On the optimality of hedge fund investment strategies: a Bayesian skew t distribution model

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Abstract

This paper presents a forward looking model for selection of hedge fund investment strategies. Given excess skewness observed in hedge funds' return distributions, we assume that the historical return distribution is a skewed student t distribution. We implement a Bayesian framework to derive the parameters of the posterior return distribution. The predictive return distribution is easily obtained once the posterior parameters are known by assuming that the unknown future expected returns are equal to the posterior distribution multiplied by the likelihood of unknown future expected returns conditional on available posterior parameters. We derive the predictive mean, predictive variance and predictive skewness from the predictive distribution after twenty-one thousand simulations using GIBS sampler, and solve a multi-objective problem using a data set of monthly returns of investment strategy indices published by the Hedge Fund Research group. Our results show that the methodology presented in this paper provides the highest rate of return (16.79%) with a risk of 2.62% compared to the mean variance, which provides 0.8% rate of return with 1.41% risk respectively.

Keywords: Predictive distribution, skew t distribution, posterior distribution, prior distribution, MCMC simulations, GIBS sampler

Introduction

Markowitz's (1952) mean-variance portfolio selection model assumes that asset returns are normally distributed and uses its historical parameters (mean and standard deviation) as key inputs to portfolio selection. Despite its theoretical importance, the mean-variance portfolio selection model doesn't provide any forward looking framework for asset allocation. Two major limitations are worth mentioning here: firstly, the use of historical standard deviation as measure of risk is inappropriate (Sharpe, 1964; Sortino and van der Meer, 1991). Secondly, the idea that asset returns can be modelled by a normal distribution is somewhat dubious, especially for hedge funds due to the structure of investment strategies they employ to exploit market inefficiencies (Gehin, 2006). There is a growing need from finance practitioners for portfolio selection models that have a forward-looking approach following the sub-prime financial crisis. Portfolio managers want to allocate their fund different investments by taking into account not only the history (historical mean and variance) as in the original Markowitz (1952) but also incorporating the future (future expected parameters) in their investment decision making.

This paper is a response to the growing need in the hedge fund industry for an allocation model that has a forward looking approach. The presence of such a forward looking allocation model is crucial in that it can help fund managers to invest their funds only in investments that will perform very well in the future by providing the highest rate of return at the lowest cost. This paper presents a Bayesian forward looking framework for the investment strategies allocation problem under skew t distribution. We first build a predictive expected return distribution based on the posterior distribution with a skew t distribution and use its predictive parameters (i.e. predictive mean, predictive standard deviation and predictive skewness) as key inputs to the portfolio selection model.

By using the predictive parameters we account for estimation risk, which arises as a result of the use of historical parameters. As Scott and Horvath (1980) pointed out, the inclusion of skewness in the selection model is also important: under non-normality assumption investors will exhibit a preference for positively skewed portfolios. We allow for different levels of attitude toward risk and skewness. In practice, most hedge fund managers are unregulated; they use unlimited leverage and short selling depending on their appetite for risk and/or skewness. The multi-objective utility function formulated in this paper is consistent with the reality in the hedge fund industry and reflects their freedom with regard to leverage and short selling behaviour.

We compare our portfolio selection model with the original Markowitz (1952) model by making use of a data set of monthly investment strategy indices published by the Hedge Fund Research group.[1] The data set extends from January 1995 to June 2010 and includes different bull and bear market trends. Our results show that the methodology presented in this paper provides the highest rate of return (16.79%) with a risk of 2.62%, compared to the mean variance, which provides 0.8% rate of return with 1.41% risk respectively.

Methodology

The effect of the uncertainty of future expected returns parameters in the hedge fund industry can be overcome by expressing the investment selection problem in terms of the predictive distribution of the future expected returns. We use the parameters of predictive return distribution instead of those of historical return distribution employed in the original mean-variance model.

Suppose that a fund manager has a holding period of length τ ; the fund manager's objective is to maximize his wealth at the end of the investment period $T+\tau$ where T is the sample period. Denote by $Y_{T+\tau}$ the unobserved next τ period's expected returns; the predictive returns distribution can be written as:

$$p(Y_{T+\tau}/Y_n) \propto \int p(Y_{T+\tau}/\mu,\Sigma,S)\,p(\mu,\Sigma,S/Y_n)\,d\mu d\Sigma dS \qquad (1)$$

where Y_n is a $(T\times N)$ matrix of historical returns of all investment strategies (N strategies) during the past T periods.

$p(\mu,\Sigma,S/Y_n)$ is the joint posterior distribution of investment strategy returns assumed to be a skewed student's t-distribution with first, second and third moments given by $\mu,\Sigma,$ and S respectively. This distribution summarizes uncertainty about the future expected returns distribution.

[1] For more details on the definition of these investment strategies we refer the reader to www.hedgefundresearch.com

$p(Y_{T+\tau} / \mu, \Sigma, S)$ is a multivariate skewed student's t-distribution for the next τ period future expected returns, and $\propto$: is a proportionality sign.

