Interest rate modeling under multiple discounting curves

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Abstract

For deals denominated in a single currency, different collateralization schemes imply different accrual rates for funds posted as collateral, so that we can end up with different current accounts that accrue at different rates and their corresponding discount factors.

In this paper we examine how to incorporate this multiple discounting curves environment in a pricing framework, presenting the different numeraires available and examining how the change of measure works when the corresponding numeraires are associated with different collateralization schemes. The simulation of a stochastic funding curve will also be tackled.

We will assume Heath Jarrow Morton dynamics for the different discounting curves and will obtain the drift restrictions on those curves under different numeraires.

Finally, we will analyze the best strategy to incorporate this multiple discounting curves framework for each single currency in a multi currency setting where different transactions following different collateral schemes are simultaneously modeled, such as a CVA pricing engine

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1 Introduction

In the classical quantitative finance literature it is assumed that there exist a unique risk free curve used by derivatives dealers to borrow and lend funds in the replication process of financial derivatives. In such a framework, interest rate modeling refers to describing the evolution of this single discounting curve in a risk neutral world. The building blocks are the risk free current account and risk free discount factors.

Nevertheless, in the current market environment, counterparty credit risk is a big concern, so that most transactions between financial counterparties are collateralized. Amounts posted in collateral accounts usually earn interest at the OIS rate of the currency of the deal. However, deals are sometimes collateralized in cash denominated in a currency different to the deal currency, or even in bonds or stocks. Each collateralization mechanism will imply a different interest rate earned on funds posted as collateral, so that for a single currency we could end up with different current accounts that accrue at different rates and their corresponding discounting curves. The funding curve of the derivatives dealer also implies an additional current account and its corresponding discount factor curve.

In this new framework some questions arise:

- Can any of the different current accounts associated to each collateral be used as numeraire?
- What about other numeraires (discount factors, annuities)? Can we use as numeraire discount factors or annuities associated to any collateralization scheme?
- If so, what are the risk neutral dynamics of the different discount factor curves in a pricing framework once a particular numeraire associated with a particular collateral has been chosen?
- How can we discount derivatives collateralized under a collateralization scheme different to that of the numeraire?
- How does the change of measure work when we change from measures associated to numeraires with different collateral schemes?
- When valuing non collateralized derivatives, their values will depend, among other factors, on the funding curve of the derivative's hedger. If we wanted to assume a stochastic funding curve, how should it be simulated?
- How should we incorporate this multiple discounting curves framework for each single currency in a multi currency setting such as a CVA pricing engine?

We will try to answer each of these questions in the remaining of the paper. The structure of the paper is as follows:

- In section 2 we will explore the effect of collateralizing deals in an asset different from cash denominated in the same currency as the deal. We will conclude that different assets used as collateral will imply different interest rates earned on funds posted as collateral.

- In section 3, we will summarize our modeling assumptions. As a particular framework, we will assume Heath Jarrow Morton dynamics for the different discount factor curves associated with the different collateralization schemes.
- In section 4 we will analyze the valuation of derivatives associated with different collateral schemes under the spot martingale measure. We will see that under this measure collateralized derivatives must be divided by the current account that accrues at the corresponding collateral rate so that the ratio behaves as a martingale.
- In section 5 we will tackle the valuation of derivatives associated with different collateral schemes under a measure associated with a generic numeraire associated with a particular collateral. We will derive the adjustment to apply to derivatives with a collateral different to that of the numeraire. We will also explore the change of measure between measures associated with different collateral schemes.
- In section 6 we apply the results obtained under Heath Jarrow Morton dynamics for the different discounting curves.
- In section 7 we address stochastic modeling of the derivative's hedger funding curve.
- In section 8 we deal with how to incorporate this multiple discounting curves environment in a multicurrency setting such as a CVA simulation engine.
- In section 9, we summarize the main conclusions obtained.

With the exception of section 7, where we tackle the valuation of uncollateralized trades, we will focus on the replication / valuation of fully collateralized credit derivatives. By fully collateralized we refer to symmetrical collateral schemes with no thresholds, no haircuts, no minimum transfer amounts and with continuous margining [1]. We also assume that the amount posted as collateral coincides with the replication value of the derivative. In section 2 we will see that the asset used as collateral has an impact in the replication cost. Hence, we assume that the amount posted as collateral reflects this impact.

We will abandon the concept of a risk free curve. First it is a theoretical concept, since nowadays no risk manager would consider an investment to be risk free. Second, if it was substituted by a proxy, there would be several candidates. Government rates would never be a valid candidate, since the rates are not accessible for a derivatives hedger while funding. Interest rates paid on collateral funds seem promising, since when a counterparty posts collateral on a fully collateralized deal, she finds herself in a situation that is close to a risk free one, since upon default of the other counterparty the value of the asset (collateral posted) will cancel the value of the liabilities [2] (NPV

[1] In practical terms, the most frequent margining frequency is daily. Therefore, continuous margining is just a theoretical concept.

[2] Unless the default of the counterparty was accompanied by a sudden jump in the derivative's value due to a sudden movement in any of the relevant market factors.

of the derivatives). Having said so, as we will see in section 2, different collateral assets will imply different interest rates earned on collateral funds and that is the reason why there are several candidates. Another problem with choosing a curve associated with a particular collateralization mechanism as a proxy for the risk free curve is that there is no guarantee that it would constitute a floor for funding rates of the different counterparties [3].

Throughout the paper, we will assume that the default of any of the counterparties does not imply a sudden change in the value of market variables or the prices of the assets used as collateral. Although this situation is not realistic, most of the times these risks are unhedgeable because of their systemic nature. We prefer to assume that both counterparties represent the lowest wrong way risk available in the market between two generic counterparties and the corresponding risk factors, and that the Armageddon event would imply the default of both counterparties, so that there is no use in trying to model what can never be hedged.

In the remaining of the paper, every equation will be in matrix form. Sometimes we will point out the dimensions of the different matrices involved. In the equations where this is done, variables with no indication are scalar variables (1×1 matrices).

2 The effect of collateral in the replication strategy

In this section we will examine the effect of collateralizing a trade with a collateral different from cash denominated in the same currency as the deal. We will follow [2].

The most general situation would be using an asset as collateral (could be a stock or bond) that was denominated in a currency different from that of the deal. The deal currency will be referred to as currency D, whereas the collateral currency will be represented by F.

We will use the following notation: r_t^D will represent the OIS rate in currency D, r_t^F the OIS rate in currency F, r_t^C the REPO rate of the collateral asset, C_t the collateral price at time t and X_t the FX rate expressed in D/F.

We assume, in line with market practice, that the counterparty that receives collateral in cash pays interest on it at the OIS rate of the corresponding currency.

We will assume V_t to be the time t derivative's value from the investor's standpoint measured in D. Assuming that V_t is positive, the hedger would have a positive amount V_t in cash in currency D available as a byproduct of the dynamic replication strategy. Nevertheless V_t should be posted by the hedger to the investor in the form of the collateral asset denominated in currency F. Therefore the hedger will have to buy the collateral asset. By doing so, the hedger will be left with a long position in an asset

[3]For example, some derivatives hedgers might fund themselves below the OIS curve

denominated in currency F. Both the FX risk and the exposure to the collateral asset price changes will have to be hedged by the derivatives hedger. Therefore, the hedger will have to enter into these transactions at a generic time step t:

- Exchange V_t in cash denominated in D for cash denominated in F in the spot FX market.
- With the cash obtained from the FX spot transaction, the hedger will buy the collateral asset spot and sell it forward (with maturity $t+dt$) through a REPO transaction. Under the REPO transaction the hedger will deliver at time t $\frac{V_t}{X_t}$ in cash denominated in F in exchange of collateral asset shares with the same value [4].
- These shares in the collateral asset will be posted as collateral to the investor.
- At time $t+dt$ the investor will give the collateral back (with a value of $\frac{V_t C_{t+dt}}{X_t C_t}$ measured in currency F) to the hedger, who will give it back to the REPO counterparty.
- At time $t+dt$ the hedger will receive $\frac{V_t}{X_t}\left(1+r_t^C dt\right)$ from the REPO counterparty in cash denominated in F.
- In order to hedge the FX risk of the last amount, since it is denominated in F, at time t the hedger should sell this amount forward (with maturity $t+dt$) receiving at time $t+dt$ cash in currency D with a value equal to the amount to be paid in currency F ($\frac{V_t}{X_t}\left(1+r_t^C dt\right)$) multiplied by the forward FX rate $X_t\frac{\left(1+r_t^D\right)}{\left(1+\left(r_t^F+b_t\right)\right)}$ seen at time t with maturity $t+dt$. We assume that forward rates cannot be inferred by the spot FX rate and the OIS rates in both currencies, so that an adjustment needs to be made in the F rate. Notice that this adjustment represents the short term cross currency basis and will be represented by b_t.

Both cash transactions (in currencies D and F) and collateral asset transactions occurring at times t and $t+dt$ are represented in figure 1. Notice that if V_t was negative, the trades will be right the opposite.

So that from t to $t+dt$ the value of the funds posted as collateral experiences a change equal to:

$$V_t\left(r_t^D+r_t^C-r_t^F-b_t\right)dt \tag{1}$$

Notice that the interest rate in (1) would be equal to:

- r_t^D if the collateral was cash in D.
- r_t^C if the collateral was an asset denominated in D.
- $r_t^D-b_t$ if the collateral was cash in F.

