The debt trap: a two-compartment train wreck...and how to avoid it

Artzrouni, Marc and Tramontana, Fabio

University of Pau, University of Pavia

2 October 2013

The debt trap: a two-compartment train wreck... and how to avoid it

Marc Artzrouni
Department of Mathematics (UMR CNRS 5142) and
Center for the Theoretical Analysis and Handling of Economic Data
University of Pau; FRANCE

Fabio Tramontana
Department of Economics and Management
University of Pavia; ITALY

20 September 2013

Abstract

We explore sustainable paths out of a debt trap with a highly stylized two-sector differential equations model for the stocks of money in Government and Society. The model fits the data for the U.S. between 1981 and 2012 with a coefficient of correlation of 0.996. The solutions provide detailed "escape conditions" from the debt trap. A primary surplus is required. Then a government can escape its debt trap either through sustained annual monetary outflows from society to the government (taxation) but with a low initial growth rate, or through annual monetary inflows into both sectors (stimulus) with higher initial growth rate. We illustrate the use of our model with simulations which show how five indebted countries can escape their debt trap in 30 (or 70) years.

Keywords: Compartmental model, debt, system of differential equations, dynamical system, fiscal policy.

JEL Classification: C51, C62, C63, E61, H63.

E-mail: marc.artzrouni@univ-pau.fr; fabio.tramontana@unipv.it

`http://web.univ-pau.fr/~artzroun;http://tramontana.altervista.org/`

1 Introduction

Many countries today have a debt to GDP ratio between 50 and 100 %. A few are above 100 % and Japan tops the chart well over 200 %. A debate rages between "hawks" who advocate rapid deficit reduction and "doves" who believe the best way out of the debt trap is to tax less and stimulate more (Cottarelli, 2013).

Those in the first group feel that debt-reduction is indeed the first priority. For example Reinhart and Rogoff (2010) warn against a debt above 90% of GDP - a finding questioned by some (Herndon et al., 2013). Those in the second group, led by Krugman urge caution and warn that austerity can lead to doubts about government solvency (see Krugman (2012) and other articles on Krugman's blog). They also fear the long-term impact of reduced growth associated with austerity measures and advocate stimulus spending instead of austerity. One can see both sides: pro-austerity economists emphasize the need to stabilize public finances while pro-stimulus ones fear that austerity can be a vicious circle that keeps an economy permanently in its debt trap.

In their attempts at finding objective solutions to these fraught questions, policy modellers have deployed impressive arsenals of sophisticated mathematical tools that were supposed to help policy-makers understand, predict and affect macroeconomic trends (Ruiz Estrada and Yap, 2013). Perhaps the best-known of these tools are the Discrete Stochastic General Equilibrium (DSGE) models. Although criticized by some (Solow, 2010; Kocherlakota, 2010; Knibbe, 2013) these models have proved useful and have informed macro-economic debates. They are used by Central Banks and the European Union among others (see Ratto et al. (2009), Marattin et al. (2011) and Annicchiarico et al. (2013) for a recent application to Italy). For example Trachanas and Katrakilidis (2013) warn of unsustainable debt in Greece, Italy, and Spain.

More generally econometric methods have been used to assess the role of taxation and find ways of restoring " fiscal sanity" (Hauptmeier et al., 2011; Saunoris and Payne, 2010). Others have studied "fiscal gaps" in order to assess the fiscal consolidation required for an escape from the debt trap (Merola and Sutherland, 2012). In the U.S. the Congressional

Budget Office issues detailed budget projections (CBO, 2013). A highly stylized model with a "good" and a "bad" equilibrium has been used to explore conditions under which the southern euro area could move from one equilibrium to the other (Padoan et al., 2013). This model relies on two variables (output Y and real government debt D). The dynamics are formulated with a simple differential equation.

The model we propose here is similar in spirit to the good/bad equilibrium model of Padoan et al. (2013): it is both extremely parsimonious (also two variables) and extremely simple. Ours is a "toy model" which aims to shed light on the macroeconomic conditions needed for an escape from the debt trap. We focus on the "very big picture", that is the flow of money in and out of government. Indeed, national governments are like households and firms: money flows in, money flows out, and the balance contributes to the stock of money in the system. A natural and simple way of describing these flows is with a two-sector economy, Government (G) and Society (S), each one characterized by its stock of money (money supply). Ours is a dynamical systems approach: accounting identities are formulated through a system of two linear differential equations which capture the essential mechanisms that drive the monetary flows between the two compartments. Such a compartmental representation is borrowed from epidemiology and has been recently introduced in economics (Tramontana, 2010).

In Section 2 we describe the model and provide simple closed-form expressions for the two stocks. In Section 3 we show that the model captures well the trajectories of the two stocks in the U.S. from 1981 to 2012. In Section 4 we formulate escape conditions which show that an escape hinges on a fine balance between a primary surplus and the inflows into (or between) the two compartments. In Section 5 we apply the model by showing how five indebted countries can theoretically escape their debt trap in 30 (or 70) years. Section 6 discusses policy prescriptions and perspectives while Section 7 closes with concluding remarks.

2 The model

A compartmental representation is used to describe a two-sector economy consisting of a Government and Society. The G and S compartments are characterized at every instant t by the stocks of money $G(t)$ and $S(t)$ in the two sectors (Figure 1). In the sequel a compartment's "stock" will always refer to its stock of money.

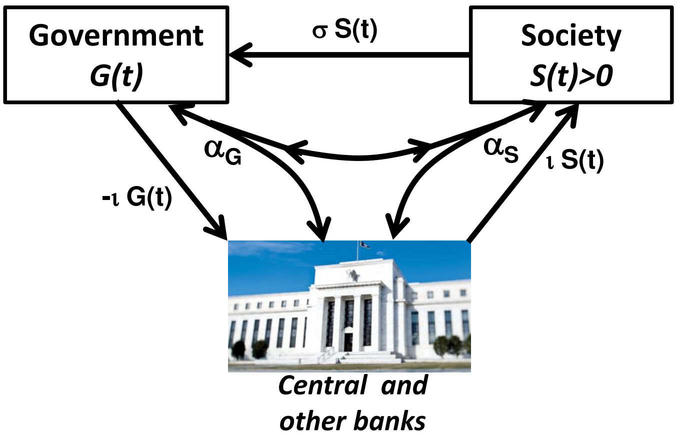

Figure 1: Compartmental model of Government and Society stocks with links to Central and other banks. If a flow is given as a positive value it is in the direction of the arrow, and vice versa. For example if the government stock $G(t)$ is negative the interest flow $-\iota G(t)$ is positive and in the direction of the arrow, towards the banks. If $G(t)$ escapes the debt trap and becomes positive the flow $-\iota G(t)$ is negative and in the opposite direction to the arrow, towards G. Bidirectional arrows reflect our uncertainty concerning the direction of the annual flows that are needed for an escape from the debt trap.