We account for estimation risk by averaging in (1) over the posterior distribution of the parameters $\mu, \Sigma,$ and S. Therefore the distribution of $Y_{T+\tau}$ will not depend on unknown parameters, but only on the past returns series Y_n assumed to be skewed student's t-distribution.

The analytical solution of (1) is computationally difficult to obtain; often numerical methods such as the MCMC simulations (Metropolis-Hasting or the Gibbs sampler algorithm) are used to obtain the predictive distribution. In this paper the Gibbs sampler algorithm is used for this purpose.

Substituting the predictive returns distribution into the fund manager's objective functions, the following multi-objective portfolio selection problem is presented:

$$\begin{cases} \max_{W} \int \omega' \tilde{\mu}_{T+\tau} \, p(Y_{T+\tau} / Y_n) dY_{T+\tau} \\ \min_{W} \lambda \int (\omega' \tilde{\Sigma}_{T+\tau} \omega) \, p(Y_{T+\tau} / Y_n) dY_{T+\tau} \\ \max_{W} \gamma \int (\omega' \tilde{S}_{T+\tau} \omega \otimes \omega) \, p(Y_{T+\tau} / Y_n) dY_{T+\tau} \\ \qquad \text{Subject to} : \omega \mathrm{I} = 1 \end{cases} \tag{2}$$

where $\tilde{\mu}_{T+\tau}, \tilde{\Sigma}_{T+\tau}, \tilde{\mathrm{S}}_{T+\tau}, \lambda, \gamma,$ and $\otimes$ represents the predictive mean, predictive covariance matrix, predictive coskewness matrix of future expected returns, aversion to change in risk, aversion to change in skewness, and the kronecker product.

To obtain the predictive moments of future expected returns, we use a skew t distribution derived from the skew elliptical class of distributions presented by Sahu et al (2003). The general form of elliptical distribution is given by

$$f(X / \mu, \Sigma, g^{(P)}) = |\Sigma|^{1/2} g^{(P)} \left\{ (X - \mu)' \Sigma^{-1/2} (X - \mu) \right\}; \quad X \in \Re^P \tag{3}$$

with $g^{(P)}(u) = \dfrac{\Gamma(p/2)g(u,p)}{\pi^{p/2}\int_0^\infty r^{\frac{P}{2}-1} g(r,p)dr}$; where; a ≥ 0; $\int_0^\infty r^{\frac{P}{2}-1} g(r,p)dr \neq 0$

Sahu et al (2003) show that when $g(u,p) = \left(1+\dfrac{u}{\upsilon}\right)^{-\frac{(\upsilon+P)}{2}}$; (with $\upsilon > 0$) equation (3) becomes a multivariate student's t-distribution under the condition that the vector of random variables X is transformed as follows:

$$X = \mu + DZ + \varepsilon \qquad (4)$$

where Z is a vector of unobservable random variables whose distribution is elliptical with mean zero and identity covariance matrix I_p ; $\mu \in \Re^P$ vector of mean; D, is a $p \times p$ matrix of skewness and co-skewness:

$$D = \begin{bmatrix} \delta_{11} & \delta_{12} & .... & \delta_{1p} \\ \delta_{21} & \delta_{22} & .... & \delta_{2p} \\ : & : & .... & : \\ \delta_{p1} & \delta_{p2} & .... & \delta_{pp} \end{bmatrix};$$

with δ_{ij} representing the co-skewness of random variable x_i and x_j for all $i \neq j$; and skewness for $i = j$; and ε, a vector of error terms defined as $\varepsilon \rightarrow st(0,\Sigma,\upsilon)$ (i.e. skew t-student random variable). Consequently Sahu et al (2003) show that the conditional distribution of random variable $Y = (X/Z > 0)$ given $\mu, \Sigma, D,$ and υ has the following multivariate skew t-distribution:

$$p(Y/\mu,\Sigma,D,\upsilon) = 2^{\alpha} t_{\alpha},\upsilon(Y/\mu,\Sigma + D^2) \qquad (5)$$

where υ is the degree of freedom for a skewed student's t distribution.

It is now possible to implement a Bayesian investment selection model under the assumption that hedge fund returns have excess skewness characteristic i.e. a skew t-distribution. This implementation is done using the MCMC simulations with a Gibbs sampler that requires us to first specify the likelihood function and the priors before computing the predictive moments of future expected returns.

