Szpilrajn-type extensions of fuzzy quasiorderings

Alcantud, José Carlos R. and Díaz, Susana

Universidad de Salamanca, Universidad de Oviedo

12 July 2013

Szpilrajn-type extensions of fuzzy quasiorderings

J. C. R. Alcantud
Facultad de Economía y Empresa
Campus Unamuno
Universidad de Salamanca, Spain
jcr@usal.es

S. Díaz
Dept. Statistics and O. R.
Faculty of Sciences
University of Oviedo, Spain
diazsusana@uniovi.es

Abstract. The problem of embedding incomplete into complete relations has been an important topic of research in the context of crisp relations. After Szpilrajn's result, several variations have been published. Alcantud studied in 2009 the case where the extension is asked to satisfy some order conditions between elements. He first studied and solved a particular formulation where conditions are imposed in terms of strict preference only, which helps to precisely identify which quasiorderings can be extended when we allow for additional conditions in terms of indifference too. In this contribution we generalize both results to the fuzzy case.

Keywords: quasiordering, order, extension of a quasiordering.

1 Introduction

As Herden and Pallack [17] put it, "[o]ne of the best known theorems in order theory, mathematical logic, computer sciences and mathematical social sciences is the Szpilrajn Theorem which states that every partial order can be refined to a linear order". Another form of the same principle states that any quasiordering has an ordering extension (cf., Arrow [2, Chapter VI], Hansson [16, Lemma 3]). Many variations and generalizations followed. Dushnik and Miller [10] prove that any partial order is the intersection of linear orderings, and then Donaldson and Weymark [9] (see also Bossert [6]) prove the corresponding result for quasiorderings, namely, that any quasiordering is the intersection of orderings. Suzumura [23], [24, Theorem A(5)] shows that a property called consistency is necessary and sufficient for the existence of an ordering extension. Bosi and Herden [4, 5], Bossert et al. [7], Herden and Pallack [17], Jaffray [19], or Yi [26] among others discuss continuity and semicontinuity issues in relation with the Szpilrajn theorem. Alcantud [1] systematizes the identification of constraints that can be imposed on the ordering extensions. This is important because Szpilrajn's theorem is not constructive thus the researcher cannot proceed by direct inspection of the resulting orderings.

In this fruitful field of research we can also name various fuzzy versions or extensions of Szpilrajn's theorem. Among them, Georgescu [13, Theorem 5.4] and [14, Corollary 4.37], Bodenhofer and Klawonn [3, Theorem 6.7] –who conduct a detailed investigation of linearity axioms for fuzzy orderings–, Gottwald [15, Proposition 2.34], Höhle and Blanchard [18] –who produce variations for *antisymmetric* quasiorderings both of the extension theorem in Theorem II.7 and of the intersection theorem in Corollary II.8–, or Zadeh [28, Theorem 8].

We contribute to the field by providing original extensions of Szpilrajn's theorem in the spirit of [1], namely an exact identification of the constraints that can be imposed on the fuzzy ordering extensions.

The remaining sections are organized as follows. In Section 2 we introduce relevant notions on crisp set theory and we recall known results about extensions in that context. Section 3 includes the corresponding definitions in the fuzzy case and concepts that permit to understand our subsequent results. In Section 4 we report on the characterizations obtained. Section 5 concludes with a discussion of our contribution, and we also pose the problem of generalizing other extension results.

2 Crisp Extensions

In this section we compile the background on crisp relations that permits to present our contribution.