[4] We assume no haircut in the REPO transaction

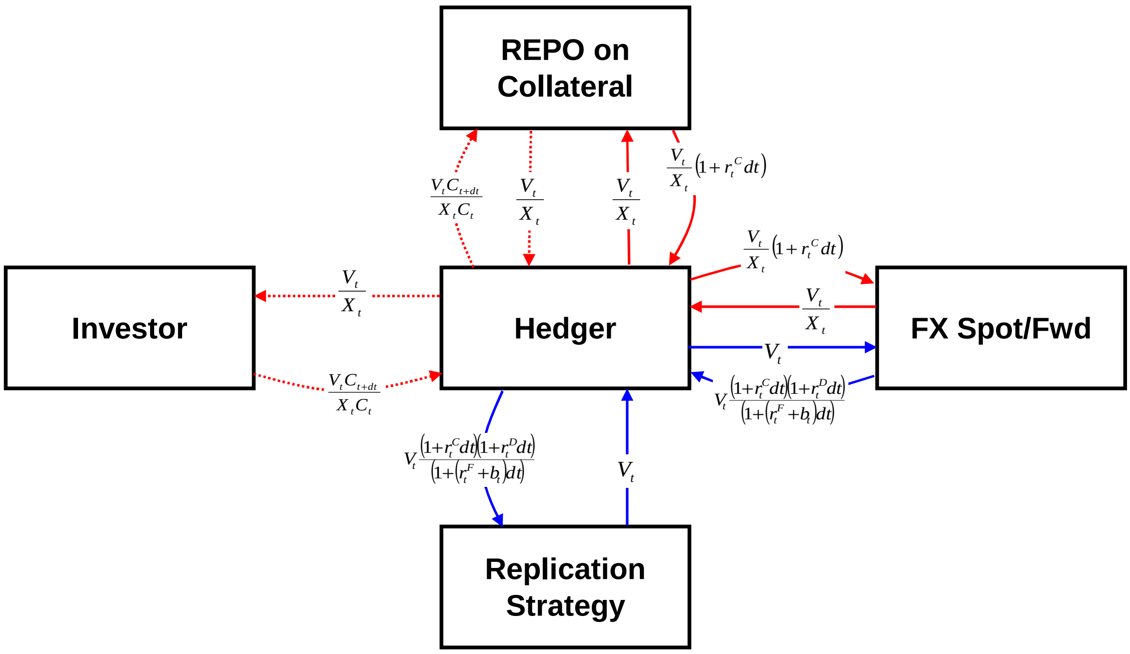

Figure 1: Continuous lines represent cash transactions whereas discontinuous ones represent asset transactions. Blue lines indicate amounts denominated in currency D, whereas red ones represent cash or asset transactions denominated in currency F. Straight lines refer to initial transactions, that take place at time t, and curved lines to final transactions taking place at time $t + dt$.

- $r_t^D + r_t^C - r_t^F - b_t$ if the collateral was an asset denominated in F.

We have seen that collateralizing deals in assets different from cash denominated in the currency of the deal implies additional risks (FX and collateral price changes risks), that once hedged imply that funds posted as collateral accrue at a rate that differs generally from the OIS rate of the deal's currency.

3 Model Assumptions

We assume that for a particular currency there are two different collateralization schemes. We will refer to one of them as the standard one and to the other as the non standard. Generally, for a given currency, the standard collateral will be cash denominated in the same currency. Nevertheless, there can be exceptions to this rule, since for a particular currency the standard collateral, which is reflected in liquid market quotes, could be different from cash denominated in that currency [5].

$B(t,T)$ represents the discount factor curve used to discount cash flows collateralized

[5] For example, cash in USD.

under the standard scheme. Therefore, $B(t,T)$ represents the value at time t of receiving one currency unit at time T but collateralized with the standard collateral.

$$B(t,T) = \exp\left(-\int_{s=t}^{T} f(t,s)ds\right) \tag{2}$$

Where $f(t,T)$ is the instantaneous forward curve for the standard collateral.

Similarly $\widetilde{B}(t,T)$ represents the discount factor curve used to discount cash flows collateralized under the non standard scheme and $\widetilde{f}(t,T)$ its instantaneous forward curve.

$$\widetilde{B}(t,T) = \exp\left(-\int_{s=t}^{T} \widetilde{f}(t,s)ds\right) \tag{3}$$

So that $r_t = f(t,t)$ and $\widetilde{r}_t = \widetilde{f}(t,T)$ are the short term interest rates at which funds posted as collateral accrue under each collateralization scheme [6].

We assume that under the real world measure $\mathbb{P}$, the evolutions of $f(t,T)$ and $\widetilde{f}(t,T)$ follow

$$\begin{aligned} df(t,T) &= \mu^f(t,T)dt + \underbrace{\sigma^f(t,T)}_{1\times n}\underbrace{dW_t^{\mathbb{P}}}_{n\times 1} \\ d\widetilde{f}(t,T) &= \mu^{\widetilde{f}}(t,T)dt + \underbrace{\sigma^{\widetilde{f}}(t,T)}_{1\times n}\underbrace{dW_t^{\mathbb{P}}}_{n\times 1} + \underbrace{\widetilde{\sigma}^{\widetilde{f}}(t,T)}_{1\times m}\underbrace{dZ_t^{\mathbb{P}}}_{m\times 1} \end{aligned} \tag{4}$$

Where $W_t^{\mathbb{P}}$ and $Z_t^{\mathbb{P}}$ represent vectors of independent Wiener processes under $\mathbb{P}$ of dimensions n and m respectively. $\mu^f(t,T)$ and $\mu^{\widetilde{f}}(t,T)$ are real world drifts of the two processes and $\sigma^f(t,T)$, $\sigma^{\widetilde{f}}(t,T)$, $\widetilde{\sigma}^{\widetilde{f}}(t,T)$ their volatilities. $W_t^{\mathbb{P}}$ and $Z_t^{\mathbb{P}}$ are also independent of each other.

The evolutions of the current accounts that accrue at r_t and $\widetilde{r}_t$ are governed by the following differential equations:

$$\begin{aligned} dC_t &= r_t C_t dt \\ d\widetilde{C}_t &= \widetilde{r}_t \widetilde{C}_t dt \end{aligned} \tag{5}$$

[6] Once the additional risks described in section 2 have been hedged.

In this paper we will only analyze the effect of multiple discounting curves, letting aside the tenor basis [7]. Therefore we assume that the tenor basis is non stochastic.

In the following sections we will try to price derivatives with standard and non standard collateral. E_t will represent the time t value of a derivative with standard collateral and $\widetilde{E}_t$ the value of a derivative with non standard collateral. We will also assume that the cashflows of E_t only depend on interest rate indexes referenced to $B(t,T)$, therefore E_t will only depend on $W_t^{\mathbb{P}}$. On the other hand, $\widetilde{E}_t$ will depend on both $W_t^{\mathbb{P}}$ and $Z_t^{\mathbb{P}}$. Hence, Itô's Lemma, together with (4) imply

$$
\begin{aligned}
dE_t &= \mu_t^E dt + \underbrace{\sigma_t^E}_{1\times n} \underbrace{dW_t^{\mathbb{P}}}_{n\times 1} \\
d\widetilde{E}_t &= \mu_t^{\widetilde{E}} dt + \underbrace{\sigma_t^{\widetilde{E}}}_{1\times n} \underbrace{dW_t^{\mathbb{P}}}_{n\times 1} + \underbrace{\widetilde{\sigma}_t^{\widetilde{E}}}_{1\times m} \underbrace{dZ_t^{\mathbb{P}}}_{m\times 1}
\end{aligned}
\tag{6}
$$

μ_t^E and $\mu_t^{\widetilde{E}}$ are the real world drifts of both processes and σ_t^E, $\sigma_t^{\widetilde{E}}$ and $\widetilde{\sigma}_t^{\widetilde{E}}$ their volatilities.

In order to replicate E_t, we will use a set of n interest rate derivatives collateralized under the standard scheme and whose cashflows only depend on $B(t,T)$. H_t will be a $n \times 1$ vector representing the prices at time t of these. The stochastic differential equation followed by H_t under the real world measure will be given by:

$$
\underbrace{dH_t}_{n\times 1} = \underbrace{\mu_t^H}_{n\times 1} dt + \underbrace{\sigma_t^H}_{n\times n} \underbrace{dW_t^{\mathbb{P}}}_{n\times 1} \tag{7}
$$

Where the size of the different matrices has been pointed out.

In order to replicate $\widetilde{E}_t$ we will use H_t plus m additional instruments collateralized under the non standard collateral [8] due to the dependence of $\widetilde{E}_t$ on $\widetilde{B}(t,T)$. $\widetilde{H}_t$ represents the t price of this set of additional hedging instruments. The stochastic differential equation followed by $\widetilde{H}_t$ under the real world measure will be given by:

$$
\underbrace{d\widetilde{H}_t}_{m\times 1} = \underbrace{\mu_t^{\widetilde{H}}}_{m\times 1} dt + \underbrace{\sigma_t^{\widetilde{H}}}_{m\times n} \underbrace{dW_t^{\mathbb{P}}}_{n\times 1} + \underbrace{\widetilde{\sigma}_t^{\widetilde{H}}}_{m\times m} \underbrace{dZ_t^{\mathbb{P}}}_{m\times 1} \tag{8}
$$

We would like to point out that μ_t^E, $\mu_t^{\widetilde{E}}$, μ_t^H and $\mu_t^{\widetilde{H}}$ are real world drifts.

[7] Basis due to different tenors of floating references.

[8] We could have assumed that $\widetilde{E}_t$ is hedged with $n+m$ derivatives collateralized with non standard collateral obtaining the same conclusions. We have chosen this alternative for didactic reasons.

4 Valuing derivatives under the spot martingale measure

In this section we will deal with the valuation of both E_t and $\widetilde{E}_t$ under the spot martingale measure, that is the measure associated with current accounts as numeraire.

4.1 Derivatives with standard collateral

The hedging formula will be the following.

$$E_t = \underbrace{\alpha_t}_{1\times n} \underbrace{H_t}_{n\times 1} + C_t \tag{9}$$

C_t represents funds posted as collateral by the hedger [9], H_t the value of the hedging instruments from the hedger's perpective, α_t is a vector that contains the amounts to invest in each one of the components of H_t in order to hedge the risks of E_t. E_t represents the value of the derivative to be replicated from the risk taker's perspective (which implies that the value from the risk hedger's perspective is $-E_t$).

Taking into account the stochastic differential equations followed by E_t and H_t, the replication equation in differential form will be given by

$$\mu_t^E dt + \sigma_t^E dW_t^{\mathbb{P}} - E_t r_t dt = \alpha_t \left(\mu_t^H dt + \sigma_t^H dW_t^{\mathbb{P}} - H_t r_t dt \right) \tag{10}$$

Where we have taken into account that fact that C_t accrues at r_t and that $C_t = E_t - \alpha_t H_t$

In order to be hedged α_t must be chosen so that the terms in $dW_t^{\mathbb{P}}$ in both sides of (9) are canceled. For this to happen, α_t must be the solution of the following system of linear equations:

$$\sigma_t^E = \alpha_t \sigma_t^H \tag{11}$$

So that the real world drifts must follow:

[9] If $E_t > 0$, the hedger receives E_t from the risk taker and posts it as collateral (and the opposite if $E_t \leq 0$) to the risk taker. The risk hedger trades $\alpha_t H_t$ with interbank counterparties paying its value and receiving it as collateral from the same interbank counterparties.

$$\mu_t^E - E_t r_t = \alpha_t \left(\mu_t^H - H_t r_t \right) \tag{12}$$

Being in a complete market [10] together with the absence of arbitrage opportunities [11] implies both (11) and (12).

