}{2}\omega'\Sigma\omega \quad | \mathbb{E}\left[R_p\right] = \omega'\mu + (1 - \omega'e)\,\mu_f\right\}. \tag{53}$$

The Lagrangian function associated to the problem above is:

$$\mathcal{L}(\omega, \lambda) = \frac{1}{2}\omega'\Sigma\omega + \lambda\left(\mathbb{E}(R_p) - \omega'\mu - (1-\omega'e)\mu_f\right). \tag{54}$$

The first order conditions for ω to be a solution are therefore:

$$\begin{cases} \frac{d\mathcal{L}(\omega,\lambda)}{d\omega} = \Sigma\omega - \lambda\left(\mu - e\mu_f\right) = 0 \\ \frac{d\mathcal{L}(\omega,\lambda)}{d\lambda} = \mathbb{E}(R_p) - \omega'\mu - \left(1-\omega'e\right)\mu_f = 0. \end{cases}$$

Solving for ω and noting that $\omega'\mu = \mu'\omega$, we have,

$$\begin{aligned} \mathbb{E}(R_p) &= \omega'\mu + \left(1-\omega'e\right)\mu_f \\ &= \omega'[\mu - e\mu_f] + \mu_f \end{aligned}$$

hence,

$$\mathbb{E}(R_p) - \mu_f = \omega'[\mu - e\mu_f].$$

Also, we have that,

$$\begin{aligned} \omega &= \Sigma^{-1}\lambda\left[\mu - e\mu_f\right] \\ \omega' &= \lambda\left[\mu - e\mu_f\right]'\Sigma^{-1} \\ \omega'\left[\mu - e\mu_f\right] &= \lambda\left[\mu - e\mu_f\right]'\Sigma^{-1}\left[\mu - e\mu_f\right]. \end{aligned}$$

Combining the two equations equal to $\omega'\left[\mu - e\mu_f\right]$, we can solve for λ

$$\begin{aligned} \mathbb{E}(R_p) - \mu_f &= \lambda\left[\mu - e\mu_f\right]'\Sigma^{-1}\left[\mu - e\mu_f\right] \\ \lambda &= \frac{\mathbb{E}(R_p) - \mu_f}{\left[\mu - e\mu_f\right]\Sigma^{-1}\left[\mu - e\mu_f\right]}, \end{aligned}$$

Substituting λ in equation we have the general solution to the portfolio weights as follows:

$$\omega = \Sigma^{-1}\left[\mu - e\mu_f\right]\frac{\mathbb{E}(R_p) - \mu_f}{\left[\mu - e\mu_f\right]'\Sigma^{-1}\left[\mu - e\mu_f\right]}. \tag{55}$$

It is easy to show that the denominator,

$$\begin{aligned} \left[\mu - e\mu_f\right]'\Sigma^{-1}\left[\mu - e\mu_f\right] &= \mu'\Sigma^{-1}\mu - \mu'\Sigma^{-1}e\mu_f - \mu_f'e'\Sigma^{-1}\mu + \mu_f'e'\Sigma^{-1}e\mu_f \\ &= \mu'\Sigma^{-1}\mu - 2\mu'\Sigma^{-1}e\mu_f + \mu_f'e'\Sigma^{-1}e\mu_f \\ &= C\mu_f^2 - 2B\mu_f + A \\ &= H. \end{aligned}$$

Hence, the portfolio variance $\mathbb{V}(R_p)$ is therefore,

$$\begin{aligned} \mathbb{V}(R_p) &= \omega'\Sigma\omega \\ &= \left(\frac{\Sigma^{-1}\left(\mu - e\mu_f\right)\left(\mathbb{E}\left(R_p\right) - \mu_f\right)}{H}\right)'\Sigma\left(\frac{\Sigma^{-1}\left(\mu - e\mu_f\right)\left(\mathbb{E}\left(R_p\right) - \mu_f\right)}{H}\right). \end{aligned}$$