The model is in continuous time and we will describe the temporal dynamics of $G(t)$ and $S(t)$ in nominal terms with a system of two linear differential equations:

$$\frac{dG}{dt} = \overbrace{\iota G + \alpha_G}^{\text{monetary policy}} + \overbrace{\sigma S}^{\text{fiscal policy}} \tag{1}$$

$$\frac{dS}{dt} = \iota S + \alpha_S - \sigma S. \tag{2}$$

The first components of the derivatives are the interests $\iota G(t)$ and $\iota S(t)$ that are received or paid out depending on the sign of $G(t)$ and $S(t)$. A simplification of the model is that the lending and borrowing are thus at the same unchanging "intrinsic interest rate" ι. When $G(t)$ is negative then $\iota G(t)$ is negative and is an interest paid. With $S(t)$ being positive the S compartment earns an interest $\iota S(t)$ per unit of time. These interest flows link the G and S compartments to an outside world of banks and other financial institutions that lend, borrow and print money. (For this reason these financial institutions must not be included in the S sector).

The constant annual flows α_G and α_S are of arbitrary sign. A positive α reflects an infusion of "fresh money" resulting from money printing, quantitative easing, government bond-buying, a positive balance of trade, etc. A negative α reflects annual expenditures which can benefit the other compartment, e.g. constant annual expenditures/stimulus from G to S or constant tax flow from S to G (more on this in the discussions). The "monetary policy" annotation in Eqs. (1) - (2) reflects the role of the interest rate ι and of the annual monetary flows α_G and α_S.

The quantity $\sigma S(t)$ is a net annual flow from S into G. This flow captures the balance of receipts (e.g. taxes) and of outlays (e.g. schools, civil servants, defense, etc.). These receipts and outlays are assumed to be intrinsically proportional to the size of the economy/population crudely measured by society's stock $S(t)$. The coefficient of proportionality σ (or "transfer rate") is positive when receipts exceed outlays and negative otherwise (always assuming $S(t)$ remains positive). The "fiscal policy" annotation in Eqs. (1) - (2) reflects the role of taxation and government spending which are captured in the transfer rate σ.

In short the government and society's stocks both change under three effects: i) a constant flow in or out of each compartment (the α's); ii) interests paid or received at rate ι; iii) a primary surplus or deficit at rate σ (i.e. the net transfer $\sigma S(t)$ of government funds in or out of the S compartment, which excludes interest payments).

The linear system (1)-(2) is simple particularly as Eq. (2) is a differential equation in

S alone. The solution is

$$G(t) = (\iota(G(0) + S(0)) + \alpha_G + \alpha_S)\,\frac{e^{\iota t}}{\iota} - \left(S(0) + \frac{\alpha_S}{\iota - \sigma}\right) e^{(\iota-\sigma)t} + \frac{\sigma(\alpha_G + \alpha_S) - \iota\alpha_G}{\iota(\iota - \sigma)}, \tag{3}$$

$$S(t) = \left(S(0) + \frac{\alpha_S}{\iota - \sigma}\right) e^{(\iota-\sigma)t} - \frac{\alpha_S}{\iota - \sigma}. \tag{4}$$

3 Model fitting to U.S. data

We illustrate the model with data on the total (federal) public debt in the U.S. ($G(t)$) which is readily available. We recognize that this is gross simplification, mostly because we ignore the role of states which provide many services. Society's stock is an even bigger challenge. The money supplies M1-M3 come to mind but we are unsure which one is the most relevant; M1 or M3 may be too narrowly or broadly defined. In the U.S. no data on the M3 money supply is available after 1986. For these reasons we chose M2.[1]

We fit the model starting in 1981, the earliest year available for the readily downloadable data on M2 from the Federal Reserve.[2] The last data point was for 2012 ($n = 32$). We fixed the initial values of the stocks $G(0)$ and $S(0)$ at their 1981 values.

We next need at least a crude estimate for a constant value of our "intrinsic interest rate" ι during the period 1981-2012. We divided for each year the "Interest Expense on National Debt" by the debt as a naive estimate of the interest rate paid by the government. This interest rate declined from roughly 7 to 4 % during the 32 years between 1981 and 2012. We thus chose to estimate ι as the average value during that period which is $\iota = 5.875\%$.

[1] If society's real stock is quite different from M2 and/or immeasurable this complicates the use of the model, although all is not necessarily lost. Indeed suppose a "real" stock S were an affine but unknown function of the M2 stock: $S = \rho_0 + \rho_1 M2$. Substituting this expression for S in Eqs. (1)-(2) yields a model in $(G(t), M2(t))$ that is structurally the same as before with $M2(t)$ as a proxy stock for society. Indeed σ becomes $\rho_1\sigma$; α_G becomes $\alpha_G + \rho_0\sigma$ and α_S becomes $\alpha_S - \rho_0\sigma$. The unknown ρ's have been folded into the model's unknown parameters.

[2] Federal Reserve Economic Data; https://research.stlouisfed.org/fred2/.

Estimates	Value
$\widehat{\alpha_G}$	-0.07447
$\widehat{\alpha_S}$	-0.02359
$\widehat{\sigma}$	-0.00561
Coeff. corr. r	0.996

Table 1: Parameter estimates and coefficient of correlation in fitting of U.S. data (1981-2012).

With ι now fixed we define the vector θ of the three remaining parameters

$$\theta = (\alpha_G, \alpha_S, \sigma). \tag{5}$$

The parameters are estimated by minimizing over θ the sum of squared deviations

$$SS_1(\theta) = \sum_k (G(k)^{actual} - G(k,\theta))^2 + \sum_k (S(k)^{actual} - S(k,\theta))^2 \tag{6}$$

where $G(k,\theta)$ and $S(k,\theta)$ are for the year k the modelled values given in Eqs. (3)-(4).

The initial values of the stocks are the debt and M2 money supply on 1/1/1981: $G(0) = -0.991; S(0) = 1.679$ (in trillions of USD). Standard numerical procedures converged nicely to the same estimated value $\widehat{\theta} = (\widehat{\alpha_G}, \widehat{\alpha_S}, \widehat{\sigma})$ for any reasonable initial values of the parameters. The three parameter estimates are given in Table 1 together with the coefficient of correlation.[3]

The solutions in Eqs. (3)-(4) are now

$$G(t) = -0.98105\,\mathrm{e}^{0.05875t} - 1.31213\,\mathrm{e}^{0.06437t} + 1.3026 \tag{7}$$

$$S(t) = 1.31213\,\mathrm{e}^{0.06437t} + 0.36657 \tag{8}$$

with $t = 0$ corresponding to Jan 1, 1981 (Figure 2). With a coefficient of correlation of 0.996 the model is able to describe with four parameters (one of which, ι, was fixed) the

[3]Ours is not an econometric model and we do not attempt to derive confidence interval for the parameters - which we estimate just for illustrative purposes.

dynamics of the U.S. debt and of the M2 stock of money over 32 years.

Equation (7) shows unsurprisingly that the U.S. economy is caught in a debt trap: extrapolating the model beyond 2012 leads to an irreversible decline of the government stock, which is neither desirable nor realistic. In the next section we will find what parameter values can reverse such a trend and bring an indebted government's stock back into positive territory.