On the other hand, Girsanov theorem guarantees that when we perform a change of measure from real world measure $\mathbb{P}$ to an equivalent measure $\mathbb{Q}$, $\mathbb{P}$ and $\mathbb{Q}$ Wiener processes are related through

$$\begin{aligned} \underbrace{dW_t^{\mathbb{Q}}}_{n\times 1} &= \underbrace{dW_t^{\mathbb{P}}}_{n\times 1} - \underbrace{\gamma_t}_{n\times 1} dt \\ \underbrace{dZ_t^{\mathbb{Q}}}_{m\times 1} &= \underbrace{dZ_t^{\mathbb{P}}}_{m\times 1} - \underbrace{\widetilde{\gamma}_t}_{m\times 1} dt \end{aligned} \tag{13}$$

Where γ_t and $\widetilde{\gamma}_t$ are non anticipative processes of dimensions n and m that describe the change of measure.

Girsanov Theorem also implies that under $\mathbb{Q}$ the drift of H_t will be given by

$$\underbrace{\mu_t^H}_{n\times 1} - \underbrace{\sigma_t^H}_{n\times n} \underbrace{\gamma_t}_{n\times 1}$$

If we wanted to change to a measure $\mathbb{Q}$ where the drift of H_t was given by $H_t r_t$, γ_t will be the solution to:

$$\underbrace{\mu_t^H}_{n\times 1} - \underbrace{\sigma_t^H}_{n\times n} \underbrace{\gamma_t}_{n\times 1} = \underbrace{H_t}_{n\times 1} r_t \tag{14}$$

Up to this point we will have no condition for $\widetilde{\gamma}_t$, although it will be revealed in the next subsection.

Now we will explore what the drift of E_t is under $\mathbb{Q}$. Girsanov theorem implies:

$$\underbrace{\mu_t^E}_{1\times 1} - \underbrace{\sigma_t^E}_{1\times n} \underbrace{\gamma_t}_{n\times 1} \tag{15}$$

Plugging (11) and (12) into (15) and taking into account (14)

[10] The value of a derivative can be replicated with a set of hedging instruments.
[11] The value of a derivative must coincide with the value of the replicating portfolio.

$$\underbrace{\mu_t^E}_{1\times 1} - \underbrace{\sigma_t^E}_{1\times n}\underbrace{\gamma_t}_{n\times 1} = \alpha_t\left(\mu_t^H - H_t r_t\right) + E_t r_t - \alpha_t \sigma_t^H \gamma_t = E_t r_t \tag{16}$$

So that under $\mathbb{Q}$ any interest rate derivative with standard collateral follows

$$dE_t = E_t r_t dt + \sigma_t^E dW_t^{\mathbb{Q}}$$

Which implies that

$$E_t = E_{\mathbb{Q}}\left[E_T \exp\left(-\int_{s=t}^{T} r_s ds\right)\Big|\mathcal{F}_t\right] \Rightarrow \frac{E_t}{\beta_t} = E_{\mathbb{Q}}\left[\frac{E_T}{\beta_T}\Big|\mathcal{F}_t\right]$$

Where $\beta_T = \exp\left(\int_{s=0}^{T} r_s ds\right)$ represents the current account that accrues at the standard collateral rate r_t.

Notice that nothing new has been obtained in this section. We have just confirmed the fundamental theorem of asset pricing in a collateralization framework as was already obtained in [4].

4.2 Derivatives with non standard collateral

In this case, the hedging equation will be

$$\widetilde{E}_t = \underbrace{\alpha_t}_{1\times n}\underbrace{H_t}_{n\times 1} + \underbrace{\epsilon_t}_{1\times m}\underbrace{\widetilde{H}_t}_{m\times 1} + C_t + \widetilde{C}_t \tag{17}$$

H_t and $\widetilde{H}_t$ are the values of the hedging instruments from the hedger's perspective. $-\widetilde{E}_t$ is the value of the derivative to be replicated also from the hedger's point of view. α_t and ϵ_t are the amounts to invest in each component of H_t and $\widetilde{H}_t$ respectively. C_t and $\widetilde{C}_t$ represent amounts posted as collateral by the hedger in the standard and non standard collateralization schemes respectively.

Notice that due to the fact that only H_t is collateralized under the standard scheme and both $\widetilde{H}_t$ and $\widetilde{E}_t$ under the non standard the following must hold:

$$\begin{aligned} C_t &= -\alpha_t H_t \\ \widetilde{C}_t &= \widetilde{E}_t - \epsilon_t \widetilde{H}_t \end{aligned} \tag{18}$$

The hedging equation under $\mathbb{P}$ in differential form will be:

$$\begin{aligned}\mu_t^{\widetilde{E}} dt + \sigma_t^{\widetilde{E}} dW_t^{\mathbb{P}} + \widetilde{\sigma}_t^{\widetilde{E}} dZ_t^{\mathbb{P}} - E_t \widetilde{r}_t &= \alpha_t \left(\mu_t^H dt + \sigma_t^H dW_t^{\mathbb{P}} - H_t r_t dt \right) \\ &\quad + \epsilon_t \left(\mu_t^{\widetilde{H}} dt + \sigma_t^{\widetilde{H}} dW_t^{\mathbb{P}} + \widetilde{\sigma}_t^{\widetilde{H}} dZ_t^{\mathbb{P}} - \widetilde{H}_t \widetilde{r}_t dt \right)\end{aligned} \tag{19}$$

In order to be hedged, terms in $dW_t^{\mathbb{P}}$ and $dZ_t^{\mathbb{P}}$ in (19) should be canceled. Therefore α_t and ϵ_t must be the solution to the following system of linear equations:

$$\begin{aligned}\sigma_t^{\widetilde{E}} &= \alpha_t \sigma_t^H + \epsilon_t \sigma_t^{\widetilde{H}} \\ \widetilde{\sigma}_t^{\widetilde{E}} &= \epsilon_t \widetilde{\sigma}_t^{\widetilde{H}}\end{aligned} \tag{20}$$

So that the condition followed by the drifts under the real world measure is

$$\mu_t^{\widetilde{E}} - \widetilde{E}_t \widetilde{r}_t = \alpha_t \left(\mu_t^H - H_t r_t \right) + \epsilon_t \left(\mu_t^{\widetilde{H}} - \widetilde{H}_t \widetilde{r}_t \right) \tag{21}$$

In the previous section we imposed a change of measure from real world measure $\mathbb{P}$ to the spot martingale measure $\mathbb{Q}$ by imposing that the $\mathbb{Q}$ drift of H_t becomes $H_t r_t$. In this section we are analyzing the hedge of $\widetilde{H}_t$, which carries a collateralization scheme different from the standard. Since we find ourselves in unexplored territory, let's leave the drift of $\widetilde{H}_t$ under $\mathbb{Q}$ as $\widetilde{H}_t z_t$, where z_t will be determined thereon. So that once z_t is known, $\widetilde{\gamma}_t$ will be given by the solution to the following system of linear equations (notice that γ_t was obtained in the last subsection):

$$\underbrace{\mu_t^{\widetilde{H}}}_{m\times 1} - \underbrace{\sigma_t^{\widetilde{H}}}_{m\times n} \underbrace{\gamma_t}_{n\times 1} - \underbrace{\widetilde{\sigma}_t^{\widetilde{H}}}_{m\times m} \underbrace{\widetilde{\gamma}_t}_{m\times 1} = \underbrace{\widetilde{H}_t}_{m\times 1} \underbrace{z_t}_{1\times 1} \tag{22}$$

So that the change of measure performed on $\widetilde{E}_t$ implies a new drift that is equal to

$$\mu_t^{\widetilde{E}} - \sigma_t^{\widetilde{E}} \gamma_t - \widetilde{\sigma}_t^{\widetilde{E}} \widetilde{\gamma}_t \tag{23}$$

Plugging (20) and (21) into (23) and taking into account both (14) and (22) imply

$$
\begin{aligned}
\mu_t^{\widetilde{E}} - \sigma_t^{\widetilde{E}}\gamma_t - \widetilde{\sigma}_t^{\widetilde{E}}\widetilde{\gamma}_t &= \widetilde{E}_t\widetilde{r}_t + \alpha_t\left(\mu_t^H - H_t r_t\right) + \epsilon_t\left(\mu_t^{\widetilde{H}} - \widetilde{H}_t\widetilde{r}_t\right) \\
&\quad - \left(\alpha_t\sigma_t^H + \epsilon_t\sigma_t^{\widetilde{H}}\right)\gamma_t - \epsilon_t\widetilde{\sigma}_t^{\widetilde{H}}\widetilde{\gamma}_t \\
&= \widetilde{E}_t\widetilde{r}_t + \epsilon_t\widetilde{H}_t\left(z_t - \widetilde{r}_t\right)
\end{aligned} \tag{24}
$$

Notice that if $z_t = \widetilde{r}_t$ the drift of $\widetilde{E}_t$ becomes $\widetilde{E}_t\widetilde{r}_t$. Any other value of z_t will imply a drift of $\widetilde{E}_t$ under $\mathbb{Q}$ that depends on the particular characteristics of the contract being replicated (which are reflected in ϵ_t) and is therefore useless from a pricing perspective. Hence, under $\mathbb{Q}$ the growth rate of every derivative with standard collateral (either H_t or E_t) becomes r_t and the growth rate of any derivative with non standard collateral (either $\widetilde{H}_t$ or $\widetilde{E}_t$) becomes $\widetilde{r}_t$.

$$
\begin{aligned}
dE_t &= E_t r_t dt + \sigma_t^E dW_t^{\mathbb{Q}} \\
d\widetilde{E}_t &= \widetilde{E}_t\widetilde{r}_t dt + \sigma_t^{\widetilde{E}} dW_t^{\mathbb{Q}} + \widetilde{\sigma}_t^{\widetilde{E}} dZ_t^{\mathbb{Q}}
\end{aligned}
$$

That are equivalent to

$$
\begin{aligned}
E_t &= E_{\mathbb{Q}}\left[E_T \exp\left(-\int_{s=t}^T r_s ds\right)\Big|\mathcal{F}_t\right] \;\Rightarrow\; \frac{E_t}{\beta_t} = E_{\mathbb{Q}}\left[\frac{E_T}{\beta_T}\Big|\mathcal{F}_t\right] \\
\widetilde{E}_t &= E_{\mathbb{Q}}\left[\widetilde{E}_T \exp\left(-\int_{s=t}^T \widetilde{r}_s ds\right)\Big|\mathcal{F}_t\right] \;\Rightarrow\; \frac{\widetilde{E}_t}{\widetilde{\beta}_t} = E_{\mathbb{Q}}\left[\frac{\widetilde{E}_T}{\widetilde{\beta}_T}\Big|\mathcal{F}_t\right]
\end{aligned} \tag{25}
$$

Where $\beta_T = \exp\left(\int_{s=0}^T r_s ds\right)$ represents the current account that accrues at the standard collateral rate r_t and $\widetilde{\beta}_T = \exp\left(\int_{s=0}^T \widetilde{r}_s ds\right)$ represents the current account that accrues at the non standard collateral rate $\widetilde{r}_t$.