Notice that $\left(\mathbb{E}\left(R_p\right) - \mu_f\right)$ and H are scalars hence, a simple algebraic manipulation we have,

$$\mathbb{V}(R_p) = \frac{\left(\mathbb{E}\left(R_p\right) - \mu_f\right)^2}{H^2}\left(\Sigma^{-1}\left(\mu - e\mu_f\right)\right)'\left(\mu - e\mu_f\right).$$

Which simplifies to,

$$= \frac{\left(\mathbb{E}\left(R_p\right) - \mu_f\right)^2}{H}.$$

.

3.4.5 Conclusion

However the inclusion of the riskfree asset into a portfolio of risky assets maps out a straight line in a (μ_p, σ_p) two dimensional space. However, for a desired portfolio return $\mathbb{E}(R_p)$ the growth optimal portfolio is constructed by computing,

$$G_{opt} = \omega'\mu - \omega'\Sigma\omega,$$

where the portfolio weight ω is defined in equation [55].

$$\omega = \Sigma^{-1}\left[\mu - e\mu_f\right] \frac{\mathbb{E}(R_p) - \mu_f}{\left[\mu - e\mu_f\right]' \Sigma^{-1}\left[\mu - e\mu_f\right]}.$$

4 Data and Results

In this section I present a practical example demonstrating the application of log optimal maximization algorithm. Five stocks from different economic sectors were randomly selected to illustrate how log optimal growth portfolios are constructed. The data used are the daily prices downloaded from I-Net bridge, covering the period from the 30 November 1999 up to 31 October 2009. That makes up nine years of monthly data, and in total 120 monthly observations. The results we will derive are then the portfolio weights the investor would allocate to each asset in order to achieve an optimal portfolio. The assets under consideration are as follows BHPBILL, WBHOVCO,NEDBANK,ABIL and SAB.

4.1 Calculation 1: Markowitz portfolio optimization and Log-Optimal Growth Portfolio When short selling is allowed

Given a target value $\mathbb{E}(R_p)$ for the mean return of a portfolio, The efficient portfolio characterized by Markowitz and the Log optimal growth portfolio for $N = 5$ is therefore derived as follows,

	BIL	WBO	NED	ABL	SAB
BIL	0.008770	0.000569	0.000252	0.001357	0.003156
WBO	0.000569	0.006594	0.001717	0.002571	0.000467
NED	0.000252	0.001717	0.006045	0.003549	0.001697
ABL	0.001357	0.002571	0.003549	0.011630	0.000881
SAB	0.003156	0.000467	0.001697	0.000881	0.004876

Table 1: Variance Covariance Matrix

Having the variance covariance matrix and the individual asset means we can derive the portfolio constants which in principle are functions of the market parameters.

A	B	C	D
$e'\Sigma^{-1}\mu$	$e'\Sigma^{-1}\mu$	$e'\Sigma^{-1}\mu$	BC-AA
5.3160	0.15451	394.7973	32.7419

Table 2: Constants

The weights for the efficient portfolio satisfy the the following equation,

$$\begin{aligned} \omega*_{eff} &= \frac{1}{D}\left[\underbrace{\left(A\Sigma^{-1}e - B\Sigma^{-1}\mu\right)}_{\omega_1} + \mathbb{E}\left(R_p\right)\underbrace{\left(A\Sigma^{-1}\mu - A\Sigma^{-1}e\right)}_{\omega_2}\right] \\ &= \omega_1 + \mathbb{E}\left(R_p\right)\omega_2 \end{aligned} \quad (56)$$