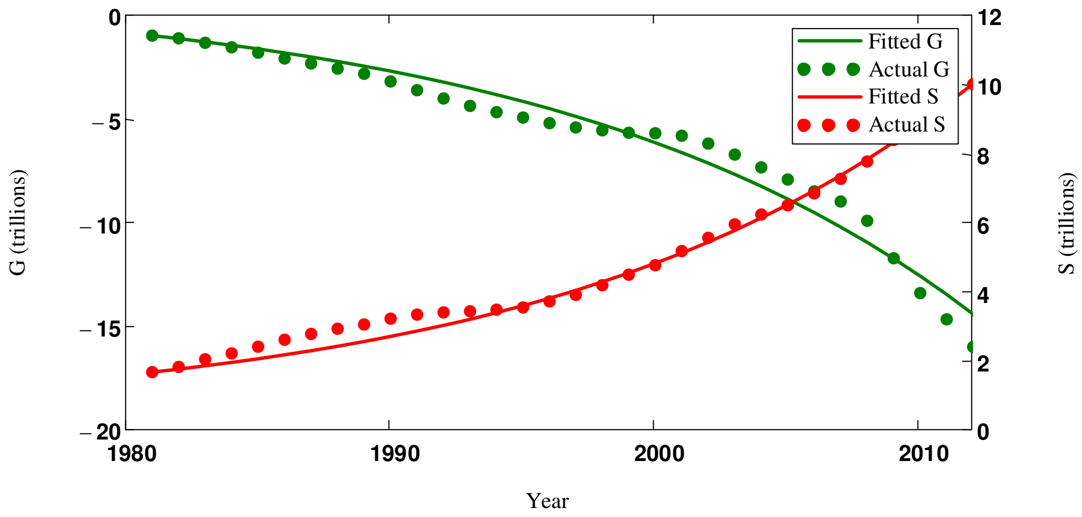

Figure 2: Fitted and actual trajectories of public debt G (left axis) and M2 money supply S (right axis) in the U.S., 1981-2012, trillions of USD (coefficient of correlation $r = 0.996$).

4 Dimensionless dynamics and escape conditions

4.1 Dimensionless dynamics

The public debt and the deficit are usually measured as percentages of GDP in order to make international comparisons with dimensionless quantities that do not depend on monetary units. We do the same here by defining the stocks of the two compartments as fractions $G^*(t)$ and $S^*(t)$ of the *initial* gross domestic product $GDP(0) = v(0)S(0)$ where $v(t)$ is the velocity of $S(t)$ money stock (e.g. the M2 money stock in the U.S. case above and the five countries below):

$$G^*(t) \overset{\text{def.}}{=} \frac{G(t)}{GDP(0)} = \frac{G(t)}{v(0)S(0)}, \tag{9}$$

$$S^*(t) \overset{\text{def.}}{=} \frac{S(t)}{GDP(0)} = \frac{S(t)}{v(0)S(0)}. \tag{10}$$

In a similar manner we define the model's annual inflows into the two compartments as fractions of initial GDP:

$$\alpha_G^* \overset{\text{def.}}{=} \alpha_G/GDP(0); \alpha_S^* \overset{\text{def.}}{=} \alpha_S/GDP(0). \tag{11}$$

With these notations the solutions are the starred equivalents of Eqs. (3)-(4), namely

$$G^*(t) = (\iota(G^*(0) + S^*(0)) + \alpha_G^* + \alpha_S^*)\,\frac{\mathrm{e}^{\iota t}}{\iota} - \left(S^*(0) + \frac{\alpha_S^*}{\iota - \sigma}\right)\mathrm{e}^{(\iota-\sigma)t} + \frac{\sigma(\alpha_G^* + \alpha_S^*) - \iota\alpha_G^*}{\iota(\iota - \sigma)}, \tag{12}$$

$$S^*(t) = \left(S^*(0) + \frac{\alpha_S^*}{\iota - \sigma}\right)\mathrm{e}^{(\iota-\sigma)t} - \frac{\alpha_S^*}{\iota - \sigma}. \tag{13}$$

The initial conditions are now the initial debt to GDP ratio $G^*(0) = G(0)/GDP(0)$ and the inverse of the initial velocity of money $S^*(0) = S(0)/GDP(0) = 1/v(0)$, two readily available statistics for most countries.

4.2 Escape conditions

We will say that the economy escapes the debt trap if in the long-run both $G^*(t)$ and $S^*(t)$ remain positive. Equations (3)-(4) show that this can only happen with $G^*(t)$ and $S^*(t)$ in the long run growing exponentially with rates $\iota > 0$ and $\iota - \sigma > 0$ respectively (i.e. $G^*(t) \sim \mathrm{e}^{\iota t}$ and $S^*(t) \sim \mathrm{e}^{(\iota-\sigma)t}$ for large t). Before giving the escape conditions in the proposition below we define

$$\omega_G \overset{\text{def.}}{=} -\iota G^*(0) - \sigma S^*(0), \;\; \omega_S \overset{\text{def.}}{=} (\sigma - \iota)S^*(0) \tag{14}$$

and

$$
t_m \overset{\text{def.}}{=} \begin{cases} \dfrac{\ln\left(\frac{\alpha_S^*+S^*(0)(\iota-\sigma)}{\alpha_G^*+\alpha_S^*+\iota(G^*(0)+S^*(0))}\right)}{\sigma} & \text{if argument of ln is } > 1, \quad (15a) \\ 0 & \text{otherwise,} \quad (15b) \end{cases}
$$

where t_m of Eq. (15a) is the root of the equation $dG^*(t)/dt = 0$ (from Eq. (12)) when the argument of the logarithm is larger than 1; $G^*(t)$ then reaches an extremum at t_m.

Proposition 1. *Given initial conditions $G^*(0)$ and $S^*(0)$ we define condition C1 as $0 < \sigma < \iota$. The conditions for the asymptotic exponential growths of $G^*(t)$ and $S^*(t)$ are:*

$$
C2\ (\text{for } G^*(t) \sim \mathrm{e}^{\iota t}): \quad \alpha_G^* + \alpha_S^* > -\iota(G^*(0) + S^*(0)) = \omega_G + \omega_S, \tag{16}
$$

$$
C3\ (\text{for } S^*(t) \sim \mathrm{e}^{(\iota-\sigma)t}): \quad \alpha_S^* > S^*(0)(\sigma - \iota) = \omega_S < 0. \tag{17}
$$

If in addition to C1-C3 the condition

$$
C4\ (\text{for } dG^*(0)/dt > 0): \quad \alpha_G^* > -\iota G^*(0) - \sigma S^*(0) = \omega_G \tag{18}
$$

is also satisfied then the government stock $G^*(t)$ starts increasing immediately and monotonically. In this case the argument of the logarithm in Eq. (15a) is ≤ 1 and we have a "Rapid Escape" with a "time of minimum" t_m equal to 0 (Eq. (15b)). If C4 is not satisfied then the argument of the logarithm is > 1; $G^*(t)$ begins by decreasing to reach a minimum at time t_m in Eq. (15a) and then increases exponentially without bounds - we have a "Delayed Escape".