The likelihood for each observation can be specified as

$$x_i / z_i, \mu, \Sigma, D, w_i \rightarrow \mathrm{N_p}\left(\mu + Dz_i, \frac{\Sigma}{w}\right) \tag{6}$$

where $z_i \rightarrow N_p(0, I_P)$; and $\mathrm{w_i} \rightarrow \Gamma\left(\frac{\upsilon}{2}, \frac{\upsilon}{2}\right)$

For the informative priors scenario we consider the conjugate priors distribution for the unknown parameter μ given $\Sigma, \upsilon,$ and D, and the unknown parameter Σ, which has a multivariate inverted Wishart distribution:

$$\begin{aligned}
&\mu \rightarrow N_p(m, \Sigma_\mu) \\
&\Sigma \rightarrow Inv - W_p(C_\Sigma, \Omega_\Sigma) \\
&D \approx \delta \rightarrow N_p(d, \Sigma_\delta) \\
&\upsilon \rightarrow \Gamma(\gamma, \Sigma_\upsilon)
\end{aligned} \tag{7}$$

Notice that δ is a parameter that adjusts the degree of our beliefs about the skewness in the distribution of the data, and a prior value of this parameter must be specified in the informative prior settings. The same goes for the mean vector d, which reflects our prior information.

Following Polson and Tew (2000), and Harvey et al (2004), we then obtain the predictive moments of future expected distribution as

$$\begin{aligned}
&\tilde{\mu}_{T+\tau} = \mu \\
&\tilde{\Sigma}_{T+\tau} = \Sigma + \mathrm{var}(m / Y) \\
&\tilde{S}_{T+\tau} = S + 3E(V \otimes m / Y) - 3E(V / Y) \otimes \tilde{\mu}_{T+\tau} - E(m - \tilde{\mu}_{T+\tau}) \otimes (m - \tilde{\mu}_{T+\tau}) / Y)
\end{aligned} \tag{8}$$

where $\tilde{\mu}_{T+\tau}, \tilde{\Sigma}_{T+\tau}, \tilde{\mathrm{S}}_{T+\tau}$ are the predictive moments, and μ, Σ, S are the posterior moments obtained with the Gibbs sampler (see Geman and Geman, 1984).

To implement the Gibbs sampler algorithm we need to be able to sample from the posterior distribution $p(\mu, \Sigma, S / Y)$. The algorithm proceeds by drawing iteratively from this distribution, starting with our informative prior set of values $(\mu^{(0)}, \Sigma^{(0)}, S^{(0)})$, and then draws

$$\begin{array}{l}
\mu^{(1)} \rightarrow p(\mu / \Sigma^{(0)}, S^{(0)}, Y) \\
\Sigma^{(1)} \rightarrow p(\Sigma / \mu^{(1)}, S^{(0)}, Y) \\
S^{(1)} \rightarrow p(S / \mu^{(1)}, \Sigma^{(1)}, Y) \\
\cdots\cdots\cdots\cdots\cdots \\
. \\
. \\
\ldots \mu^{(N)}, \Sigma^{(N)}, S^{(N)}
\end{array} \tag{9}$$

Geman and Geman (1984) showed that for the $(\mu^{(t)}, \Sigma^{(t)}, S^{(t)})$ sample obtained after N iterations we need:

$$(\mu^{(t)}, \Sigma^{(t)}, S^{(t)}) \xrightarrow{\textit{converge to}} (\mu, \Sigma, S) \xrightarrow{\textit{In}\text{ Probability to}} p(\mu, \Sigma, S / Y) \text{ as t} \rightarrow \infty$$

Once the predictive parameters are computed, the optimization problem in equation (2) can be solved with a different level of aversion to risk and skewness (k and c) using a numerical method such as the genetic algorithm.

Empirical Results

We consider a set of returns on hedge fund indices provided by Hedge Fund Research Inc. (HFRI). The data is drawn from a database containing more than 6 500 hedge funds from all over the world. The monthly returns series are HFRI strategy indices representing the equally weighted returns, net of fees, of hedge funds classified in each strategy. The database is updated bi-weekly with new funds information (removed and/or newly included funds).

The data set on these strategy indices spans January 1995 to June 2010; to account for survivorship bias we consider only the sample periods of after 1994. Following Capocci and Hubner (2004), hedge fund data starting after 1994 is more reliable and does not contain any survivorship bias.

For the purpose of this study we re-categorize HFRI indices into seven main investment strategies: equity hedge (EH), event-driven (ED), macro (MCRO), relative value RV), fund of funds (FOF), emerging markets (EM), and the fund of weighted composite index (FWC). The weighted composite index category is an equal-weighted index with no fund of funds. Table 1 reflects the first, second and third moments of the historical return distribution.

Table 1: Sample first, second and third moments

	ED	EH	EM	FOF	FWC	MCRO	RV
Mean	0.914	0.956	0.876	0.530	0.810	0.8086	0.727
Variance	2.046	2.778	4.208	1.805	2.136	1.8932	1.299
Skewness	1.374	-0.225	-1.027	-0.752	-0.691	0.416	-3.069

This table shows that event-driven (ED) and macro (MCRO) investment strategies have positive skewness, while the rest of the investment strategies exhibit negative skewness. The higher historical return is observed with event-driven investment strategies.