Which yields,

$$\omega_1 = 5.316 * 10^{-3} \begin{pmatrix} 8.77 & 0.57 & 0.25 & 1.36 & 3.16 \\ 0.57 & 6.59 & 1.72 & 2.57 & 0.47 \\ 0.25 & 1.72 & 6.04 & 3.55 & 1.70 \\ 1.36 & 2.57 & 3.55 & 11.63 & 0.88 \\ 3.16 & 0.47 & 1.70 & 0.88 & 4.88 \end{pmatrix}^{-1} \left[5.316 \times \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 0.15451 \times \begin{pmatrix} 1.64\% \\ 2.78\% \\ 0.05\% \\ 0.77\% \\ 0.96\% \end{pmatrix} \right] \tag{57}$$

and,

$$\omega_2 = \mathbb{E}\left(R_p\right) 5.316 * 10^{-3} \begin{pmatrix} 8.77 & 0.57 & 0.25 & 1.36 & 3.16 \\ 0.57 & 6.59 & 1.72 & 2.57 & 0.47 \\ 0.25 & 1.72 & 6.04 & 3.55 & 1.70 \\ 1.36 & 2.57 & 3.55 & 11.63 & 0.88 \\ 3.16 & 0.47 & 1.70 & 0.88 & 4.88 \end{pmatrix}^{-1} \left[394.79 \times \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 0.154 \times \begin{pmatrix} 1.64\% \\ 2.78\% \\ 0.05\% \\ 0.77\% \\ 0.96\% \end{pmatrix} \right] \tag{58}$$

Hence the efficient frontier can be mapped as follows

$$\omega^*_{eff} = \begin{pmatrix} 0.08187 \\ (0.21544) \\ 0.66051 \\ 0.10715 \\ 0.36592 \end{pmatrix} + \mathbb{E}\left(R_p\right) \begin{pmatrix} 4.5709 \\ 36.2866 \\ (32.9226) \\ (4.2105) \\ (3.7244) \end{pmatrix} \tag{59}$$

From the efficient set we derive the portfolio variance,

$$\sigma_p^2 = \frac{1}{32.7419} \left(C\mu_p^2 - 2B\mu_p + A = 394.7973 \times \mu_p^2 - 2 \times 5.3160 + 0.1545 \right) \tag{60}$$

	BIL	WBO	NED	ABL	SAB	E[R]	V[R]	**S[R]**	**Growth**
Portfolio1	8.2	-21.5	66.1	10.7	36.6	0.0	0.5	6.9	-0.2
Portfolio2	10.5	-3.4	49.6	8.6	34.7	0.5	0.3	5.8	0.3
Portfolio3	12.8	14.7	33.1	6.5	32.9	1.0	0.3	5.2	0.9
Portfolio4	15.0	32.9	16.7	4.4	31.0	1.5	0.3	5.1	1.4
Portfolio5	17.3	51.0	0.2	2.3	29.1	2.0	0.3	5.5	1.8
Portfolio6	19.6	69.2	-16.3	0.2	27.3	2.5	0.4	6.4	2.3
Portfolio7	21.9	87.3	-32.7	-1.9	25.4	3.0	0.6	7.6	2.7
Portfolio8	24.2	105.5	-49.2	-4.0	23.6	3.5	0.8	9.0	3.1
Portfolio9	26.5	123.6	-65.6	-6.1	21.7	4.0	1.1	10.5	3.4
Portfolio10	28.8	141.7	-82.1	-8.2	19.8	4.5	1.5	12.1	3.8

Table 3: A Combination of 20 Portfolios

By varying the portfolio desired return $\mathbb{E}(R_p)$, we can trace out the efficient frontier and this is depicted by the continuous line in figure 1. This demonstrates the diversification benefit of the growth portfolio and the mean variance optimization.