Proof. Equation (4) shows that both $\iota - \sigma$ and $S(0) + \alpha_S/(\iota - \sigma)$ must be positive (conditions C1, C3) in order to have $S(t) \sim \mathrm{e}^{(\iota-\sigma)t}$. Equation (3) shows that we need $0 < \sigma < \iota$ and $\iota(G(0) + S(0)) + \alpha_G + \alpha_S > 0$ (C1, C2) in order to have $G^*(t) \sim \mathrm{e}^{\iota t}$. Under condition C4 the stock $G^*(t)$ initially increases and this increase can only be monotone. If C4 is not satisfied $G^*(t)$ initially decreases. Given that $G^*(t) \sim \mathrm{e}^{\iota t}$ the trajectory must reach a minimum that can only by at time t_m of Eq. (15a). Then $G^*(t)$ increases without bounds. □

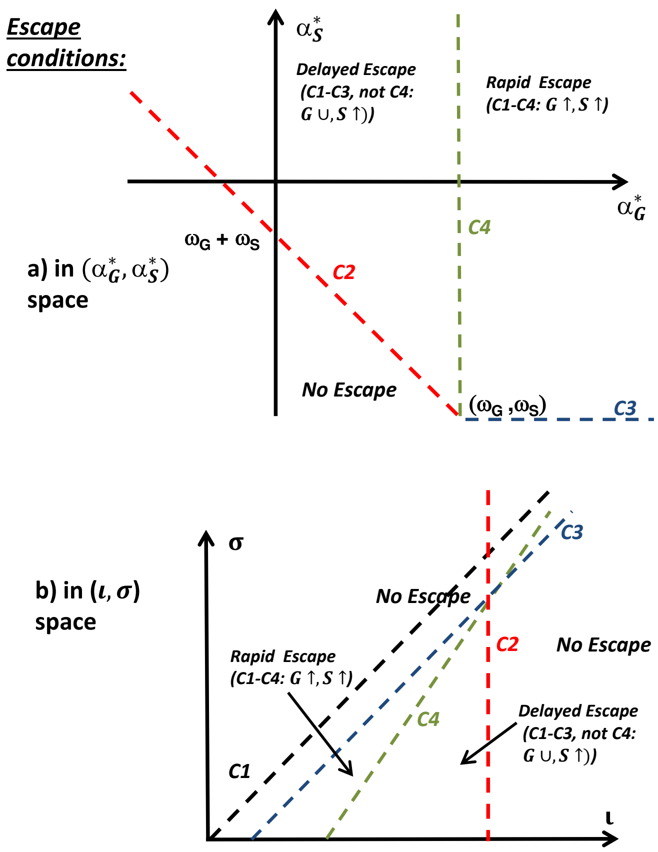

Figure 3: Typical escape conditions in the (α_G^*, α_S^*) space and in the (ι, σ) space. Each one of the straight-line conditions C1-C4 is plotted with the same color in both spaces (no visible C1 in (α_G^*, α_S^*)). The rapid escape region is between the blue and green lines. The critical case of both stocks remaining constant arises at the intersection of all three constraints C2-C4 (red, green and blue line).

In Figure 3 we plotted typical escape conditions/regions in the (α_G^*, α_S^*) space (with ι and σ assumed fixed and $0 < \sigma < \iota$) and in the (ι, σ) space (with α_G^* and α_S^* assumed fixed). Although the plots depend somewhat on the values of the fixed parameters Figure 3a shows that an escape hinges on a sufficiently large sum of annual flow (red condition/constraint C2 at a 45^o angle). A rapid escape requires each flow to be larger than a certain minimum (C3 (blue), C4 (green)). Figure 3b sheds light on the roles of ι and σ. In this example α_S^* is assumed negative and a rapid escape hinges on the pair (ι, σ) being in the narrow triangle defined by the C3 (blue) and C4 (green) borders.

4.3 Parameterization of escape

Given initial conditions $G^*(0)$ and $S^*(0)$ we want to explore realistic parameter values for which an escape can occur. Once we have chosen an interest rate ι and a positive transfer rate $\sigma < \iota$ we need to choose annual flows α^*_G and α^*_S in the delayed or rapid escape regions given in Figure 3. We will see in the Applications section that even countries that are considered seriously indebted can have negative sums $\omega_G + \omega_S$, meaning that the initial debt $|G(0)|$ is less than society's initial stock of money $S(0)$ (as in Figure 3). In this case zero flows α^*_G and α^*_S bring about at least a delayed escape since then conditions C1-C3 are satisfied. However this would be of little use if it takes 50 years for $G(t)$ to reach its minimum or 200 years for $G(t)$ to become positive.

For this reason we parametrize the annual flows required for an escape with two economically meaningful parameters that specify the timing and tempo of the escape. The first is the time until the escape t_e defined by $G^*(t_e) = 0$. Equation (12) shows that t_e is the root of the equation

$$0 = (\iota(G^*(0) + S^*(0)) + \alpha^*_G + \alpha^*_S)\,\frac{\mathrm{e}^{\iota t_e}}{\iota} - \left(S^*(0) + \frac{\alpha^*_S}{\iota - \sigma}\right)\mathrm{e}^{(\iota-\sigma)t_e} + \frac{\sigma(\alpha^*_G + \alpha^*_S) - \iota\alpha^*_G}{\iota(\iota - \sigma)}. \tag{19}$$

Although a root can always be found numerically, there is no closed form solution for t_e. However with a specified escape time t_e Eq. (19) can be viewed as a linear constraint between the two flows α^*_G and α^*_S. This "iso-t_e" constraint will therefore find a straight line expression in the (α^*_G, α^*_S) space of Figure 4 (solid red lines). In order to decide which point on the line to choose, we define a second parameter, namely the initial *growth rate* γ of society's stock, i.e.

$$\gamma \overset{\text{def.}}{=} \frac{dS(0)/dt}{S(0)} = \iota - \sigma + \frac{\alpha^*_S}{S^*(0)}. \tag{20}$$

We recall that $GDP(t) = v(t)S(t)$. If the velocity of money $v(t)$ remains constant and equal to its initial value $v(0)$ (at least for a short initial period) then $GDP(t)$, $S(t)$ and $S^*(t)$ are multiples of one another during that period. This means that the initial growth rates of $S^*(t)$ and $GDP(t)$ are the same. The specified γ can therefore be interpreted as an initial value of the growth rate of the GDP, which is a meaningful economic variable. Equation (20) determines α_S^* as an affine function $\alpha_S^*(\gamma)$ of $\gamma > 0$:

$$\alpha_S^*(\gamma) \stackrel{\text{def.}}{=} (\gamma + \sigma - \iota)S^*(0). \tag{21}$$

We use Eqs. (19)-(21) to express α_G^* as an affine function $\alpha_G^*(\gamma, t_e)$ of γ:

$$\alpha_G^*(\gamma, t_e) \stackrel{\text{def.}}{=} \gamma \frac{S^*(0)(\sigma - \iota\, \mathrm{e}^{t_e(\iota-\sigma)} + \mathrm{e}^{\iota t_e}(\iota - \sigma))}{(\iota - \sigma)(1 - \mathrm{e}^{\iota t_e})} - \sigma S^*(0) + \frac{\iota\, \mathrm{e}^{\iota t_e}\, G^*(0)}{1 - \mathrm{e}^{\iota t_e}}. \tag{22}$$

Equations (21)-(22) define parametrically (with parameter γ) a straight-line "iso-t_e" locus of values $(\alpha_G^*(\gamma, t_e), \alpha_S^*(\gamma))$ for which the time of escape is t_e (solid red lines of Figure 4 for three values of t_e). Specifying a small t_e, say $t_e = 1$, and some desired initial growth rate γ yields values $\alpha_G^*(\gamma, 1)$ and $\alpha_S^*(\gamma)$ which bring about an escape in one year. However if the debt is severe the resulting $\alpha_G^*(\gamma, 1)$ would be unrealistically large. The challenge is to find a time of escape t_e that yields *realistic* annual flows $\alpha_G^*(\gamma, t_e)$ and $\alpha_S^*(\gamma)$. (Note however that the flow $\alpha_S^*(\gamma)$ (Eq. (21)) in (or out of) S does not depend on t_e).