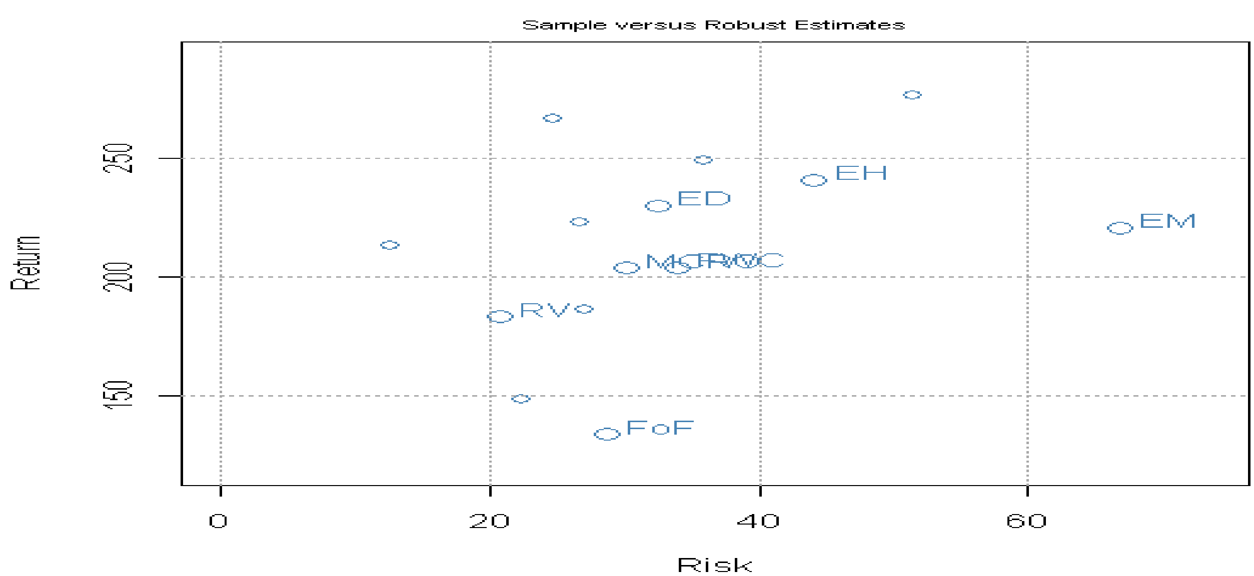

Figure 1: Risk reward trade-off

However, in the risk-reward trade-off analysis carried out in Figure 1 above we find that emerging market investment (EM) is more risky than any investment strategy that exists in the hedge fund universe: it has the highest annualized risk during our sample period, and is ranked third in terms of return. The relative value (RV) investment strategy is the least risky investment strategy during the same period, followed by fund of funds (FOF), which has the lowest rate of return. The equity hedge (EH) investment strategy has the highest rate of return during this period. The macro and weighted composite in currencies (FWC) has almost the same rate of return but with a different risk level, and the lowest is macro investments.

To obtain the first, second and third posterior moments we use 21 000 MCMC Gibbs sampler simulations using WinBUGS package[2]. The posterior means, MCMC error, 2.5 percentile, median, and 97.5 percentiles of the posterior parameters are shown in Tables 4, 5 and 6 of the appendix respectively.

Predictive mean, predictive skewness and predictive covariance are obtained using expressions in equation (8) above. In fact, the predictive mean is equal to the posterior mean, and the predictive variance and predictive skewness equal the posterior means of variance and skewness plus additional terms that account for uncertainty about the unknown future true parameters. We use these predictive parameters as proxy for the unknown future expected returns to solve the investment selection problem in equation (2) using a numerical optimization technique known as the genetic algorithm technique. The predictive optimal weights are shown in Table 2 below, where k and c are aversion to risk and skewness respectively.

We distinguish aggressive fund managers from moderate and conservative fund managers. This categorization follows Waggle et al (2005), who showed that reasonable values of aversion should be in the range of 1 to 10. They classify an aggressive investor as having an aversion coefficient between 1 and 2. A moderate investor has a coefficient of aversion between 2 and 5. They argue that a conservative investor would have a coefficient of aversion between 5 and 10. They call an investor with a coefficient of aversion of 3 an average investor.

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[2] WinBUGS is a statistical package for robust Bayesian MCMC simulation using GIBS sampler. The package is freely available at: www.mrc-bsu.cam.ac.uk/bugs

Table 2: Predictive optimal allocations for the aggressive, moderate and conservative fund manager

AVERSION	ED	EH	EM	FoF	FWC	MCRO	RV
$k = c =$**0.5**	0.1350	0.3861	0.1856	0.0117	0.1221	0.1000	0.0599
k **=0.5&** c **=1**	0.4408	0.1607	0.0977	0.0679	0.0678	0.0670	0.0977
k **=1&** c **=0.5**	0.4407	0.1605	0.0979	0.0676	0.0677	0.0667	0.0979
$k = c =$**1**	0.0449	0.3224	0.1355	0.1046	0.1549	0.1036	0.1333
k **=1&** c **=2**	0.0244	0.3790	0.1375	0.1071	0.1073	0.1062	0.1375
k **=2&** c **=1**	0.9980	0.0010	0.0000	0.0000	0.0020	0.0000	0.0000
$k = c =$**2**	0.9980	0.0010	0.0000	0.0000	0.0020	0.0000	0.0000
$k = c =$**3**	0.1904	0.1623	0.1809	0.1309	0.1345	0.1326	0.0677
$k = c =$**9**	0.3173	0.1291	0.1289	0.1298	0.1131	0.1156	0.0652
$k = c =$**10**	0.7847	0.0357	0.0357	0.0357	0.0357	0.0357	0.0357