The notable properties of the efficient set is that there exist a portfolio on the frontier which has the minimum variance known as the global minimum variance portfolio and the mean, variance and standard deviation are as follows;

$$\mu_{gmv} = \frac{\mu' \Sigma^{-1} e}{e' \Sigma^{-1} e} = \frac{5.316045}{394.79733} = 1.3465 \tag{61}$$

$$\sigma^2_{gmv} = \frac{1}{e' \Sigma^{-1} e} = \frac{1}{394.7973} = 0.2533 \tag{62}$$

$$\sigma_{gmv} = 5.033 \tag{63}$$

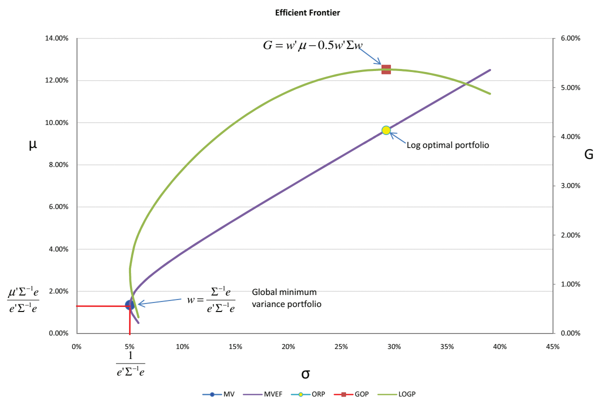

Figure 1: Feasible Region MVO and LOGP

The portfolio weights for the global minimum variance portfolio is therefore;

$$\omega_{gmv} = \frac{\Sigma^{-1}}{e'\Sigma^{-1}e} = 394.797 \times \begin{pmatrix} 56.622 \\ 107.84 \\ 85.74811 \\ 19.9176 \\ 124.663 \end{pmatrix} = \begin{pmatrix} 14.3\% \\ 27.3\% \\ 21.7\% \\ 5.0\% \\ 31.6\% \end{pmatrix} \tag{64}$$

	BIL	WHO	NED	ABL	SAB	μ_{gmv}	σ^2_{gmv}	σ	G
PortGMV	14.342	27.317	21.720	5.045	31.577	1.3465	0.2533	5.0	1.220

Table 4: Global minimum Variance Portfolio

4.1.1 LOG-Optimal Portfolio

In case of the log optimal portfolio we can represent the the path followed by each asset as follows,

$$\begin{aligned} \frac{dS_1}{S_1} &= 0.016dt + 0.094dw_1 \\ \frac{dS_2}{S_2} &= 0.028dt + 0.081dw_2 \\ \frac{dS_3}{S_3} &= 0.0005dt + 0.078dw_3 \\ \frac{dS_4}{S_4} &= 0.008dt + 0.108dw_4 \\ \frac{dS_5}{S_5} &= 0.010dt + 0.07dw_5 \end{aligned} \tag{65}$$

Assuming that the assets are correlated through the wiener term, then the solution to equation [36] is exactly equal to the solution of equation [42]. The feasible region of the growth optimal portfolio and mean variance portfolio

4.2 Calculations2:Efficient Portfolios with a Riskless Asset

With the inclusion of a risky asset and using $\mu_f = 2\%$

$$(\mu - e\mu_f) = \begin{pmatrix} 1.64 \\ 2.78 \\ 0.05 \\ 0.77 \\ 0.96 \end{pmatrix} - 2.00\% \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.004 \\ 0.008 \\ -0.020 \\ -0.012 \\ -0.010 \end{pmatrix} \tag{66}$$

$$\Sigma^{-1}(\mu - e\mu_f) = \begin{pmatrix} 0.0088 & 0.0006 & 0.0003 & 0.0014 & 0.0032 \\ 0.0006 & 0.0066 & 0.0017 & 0.0026 & 0.0005 \\ 0.0003 & 0.0017 & 0.0060 & 0.0035 & 0.0017 \\ 0.0014 & 0.0026 & 0.0035 & 0.0116 & 0.0009 \\ 0.0032 & 0.0005 & 0.0017 & 0.0009 & 0.0049 \end{pmatrix}^{-1} \begin{pmatrix} -0.004 \\ 0.008 \\ -0.020 \\ -0.012 \\ -0.010 \end{pmatrix} = \begin{pmatrix} 0.0091 \\ 2.3046 \\ -3.2907 \\ -0.4793 \\ -1.1235 \end{pmatrix} \tag{67}$$

$$\begin{aligned} H &= C\mu_f^2 - 2 \times \mu_f + A = 394.7973 \times 2.00^2 - 2 \times 0.15451 \times 2.00 + 5.3160 && (68) \\ &= 9.9792 && (69) \\ & && (70) \end{aligned}$$