Notice that under measure $\mathbb{Q}$ there seems to be two different numeraires: the standard collateral current account β_t used to deflate derivatives with standard collateral and the non standard collateral current account $\widetilde{\beta}_t$ used to deflate derivatives with non standard collateral. This result was obtained, for example, in [5].

We could also have written

$$
\begin{aligned}
\frac{\widetilde{E}_t}{\beta_t} &= E_{\mathbb{Q}}\left[\exp\left(\int_{s=t}^T \left(r_s - \widetilde{r}_s\right) ds\right)\frac{\widetilde{E}_T}{\beta_T}\Big|\mathcal{F}_t\right] \\
\frac{E_t}{\widetilde{\beta}_t} &= E_{\mathbb{Q}}\left[\exp\left(\int_{s=t}^T \left(\widetilde{r}_s - r_s\right) ds\right)\frac{E_T}{\widetilde{\beta}_T}\Big|\mathcal{F}_t\right]
\end{aligned} \tag{26}
$$

This last expression will be analyzed in subsection 5.4.

In the next section we will generalize the results obtained so far to a numeraire different from current accounts (such as discount factors, annuities...)

5 Change of numeraire

In this section we assume that we use as numeraire a derivative with standard collateral whose cashflows are referenced to the curve $B(t,T)$. Therefore we will assume that any of the components of H_t whose value cannot vanish is used as numeraire, so that under the real world measure $\mathbb{P}$ the evolution of the numeraire N_t will be governed by

$$dN_t = \mu_t^N N_t dt + N_t \underbrace{\sigma_t^N}_{1\times n} \underbrace{dW_t^{\mathbb{P}}}_{n\times 1}$$

N_t could, for example, be annuities or discount factors collateralized under the standard scheme.

5.1 Derivatives with standard collateral

Again, the hedging equation will be

$$E_t = \underbrace{\alpha_t}_{1\times n} \underbrace{H_t}_{n\times 1} + C_t \tag{27}$$

We divide every term by the numeraire N_t, so that we define

$$\begin{aligned} e_t &:= \tfrac{E_t}{N_t} \\ \underbrace{h_t}_{n\times 1} &:= \underbrace{H_t}_{n\times 1} \underbrace{\frac{1}{N_t}}_{1\times 1} \\ c_t &:= \tfrac{C_t}{N_t} \end{aligned} \tag{28}$$

So that the hedging equation, once every term has been divided by the numeraire, is

$$e_t = \alpha_t h_t + c_t \tag{29}$$

And in differential form

$$\mu_t^e dt + \underbrace{\sigma_t^e}_{1\times n}\underbrace{dW_t^{\mathbb{P}}}_{n\times 1} = \underbrace{\alpha_t}_{1\times n}\left(\underbrace{\mu_t^h}_{n\times 1} dt + \underbrace{\sigma_t^h}_{n\times n}\underbrace{dW_t^{\mathbb{P}}}_{n\times 1}\right) + \mu_t^c dt + \underbrace{\sigma_t^c}_{1\times n}\underbrace{dW_t^{\mathbb{P}}}_{n\times 1} \tag{30}$$

μ_t^e, μ_t^h and μ_t^c are the $\mathbb{P}$ drifts of the deflated processes and σ_t^e, σ_t^h and σ_t^c their volatilities.

Notice that c_t has a diffusion different from 0 since C_t has been divided by a numeraire with non zero diffusion.

In order to be hedged, α_t must be the solution to

$$\sigma_t^e = \alpha_t \sigma_t^h + \sigma_t^c \tag{31}$$

So that the real world drifts must follow in this complete market / no arbitrage environment

$$\mu_t^e = \alpha_t \mu_t^h + \mu_t^c \tag{32}$$

Let's now apply a change of measure from $\mathbb{P}$ to an equivalent martingale measure $\mathbb{N}$ associated with N_t that vanishes the drift of every component of h_t

$$\begin{aligned} \underbrace{\mu_t^h}_{n\times 1} - \underbrace{\sigma_t^h}_{n\times n}\underbrace{\gamma_t}_{n\times 1} &= \underbrace{0}_{n\times 1} \\ \underbrace{\mu_t^c}_{1\times 1} - \underbrace{\sigma_t^c}_{1\times n}\underbrace{\gamma_t}_{n\times 1} &= \underbrace{0}_{1\times 1} \end{aligned} \tag{33}$$

Notice that the first equation in (33) will not be enough to determine γ_t, since N_t will be a component of H_t, so that $\frac{N_t}{N_t}$ will have null drift under every measure. We must also impose that the current account that accrues at the collateral rate r_t divided by the numeraire has also zero drift. This is reflected in the second equation in (33), so that both expressions help us determine γ_t.

The drift of e_t under $\mathbb{N}$ will be given by

$$\mu_t^e - \sigma_t^e \gamma_t \tag{34}$$

Plugging (31) and (32) into (34) and taking into account (33) implies

$$\mu_t^e - \sigma_t^e \gamma_t \;=\; \alpha_t \mu_t^h + \mu_t^c - \left(\alpha_t \sigma_t^h + \sigma_t^c\right)\gamma_t = 0 \tag{35}$$

So that μ_t^e has also zero drift under $\mathbb{N}$. This implies that

$$\frac{E_t}{N_t} = E_{\mathbb{N}}\left[\frac{E_T}{N_T}\Big|\mathcal{F}_t\right] \tag{36}$$

Notice that in this subsection we have just confirmed the change of numeraire result in a collateralization framework. In the next subsection we analyze the effect of the change of measure introduced in this section in derivatives collateralized with the non standard collateral.

5.2 Derivatives with non standard collateral

The hedging equation will be given by

$$\widetilde{E}_t = \underbrace{\alpha_t}_{1\times n}\underbrace{H_t}_{n\times 1} + \underbrace{\epsilon_t}_{1\times m}\underbrace{\widetilde{H}_t}_{m\times 1} + C_t + \widetilde{C}_t \tag{37}$$

We divide every component in (37) by N_t, so that we define the following terms

$$\begin{aligned}
\widetilde{e}_t &:= \tfrac{\widetilde{E}_t}{N_t} \\
\underbrace{h_t}_{n\times 1} &:= \underbrace{H_t}_{n\times 1}\underbrace{\frac{1}{N_t}}_{1\times 1} \\
\underbrace{\widetilde{h}_t}_{m\times 1} &:= \underbrace{\widetilde{H}_t}_{m\times 1}\underbrace{\frac{1}{N_t}}_{1\times 1} \\
c_t &:= \tfrac{C_t}{N_t} \\
\widetilde{c}_t &:= \tfrac{\widetilde{C}_t}{N_t}
\end{aligned} \tag{38}$$

So that once it has been divided by N_t, the hedging equation becomes

$$\widetilde{e}_t = \alpha_t h_t + \epsilon_t \widetilde{h}_t + c_t + \widetilde{c}_t \tag{39}$$

And in differential form

$$\mu_t^{\widetilde{e}} dt + \underbrace{\sigma_t^{\widetilde{e}}}_{1\times n} \underbrace{dW_t^{\mathbb{P}}}_{n\times 1} + \underbrace{\widetilde{\sigma}_t^{\widetilde{e}}}_{1\times m} \underbrace{dZ_t^{\mathbb{P}}}_{m\times 1} = \underbrace{\alpha_t}_{1\times n} \left(\underbrace{\mu_t^h}_{n\times 1} dt + \underbrace{\sigma_t^h}_{n\times n} \underbrace{dW_t^{\mathbb{P}}}_{n\times 1} \right)$$

$$+ \underbrace{\epsilon_t}_{1\times m} \left(\underbrace{\mu_t^{\widetilde{h}}}_{m\times 1} dt + \underbrace{\sigma_t^{\widetilde{h}}}_{m\times n} \underbrace{dW_t^{\mathbb{P}}}_{n\times 1} + \underbrace{\widetilde{\sigma}_t^{\widetilde{h}}}_{m\times m} \underbrace{dZ_t^{\mathbb{P}}}_{m\times 1} \right) \tag{40}$$

$$+\mu_t^c dt + \underbrace{\sigma_t^c}_{1\times n} \underbrace{dW_t^{\mathbb{P}}}_{n\times 1}$$

$$+\mu_t^{\widetilde{c}} dt + \underbrace{\sigma_t^{\widetilde{c}}}_{1\times n} \underbrace{dW_t^{\mathbb{P}}}_{n\times 1}$$

Again, the drifts in the last equation are the real world measure drifts of the deflated processes.

Notice that both c_t and $\widetilde{c}_t$ have non zero diffussions. Also notice that due to the fact that N_t solely depends on $W_t^{\mathbb{P}}$, neither c_t nore $\widetilde{c}_t$ depend on $Z_t^{\mathbb{P}}$.