Table 2 shows that whenever the aversion to risk is higher than the aversion to skewness (i.e. $k=2\& c=1$ or $k=c=2$), an aggressive fund manager would have to invest heavily in event-driven (ED) investments. However, his expected return will be maximized only if his skew aversion is higher than his risk aversion (i.e. $k=1\& c=2$) (see Table 3 below); in this case he'd largely attempt to increase his holdings in equities (EH).

The computed predictive portfolio mean return, predictive portfolio risk and predictive portfolio skewness are reported in Table 3 below. These are estimates of portfolio mean return, portfolio risk and portfolio skewness of unknown future expected returns.

Table 3: Portfolio predictive mean returns, risk and skewness

Aversion	**Pred.Port.Mean Ret**	**Pred. Portf Risk**	**Pred.Portf.Skew**
k = c =**0.5**	15.1202%	2.7267%	-0.6072%
k =**0.5&** c =**1**	1.2808%	2.6505%	-7.9655%
k =**1&** c =**0.5**	1.2548%	2.6197%	-7.9801%
k = c =**1**	14.7576%	2.6284%	5.3038%
k =**1&** c =**2**	16.7917%	2.6196%	6.3887%
k =**2&** c =**1**	-12.8618%	2.7554%	-24.2254%
k = c =**2**	-12.8618%	2.7554%	-24.2254%
k = c =**3**	6.9979%	2.6083%	-2.3831%
k = c =**9**	4.0908%	2.5990%	-4.6545%
k = c =**10**	-8.0860%	2.6308%	-18.1032%

Clearly, a more aggressive fund manager (with a risk aversion equal to 1 and a skewness aversion of 2) will expect 16.8% of portfolio predictive return, with an overall portfolio predictive risk of 2.6% and a positive predictive skewness of 6.4%. This result is interesting in the sense that positive skewness means that the likelihood of extreme positive returns is possible.

Table 3 shows only two possible investment options that can produce positive skewness: the first is the case where both risk and skewness aversions are equal to unity; in this case the overall portfolio predictive rate of return is 14,8%, with 2,6% predictive risk. The second case is where the skewness aversion is greater than the risk aversion (risk aversion equals one and skewness aversion equals two); in this case one would expect fund managers who always attempt to generate abnormal rates of return to be risk lovers and to be more skewness averse.

In other words, changes that can affect the skewness are likely to affect the occurrence of extreme positive returns; hence the possibility of generating abnormal rates of return becomes difficult. The message here is clear: a fund manager would take any risky position as long as it doesn't alter his/her aversion to the portfolio skewness.

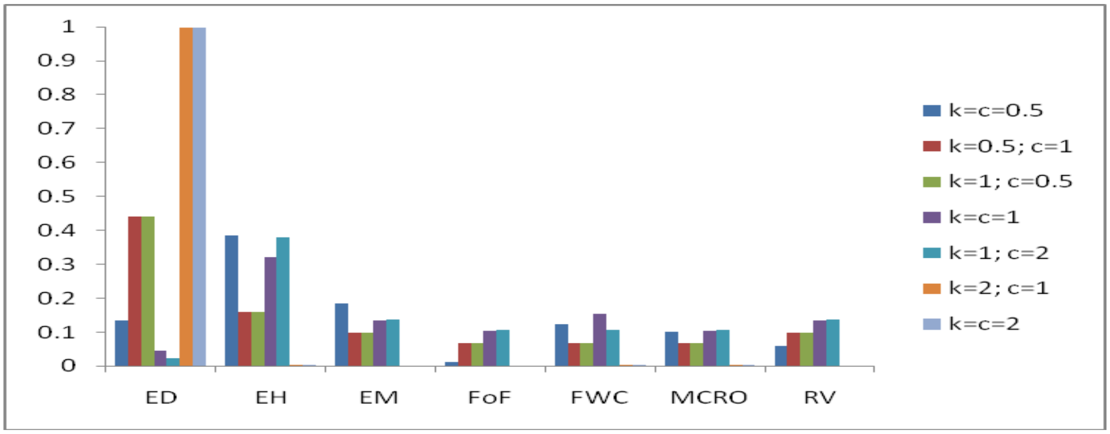

Figure 2: Histogram of weights per risk and skewness aversion

Figure 2 shows different investment allocations corresponding to each investment strategy: for example, if the risk aversion is greater than the skewness aversion ($k=2$ and $c=1$) then the optimal investment is to allocate 100% of capital to ED. One explanation for this allocation is that ED managers are capable of taking advantage of private information that they may have obtained during merger and acquisitions events or during the acquisition of a distressed company and trading on this information in order to make abnormal rates of return.