Let X be a non empty universe of alternatives. A binary relation Q on X is a subset of $X \times X$. The notation aQb is more common than $(a, b) \in Q$. A binary relation can also be identified with the mapping $Q : X \times X \to \{0, 1\}$. Thus the notation $Q(a, b) = 1$ is often used to mean $(a, b) \in Q$. A binary relation Q is reflexive if aQa or equivalently, if $Q(a, a) = 1$ for every alternative a. It is symmetric if $Q(a, b) = Q(b, a)$ throughout.
A weak preference relation, also called large preference relation, is a reflexive binary relation and it is usually denoted by R. If R is a weak preference relation, aRb is interpreted as "alternative a is at least as good as b".
We can associate three standard binary relations with every weak preference relation:

- The strict preference relation P defined as $P = \{(a, b) \in X \times X \mid (a, b) \in R \wedge (b, a) \notin R\}$.
- The indifference relation I defined as $I = \{(a, b) \in X \times X \mid (a, b) \in R \wedge (b, a) \in R\}$.
- The incomparability relation J defined as $J = \{(a, b) \in X \times X \mid (a, b) \notin R \wedge (b, a) \notin R\}$.

It is easy to check that these relations are mutually disjoint and that they cover all the possible answers of a decision maker. The strict preference relation P connects alternative a to alternative b if a is strictly better than b. The notation aIb means that a and b are equally good/bad, and aJb stands for incomparability: the decision maker cannot compare or relatively order alternatives a and b. The existence of incomparable alternatives is one of the drawbacks that makes it difficult to make a decision over a set of alternatives.
A binary relation R is total or complete if for any pair of alternatives a and b, at least aRb or bRa holds true. It is easy to check that a weak preference relation is total if and only if its associated incomparability relation is empty. This is to say, R is total when all pairs of alternatives can be compared or relatively ordered.

In addition to completeness, transitivity is the classical property requested when coherence is considered. A binary relation is transitive if for any three alternatives a, b, c in X it holds that

$$R(a, b) \wedge R(b, c) \quad \Rightarrow \quad R(a, c).$$

However transitivity is a very demanding condition in many instances. A weaker property that partially captures a similar type of consistency is acyclicity.
The weak preference relation R is acyclic if for any chain of alternatives $a_1, \ldots a_n$ in X,

$$a_1 R a_2 \ \wedge \quad \ldots \quad \wedge \ a_{n-1} R a_n \quad \Rightarrow \quad a_n \not{P} a_1.$$

It is easy to check that transitivity is stronger than acyclicity, i.e., every transitive weak preference relation is acyclic.

A transitive relation is called a partial order. A linear order is a complete partial order.

A reflexive and transitive relation is called a quasiordering. An order is a complete or total quasiordering.

Consider two weak preference relations $\tilde{R}$ and R. The relation $\tilde{R}$ extends R if $R \subseteq \tilde{R}$ and $P \subseteq \tilde{P}$, where P and $\tilde{P}$ are the strict preference relations associated with R and $\tilde{R}$ respectively.

In the crisp context Szpiljrajn proved in [25] that every partial order can be extended to a linear order. Concerning extensions of incomplete relations, it was proved that every quasiordering has an order extension (see Arrow [2], Hansson [16]).

Alcantud [1] contributes to an original approach to conditional ordering extensions in the crisp setting. It raises and solves the question: what do we need to check in order for a finite list of comparisons to be realized by an extension of a given quasiordering? The solutions are stated in terms of consistency properties for certain auxiliary binary relations.

Definition 2.1 [1] *Let $X_I = \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ be an ordered list of possibly repeated elements of X, and R a quasiordering on X. The R^A relation associated with X_I and R is given by $a_i R^A a_j$ if and only if $a_i R b_j$.*

Theorem 2.2 [1] *Let R be a quasiordering on a set X. Let $X_I = \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ be an ordered list of possibly repeated elements of X. The following statements are equivalent:*

(a) There is $\tilde{R}$ ordering extension of R such that $b_i \tilde{P} a_i$ for each $i = 1, \ldots, n$, where $\tilde{P}$ denotes the asymmetric part of $\tilde{R}$.

(b) R^A associated with $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ and R is acyclic.

Next we introduce some definitions to prove the more general result: a characterization for extensions satisfying some conditions expressed in terms of strict preference but also some others based on the indifference between some pairs of options.