In order to be hedged, α_t and ϵ_t must be obtained from

$$\begin{aligned} \sigma_t^{\widetilde{e}} &= \alpha_t \sigma_t^h + \epsilon_t \sigma_t^{\widetilde{h}} + \sigma_t^c + \sigma_t^{\widetilde{c}} \\ \widetilde{\sigma}_t^{\widetilde{e}} &= \epsilon_t \widetilde{\sigma}_t^{\widetilde{h}} \end{aligned} \tag{41}$$

So that terms in $dW_t^{\mathbb{P}}$ and $dZ_t^{\mathbb{P}}$ are canceled, which yields a relationship between the real world drifts

$$\mu_t^{\widetilde{e}} = \alpha_t \mu_t^h + \epsilon_t \mu_t^{\widetilde{h}} + \mu_t^c + \mu_t^{\widetilde{c}} \tag{42}$$

Now let's assume that we perform the same change of measure that was discussed in the last subsection and that produced zero drifts for both h_t and e_t. Since we are again in an unexplored territory, due to the fact that both $\widetilde{E}_t$ and $\widetilde{H}_t$ are collateralized with the non standard collateral, we assume that $\mathbb{N}$ implies a drift of $\widetilde{h}_t z_t$ in $\widetilde{h}_t$, where z_t will again be determined thereon.

$$\mu_t^{\widetilde{h}} - \sigma_t^{\widetilde{h}} \gamma_t - \widetilde{\sigma}_t^{\widetilde{h}} \widetilde{\gamma}_t = \widetilde{h}_t z_t \tag{43}$$

Notice that (43) will help us determine $\widetilde{\gamma}_t$ once z_t is known (γ_t has already been determined in subsection 5.1).

Let's analyze the relationship between the drifts of c_t and $\widetilde{c}_t$ under $\mathbb{N}$. If we apply Itô's Lemma to c_t under $\mathbb{N}$

$$c_t = \frac{C_t}{N_t} \;\Rightarrow\; dc_t = c_t \left(r_t dt - \mu_t^{N,\mathbb{N}} dt - \sigma_t^N dW_t^{\mathbb{N}} + (\sigma_t^N)^2 dt \right) \tag{44}$$

Where $\mu_t^{N,\mathbb{N}}$ is the $\mathbb{N}$ drift of N_t.

Doing the same to $\widetilde{c}_t$

$$\widetilde{c}_t = \frac{\widetilde{C}_t}{N_t} \;\Rightarrow\; d\widetilde{c}_t = \widetilde{c}_t \left(\widetilde{r}_t dt - \mu_t^{N,\mathbb{N}} dt - \sigma_t^N dW_t^{\mathbb{N}} + (\sigma_t^N)^2 dt \right) \tag{45}$$

So that if $\mu_t^{c,\mathbb{N}} = 0$ (as imposed in 5.1), $\mu_t^{\widetilde{c},\mathbb{N}}$ will be given by

$$\mu_t^{c,\mathbb{N}} = 0 \;\Rightarrow \mu_t^{N,\mathbb{N}} = r_t + (\sigma_t^N) \Rightarrow \mu_t^{\widetilde{c},\mathbb{N}} = \widetilde{c}_t \left(\widetilde{r}_t - r_t \right) \tag{46}$$

If we apply Girsanov's theorem to $\widetilde{e}_t$, its drift under $\mathbb{N}$ is given by

$$\mu_t^{\widetilde{e}} - \sigma_t^{\widetilde{e}} \gamma_t - \widetilde{\sigma}_t^{\widetilde{e}} \widetilde{\gamma}_t \tag{47}$$

Plugging (41) and (42) in the last equation and taking into account (33) and (43)

$$\begin{aligned}
\mu_t^{\widetilde{e}} - \sigma_t^{\widetilde{e}} \gamma_t - \widetilde{\sigma}_t^{\widetilde{e}} \widetilde{\gamma}_t &= \alpha_t \mu_t^h + \epsilon_t \mu_t^{\widetilde{h}} + \mu_t^c + \mu_t^{\widetilde{c}} \\
&\quad - \left(\alpha_t \sigma_t^h + \epsilon_t \sigma_t^{\widetilde{h}} + \sigma_t^c + \sigma_t^{\widetilde{c}} \right) \gamma_t - \epsilon_t \widetilde{\sigma}_t^{\widetilde{h}} \widetilde{\gamma}_t \\
&= \underbrace{\mu_t^{\widetilde{c}} - \sigma_t^{\widetilde{c}} \gamma_t}_{\mu_t^{\widetilde{c},\mathbb{N}}} + \epsilon_t \widetilde{h}_t z_t
\end{aligned} \tag{48}$$

And taking into account (46)

$$\mu_t^{\widetilde{e}} - \sigma_t^{\widetilde{e}} \gamma_t - \widetilde{\sigma}_t^{\widetilde{e}} \widetilde{\gamma}_t = \widetilde{c}_t \left(\widetilde{r}_t - r_t \right) + \epsilon_t \widetilde{h}_t z_t \tag{49}$$

Since $\widetilde{C}_t = \widetilde{E}_t - \epsilon_t \widetilde{H}_t$, then $\widetilde{c}_t = \widetilde{e}_t - \epsilon_t \widetilde{h}_t$, so that

$$\mu_t^{\widetilde{e}} - \sigma_t^{\widetilde{e}} \gamma_t - \widetilde{\sigma}_t^{\widetilde{e}} \widetilde{\gamma}_t = \widetilde{e}_r \left(\widetilde{r}_t - r_t \right) + \epsilon_t \widetilde{h}_t \left(z_t - \left(\widetilde{r}_t - r_t \right) \right) \tag{50}$$

Notice that unless $z_t = \widetilde{r}_t - r_t$, the drift of $\widetilde{e}_t$ would depend on the particular characteristics of $\widetilde{E}_t$, so that the only valid drift for valuation purposes would be

$$\mu_t^{\widetilde{e}} - \sigma_t^{\widetilde{e}} \gamma_t - \widetilde{\sigma}_t^{\widetilde{e}} \widetilde{\gamma}_t = \widetilde{e}_r \left(\widetilde{r}_t - r_t \right) \tag{51}$$

Which implies

$$\frac{\widetilde{E}_t}{N_t} = E_{\mathbb{N}} \left[\exp \left(\int_{s=t}^{T} \left(r_s - \widetilde{r}_s \right) ds \right) \frac{\widetilde{E}_T}{N_T} \Big| \mathcal{F}_t \right] \tag{52}$$

5.3 Using numeraires with non standard collateral

In subsections 5.1 and 5.2 we chose as numeraire a derivative whose payments depend on the standard collateral discounting curve $B(t,T)$, that was collateralized with the standard collateral and whose price cannot vanish.

Notice that if we had assumed that the numeraire is collateralized under the non standard scheme and that its cashflows just depended on $\widetilde{B}(t,T)$, the situation would be exactly symmetrical as the one analyzed in 5.1 and 5.2, so that we would have obtained:

$$\frac{\widetilde{E}_t}{\widetilde{N}_t} = E_{\widetilde{\mathbb{N}}} \left[\frac{\widetilde{E}_T}{\widetilde{N}_T} \Big| \mathcal{F}_t \right] \tag{53}$$

$$\frac{E_t}{\widetilde{N}_t} = E_{\widetilde{\mathbb{N}}} \left[\exp \left(\int_{s=t}^{T} \left(\widetilde{r}_s - r_s \right) ds \right) \frac{E_T}{\widetilde{N}_T} \Big| \mathcal{F}_t \right] \tag{54}$$

5.4 The zero vol FX analogy

So far, we have obtained the following

$$
\begin{aligned}
&\frac{E_t}{\beta_t} = E_{\mathbb{Q}}\left[\frac{E_T}{\beta_T}\middle|\mathcal{F}_t\right] && \frac{\widetilde{E}_t}{\beta_t} = E_{\mathbb{Q}}\left[\exp\left(\int_{s=t}^{T}(r_s-\widetilde{r}_s)\,ds\right)\frac{\widetilde{E}_T}{\beta_T}\middle|\mathcal{F}_t\right]\\
&\frac{\widetilde{E}_t}{\widetilde{\beta}_t} = E_{\mathbb{Q}}\left[\frac{\widetilde{E}_T}{\widetilde{\beta}_T}\middle|\mathcal{F}_t\right] && \frac{E_t}{\widetilde{\beta}_t} = E_{\mathbb{Q}}\left[\exp\left(\int_{s=t}^{T}(\widetilde{r}_s-r_s)\,ds\right)\frac{E_T}{\widetilde{\beta}_T}\middle|\mathcal{F}_t\right]\\
&\frac{E_t}{N_t} = E_{\mathbb{N}}\left[\frac{E_T}{N_T}\middle|\mathcal{F}_t\right] && \frac{\widetilde{E}_t}{N_t} = E_{\mathbb{N}}\left[\exp\left(\int_{s=t}^{T}(r_s-\widetilde{r}_s)\,ds\right)\frac{\widetilde{E}_T}{N_T}\middle|\mathcal{F}_t\right]\\
&\frac{\widetilde{E}_t}{\widetilde{N}_t} = E_{\widetilde{\mathbb{N}}}\left[\frac{\widetilde{E}_T}{\widetilde{N}_T}\middle|\mathcal{F}_t\right] && \frac{E_t}{\widetilde{N}_t} = E_{\widetilde{\mathbb{N}}}\left[\exp\left(\int_{s=t}^{T}(\widetilde{r}_s-r_s)\,ds\right)\frac{E_T}{\widetilde{N}_T}\middle|\mathcal{F}_t\right]
\end{aligned} \tag{55}
$$

Notice that the expressions in (55) would appear in a cross currency setting where deals with standard collateral were denominated in the local currency and deals with non standard collateral in a foreign currency, such that r_t is the domestic short rate, $\widetilde{r}_t$ the foreign short rate and the spot FX rate ζ_t expressed in D/F followed under any measure [12] the following stochastic differential equation:

$$
d\zeta_t = (r_t - \widetilde{r}_t)\,\zeta_t dt \Rightarrow \zeta_T = \zeta_t \exp\left(\int_{s=t}^{T}(r_s-\widetilde{r}_s)\,ds\right)
$$

In such a framework, the change of measure between the two spot martingale measures $\mathbb{Q}$ (domestic) and $\widetilde{\mathbb{Q}}$ (foreign) would be innocuous, since the Radon-Nikodym derivative would be given by:

$$
\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}(t,T) = \frac{\widetilde{\beta}_T\zeta_T}{\beta_T}\frac{\beta_t}{\widetilde{\beta}_t\zeta_t} = \frac{\widetilde{\beta}_T\beta_t}{\widetilde{\beta}_t\beta_T}\exp\left(\int_{s=t}^{T}(r_s-\widetilde{r}_s)\,ds\right) = 1
$$