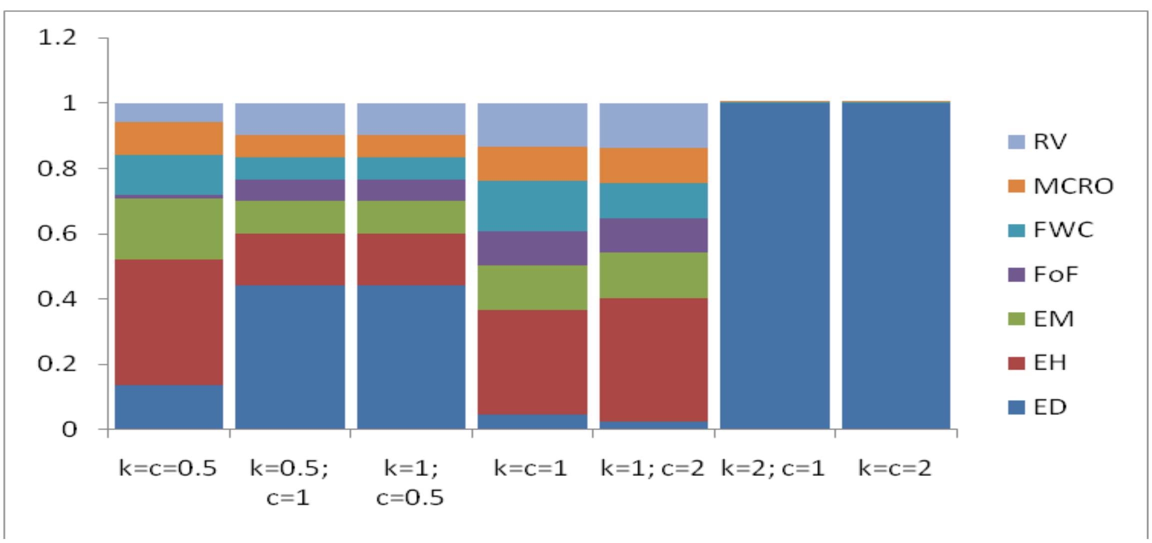

Figure 3: Stacked bar aggressive fund manager

Figure 3 exhibits a stacked bar chart for an aggressive fund manager: for instance, for a fund manager with $k=c=2$, the optimal allocation would be to investment in ED only. If the principle of diversification matters, then the optimal allocation obtained when $k=2$ and $c=1$ with positive predictive skewness would be a clever allocation. The stacked bar chart shows that the optimal investment option allocates more capital to equities (EH), followed by emerging markets (EM); less capital is allocated to ED. As mentioned earlier, EH and ED are two of the most risky investments and one would expect a risk-taker fund manager to have such positions as long as his predictive portfolio skewness is not altered i.e. remains positive and according to his expectations.

The Markowitz (1952) mean-variance analysis has also been carried out for comparison purposes; Figure 3 below shows that the optimal portfolio is made up of 32.5% event-driven, 32.2% macro and 35.32% relative value strategy only. The portfolio expected mean return is 0.80% with the portfolio risk of 1.41%, which is far less than the 16,79% of the predictive portfolio mean return (with predictive 2.62% risk) obtained with our forward looking selection model.

