Simple Fractional Dickey Fuller Test

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This paper proposes a new testing procedure for the degree of fractional integration of a time series inspired on the unit root test of Dickey-Fuller (1979). The composite null hypothesis is that of $d \geq d_0$ against $d < d_0$. The test statistics is the same as in Dickey-Fuller test using as output $\Delta^{d_0} y_t$ instead of Δy_t and as input $\Delta^{-1+d_0} y_{t-1}$ instead of y_{t-1}, exploiting the fact that if y_t is $I(d)$ then $\Delta^{-1+d_0} y_t$ is $I(1)$ under the null $d = d_0$. If $d \geq d_0$, using the generalization of Sowell's results (1990), we propose a test based on the least favorable case $d = d_0$, to control type I error and when $d < d_0$ we show that the usual tests statistics diverges to $-\infty$, providing consistency. By noting that $d - d_0$ can always be decomposed as $d - d_0 = m + \delta$, where $m \in \mathbb{N}$ and $\delta \in]-0.5, 0.5]$, the asymptotic null and alternative of the Dickey-Fuller, normalized bias statistic $n\widehat{\rho}_n$ and the Dickey-Fuller t-statistic $t_{\widehat{\rho}_n}$ are provided by the theorem 1.

Theorem 1. *Let $\{y_t\}$ be generated according DGP $\Delta^d y_t = \varepsilon_t$. If regression model $\Delta^{d_0} y_t = \widehat{\rho}_n \Delta^{-1+d_0} y_{t-1} + \widehat{\epsilon}_t$ is fitted to a sample of size n then, as $n \uparrow \infty$, $n\widehat{\rho}_n$* and $t_{\widehat{\rho}_n}$ *verifies that*

$$\widehat{\rho}_n = O_p(\log^{-1} n) \text{ and } (\log n)\,\widehat{\rho}_n \xrightarrow{p} -\infty, \text{ if } d - d_0 = -0.5, \tag{1}$$

$$\widehat{\rho}_n = O_p(n^{-1-2\delta}) \text{ and } n\widehat{\rho}_n \xrightarrow{p} -\infty, \text{ if } -0.5 < d - d_0 < 0, \tag{2}$$

$$\widehat{\rho}_n = O_p(n^{-1}) \text{ and } n\widehat{\rho}_n \Rightarrow \frac{\frac{1}{2}\left\{\mathbf{w}^2(1) - 1\right\}}{\int_0^1 \mathbf{w}^2(r)dr}, \text{ if } d - d_0 = 0, \tag{3}$$

$$\widehat{\rho}_n = O_p(n^{-1}) \text{ and } n\widehat{\rho}_n \Rightarrow \frac{\frac{1}{2}\mathbf{w}^2_{\delta,m+1}(1)}{\int_0^1 \mathbf{w}^2_{\delta,m+1}(r)dr}, \text{ if } d - d_0 > 0. \tag{4}$$

$$t_{\widehat{\rho}_n} = O_p(n^{-0.5}\log^{-0.5} n) \text{ and } t_{\widehat{\rho}_n} \xrightarrow{p} -\infty, \text{ if } d - d_0 = -0.5, \tag{5}$$

$$t_{\widehat{\rho}_n} = O_p(n^{-\delta}) \text{ and } t_{\widehat{\rho}_n} \xrightarrow{p} -\infty, \text{ if } -\frac{1}{2} < d - d_0 < 0, \tag{6}$$

$$t_{\widehat{\rho}_n} = O_p(1) \text{ and } t_{\widehat{\rho}_n} \Rightarrow \frac{\frac{1}{2}\left\{\mathbf{w}^2(1) - 1\right\}}{\left[\int_0^1 \mathbf{w}^2(r)dr\right]^{1/2}}, \text{ if } d - d_0 = 0, \tag{7}$$

$$t_{\widehat{\rho}_n} = O_p(n^{\delta}) \text{ and } t_{\widehat{\rho}_n} \xrightarrow{p} +\infty, \text{ if } 0 < d - d_0 < 0.5, \tag{8}$$

$$t_{\widehat{\rho}_n} = O_p(n^{0.5}) \text{ and } t_{\widehat{\rho}_n} \xrightarrow{p} +\infty, \text{ if } d - d_0 \geq 0.5. \tag{9}$$

where $\mathbf{w}_{\delta,m}(r)$ is $(m-1)-$fold integral of $\mathbf{w}_\delta(r)$ recursively defined as $\mathbf{w}_{\delta,m}(r) = \int_0^r \mathbf{w}_{\delta,m-1}(s)ds$, with $\mathbf{w}_{\delta,1}(r) = \mathbf{w}_\delta(r)$ and $\mathbf{w}(r)$ is the standard Brownian motion.

These properties and distributions are the generalization of those established by Sowell (1990) for the cases $-\frac{1}{2} < d - 1 < 0$, $d - 1 = 0$ and $0 < d - 1 < \frac{1}{2}$.

References

[A. BENSALMA. (2012)] BENSALMA, A. 2012: Unified theoretical framework for unit root and fractional unit root, arXiv:1209.1031 (September 2012).