So that we could rewrite (55)

$$
\begin{aligned}
&\frac{E_t}{\beta_t} = E_{\mathbb{Q}}\left[\frac{E_T}{\beta_T}\middle|\mathcal{F}_t\right] && \frac{\widetilde{E}_t}{\beta_t} = E_{\mathbb{Q}}\left[\exp\left(\int_{s=t}^{T}(r_s-\widetilde{r}_s)\,ds\right)\frac{\widetilde{E}_T}{\beta_T}\middle|\mathcal{F}_t\right]\\
&\frac{\widetilde{E}_t}{\widetilde{\beta}_t} = E_{\widetilde{\mathbb{Q}}}\left[\frac{\widetilde{E}_T}{\widetilde{\beta}_T}\middle|\mathcal{F}_t\right] && \frac{E_t}{\widetilde{\beta}_t} = E_{\widetilde{\mathbb{Q}}}\left[\exp\left(\int_{s=t}^{T}(\widetilde{r}_s-r_s)\,ds\right)\frac{E_T}{\widetilde{\beta}_T}\middle|\mathcal{F}_t\right]\\
&\frac{E_t}{N_t} = E_{\mathbb{N}}\left[\frac{E_T}{N_T}\middle|\mathcal{F}_t\right] && \frac{\widetilde{E}_t}{N_t} = E_{\mathbb{N}}\left[\exp\left(\int_{s=t}^{T}(r_s-\widetilde{r}_s)\,ds\right)\frac{\widetilde{E}_T}{N_T}\middle|\mathcal{F}_t\right]\\
&\frac{\widetilde{E}_t}{\widetilde{N}_t} = E_{\widetilde{\mathbb{N}}}\left[\frac{\widetilde{E}_T}{\widetilde{N}_T}\middle|\mathcal{F}_t\right] && \frac{E_t}{\widetilde{N}_t} = E_{\widetilde{\mathbb{N}}}\left[\exp\left(\int_{s=t}^{T}(\widetilde{r}_s-r_s)\,ds\right)\frac{E_T}{\widetilde{N}_T}\middle|\mathcal{F}_t\right]
\end{aligned} \tag{56}
$$

And if we take into account that current accounts are particular cases of numeraires with standard and non standard collaterals:

[12]Having zero diffusion implies no drift change due to a measure change.

$$
\begin{aligned}
\tfrac{E_t}{N_t} &= E_{\mathbb{N}}\left[\tfrac{E_T}{N_T}\Big|\mathcal{F}_t\right] \qquad & \tfrac{\widetilde{E}_t}{N_t} &= E_{\mathbb{N}}\left[\exp\left(\textstyle\int_{s=t}^{T}(r_s-\widetilde{r}_s)\,ds\right)\tfrac{\widetilde{E}_T}{N_T}\Big|\mathcal{F}_t\right] \\
\tfrac{\widetilde{E}_t}{\widetilde{N}_t} &= E_{\widetilde{\mathbb{N}}}\left[\tfrac{\widetilde{E}_T}{\widetilde{N}_T}\Big|\mathcal{F}_t\right] \qquad & \tfrac{E_t}{\widetilde{N}_t} &= E_{\widetilde{\mathbb{N}}}\left[\exp\left(\textstyle\int_{s=t}^{T}(\widetilde{r}_s-r_s)\,ds\right)\tfrac{E_T}{\widetilde{N}_T}\Big|\mathcal{F}_t\right]
\end{aligned}
\tag{57}
$$

For generic numeraires, the Radon-Nikodym derivative expression can be obtained from either

$$
E_t = N_t E_{\mathbb{N}}\left[\frac{E_T}{N_T}\Big|\mathcal{F}_t\right] = \widetilde{N}_t E_{\widetilde{\mathbb{N}}}\left[\exp\left(\int_{s=t}^{T}(\widetilde{r}_s - r_s)\,ds\right)\frac{E_T}{\widetilde{N}_T}\Big|\mathcal{F}_t\right] \tag{58}
$$

or

$$
\widetilde{E}_t = N_t E_{\mathbb{N}}\left[\exp\left(\int_{s=t}^{T}(r_s - \widetilde{r}_s)\,ds\right)\frac{\widetilde{E}_T}{N_T}\Big|\mathcal{F}_t\right] = \widetilde{N}_t E_{\widetilde{\mathbb{N}}}\left[\frac{\widetilde{E}_T}{\widetilde{N}_T}\Big|\mathcal{F}_t\right] \tag{59}
$$

and would be given by

$$
\frac{d\widetilde{\mathbb{N}}}{d\mathbb{N}}(t,T) = \frac{\widetilde{N}_T\zeta_T}{N_T}\frac{N_t}{\widetilde{N}_t\zeta_t} = \frac{\widetilde{N}_T N_t}{\widetilde{N}_t N_T}\exp\left(\int_{s=t}^{T}(r_s-\widetilde{r}_s)\,ds\right)
$$

The zero volatility FX analogy has already been found in [1] using other arguments.

6 Collateral basis modeling in a HJM framework

In this section we apply the results obtained so far in a HJM framework, where we assume that the dynamics of both discounting curves (the one for standard collateral and the one for non standard collateral) is driven by the following SDEs under the real world measure $\mathbb{P}$ for the instantaneous forward curves

$$
\begin{aligned}
df(t,T) &= \mu^{f,\mathbb{P}}(t,T)dt + \underbrace{\sigma^f(t,T)}_{1\times n}\underbrace{dW_t^{\mathbb{P}}}_{n\times 1} \\
d\widetilde{f}(t,T) &= \mu^{\widetilde{f},\mathbb{P}}(t,T)dt + \underbrace{\sigma^{\widetilde{f}}(t,T)}_{1\times n}\underbrace{dW_t^{\mathbb{P}}}_{n\times 1} + \underbrace{\widetilde{\sigma}^{\widetilde{f}}(t,T)}_{1\times m}\underbrace{dZ_t^{\mathbb{P}}}_{m\times 1}
\end{aligned}
\tag{60}
$$

Regarding the discount factors

$$
\begin{aligned}
\frac{dB(t,T)}{B(t,T)} &= \mu^{B,\mathbb{P}}(t,T)dt - \underbrace{\sigma^B(t,T)}_{1\times n}\underbrace{dW_t^{\mathbb{P}}}_{n\times 1} \\
\frac{d\widetilde{B}(t,T)}{\widetilde{B}(t,T)} &= \mu^{\widetilde{B},\mathbb{P}}(t,T)dt - \underbrace{\sigma^{\widetilde{B}}(t,T)}_{1\times n}\underbrace{dW_t^{\mathbb{P}}}_{n\times 1} - \underbrace{\widetilde{\sigma}^{\widetilde{B}}(t,T)}_{1\times m}\underbrace{dZ_t^{\mathbb{P}}}_{m\times 1}
\end{aligned} \tag{61}
$$

The drift restrictions of $f(t,T)$ and $B(t,T)$ under numeraires with standard collateral will be the same as the ones obtained in the classical quantitative finance literature. Therefore, we will focus on the drift restrictions of $\widetilde{f}(t,T)$ and $\widetilde{B}(t,T)$ under $\mathbb{Q}$ (standard collateral spot martingale measure) and $\mathbb{M}$ (standard collateral terminal measure corresponding to numeraire $B(t,M)$).

6.1 Drift restrictions under the spot martingale measure

In subsection 5.4 we saw that the change of measure between $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$ had no effect. Therefore the drifts of $\widetilde{B}(t,T)$ and $\widetilde{f}(t,T)$ under $\mathbb{Q}$ will be the same as under $\widetilde{\mathbb{Q}}$.

We have summarized in subsection 5.4 that under $\widetilde{\mathbb{Q}}$ the ratio $\frac{\widetilde{B}(t,T)}{\widetilde{\beta}_t}$ is a martingale, therefore

$$
\frac{d\widetilde{B}(t,T)}{\widetilde{B}(t,T)} = \widetilde{r}_t dt - \sigma^{\widetilde{B}}(t,T)dW_t^{\widetilde{\mathbb{Q}}} - \widetilde{\sigma}^{\widetilde{B}}(t,T)dZ_t^{\widetilde{\mathbb{Q}}}
$$

Now, applying Itô's Lemma to $\widetilde{B}(t,T) = \exp\left(-\int_{s=t}^{T}\widetilde{f}(t,s)ds\right)$ and relating drifts and volatilities of $\widetilde{B}(t,T)$ and $\widetilde{f}(t,T)$, we obtain:

$$
\begin{aligned}
\sigma^{\widetilde{f}}(t,T) &= \frac{\partial\sigma^{\widetilde{B}}(t,T)}{\partial T} \\
\widetilde{\sigma}^{\widetilde{f}}(t,T) &= \frac{\partial\widetilde{\sigma}^{\widetilde{B}}(t,T)}{\partial T} \\
\mu^{\widetilde{f},\widetilde{\mathbb{Q}}}(t,T) &= \sigma^{\widetilde{f}}(t,T)\sigma^{\widetilde{B}}(t,T)^{\top} + \widetilde{\sigma}^{\widetilde{f}}(t,T)\widetilde{\sigma}^{\widetilde{B}}(t,T)^{\top}
\end{aligned} \tag{62}
$$

Which is nothing but the standard HJM results taking into account the particular way in which the diffusion of $\widetilde{f}(t,T)$ and $\widetilde{B}(t,T)$ have been expressed in order to make the correlation between $f(t,T)$ and $\widetilde{f}(t,T)$ explicit.

Since the drifts under $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$ are the same, then

$$\begin{aligned}
&\frac{d\widetilde{B}(t,T)}{\widetilde{B}(t,T)} = \widetilde{r}_t dt - \sigma^{\widetilde{B}}(t,T)dW_t^{\mathbb{Q}} - \widetilde{\sigma}^{\widetilde{B}}(t,T)dZ_t^{\mathbb{Q}} \\
&\mu^{\widetilde{f},\mathbb{Q}}(t,T) = \sigma^{\widetilde{f}}(t,T)\sigma^{\widetilde{B}}(t,T)^\top + \widetilde{\sigma}^{\widetilde{f}}(t,T)\widetilde{\sigma}^{\widetilde{B}}(t,T)^\top
\end{aligned} \tag{63}$$

6.2 Drift restrictions under the terminal measure

In this subsection we will explore the drift of $\widetilde{B}(t,T)$ and $\widetilde{f}(t,T)$ under $\mathbb{M}$ (martingale measure that corresponds to the numeraire $B(t,M)$).