Definition 2.3 *Let $X_I = \{a_1, \ldots, a_p, b_1, \ldots, b_p\}$ be an ordered list of possibly repeated elements of X. For each $x \in X_I$, let*

$$\delta(x) = \begin{cases} b_i & \text{if } x = a_i \\ a_i & \text{if } x = b_i \end{cases}$$

Let $X_I^n = \{a_{n+1}, \ldots, a_p, b_{n+1}, \ldots, b_p\}$. If $n = p$, then $X_I^n = \emptyset$.

Definition 2.4 *The R^I relation associated with R and X_I is defined by:*

$$\text{for each } x, y \in X_I : \quad x R^I y \;\Leftrightarrow\; x R \delta(y).$$

We say that R^I is δ-cyclic along X_I^n when $x_1 R^I x_2 R^I \ldots R^I x_k R^I x_1$ implies that $\delta(x_1) R^I \delta(x_k) R^I \ldots R^I \delta(x_2) R^I \delta(x_1)$ holds too, provided that $x_1, \ldots, x_k \in X_I^n$.

Definition 2.5 *The R^G relation associated with R, $n \leq p$ and X_I is defined by: for each $i, j \in \{1, \ldots, p\}$,*

$$a_i R^G a_j \Leftrightarrow \begin{cases} a_i R^A a_j, & \qquad\qquad i.e., a_i R b_j = \delta(a_j) \\ or & \\ a_i R \delta(y_1^t), y_{k_t}^t R b_j, \;\; where y_1^t R^I y_2^t R^I \ldots y_{k_t}^t \;\; and \; y_1^t, y_2^t, \ldots, y_{k_t}^t \in X_I^n. & \end{cases}$$

The relation R^G is δ-consistent with X_I and $n \leq p$ if:

$$a_{i_1} R^G a_{i_2} R^G \ldots R^G a_{i_k} R^G a_{i_1} \text{ entails } i(t) > n \quad \text{for some } t = 1, \ldots, k.$$

We are ready to recall the second result of Alcantud [1].

Theorem 2.6 [1] *Let R be a quasiordering on a set X. Let $X_I = \{a_1, \ldots, a_p, b_1, \ldots, b_p\}$ be an ordered list of possibly repeated elements of X and let $n \leq p$. The following statements are equivalent:*

(a) There is $\tilde{R}$ ordering extension of R such that $b_i \tilde{P} a_i$ for each $i = 1, \ldots, n$, and $b_i \tilde{I} a_i$ for each $i = n+1, \ldots, p$.

(b) R^G is δ-consistent with X_I and n, and R^I is δ-cyclic along X_I^n.

In Section 4 we extend these results to the fuzzy case.

3 Fuzzy Extension

In this section we introduce basic definitions on fuzzy set theory.

Fuzzy set theory was introduced to describe human behavior in a more accurate way than classical sets and relations. When dealing with crisp relations, alternatives are connected (value 1) or not (value 0). Although human decisions are hardly ever categorical, intermediate degrees are not allowed in crisp decision theory. Fuzzy sets were introduced by Zadeh in [27] to catch those nuances that crisp sets cannot express. Fuzzy relations can take any value in the $[0,1]$ interval. The value shows the strength of the connection between the alternatives.

A fuzzy relation Q defined on a universe X is a mapping $Q : X \times X \to [0,1]$ such that for every $a, b \in X$, $Q(a,b)$ indicates the degree with which a is connected to b by the relation Q.

A reflexive fuzzy relation is a fuzzy relation R satisfying $R(a,a) = 1$ for all $a \in X$. A fuzzy weak preference relation is a reflexive fuzzy relation.

Union and intersection of fuzzy sets are usually based on t-norms and t-conorms. A t-norm T is a binary mapping $T : [0,1] \times [0,1] \to [0,1]$ satisfying the following four properties: commutativity, associativity, monotonicity (in each component) and neutral element 1. The greatest t-norm is the minimum operator $T_{\mathbf{M}}(x,y) = \min(x,y)$, and the smallest one is the drastic t-norm, namely

$$T_{\mathbf{D}}(x,y) = \begin{cases} \min(x,y), & \text{if } \max(x,y) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

A t-norm T has zero divisors if there is a pair of values $(x,y) \in]0,1[^2$ such that $T(x,y) = 0$. In this case x and y are called zero divisors of T. The minimum t-norm does not admit zero divisors. For the drastic t-norm every pair of values $(x,y) \in]0,1[^2$ are zero divisors.