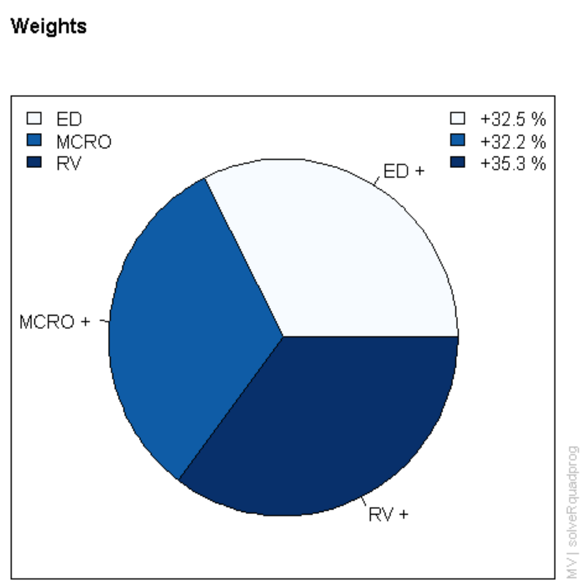

Figure 3: Pie of optimal mean-variance weights

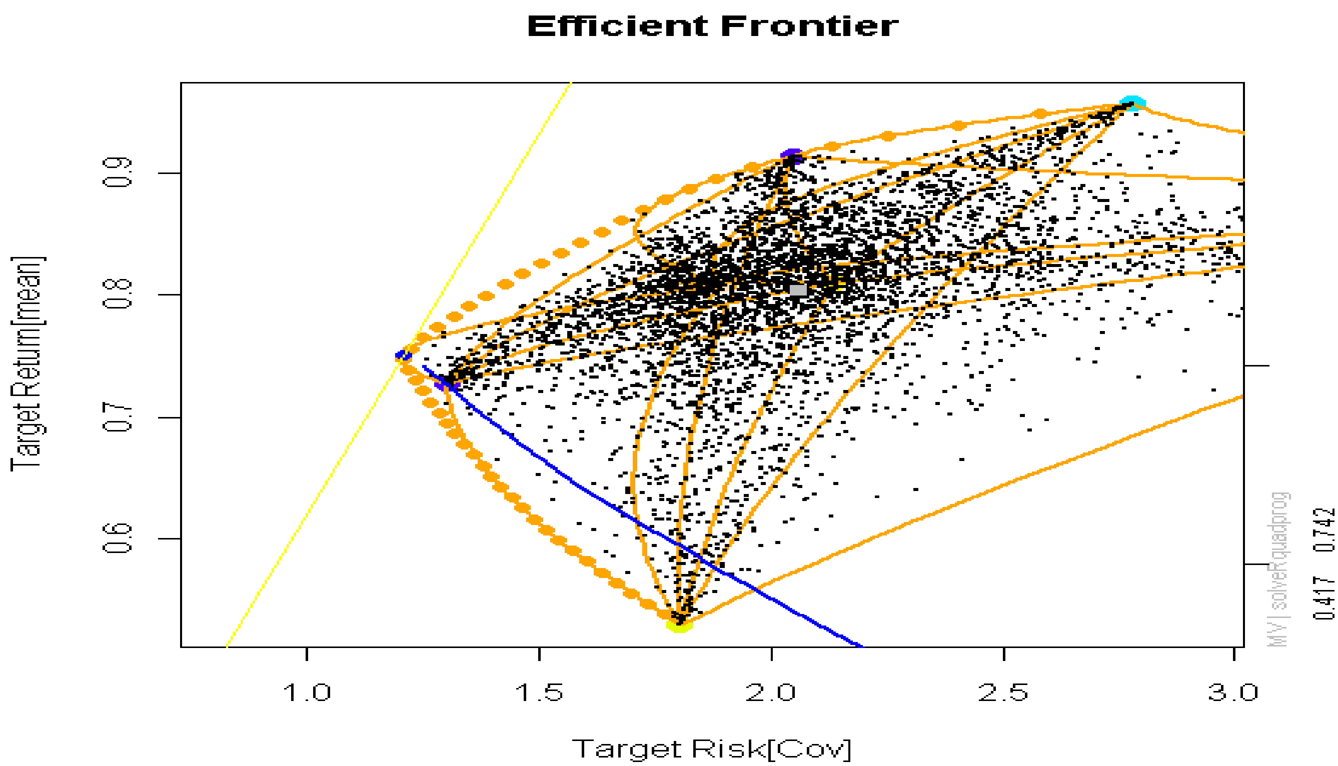

Figure 4: Corresponding mean-variance efficient frontier

Figure 4 shows the mean-variance efficient frontier with a negatively sloped down Sharpe ratio (blue line), meaning that as the manager's targeted return increases, the ratio of the return to risk decreases inversely. The efficient frontier has only three points: these correspond to 32.5% of event-driven, 32.2% of macro and 35.32% of relative value investments. This allocation doesn't consider the diversification principle according to which funds must be allocated across all available investments in order to spread the risk.

Conclusion

This paper presents a forward looking way of selecting hedge fund investment strategies by taking into account the skewness, variance and mean of the predictive of future expected returns. Based on monthly return indices, we have shown that a predictive return distribution can be built in Bayesian settings by first assuming that the historical distribution is a student t distribution, and that the predictive return distribution is equal to the posterior distribution multiplied by the likelihood of unknown future expected returns conditional on available posterior parameters. We generate 21 000 simulations from this predictive distribution using GIBS sampler to obtain the predictive mean, predictive variance and predictive skewness that are used as key inputs to the portfolio optimization process. Based on different levels of risk and skewness aversion, we found that our portfolio selection model provides a higher rate of return than the mean variance model. In financial markets past performance is not indicative of future performance; hence the use of predictive rather than historical parameters is of great importance in asset allocation.

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APPENDIX

Table 4: Posterior mean

node	mean	MC error	2.50%	median	97.50%
Mean[1]	-0.130	0.210	-61.810	0.037	61.710
Mean[2]	0.438	0.222	-61.760	0.760	62.470
Mean[3]	-0.158	0.220	-61.830	-0.361	62.080
Mean[4]	0.136	0.207	-61.950	-0.030	61.510
Mean[5]	0.145	0.212	-61.740	-0.004	62.330
Mean[6]	0.205	0.216	-61.160	-0.087	62.260
Mean[7]	-0.18	0.21	-62.06	0.02	61.95

Table 5: Posterior skewness

node	mean	MC error	2.50%	median	97.50%
Skewness[1]	-0.2431	0.2429	-63.44	-0.1829	62.35
Skewness[2]	0.1772	0.2158	-60.74	0.2247	61.25