The portfolio weights given a desired level of portfolio return $\mathbb{E}(R_p)$

$$\omega^* = \begin{pmatrix} 0.0091 \\ 2.3046 \\ -3.2907 \\ -0.4793 \\ -1.1235 \end{pmatrix} \times \frac{\mathbb{E}(R_p) - 2.00}{H} \tag{71}$$

The optimal portfolio mean,

$$\omega'\mu + (1 - \omega' e)\mu_f = \begin{pmatrix} 0.0091 \\ 2.3046 \\ -3.2907 \\ -0.4793 \\ -1.1235 \end{pmatrix} \begin{pmatrix} 1.64 \\ 2.78 \\ 0.05 \\ 0.77 \\ 0.96 \end{pmatrix} = 0.1198 \tag{72}$$

and the optimal portfolio variance,

$$\omega'\Sigma^{-1}\omega = \begin{pmatrix} 0.0091 \\ 2.3046 \\ -3.2907 \\ -0.4793 \\ -1.1235 \end{pmatrix}' \begin{pmatrix} 0.0088 & 0.0006 & 0.0003 & 0.0014 & 0.0032 \\ 0.0006 & 0.0066 & 0.0017 & 0.0026 & 0.0005 \\ 0.0003 & 0.0017 & 0.0060 & 0.0035 & 0.0017 \\ 0.0014 & 0.0026 & 0.0035 & 0.0116 & 0.0009 \\ 0.0032 & 0.0005 & 0.0017 & 0.0009 & 0.0049 \end{pmatrix}^{-1} \begin{pmatrix} 0.0091 \\ 2.3046 \\ -3.2907 \\ -0.4793 \\ -1.1235 \end{pmatrix} = 9.98\% \tag{73}$$

A mapping of different portfolios is therefore tabulated below in table 5 and the graphical representation is shown in

	BIL	WBO	NED	ABL	SAB	Risk-Free	V[R]	σ	Growth	E[Rp]
		-	-	-	-	1.00	0.000%	0	0.0200	0.0200
Portfolio1	0.000	0.115	-0.165	-0.024	-0.056	1.13	0.025%	0.01583	0.0249	0.0250
Portfolio2	0.001	0.231	-0.330	-0.048	-0.113	1.26	0.100%	0.03166	0.0295	0.0300
Portfolio3	0.001	0.346	-0.495	-0.072	-0.169	1.39	0.225%	0.04748	0.0339	0.0350
Portfolio4	0.002	0.462	-0.660	-0.096	-0.225	1.52	0.401%	0.06331	0.0380	0.0400
Portfolio5	0.002	0.577	-0.824	-0.120	-0.281	1.65	0.626%	0.07914	0.0419	0.0450
Portfolio6	0.003	0.693	-0.989	-0.144	-0.338	1.78	0.902%	0.09497	0.0455	0.0500
Portfolio7	0.003	0.808	-1.154	-0.168	-0.394	1.90	1.228%	0.11079	0.0489	0.0550
Portfolio8	0.004	0.924	-1.319	-0.192	-0.450	2.03	1.603%	0.12662	0.0520	0.0600
Portfolio9	0.004	1.039	-1.484	-0.216	-0.507	2.16	2.029%	0.14245	0.0549	0.0650
Portfolio10	0.005	1.155	-1.649	-0.240	-0.563	2.29	2.505%	0.15828	0.0575	0.0700