The intersection of fuzzy relations is defined using t-norms: the intersection of the fuzzy relations R and Q defined on the universe X is denoted by $R \cap_T Q$, and it is defined by $R \cap_T Q(a,b) = T(R(a,b), Q(a,b))$ for all $a, b \in X$.

A t-conorm S is a mapping $S : [0,1] \times [0,1] \to [0,1]$ satisfying similar properties to t-norms: commutativity, associativity, monotonicity (in each component) and neutral element 0. The smallest t-conorm is the maximum operator $S_{\mathbf{M}}(x,y) = \max(x,y)$ and the greatest t-conorm is the drastic t-conorm

$$S_{\mathbf{D}}(x,y) = \begin{cases} \max(x,y), & \text{if } \min(x,y) = 0, \\ 1, & \text{otherwise.} \end{cases}$$

The union of fuzzy relations is defined using t-conorms: the union of the fuzzy relations R and Q defined on the universe X is denoted $R \cup_S Q$ and defined as $R \cup_S Q(a,b) = S(R(a,b), Q(a,b))$ for all $a, b \in X$.
For a complete study on t-norms and t-conorms see [20].

Let T be a t-norm. A fuzzy relation R is called T-transitive if

$$T(R(a,b), R(b,c)) \leq R(a,c), \qquad \text{for all } a,b,c \in X.$$

Thus there is not a unique definition of transitivity for fuzzy relations, since the concept is conditional on the t-norm. The T-transitivity of a fuzzy relation R is usually denoted by $R \circ_T R \subseteq R$.
In this contribution we focus on min-transitivity: $\min(R(a,b), R(b,c)) \leq R(a,c)$.

The notion of completeness also admits different generalizations in the fuzzy case based on the notion of t-conorm. In this contribution we consider the weakest type of completeness. We say that a fuzzy relation R is total if for all $a, b \in A$ it holds that $R(a,b) \vee R(b,a) > 0$, i.e. if the completeness condition is based on the drastic t-conorm: $S_{\mathbf{D}}(R(a,b), R(b,a)) = 1$.

We say that a fuzzy relation R is a fuzzy quasiordering if it is reflexive and min-transitive. A fuzzy order R is a total fuzzy quasiordering.

As in the crisp case, given a fuzzy weak preference relation R we can define the asymmetric and symmetric components of R. However unlike the crisp case, in the fuzzy case there is not a unique way of building strict preference and indifference. Several proposals can be found in the literature [8, 11, 21, 22]. In this contribution we consider the following common definitions of strict preference and indifference relation.

$$I(a,b) = \min(R(a,b), R(b,a)) \qquad\qquad P(a,b) = \begin{cases} R(a,b) & \text{if } R(b,a) = 0, \\ 0 & \text{if } R(b,a) > 0. \end{cases}$$

The fuzzy weak preference relation R is T-transitive-consistent if for all $a_1, \ldots, a_n$,

$$T(P(a_1,a_2), R(a_2,a_3), \ldots, R(a_n,a_1)) = 0$$

For t-norms without zero divisors it holds that R is T-transitive-consistent if and only if

$$P(a_1,a_2) > 0 \wedge R(a_2,a_3) > 0 \wedge \ldots \wedge R(a_n,a_1) > 0$$

cannot hold.