We first obtain $\mu^{B,\mathbb{M}}(t,M)$ by imposing that $\frac{\beta_t}{B(t,M)}$ is a $\mathbb{M}$-martingale.

$$X_t := \frac{\beta_t}{B(t,T)}$$

$$dX_t = r_t X_t dt - X_t \frac{dB(t,M)}{B(t,M)} + X_t \left(\frac{dB(t,M)}{B(t,M)}\right)^2 = X_t \left(r_t - \mu^{B,\mathbb{M}}(t,M) + \sigma^B(t,M)^2\right) dt + \dots dW_t^{\mathbb{M}}$$

And for it to have zero drift

$$\mu^{B,\mathbb{M}}(t,M) = r_t + \sigma^B(t,M)^2 \tag{64}$$

And in order to determine $\mu^{\widetilde{B},\mathbb{M}}(t,T)$, we impose that $\frac{\widetilde{B}(t,T)\exp\left(\int_{s=0}^t (r_s - \widetilde{r}_s)ds\right)}{B(t,M)}$ is a $\mathbb{M}$-martingale.

$$Y_t := \frac{\widetilde{B}(t,T)\exp\left(\int_{s=0}^t (r_s - \widetilde{r}_s)\, ds\right)}{B(t,M)}$$

$$\begin{aligned}
dY_t &= Y_t \left((r_t - \widetilde{r}_t)\, dt + \frac{d\widetilde{B}(t,T)}{\widetilde{B}(t,T)} - \frac{dB(t,M)}{B(t,M)} + \left(\frac{dB(t,M)}{B(t,M)}\right)^2 - \frac{d\widetilde{B}(t,T)}{\widetilde{B}(t,T)}\frac{dB(t,M)}{B(t,M)} \right) \\
&= Y_t \left(r_t - \widetilde{r}_t + \mu^{\widetilde{B},\mathbb{M}}(t,T) - \mu^{B,\mathbb{M}}(t,M) + \sigma^B(t,M)^2 - \sigma^{\widetilde{B}}(t,T)\sigma^B(t,M)^\top \right) dt \\
&\quad + \dots dW_t^{\mathbb{M}} + \dots dZ_t^{\mathbb{M}}
\end{aligned} \tag{65}$$

And plugging (64) into the last equation

$$\begin{aligned} dY_t &= Y_t\left(-\widetilde{r}_t + \mu^{\widetilde{B},\mathbb{M}}(t,T) - \sigma^{\widetilde{B}}(t,T)\sigma^{B}(t,M)^{\top}\right)dt \\ &\quad + \ldots dW_t^{\mathbb{M}} + \ldots dZ_t^{\mathbb{M}} \end{aligned} \tag{66}$$

So that for Y_t to be a martingale

$$\mu^{\widetilde{B},\mathbb{M}}(t,T) = \widetilde{r}_t + \sigma^{\widetilde{B}}(t,T)\sigma^{B}(t,M)^{\top} \tag{67}$$

In order to obtain $\mu^{\widetilde{f},\mathbb{M}}(t,T)$ we apply Itô's Lemma to the following expression

$$\log \widetilde{B}(t,T) = -\int_{s=t}^{T} \widetilde{f}(t,s)ds$$

Obtaining

$$\begin{aligned} &\mu^{\widetilde{B},\mathbb{M}}(t,T)dt - \sigma^{\widetilde{B}}(t,T)dW_t^{\mathbb{M}} - \widetilde{\sigma}^{\widetilde{B}}(t,T)dZ_t^{\mathbb{M}} - \tfrac{1}{2}\sigma^{\widetilde{B}}(t,T)^2dt - \tfrac{1}{2}\widetilde{\sigma}^{\widetilde{B}}(t,T)^2dt \\ &= \underbrace{\widetilde{f}(t,t)}_{\widetilde{r}_t}dt - \textstyle\int_{s=t}^{T}\mu^{\widetilde{f},\mathbb{M}}(t,s)dtds - \int_{s=t}^{T}\sigma^{\widetilde{f}}(t,s)dW_t^{\mathbb{M}}ds - \int_{s=t}^{T}\widetilde{\sigma}^{\widetilde{f}}(t,s)dZ_t^{\mathbb{M}}ds \end{aligned} \tag{68}$$

Since the terms in dt in both sizes of the last equation must be equal, and plugging (67)

$$\int_{s=t}^{T}\mu^{\widetilde{f},\mathbb{M}}(t,s)ds = \frac{1}{2}\sigma^{\widetilde{B}}(t,T)^2 + \frac{1}{2}\widetilde{\sigma}^{\widetilde{B}}(t,T)^2 - \sigma^{\widetilde{B}}(t,T)\sigma^{B}(t,M)^{\top}$$

And if we take the derivative with respect to T

$$\mu^{\widetilde{f},\mathbb{M}}(t,T) = \sigma^{\widetilde{B}}(t,T)\sigma^{\widetilde{f}}(t,T)^{\top} + \widetilde{\sigma}^{\widetilde{B}}(t,T)\widetilde{\sigma}^{\widetilde{f}}(t,T)^{\top} - \sigma^{\widetilde{f}}(t,T)\sigma^{B}(t,M)^{\top} - \sigma^{\widetilde{B}}(t,T)\sigma^{f}(t,M)^{\top}$$

7 Stochastic funding curve modeling

In this section we assume that we want to price a non collateralized interest rate transaction. Its value at time t from the investor's (risk taker) perspective will be denoted by $\hat{E}_t$.

In pricing the non collateralized deal, we will make the following assumptions:

- The non collateralized derivative is closed with a counterparty with no default risk, so that funding issues are analyzed in isolation from counterparty credit risk.
- As assumed in [3], the hedger is not concerned about the changes in the derivative upon his own default, but is concerned about the changes experienced by the derivative due to changes in his own funding curve.

$\hat{B}(t,T)$ represents the value at time t of a zero coupon bond issued by the derivative's hedger.

$$\hat{B}(t,T) = 1_{\{\tau>t\}} \exp\left(-\int_{s=t}^{T} \hat{f}(t,s)ds\right) + R(t,T)1_{\{\tau\leq t\}} \tag{69}$$

τ represents the default time of the derivative's hedger, $\hat{f}(t,T)$ the instantaneous forward curve associated to the hedger's funding curve and $R(t,T)$ the recovery rate for a zero coupon bond maturing at T.

We assume that under the real world measure $\mathbb{P}$, the evolution of $\hat{f}(t,T)$ is given by

$$d\hat{f}(t,T) = \mu^{\hat{f}}(t,T)dt + \underbrace{\sigma^{\hat{f}}(t,T)}_{1\times n}\underbrace{dW_t^{\mathbb{P}}}_{n\times 1} + \underbrace{\hat{\sigma}^{\hat{f}}(t,T)}_{1\times m}\underbrace{dZ_t^{\mathbb{P}}}_{m\times 1} \tag{70}$$

Obviously, after τ, $\hat{f}(t,T)$ is no longer meaningful. Therefore, (70) only makes sense before default.

Regarding the short term financing of the derivative's hedger, its evolution will be given by

$$d\hat{C}_t = \hat{r}_t\hat{C}_t dt + (1-R_t)C_t dN_t^{\mathbb{P}} \tag{71}$$

Where $\hat{r}_t$ is the short term funding rate, R_t is the recovery rate for short term debt and $N_t^{\mathbb{P}} = 1_{\{\tau<t\}}$ a Poisson counting process with real world intensity $\lambda_t^{\mathbb{P}}$.

As hedging instruments the hedger will use the set of vanilla instruments H_t since the product cash flows could depend on $B(t,T)$ and also a set of discount factors associated with his funding curve. The set of funding discount factors will be denoted by $\hat{H}_t$. The set of funding discount factors is necessary for the hedger to become immune to changes in his funding curve.

$$\hat{E}_t = \alpha_t H_t + C_t + \epsilon_t \hat{H}_t + \hat{C}_t$$

Notice that $\alpha_t H_t + C_t = 0$ since every component in H_t is collateralized.

The fact that $\hat{E}_t = \epsilon_t \hat{H}_t + \hat{C}_t$ is what in [3] is called the self financing condition. That is, incoming funds from uncollateralized derivatives are used to buy back issued debt and outgoing funds from uncollateralized derivatives need to be funded. In either case, the net issuance or buy back is such that the spread sensitivity of the uncollateralized derivative matches the sensitivities with respect to the funding curve of the debt issuance / buy back.

So that in every path in which the hedger remains not defaulted

$$\begin{aligned} \mu_t^{\hat{E}} dt + \sigma_t^{\hat{E}} dW_t^{\mathbb{P}} + \hat{\sigma}_t^{\hat{E}} dZ_t^{\mathbb{P}} - E_t \hat{r}_t &= \alpha_t \left(\mu_t^H dt + \sigma_t^H dW_t^{\mathbb{P}} - H_t r_t dt \right) \\ &+ \epsilon_t \left(\mu_t^{\hat{H}} dt + \sigma_t^{\hat{H}} dW_t^{\mathbb{P}} + \hat{\sigma}_t^{\hat{H}} dZ_t^{\mathbb{P}} - \hat{H}_t \hat{r}_t dt \right) \end{aligned} \tag{72}$$

Notice that (72) is equivalent to the hedging formula obtained is previous sections for derivatives with non standard collateral. Therefore, $\hat{\beta}_t$ (funding current account) and $\hat{B}(t,T)$ can also be seen as self financing portfolios denominated in a fictitious foreign currency with the spot FX rate (ζ_t) expressed in D/F following

$$d\zeta_t = (r_t - \hat{r}_t)\, \zeta_t dt$$

Notice that under the assumption of the hedger not being concerned to what happens upon his own default, $\hat{B}(t,T)$ and $\hat{\beta}_t$ behave as risk free (there is no default dependence in their risk neutral dynamics) and can be used as numeraires.

8 Incorporating multiple discounting curves in a multicurrency setting

In this section we discuss the best strategy to incorporate this multiple discounting curves environment in a multi currency setting where different transactions following different collateral schemes are simultaneously modeled, such as a CVA pricing engine.

