Simple Fractional Dickey Fuller test

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Simple Fractional Dickey-Fuller Test

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This paper proposes a new testing procedure for the degree of fractional integration of a time series inspired on the unit root test of Dickey-Fuller (1979). The composite null hypothesis is that of \( d \geq d_0 \) against \( d < d_0 \). The test statistics is the same as in Dickey-Fuller test using as output \( \Delta^{d_0} y_t \) instead of \( \Delta y_t \) and as input \( \Delta^{-1+d_0} y_{t-1} \) instead of \( y_{t-1} \), exploiting the fact that if \( y_t \) is \( I(d) \) then \( \Delta^{-1+d_0} y_t \) is \( I(1) \) under the null \( d = d_0 \). If \( d \geq d_0 \), using the generalization of Sowell’s results (1990), we propose a test based on the least favorable case \( d = d_0 \), to control type I error and when \( d < d_0 \) we show that the usual tests statistics diverges to \(-\infty\), providing consistency. By noting that \( d - d_0 \) can always be decomposed as \( d - d_0 = m + \delta \), where \( m \in \mathbb{N} \) and \( \delta \in [-0.5, 0.5] \), the asymptotic null and alternative of the Dickey-Fuller, normalized bias statistic \( n\hat{\rho}_n \) and the Dickey-Fuller t-statistic \( t_{\hat{\rho}_n} \) are provided by the theorem 1.

**Theorem 1.** Let \( \{y_t\} \) be generated according DGP \( \Delta^{d_0} y_t = \varepsilon_t \). If regression model \( \Delta^{d_0} y_t = \hat{\rho}_n \Delta^{-1+d_0} y_{t-1} + \hat{\varepsilon}_t \) is fitted to a sample of size \( n \) then, as \( n \uparrow \infty \), \( n\hat{\rho}_n \) and \( t_{\hat{\rho}_n} \) verifies that

\[
\hat{\rho}_n = O_p(\log^{-1} n) \quad \text{and} \quad (\log n)\hat{\rho}_n \xrightarrow{p} -\infty, \quad \text{if} \quad d - d_0 = -0.5, \quad (1)
\]

\[
\hat{\rho}_n = O_p(n^{-1-2\delta}) \quad \text{and} \quad n\hat{\rho}_n \xrightarrow{p} -\infty, \quad \text{if} \quad -0.5 < d - d_0 < 0, \quad (2)
\]

\[
\hat{\rho}_n = O_p(n^{-1}) \quad \text{and} \quad n\hat{\rho}_n \xrightarrow{p} \frac{1}{2} \left\{ w^2(1) - 1 \right\}, \quad \text{if} \quad d - d_0 = 0, \quad (3)
\]

\[
\hat{\rho}_n = O_p(n^{-1}) \quad \text{and} \quad n\hat{\rho}_n \xrightarrow{p} \frac{1}{2} w^2_{m+1}(1), \quad \text{if} \quad d - d_0 > 0. \quad (4)
\]

\[
t_{\hat{\rho}_n} = O_p(n^{-0.5 \log^{-0.5} n}) \quad \text{and} \quad t_{\hat{\rho}_n} \xrightarrow{p} -\infty, \quad \text{if} \quad d - d_0 = -0.5, \quad (5)
\]

\[
t_{\hat{\rho}_n} = O_p(n^{-\delta}) \quad \text{and} \quad t_{\hat{\rho}_n} \xrightarrow{p} -\infty, \quad \text{if} \quad -\frac{1}{2} < d - d_0 < 0, \quad (6)
\]

\[
t_{\hat{\rho}_n} = O_p(1) \quad \text{and} \quad t_{\hat{\rho}_n} \xrightarrow{p} \frac{1}{2} \left\{ w^2(1) - 1 \right\}, \quad \text{if} \quad d - d_0 = 0, \quad (7)
\]

\[
t_{\hat{\rho}_n} = O_p(\delta) \quad \text{and} \quad t_{\hat{\rho}_n} \xrightarrow{p} +\infty, \quad \text{if} \quad 0 < d - d_0 < 0.5, \quad (8)
\]

\[
t_{\hat{\rho}_n} = O_p(n^{0.5}) \quad \text{and} \quad t_{\hat{\rho}_n} \xrightarrow{p} +\infty, \quad \text{if} \quad d - d_0 \geq 0.5. \quad (9)
\]

where \( w_{m}(r) \) is \((m-1)\)-fold integral of \( w_{\delta}(r) \) recursively defined as

\[
w_{\delta,m}(r) = \int_{0}^{r} w_{\delta,m-1}(s)ds, \quad \text{with} \quad w_{\delta,1}(r) = w_{\delta}(r) \quad \text{and} \quad w(r) \quad \text{is the standard Brownian motion}.
\]

These properties and distributions are the generalization of those established by Sowell (1990) for the cases \(-\frac{1}{2} < d - 1 < 0, \quad d - 1 = 0 \quad \text{and} \quad 0 < d - 1 < \frac{1}{2}\).

**References**