Table 5: Portfolios with the inclusion of a Risk Free Asset

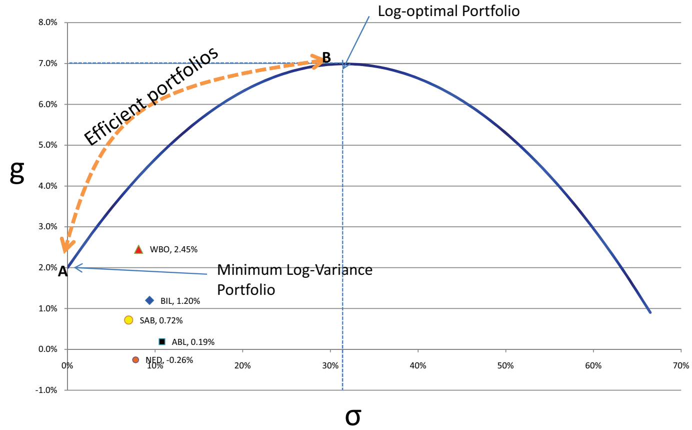

Figure 2: Log Optimal Growth Portfolio

5 Conclusions

In portfolio theory, optimization models play a critical role in determining portfolio investment strategies for investors. Any investment strategy that maximizes portfolio growth has an intuitive appeal for both the professional and non-professional investor. In Markowizt framework one works on diversifying investment assets in order to minimize risk for a given level of return. In this paper we have looked at another portfolio characteristic which is equally important in investment decision process that maximizes the portfolio long term growth rate over a specified time period-the log optimal growth portfolio. We applied our model to five randomly selected JSE stocks. While the Markowizt mean variance strategy is static one period strategy(buy and hold) and a has a fixed time horizon, the log–optimal strategy is dynamic and can be applied to any rebalancing period such as a year, a month a week or a day. The model assumes that stock prices follow a geometric Brownian process and utilizes a stochastic differential equation to quantify the distribution of asset returns. The stochastic process describe the probabilistic evolution of the asset returns through the passage of time. Our model further assumes that asset prices follow a log normal distribution. The solution to the differential equation leads us to conclude that continuously compounded asset return $r_{0,T} = \frac{1}{T}\ln(\frac{S_T}{S_0})$ are normally distributed $\mathcal{N} \sim (\mu - \frac{1}{2}\sigma^2, \frac{\sigma^2}{T})$. This further implies that the future asset price S_T is log normally distributed with $\mathbb{E}(S_T) = e^{\mu t}$. This is a realistic approach to modeling asset prices since a variable that is normally distributed can on negative values something that most financial prices can never do. The distribution of asset returns yields an expected return of $\mu - \frac{1}{2}\sigma^2$ rather than μ hence the log optimal aims at maximizing this geometric mean representing the long term growth rate $G = \mu - \frac{1}{2}\sigma^2$. There is however a subtle difference between the two expected value of assets returns[7]. In our case the portfolio has been formulated in a continuous time framework.If an investor strategy is to maximize long term capital growth then adopting a strategy that maximizes the expected logarithm of returns is considered to be an optimal strategy.

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[7]see Hull JC page 281,7th edition

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6 Appendix

A Example

Lets define a simple asset returns as $R_t = \frac{S_t - S_{t-1}}{S_{t-1}}$ where $R_t \sim N\left[\mu, \sigma^2\right]$. Since asset prices are greater than zero, we have $S_t \geq 0$. By implication then:

$$1 + R_t = \frac{S_t}{S_{t-1}} \geq 0, \quad \text{this implies that} \quad R_t \geq -1$$

Then since R_t is assumed to be normally distributed, by symmetry, the probability of $R_t \leq -1$ given that $\mu = 0.5$ and $\sigma = 0.6$ is;

$$P\left(\frac{R_t - \mu}{\sigma}\right) = P(-1.75) = 4.00\%$$

This implies that there is a 4% probability that prices can be negative which is contrary to our belief.