Let's assume that we find ourselves pricing derivatives denominated in two different currencies D (that we arbitrary assume as domestic) and F (that we arbitrary assume as foreign). Let's first assume that we have derivatives denominated in F that are collateralized in cash either denominated in F or D. As a particular example, we assume that FX forwards involving D and F are collateralized in cash denominated in D.

In this setting there will be different short term rates at which collateral funds accrue:

- r_t^{DD}: Deals denominated in D and cash collateral denominated in D.
- r_t^{FD}: Deals denominated in F and cash collateral denominated in D.
- r_t^{FF}: Deals denominated in F and cash collateral denominated in F.

With their corresponding current accounts β_t^{DD}, β_t^{FD}, β_t^{FF} and discount factor curves $B^{DD}(t,T)$, $B^{FD}(t,T)$, $B^{FF}(t,T)$. Notice that β_t^{DD} and $B^{DD}(t,T)$ are denominated in D, whereas β_t^{FD}, β_t^{FF}, $B^{FD}(t,T)$, and $B^{FF}(t,T)$ are denominated in F.

In order to model the evolution of X_t (spot FX rate expressed in D/F), there are two possibilities:

FD is blindly assumed to be the standard collateral scheme in currency F

Under this assumption, the price of a derivative denominated in F and cash collateralized in D would be given by

$$V_t^{FD} X_t = N_t^{DD} E_{\mathbb{N}^{DD}} \left[\frac{V_T^{FD} X_T}{N_T^{DD}} \middle| \mathcal{F}_t \right] \tag{73}$$

Where V_t^{FD} is the t value of a derivative denominated in F cash collateralized in D and N_t^{DD} a generic numeraire denominated in D and cash collateralized in D.

If $V_T^{FD} = 1 \Rightarrow V_t^{FD} = B^{FD}(t,T)$ and $N_T^{DD} = 1 \Rightarrow N_t^{DD} = B^{DD}(t,T)$, then

$$E_{\mathbb{T}^{DD}} \left[X_T \middle| \mathcal{F}_t \right] = X_t \frac{B^{FD}(t,T)}{B^{DD}(t,T)} \tag{74}$$

Whereas if $V_T^{FD} = \beta_t^{FD}$ and $N_T^{DD} = \beta_t^{DD}$

$$E_{\mathbb{Q}^{DD}} \left[X_T \exp\left(\int_{s=t}^{T} \left(r_s^{FD} - r_s^{DD} \right) \right) \middle| \mathcal{F}_t \right] = X_t \tag{75}$$

Notice that this alternative (already suggested in [5]) might be tempting since under the domestic terminal measure the expected value of the FX rate would be equal to the forwards observed in the market (since we have assumed that they are collateralized in D) and under the domestic spot measure the drift of X_t would be equal to $r_t^{DD} - r_t^{FD}$.

Nevertheless, this alternative has a drawback, since if the true standard collateral in currency F is different from cash in currency D (if it was, for example, cash denominated in F), we would have to assume that deals in F under the standard collateral are denominated in a theoretical currency F' such that the spot FX rate expressed in F/F' followed:

$$\zeta_T = \zeta_t \exp\left(\int_{s=t}^{T} \left(r_s^{FD} - r_s^{FF}\right) ds\right)$$

So that in order to value a deal denominated in F and cash collateralized in F (which we have assumed to be the standard collateral for deals denominated in F) we would have to calculate the following expected value

$$\underbrace{\underbrace{V_t^{FF}}_{F'} \underbrace{X_t}_{D/F} \underbrace{\zeta_t}_{F/F'}}_{D} = N_t^{DD} E_{\mathbb{N}^{DD}} \left[\frac{V_T^{FF} X_T \zeta_T}{N_T^{DD}} \Big| \mathcal{F}_t\right] \tag{76}$$

That is equal to

$$V_t^{FF} X_t = N_t^{DD} E_{\mathbb{N}^{DD}} \left[\frac{V_T^{FF} X_T \exp\left(\int_{s=t}^{T} \left(r_s^{FD} - r_s^{FF}\right) ds\right)}{N_T^{DD}} \Big| \mathcal{F}_t\right] \tag{77}$$

So that for every derivative in F under the standard (and obviously more numerous) collateralization scheme in that currency, we will have to apply the correction factor $\exp\left(\int_{s=t}^{T} \left(r_s^{FD} - r_s^{FF}\right) ds\right)$.

FF is treated as the standard collateral scheme in currency F, in line with what is observed in the market

In that case the value of V_t^{FF} would be given by

$$V_t^{FF} X_t = N_t^{DD} E_{\mathbb{N}^{DD}} \left[\frac{V_T^{FF} X_T}{N_T^{DD}} \Big| \mathcal{F}_t\right] \tag{78}$$

So that if $V_T^{FF} = 1 \;\Rightarrow V_t^{FF} = B^{FF}(t,T)$ and $N_T^{DD} = 1 \;\Rightarrow N_t^{DD} = B^{DD}(t,T)$, then

$$E_{\mathbb{T}^{DD}}\left[X_T \middle| \mathcal{F}_t\right] = X_t \frac{B^{FF}(t,T)}{B^{DD}(t,T)} \tag{79}$$

Which is not equal to the forwards observed in the market, but that is equal to the forwards if each leg of the currency forward was collateralized in its own currency (the D leg in cash in D and the F leg in cash in F).

Under this assumption, if $V_T^{FF} = \beta_t^{FF}$ and $N_T^{DD} = \beta_t^{DD}$

$$E_{\mathbb{Q}^{DD}}\left[X_T \exp\left(\int_{s=t}^{T} \left(r_s^{FF} - r_s^{DD}\right)\right) \middle| \mathcal{F}_t\right] = X_t \tag{80}$$

So that the drift of the spot rate would be $r_t^{DD} - r_t^{FF}$.

Notice that under this assumption, the price of derivatives denominated in F that are collateralized under the standard collateral scheme would follow (78) with no adjustment to be made as was the case in the previous subsection.

According to the results obtained, any derivative denominated in F and collateralized in D can be considered as if it was denominated in a currency F' such that the spot FX rate ζ_t expressed in F/F' followed

$$\zeta_T = \zeta_t \exp\left(\int_{s=t}^{T} \left(r_s^{FF} - r_s^{FD}\right) ds\right)$$

So that its value would be given by

$$\underbrace{\underbrace{V_t^{FD}}_{F'} \underbrace{X_t}_{D/F} \underbrace{\zeta_t}_{F/F'}}_{D} = N_t^{DD} E_{\mathbb{N}^{DD}}\left[\frac{V_T^{FF} X_T \zeta_T}{N_T^{DD}} \middle| \mathcal{F}_t\right] \tag{81}$$

That is equal to

$$V_t^{FD} X_t = N_t^{DD} E_{\mathbb{N}^{DD}}\left[\frac{V_T^{FD} X_T \exp\left(\int_{s=t}^{T} \left(r_s^{FF} - r_s^{FD}\right) ds\right)}{N_T^{DD}} \middle| \mathcal{F}_t\right] \tag{82}$$

If $N_t = B^{DD}(t,T)$ and $V_T^{FD} = 1 \;\Rightarrow\; V_t^{FD} = B^{FD}(t,T)$

$$B^{FD}(t,T)X_t = B^{DD}(t,T)E_{\mathbb{N}^{DD}}\left[X_T \exp\left(\int_{s=t}^{T}\left(r_s^{FF} - r_s^{FD}\right)ds\right)\Big|\mathcal{F}_t\right] \tag{83}$$

$$\Downarrow$$

$$E_{\mathbb{N}^{DD}}\left[X_T \exp\left(\int_{s=t}^{T}\left(r_s^{FF} - r_s^{FD}\right)ds\right)\Big|\mathcal{F}_t\right] = \frac{B^{FD}(t,T)X_t}{B^{DD}(t,T)} \tag{84}$$

Regarding the valuation of FX forwards (remember that we assumed that FX forward are collateralized in D), we will have to take into account that the foreign leg is collateralized under a collateral scheme different from the one assumed as the standard. Therefore, the ratio of notionals that makes the NPV of the forward to be zero is:

$$0 = N^D B^{DD}(t,T) - N^F B^{DD}(t,T)E_{\mathbb{N}^{DD}}\left[X_T \exp\left(\int_{s=t}^{T}\left(r_s^{FF} - r_s^{FD}\right)ds\right)\Big|\mathcal{F}_t\right]$$

And taking into account (83)

$$0 = N^D B^{DD}(t,T) - N^F X_t B^{FD}(t,T) \;\Rightarrow\; \frac{N^D}{N^F} = X_t \frac{B^{FD}(t,T)}{B^{DD}(t,T)}$$

So that we would of course be calibrated to the FX forward market.

Having analyzed both possibilities, we believe that in a multi currency setting where different transactions following different collateral schemes are simultaneously modeled, such as a CVA pricing engine, the best option is for the FX rate to relate cash flows under the most common collateral schemes in both currencies in order to minimize the number of adjustments such as the ones described in this section.

9 Conclusions

The conclusions obtained can be summarized as follows:

- Multiple collateral schemes for deal denominated in a given currency imply multiple discounting curves (and their corresponding current accounts) for that currency.
- Current accounts, annuities and discount factors belonging to any collateral scheme can be used as numeraires.

- From a pricing perspective, having N different collateral schemes for the same currency is equivalent to having $N-1$ additional currencies with zero volatilities and whose drift is equal to the difference between the collateral rate chosen as the standard one and the collateral rate of each of the other schemes.
- Assuming that the derivatives hedger is not concerned with his own default while hedging derivatives, the hedger's funding curve represents an additional discounting curve that can be modeled the same way as the other discounting curves.

References

[1] FRIES, CHRISTIAN P. Funded Replication: Fund Exchange Process and the Valuation with Different Funding-Accounts (Cross-Currency Analogy to Funding Revisited). July 2012.
http://papers.ssrn.com/abstract=2115839.

[2] García Muñoz, Luis Manuel. Collateral choice and the fundamental theorem of asset pricing. October 2012.
http://mpra.ub.uni-muenchen.de/42451/

[3] García Muñoz, Luis Manuel. CVA, FVA (and DVA?) with stochastic spreads. A feasible replication approach under realistic assumptions. July 2013.
http://mpra.ub.uni-muenchen.de/48452/

[4] V. Piterbarg. Funding beyond discounting: Collateral agreements and derivatives pricing.
Risk, February, 97-102, 2010.

[5] V. Piterbarg. Cooking with collateral.
Risk, August, 2012.