With an initial growth rate γ equal to 0 the point $(\alpha_G^*(0, t_e), \alpha_S^*(0))$ is at the bottom of the "iso-t_e" line, at the intersection with the blue horizontal borderline. As γ increases the point $(\alpha_G^*(\gamma, t_e), \alpha_S^*(\gamma))$ travels up on the straight line and crosses the border between the rapid and delayed escape regions at a value $\gamma^*(t_e)$[4].

The slope of the "iso-t_e" line does not depend on initial conditions $G^*(0)$ and $S^*(0)$. This means that the "iso-t_e" lines obtained in the Application section below for different countries and common values of ι, σ and t_e will be parallel.

[4] $\gamma^*(t_e)$ can be found in closed form by setting the right-hand side of Eq. (22) to ω_G and solving for γ.

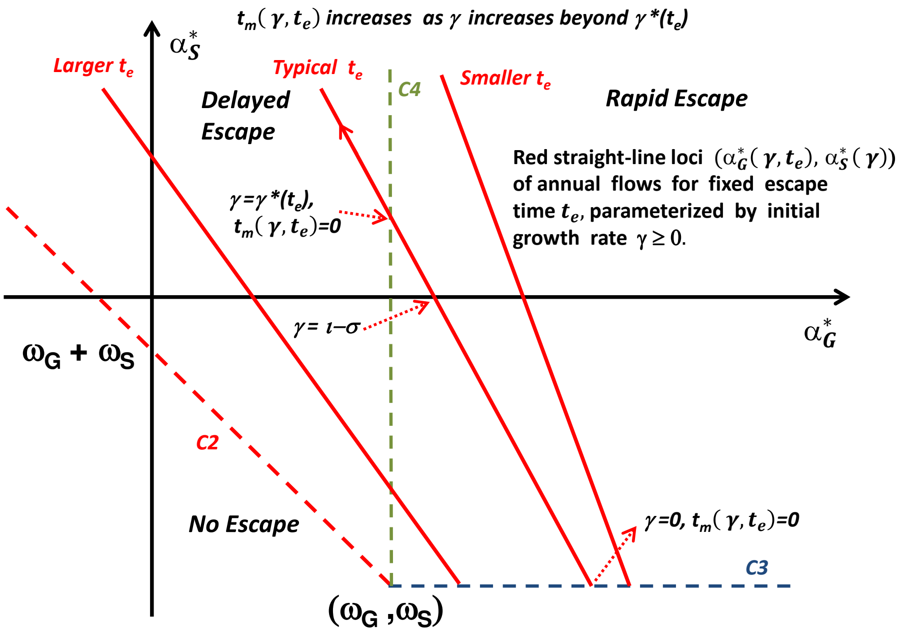

Figure 4: Space of annual flows (α_G^*, α_S^*) with three red straight-line "iso-t_e" loci of flows $(\alpha_G^*(\gamma, t_e), \alpha_S^*(\gamma))$ needed for $G(t)$ to becomes positive at time t_e (three different values of t_e). A smaller escape time t_e requires larger annual inflows $\alpha_G^*(\gamma, t_e)$ into G. As the value of the initial growth rate γ (which parametrizes the "iso-t_e" line) increases from 0 the point $(\alpha_G^*(\gamma, t_e), \alpha_S^*(\gamma))$ moves up the line starting at the horizontal blue borderline (see "Typical t_e" red line). The time $t_m(\gamma, t_e)$ at which the G stock reaches its minimum is zero as long as $(\alpha_G^*(\gamma, t_e), \alpha_S^*(\gamma))$ is in the rapid escape region (Eq.(23b)); $t_m(\gamma, t_e)$ starts increasing (Eq.(23a)) when $(\alpha_G^*(\gamma, t_e), \alpha_S^*(\gamma))$ enters the delayed escape region, i.e. for $\gamma > \gamma^*(t_e)$.

When an economy is in debt ($G^*(0) < 0$) the "iso-t_e" line always intersects the blue C3 horizontal borderline on the right of the point (ω_G, ω_S) as in Figure 4. A shorter escape time t_e means an "iso-t_e" line shifted/rotated to the right with a steeper negative slope: the escape requires larger annual inflows into G. Conversely, a longer escape time shifts/rotates the line to the left with a slope that is less steep but always < -1: more time for an escape requires smaller annual inflows.

For all values of γ the time t_m at which $G(t)$ will reach its minimum, obtained from

(15), can now be written as

$$t_m(\gamma, t_e) \overset{\text{def.}}{=} \begin{cases} \dfrac{\ln\left(\frac{\alpha_S^*(\gamma)+S^*(0)(\iota-\sigma)}{\alpha_G^*(\gamma,t_e)+\alpha_S^*(\gamma)+\iota(G^*(0)+S^*(0))}\right)}{\sigma} & \text{if argument of ln is} > 1, \quad (23a) \\ 0 & \text{otherwise.} \quad (23b) \end{cases}$$

Equation (23b) (or Eq. (23a)) applies when $(\alpha_G^*(\gamma, t_e), \alpha_S^*(\gamma))$ is in the rapid (or delayed) escape region. In the next section we illustrate the model with projection scenarios that show how five countries can escape their debt traps.

5 Appplication

We apply the model to five countries in various states of indebtedness with the initial time set in 2011. The World Bank database provides the (initial) debt to GDP ratio $G^*(0)$, which is routine. They also provide the ratio of the M2 money supply ("quasi money") to GDP. Conveniently this is our $S^*(0) = 1/v(0)$ with the M2 definition we chose for the money supply. (Data in Table 2 with countries listed in increasing order of initial debt to GDP ratio).

We set $\iota = 0.05$, a plausible interest rate for our illustrative purpose. The parameter σ is trickier and of course vital since an escape hinges on a positive value of σ. Our only indication on its value is provided by the data fitting exercise for the U.S. which yielded $\widehat{\sigma} = -0.00561$, a value pretty close to being positive. For this reason we somewhat arbitrarily choose $\sigma = 0.01$, which combined with $\iota = 0.05$ and a realistic escape time $t_e = 30$ will provide plausible parameters and trajectories. (A smaller t_e would require implausibly high annual inflows).

The big picture is given in Figure 5 where we have plotted the five "iso-t_e" segments (t_e=30) parametrized by values of γ ranging from 0.02 to 0.04 and to 0.06. Based on past experience these values represent a plausible range of initial growth rates (Salvatore, 2010). The dotted lines at a right angle are for each country the borders C3 and C4 of the rapid escape region. Numerical values $\alpha_G^*(\gamma, t_e)$ and $\alpha_S^*(\gamma)$ are given in Table 2 for

$\gamma = 0.02, 0.04, 0.06$.