B Derivation of Maximum Log Optimal Portfolio

$$\underset{\omega}{\operatorname{argmax}}\left\{\omega'\mu\frac{\theta}{2}\omega'\Sigma\omega \quad |\omega'\mu = \mathbb{E}\left(R_p\right) \quad \omega'e = 1\right\}.$$

The Lagrangian function with λ_1 and γ_1 as multipliers thus.

$$\mathcal{L} = \omega'\mu - \frac{\theta}{2}\omega'\Sigma\omega - \lambda_1\left[\omega'\mu - \mu_p\right] - \gamma_1\left[\omega'e - 1\right].$$

Differentiating and setting the matrix equations to zero we have,

$$\begin{aligned}
\frac{d\mathcal{L}}{dw} &= \mu - \Sigma\omega - \lambda_1\mu - \gamma_1 e = 0, \\
\frac{d\mathcal{L}}{d\lambda_1} &= w'\mu - \mu_p = 0, \\
\frac{d\mathcal{L}}{d\gamma_1} &= w'e - 1 = 0.
\end{aligned}$$

Notice that we can rewrite the first equation with ω as a subject of the formula as follows,

$$\omega = \Sigma^{-1}\left[\mu - \lambda_1\mu - \gamma_1 e\right].$$

The last two matrix equations are therefore written as,

$$\begin{aligned}
\omega'e = \Sigma^{-1}\left[\mu - \lambda_1\mu - \gamma_1 e\right]' e = 1 \\
1 = \mu'\Sigma^{-1}e - \lambda_1\mu'\Sigma^{-1}e - \gamma_1 e'\Sigma^{-1}e
\end{aligned} \tag{74}$$

and similarly, we know that $\mu_p = \omega'\mu$, then

$$\mu_p = \mu'\Sigma^{-1}e - \lambda_1\mu'\Sigma^{-1}\mu - \gamma_1 e'\Sigma^{-1}\mu \tag{75}$$

The system above can be represented in a more elaborate matrix form as follows:

$$\begin{bmatrix}\mu\Sigma^{-1}\mu - \mu_p \\ \mu\Sigma^{-1}e - 1\end{bmatrix} = \begin{bmatrix}\mu'\Sigma^{-1}\mu & e'\Sigma^{-1}\mu \\ \mu'\Sigma^{-1}e & e'\Sigma^{-1}e\end{bmatrix}\begin{bmatrix}\lambda_1 \\ \gamma_1\end{bmatrix} \tag{76}$$

Which has a solution of the form,

$$\frac{1}{AC - BB}\begin{bmatrix}A & -B \\ -B & C\end{bmatrix}^{-1}\begin{bmatrix}\lambda_1 \\ \gamma_1\end{bmatrix} = \begin{bmatrix}\mu'\Sigma^{-1}\mu - \mu_p \\ \mu'\Sigma^{-1}e - 1\end{bmatrix} \tag{77}$$

Where we have defined:

$$A = \mu'\Sigma^{-1}\mu, B = \mu'\Sigma^{-1}e \text{ and } C = e'\Sigma^{-1}e$$

Therefore the optimal weights

$$\begin{aligned}
\lambda_1 &= \frac{CX - BY}{D} \\
\gamma_1 &= \frac{AY - BX}{D}
\end{aligned}$$

where $D = AC - B^2$ and $X = \mu'\Sigma^{-1}\mu - \mu_p$ and $Y = e'\Sigma^{-1}\mu - 1$.

$$\begin{aligned}
\omega* &= \Sigma^{-1}\mu - \frac{CX - BY}{D}\Sigma^{-1}\mu - \frac{AY - BX}{D}\Sigma^{-1}e \\
&= \Sigma^{-1}\mu - \lambda_1\Sigma^{-1}\mu - \gamma_1\Sigma^{-1}e \\
&.
\end{aligned} \tag{78}$$