T-transitivity is stronger than T-transitive-consistency:

Lemma 3.1 *Let T be a t-norm without zero divisors. Every T-transitive fuzzy weak preference relation R is also T-transitive-consistent.*

Definition 3.2 *A fuzzy relation Q is T-acyclic where T is a t-norm if for any $a_1, \ldots, a_n$ it holds that*

$$T(Q(a_1,a_2), Q(a_2,a_3), \ldots, Q(a_n,a_1)) = 0$$

For t-norms without zero divisors, in particular for the minimum t-norm, this means that

$$\min(Q(a_1,a_2), Q(a_2,a_3), \ldots, Q(a_n,a_1)) = 0.$$

Definition 3.3 *A fuzzy relation $\tilde{R}$ extends the fuzzy weak preference relation R defined on X if $R(a,b) \leq \tilde{R}(a,b)$ and $P(a,b) \leq \tilde{P}(a,b)$ for all $a,b \in X$.*

Before presenting our generalizations, let us recall an inspiring result that is used along the paper. It extends Szpilrajn's theorem to fuzzy relations as follows:

Theorem 3.4 [13, Theorem 5.4] *For a fuzzy relation R on X the following are equivalent:*

1. *R has a total and* min*-transitive compatible extension Q.*
2. *R has a* min*-transitive compatible extension Q.*
3. *R is* min*-transitive-consistent.*

4 Fuzzy extensions with several pairwise restrictions

In this section we expand the analysis of Alcantud [1] to account for fuzzy relations too.

Firstly we consider the simpler case where we request that a number of elements are in strict relation (i.e., in any non-zero degree) with respective elements in the extended ordering. To that purpose we need the following concept:

Definition 4.1 *Let R be a fuzzy relation defined on a set of alternatives X. Let $X_I = \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ be an ordered set of possibly repeated elements of X. The fuzzy relation R^A associated with X_I and R is defined by $R^A(a_i, a_j) = R(a_i, b_j)$.*

Now we can state the solution to the aforementioned problem in the following terms:

Theorem 4.2 *Let R be a reflexive and* min*-transitive relation. Then the following statements are equivalent:*

a) There exists an order $\tilde{R}$ that extends R and such that $\tilde{P}(b_i, a_i) > 0$ for all $i = 1, \ldots, n$.

b) The relation R^A associated with $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ is min*-acyclic.*

Theorem 4.2 extends Alcantud [1, Theorem 1]. Now we proceed to consider the more general case where some elements are asked to be in strict relation (i.e., in any non-zero degree) with respective elements in the extended ordering, while some other are asked to be indifferent (i.e., with full relationship) to their companions. This is a fuzzy counterpart to [1, Theorem 2]. We need some further concepts in order to state the solution to that question.

Let $X_I^n = \{a_{n+1}, \ldots, a_p, b_{n+1}, \ldots, b_p\}$. If $n = p$, then $X_I^n = \emptyset$. And let $\delta(x)$ be the function given in Definition 2.3.

The relation R^I associated with R and X_I is defined as follows: $R^I(x, y) = R(x, \delta(y))$, for all $x, y \in X_I$.

We say that R^I is δ-cyclic along X_I^n if $R^I(x_1, x_2) > 0$ and $R^I(x_2, x_3) > 0$ and $\ldots$ $R^I(x_k, x_1) > 0$ implies that

$$R^I(\delta(x_1), \delta(x_k)) > 0, \quad \ldots \quad R^I(\delta(x_2), \delta(x_1)) > 0, \qquad \forall x_1, \ldots, x_k \in X_I^n.$$

Definition 4.3 *The relation R^G associated with R, $n \leq p$ and X_I is defined by*

$$R^G(a_i, a_j) = \max(R(a_i, b_j), \min(R(a_i, \delta(y_1^t)), R(y_1^t, \delta(y_2^t)), \ldots, R(y_k^t, b_j)))$$

where $y_1^t, \ldots, y_{k_t}^t \in X_I^n$.

The relation R^G is δ-consistent with X_I and $n \leq p$ if

$$\min(R^G(a_{i_1}, a_{i_2}), \ldots, R^G(a_{i_k}, a_{i_1})) > 0$$

implies $i_t > n$ for some $t \in \{1, \dots, k\}$.

With all these new concepts we can extend [1, Theorem 2].