In order to gauge however crudely the plausibility of the results for the α's in Table 2 we can compare their maximum 0.071 (7.1 %) to the planned one-off injection of \$ 0.475 trillion USD in the U.S. economy in 2008 (known as TARP for Troubled Asset Relief Program). This amount was roughly 3 % of the year's GDP. This was a one-off program (actually spread over a couple of years) that was considered massive and we are talking here of annual inflows continuing over decades. Still our percentages in the 0 to 7.1 % range seem plausible for the purpose of our illustrative examples.

With a low initial growth rate of $\gamma = 0.02$ the escape occurs with annual outflows $\alpha^*_S(0.02)$ between -1.7 % for Sweden and -3.2 % for France. When the absolute values of these outflows are smaller than the corresponding flow $\alpha^*_G(30, 0.02)$ into G these outflows can be channelled into G and viewed as annual constant "adjustment" taxes. This reduces the requirement for fresh funds into G accordingly. For example if $\gamma = 0.02$ then in the U.S. an escape in 30 years is achieved with a (constant) diversion of $|\alpha^*_S(0.02)| = 1.8\%$ of initial GDP from S to G. The inflow of fresh funds into G is reduced from 4.1 % to 2.3% which is the (algebraic) sum of flows $\alpha^*_G(30, 0.02) + \alpha^*_S(0.02)$. In order for this interpretation of the flows to be valid this sum of flows must be positive. However this will always be the case with a sufficiently small t_e - which we are seeking anyway. (The negative value -0.004 of the sum $\alpha^*_G(30, 0.02) + \alpha^*_S(0.02)$ for Sweden with $\gamma = 0.02$ means that the country could escape the debt trap in less than 30 years with a realistic net sum of incoming flows just larger than 0).

The intermediate initial growth rate $\gamma = 0.04$ corresponds to the special case $\gamma = \iota - \sigma$. The $\alpha^*_S(0.04)$ flows are then 0 for all countries. The corresponding flows $\alpha^*_G(0.04, 30)$ of fresh money into the government sector range from 1% of initial GDP in the least indebted country (Sweden) to 5.2 and 4.6 % in the most indebted ones (Greece and Italy).

	Sweden	U.S.	France	Greece	Italy
Initial conditions					
$G^*(0) = \frac{G(0)}{GDP(0)}$	-0.383	-0.818	-0.937	-1.065	-1.109
$S^*(0) = \frac{1}{v(0)}$	0.867	0.897	1.585	0.979	1.538
$\boldsymbol{\omega}$**'s**					
ω_G	0.010	0.032	0.031	0.043	0.040
ω_S	-0.035	-0.036	-0.063	-0.039	-0.062
$\omega_G + \omega_S$	-0.024	-0.004	-0.032	0.004	-0.021
$\boldsymbol{\gamma}$**=0.02**					
$\alpha^*_G(30, 0.02)$	0.013	0.041	0.039	0.055	0.051
$\alpha^*_S(0.02)$	-0.017	-0.018	-0.032	-0.020	-0.031
$\alpha^*_G(30, 0.02) + \alpha^*_S(0.02)$	-0.004	0.023	0.007	0.036	0.020
$\boldsymbol{\gamma}$ **=0.04**					
$\alpha^*_G(30, 0.04)$	0.010	0.038	0.034	0.052	0.046
$\alpha^*_S(0.04)$	0	0	0	0	0
$\alpha^*_G(30, 0.04) + \alpha^*_S(0.04)$	0.010	0.038	0.034	0.052	0.046
$\boldsymbol{\gamma}$**=0.06**					
$\alpha^*_G(30, 0.06)$	0.007	0.035	0.029	0.049	0.041
$\alpha^*_S(0.06)$	0.017	0.018	0.032	0.020	0.031
$\alpha^*_G(30, 0.06) + \alpha^*_S(0.06)$	0.025	0.053	0.06	0.069	0.071

Table 2: Initial conditions (five countries, 2011), ω's (Eq. (14)), annual inflows $\alpha^*_G(t_e, \gamma)$ (Eq. (22)) into G and $\alpha^*_S(\gamma)$ (Eq. (21)) into S for an escape at $t_e = 30$ and initial growth rate $\gamma = 0.02, 0.04, 0.06$ ($\iota = 0.05$, $\sigma = 0.01$). (Inconsistencies in the sums are due to rounding errors). *Source of initial conditions*: `http://data.worldbank.org/indicator`.

These are values in the middle of each "iso-t_e" segment at which the line crosses the $\alpha^*_G(\gamma, t_e)$ axis of Figure 4. This Figure shows that for all countries the escape is rapid with both $\gamma = 0.02$ and $\gamma = 0.04$. (For Sweden the point $(\alpha^*_G(30, 0.04), \alpha^*_S(0.04))$ in the middle of the segment is just on the border between the rapid and delayed escape regions).

A high initial growth rate of $\gamma = 0.06$ requires for all five countries annual inflows of fresh money into both compartments between 1.7% and 7.1% of initial GDP. The escape is still rapid for all countries except France and Sweden whose escape is delayed with minimum stocks $G^*(t)$ reached at times t_m= 2.6 and 6.3 years, respectively.

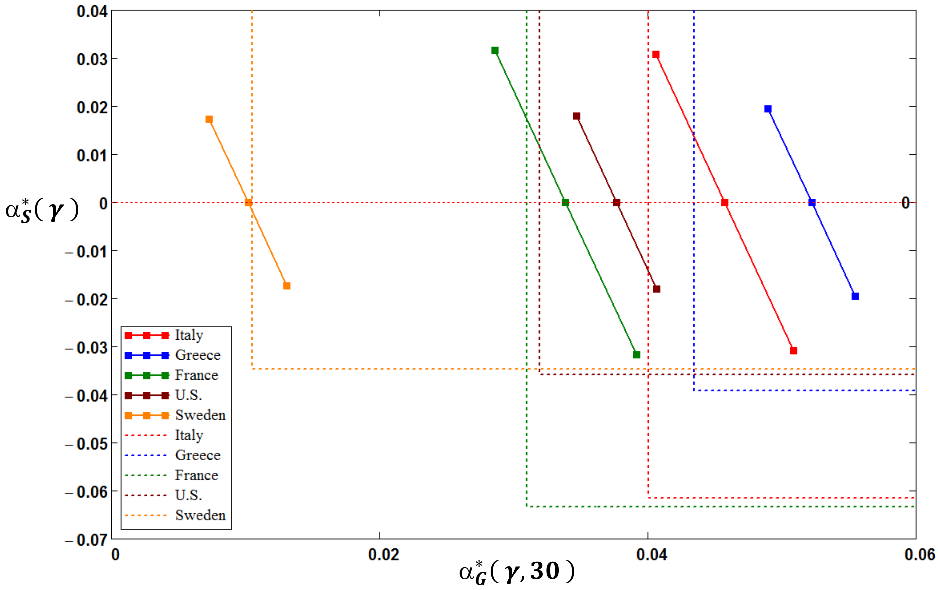

Figure 5: Straight-line loci of $(\alpha^*_G(\gamma, 30), \alpha^*_S(\gamma))$ annual flows for five countries escaping in $t_e = 30$ years. Initial growth rates γ range from 0.02 (low point of segment) to 0.04 (middle point) and 0.06 (high point). The dotted lines are for each country the right angle borders of the rapid escape region in Figure 4 (conditions C3, C4).