Skewness[3]	-0.3082	0.2261	-63.56	-0.3862	61.85
Skewness[4]	0.2781	0.2208	-61.51	0.4952	62.81
Skewness[5]	0.09323	0.2285	-61.64	-0.2194	61.97
Skewness[6]	-0.0258	0.2255	-62.17	0.08094	62.6
Skewness[7]	0.05794	0.2101	-62.03	-0.1172	62.4

Table 6: Posterior covariance matrix

node	**mean**	**MC error**	**2.50%**	**median**	**97.50%**
tau[1,1]	0.9999	0.0036	0.2447	0.9051	2.286
tau[1,2]	-0.0057	0.0026	-0.7797	-0.0034	0.7596
tau[1,3]	-0.0039	0.0027	-0.7574	-0.0017	0.7485
tau[1,4]	-2.7E-04	0.0027	-0.7677	9.7E-04	0.7717
tau[1,5]	8.4E-04	0.0023	-0.7727	0.0021	0.7777
tau[1,6]	-0.0046	0.0024	-0.7785	-0.0026	0.7497
tau[1,7]	0.0019	0.0025	-0.7482	2.4E-04	0.7794
tau[2,1]	-0.0058	0.0026	-0.7797	-0.0034	0.7596
tau[2,2]	1.002	0.0038	0.2399	0.9132	2.287
tau[2,3]	-0.0038	0.0027	-0.7645	-0.0034	0.7687
tau[2,4]	0.0017	0.0026	-0.7635	0.0024	0.7536
tau[2,5]	0.0018	0.0024	-0.7471	-0.0027	0.7805

tau[2,6]	0.0032	0.0022	-0.7646	0.0033	0.7644
tau[2,7]	0.0032	0.0024	-0.7678	0.0052	0.7769
tau[3,1]	-0.0039	0.0027	-0.7574	-0.0017	0.7485
tau[3,2]	-0.0038	0.0027	-0.7645	-0.0032	0.7687
tau[3,3]	0.9953	0.0036	0.2437	0.9036	2.288
tau[3,4]	2.3E-04	0.0024	-0.7679	2.3E-05	0.7417
tau[3,5]	0.0017	0.0026	-0.7505	5.9E-04	0.7672
tau[3,6]	9.0E-04	0.0025	-0.7566	0.0013	0.7546
tau[3,7]	-8.7E-04	0.0026	-0.7901	-0.0010	0.7719
tau[4,1]	-2.7E-04	0.0027	-0.7677	9.7E-04	0.7717
tau[4,2]	0.0017	0.0026	-0.7635	0.0024	0.7536
tau[4,3]	2.5E-04	0.0024	-0.7679	2.3E-05	0.7417
tau[4,4]	0.9925	0.0038	0.2455	0.9045	2.274
tau[4,5]	-0.0016	0.0027	-0.7655	-0.0036	0.7554
tau[4,6]	-8.1E-04	0.0027	-0.7578	6.6E-05	0.7586
tau[4,7]	-6.1E-04	0.0028	-0.7483	-3.4E-04	0.7462
tau[5,1]	8.4E-04	0.0023	-0.7727	0.0021	0.7777
tau[5,2]	0.0018	0.0024	-0.7471	-0.0027	0.7805
tau[5,3]	0.0017	0.0026	-0.7505	5.9E-04	0.7672
tau[5,4]	-0.0016	0.0027	-0.7655	-0.0036	0.7554
tau[5,5]	1.001	0.0038	0.2416	0.9115	2.296

tau[5,6]	0.0019	0.0025	-0.7804	0.0016	0.7818
tau[5,7]	0.0018	0.0029	-0.75	0.0023	0.7628
tau[6,1]	-0.0046	0.0024	-0.7785	-0.0026	0.7497
tau[6,2]	0.0032	0.0023	-0.7646	0.0033	0.7644
tau[6,3]	9.0E-04	0.0025	-0.7566	0.0013	0.7546
tau[6,4]	-8.1E-04	0.0027	-0.7578	6.6E-05	0.7586
tau[6,5]	0.0019	0.0025	-0.7804	0.0016	0.7818
tau[6,6]	1.002	0.0035	0.2379	0.9055	2.297
tau[6,7]	-0.002	0.0026	-0.7557	-0.0034	0.7583
tau[7,1]	0.0019	0.0025	-0.7482	2.4E-04	0.7794
tau[7,2]	0.0032	0.0024	-0.7678	0.0052	0.7769
tau[7,3]	-8.7E-04	0.0027	-0.7901	-0.0011	0.7719
tau[7,4]	-6.1E-04	0.0028	-0.7483	-3.4E-04	0.7462
tau[7,5]	0.002	0.0029	-0.75	0.0023	0.7628
tau[7,6]	-0.002	0.0026	-0.7557	-0.0034	0.7583
tau[7,7]	0.9971	0.0039	0.2383	0.907	2.258