We emphasize the purely illustrative nature of these projections which are probably too optimistic. Indeed, in its recent "Long-Term Budget Outlook" the Congressional Budget Office predicts a much more protracted increase in the U.S. debt (CBO, 2013). In their best-case scenario the federal debt held by the public is 65 % of GDP in 2038, a date close to the year 2041 at which the exercise above predicts a return into positive territory.

Our model can easily produce realistic delayed escape scenarios in line with the CBO projections. As an example we specify a time of escape of 70 years instead of 30, an initial growth rate γ of 0.02 and the same σ and ι of 0.01 and 0.05. Figure 6 shows the resulting delayed escape with the debt bottoming out in $t_m = 40$ years (2051) at a value $G^*(t_m) = -1.05$, i.e. 105% of *initial* (2011) GDP (which cannot be compared to *current* GDP, whether total of held by the public). The annual flow out of S is $\alpha^*_S(0.02) = -0.018$ as in Table 2 with $\gamma = 0.02$. The annual flow $\alpha^*_G(0.02, 70)$ into G has dropped from 0.041

to 0.026: a later time of escape requires a smaller annual infusion of funds into G, 0.018 of which can be drawn from S.

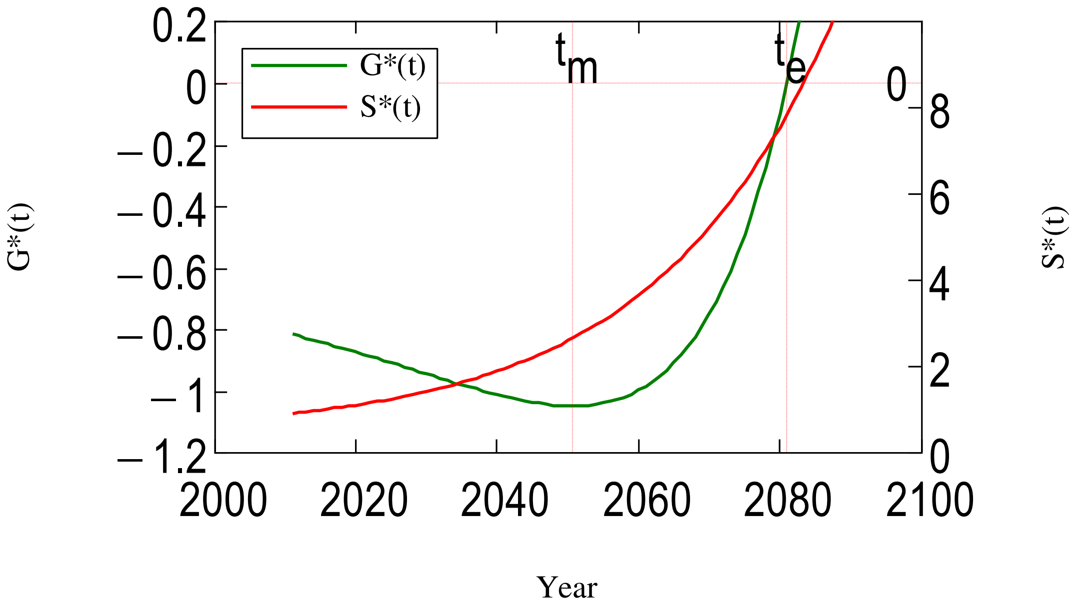

Figure 6: A possible delayed escape for the U.S. government stock $G^*(t)$, with $S^*(t)$ on right axis. Transfer rate $\sigma = 0.01$, interest rate $\iota = 0.05$, initial growth rate $\gamma = 0.02$ and specified time of escape $t_e = 70$ years (2081).

6 Policy prescriptions and caveats

6.1 Prescriptions

We now bring together guidelines policy-makers can follow if they wish to use our model to devise strategies toward fiscal consolidation. Our model may be very simple but its initial conditions reflect the fact that the prospects for fiscal consolidation depend not only on the debt to GDP ratio but also, not unreasonably, on the velocity of money, an indicator of economic activity. Given these initial conditions policy-makers first need to specify realistic interest and transfer rates satisfying $0 < \sigma < \iota$. They also decide on an escape horizon t_e they think is realistic. The required annual flows $\alpha^*_G(\gamma, t_e)$ and $\alpha^*_S(\gamma)$ then depend on the specified initial growth rate γ.

If policy-makers want the government's stock to start increasing right away ("rapid escape") this can happen only with a small γ and will lead to an outflow from S. One can put this flow to good use by diverting it into G which then requires a smaller infusion of fresh funds. We can think of this diversion of funds from S to G as an annual constant "adjustment" tax which then decreases in real terms over time. This "adjustment" tax must not be confused with the (main) taxation implicit in σ, i.e. income and corporate taxes which are proportional to the size of the economy (measured by the stock in S). Policy-makers may prefer an escape with a larger initial growth rate γ. The higher initial growth will require non-negative infusions of funds into both compartments (stimulus) and cause the debt to increase temporarily ("delayed escape") while still recovering in time for an escape at the same time horizon t_e. If the annual flows corresponding to the prescribed σ, ι, t_e and γ are plausible the corresponding policy is implemented. Otherwise the policy-maker re-assesses the parameter values until he obtains plausible flows - as we did for the five countries considered above.

In short, the model provides answers to policy-makers on both sides of the ideological divide who may have agreed on a desirable time horizon t_e for fiscal consolidation. The hawks will find solutions that reverse immediately the trajectory of the government's stock but at a cost of lower growth and higher annual constant "adjustment" taxes which will also mean less borrowing by the government. The doves will emphasize growth and lower taxes - which is no doubt desirable but will come at the cost of a delayed recovery and more government borrowing. There are complications however.

6.2 Caveats

The prescriptions described above illustrate the fact that within our model quite different policies can lead to the same long-term results. However we recognize that our model is still mechanistic and ignores intangible factors and complex feedback mechanisms between fiscal and monetary policies that may render some policies more realistic, desirable, or attainable than others. For example it may be unreasonable to assume that fiscal and monetary parameters in the model can vary independently of one another (or of initial conditions, as we will see below). Although beyond the scope of this paper such mechanisms can be fruitfully explored through a "Russian dolls"-type modelling that would take us deeper and deeper into the heart of an increasingly detailed model. We would start by taking apart (or "deconstructing") the neglected but important transfer rate σ. The obvious first step is to model the flow $\sigma S(t)$ as receipts minus outlays. Under the assumption of a constant velocity of money v, receipts are crudely obtained as a (possibly time-varying) tax rate τ applied to the gross domestic product $GDP(t) = vS(t)$. Outlays are equally crudely modelled as proportional to society's size measured by its stock $S(t)$ with coefficient of proportionality ϵ. The transfer rate then becomes $\sigma = \tau v - \epsilon$. Within this framework the initial $S^*(0)$ which is now the inverse of the (constant) velocity of money v cannot change without causing a change in σ. More generally, a time-varying constraint of the form $\sigma = \tau(t)v(t) - \epsilon(t)$ can be used to capture fluctuations of GDP or diverse feedbacks resulting for example from Laffer-curve or fiscal multipliers effects. These effects and feedbacks should not doubt incorporate other components of the model such as the interest rate and the annual flows.

7 Conclusion

The uses and abuses of mathematics in economics have attracted criticism, much of it justified (Ruiz Estrada and Yap, 2013). Mindful of this criticism we do agree with Krugman (2000) however that simple economic models must not be neglected, particularly those with modest but well defined ambitions. Indeed ours is a "toy model" which attempts to capture the essence of the mechanisms required to keep a government "afloat" - or to bring it back to solvency when it is in debt. We conceptualized the resulting fiscal consolidation not as an equilibrium, the standard approach in economics, but as an asymptotically exponential growth of money supplies. Endless exponential growth is a mathematical construct however. Once the system has escaped the debt trap we trust policy-makers to escape the confines of our model and plan their masters' next election by spending more in Society - perhaps even making σ at least temporarily negative. The government's long-term aim then is to keep its budget at or near an equilibrium. This "financial" equilibrium is not unlike the nutritional equilibrium experienced by populations of the past once they had escaped the Malthusian trap (Komlos and Artzrouni, 1990).

A highly simplified model has shortcomings. One can dispute the central role played in our model by society's slightly nebulous "money supply" $S(t)$ (measured here by the M2 money supply). Many important factors are omitted such as inflation, population growth, consumption, savings, investments, to name just a few. Also a positive "fiscal flow" $\sigma S(t)$ (i.e. a primary surplus) may not be a surprising escape condition. We still need to know how realistic the model is and what combination of the "fiscal flow" $\sigma S(t)$ and of the annual "monetary flows" α_G and α_S are needed in order to achieve an escape at some reasonable prescribed time.

We have provided answers to these questions on two levels: first by showing that at least for the U.S. there were parameter values for which the model's trajectory approximated well the evolution over 32 years of the government's and society's stocks of money; second, by showing that for five countries with debts to GDP ratios in the 38 % (Sweden) to 111 % (Italy) range there are realistic parameter values which yield optimistic escapes in 30 years. A realistic delayed escape is obtained for the U.S. in 70 years. Unrealistic parameter values are required for an escape in 10 or 20 years. We think this could well be the case for any projection model, however complex or simple. In that sense the usefulness of our (or any other) simple model lies in its ability to capture fairly accurately and robustly the salient features of the system under study. For these reasons we believe that despite its simplicity our model provides policy-makers with general, quantitative guidelines that might help them find sensible solutions to the burning problem of preventing the train wreck facing many economies caught in a debt trap.

Acknowledgement

The authors wish to thank John Komlos, Patrice Cassagnard and three anonymous referees for their comments on a previous version of this paper.

References

Annicchiarico, B., Di Dio, F., Felici, F. 2013. Structural reforms and the potential effects on the Italian economy. *Journal of Policy Modeling*, 35, 88-109.

Bernanke, B. 2010. Implications of the Financial Crisis for Economics, Conference Co-sponsored by the Center for Economic Policy Studies and the Bendheim Center for Finance, Princeton University, Princeton, New Jersey, 24 Septmber 2010.

Cecchetti, S., Mohanty, M.S. and Zampolli, F. (2010) The future of public debt. *BIS Working Papers No 300.* Bank for International Settlements, Basel, Switzerland.

Congressional Budget Office. 2013. The 2013 Long-Term Budget Outlook. (September 2013), Pub. No 4713. http://www.cbo.gov/publication/44521.

Cottarelli, P. 2013. The austerity debate, in L. Paganetto (ed.) *Public Debt, Global Governance and Economic Dynamism*, Springer Verlag, Italy, 301-308.

Hauptmeier, S., Sanchez-Fuentes, A.J. and Schuknecht, L. 2011. Towards expenditure rules and fiscal sanity in the euro area. *Journal of Policy Modeling*, 33, 597-617.

Herndon, T., Ash, M., and Pollin, R. 2013. Does High Public Debt Consistently Stifle Economic Growth? A Critique of Reinhart and Rogoff. *Working Paper No 322.* Political Economy Research Institute, University of Massachsetts, Amherts.

Knibbe, M. 2013. Looking at the right metrics in the right way: a tale of two kinds of models, *Real-world economics review*, 63, 25, 73-97. http://www.paecon.net/PAEReview/issue63/knibbe63.pdf

Kocherlakota, N. 2010. Modern Macroeconomic Models as Tools for Economic Policies (2009 Annual Report). *Banking and Policy Issues Magazines*, The Federal Reserve Bank of Minneapolis.

Komlos, J. Artzrouni M. 1990. Mathematical investigation of the escape from the Malthusian trap, *Mathematical Population Studies*, 2(4), 269-287.

Krugman, P. 2000. How complicated does the model have to be? Oxford Review of Economic Policy, 16(4), 33-42.

Krugman, P. 2012. Blunder of Blunders. *New York Times* Blog, 22 March.

Marattin, L. Marzo, M., and Zagaglia, P. 2011. A welfare perspective on the fiscal-monetary mix: The role of alternative fiscal instruments. *Journal of Policy Modeling*, 33, 920-952.

Merola, R., and Sutherland D. 2012. Fiscal Consolidation: Part 3. Long-Run Projections and Fiscal Gap Calculations. *OECD Economics Department Working Papers, No 934*, OECD Publishing.

Padoan, P.S., Sila, U., van den Noord, P. 2013. The sovereign debt crisis in Europe: how to move from bad to good equilibrium. in L. Paganetto (ed.) *Public Debt, Global Governance and Economic Dynamism*, Springer Verlag, Italy, 311-326.

Ratto, M., Roeger, W. and in't Veld, J. 2009. QUEST III: An estimated open economy DSGE model of the Euro area with fiscal and monetary policy. *Economic Modelling*, 26(1), 222-233.

Reinhart, C.M. and Rogoff, K.S. 2010. Growth in a time of debt. *American Economic Review*, 100, 573-578.

Ruiz Estrada, M.A. and Yap, S.F. 2013. The origins and evolution of policy modeling, *Journal of Policy Modeling*, 35, 170-182.

Salvatore, D. 2010. Growth or stagnation after recession for the U.S. and other large advanced economies. *Journal of Policy Modeling*, 32, 637-647.

Saunoris, J.W. and Payne, J.E. 2010. Tax more or spend less? Asymmetries in the UK revenue-expenditure nexus. *Journal of Policy Modeling*, 32, 478-487.

Solow, R. 2010. Building a Science of Economics for the Real World. House Committee on Science and Technology. Subcommittee on Investigation and Oversight. July 20, 2010.

Trachanas E. and Katrakilidis C. 2013. Fiscal deficits under financial pressure and insolvency: Evidence for Italy, Greece and Spain, *Journal of Policy Modeling*, 35, 730-749.

Tramontana, F. 2010. Economics as a compartmental system: a simple macroeconomic example, *International Review of Economics*, 57(4), 347-360.