Theorem 4.4 *Let R be a fuzzy quasiordering on a set X. Let $X_I = \{a_1, \dots, a_p, b_1, \dots, b_p\}$ be an ordered list of possibly repeated elements of X and let $n \leq p$. The following statements are equivalent:*

a) There exists a fuzzy order $\tilde{R}$ extending R such that $\tilde{P}(b_i, a_i) > 0$ for each $i = 1, \dots, n$ and $\tilde{I}(b_i, a_i) = 1$ for $i \in \{n+1, \dots, p\}$.

b) The relation R^G is δ-consistent with X^I and n, and R^I is δ-cyclic along X_I^n.

5 Conclusion

The problem of embedding incomplete into complete relations has been a topic of marked interest in the context of decision making and therefore in its many areas of application. In this contribution we consider the fuzzy analysis of an interesting variant of the original problem solved by Szpilrajn: namely, the case where the extension is asked to satisfy some order conditions between elements. The problem was formally stated and solved by Alcantud [1] in the crisp case. He first studied and solved a particular formulation where conditions are imposed in terms of strict preference only. Then he used this case to precisely identify which quasiorderings can be extended when we allow for additional conditions in terms of indifference too. In this contribution we generalize these results to the fuzzy case. By using a similar approach, we first consider the case where a finite number of conditions in terms of strict preference are required to hold true. Then we give a characterization for the case where conditions both of type strict preference and indifference are required.

While the extension problem has been widely studied in the crisp case from many viewpoints, as far as we know the problem has not been treated in depth in the fuzzy case. Thus for example, it seems pertinent to check if some type of 'intersection theorem' holds true for multivariate relations too. Donaldson and Weymark [9], and afterwards Bossert [6], proved that every quasiordering is the intersection of a set of orders that extend it. Is this true in the fuzzy case? As is apparent, the results reported here permit to give a partial result in the form of a direct inclusion. We expect to give a complete answe to that question in the near future.

Acknowledgements

The research reported in this paper has been supported by Project ECO2012-31933 (Spanish Ministerio de Economía y Competitividad) and partially supported by Project MTM2010-17844.

References

[1] J. C. R. Alcantud. Conditional ordering extensions. *Economic Theory, 39*, 495–503 (2009).

[2] K. J. Arrow. Social Choice and Individual Values, 2nd ed. John Wiley & Sons, 1963.

[3] U. Bodenhofer, F. Klawonn. A formal study of linearity axioms for fuzzy orderings. *Fuzzy Sets and Systems, 145*, 323–354 (2004).

[4] G. Bosi, G. Herden. On a strong continuous analogue of the Szpilrajn theorem and its strengthening by Dushnik and Miller. *Order, 22*, 329–342 (2005).

[5] G. Bosi, G. Herden. On a possible continuous analogue of the Szpilrajn theorem and its strengthening by Dushnik and Miller. *Order, 23*, 271–296 (2006).

[6] W. Bossert. Intersection quasi-orderings: An alternative proof. *Order, 16*, 221–225 (1999).

[7] W. Bossert, Y. Sprumont, K. Suzumura. Upper semicontinuous extensions of binary relations. *Journal of Mathematical Economics, 37*, 231–246 (2002).

[8] B. De Baets, J. Fodor. Twenty years of fuzzy preference structues (1978-1997). *JORBEL, 37*, 61–82 (1997).

[9] D. Donaldson and J. A. Weymark. A quasiordering is the intersection of orderings. *Journal of Economic Theory, 78*, 382–387 (1998).

[10] B. Dushnik and E. W. Miller. Partially ordered sets. *American Journal of Mathematics, 63,* 600–610 (1941).

[11] J. Fodor. An axiomatic approach to fuzzy preference modelling. *Fuzzy Sets and Systems, 52*, 47–52 (1992).

[12] B. Hansson, Choice structures and preference relations. *Synthese, 18*, 443–458 (1968).

[13] I. Georgescu. Compatible extensions of fuzzy relations. *Journal of Systems Science and Systems Engineering, 12*, 332–349 (2003).

[14] I. Georgescu. Fuzzy Choice Functions: A Revealed Preference Approach. Springer, 2007.

[15] S. Gottwald. Fuzzy Sets and Fuzzy Logic. Vieweg, 1993.

[16] B. Hansson, Choice structures and preference relations. *Synthese, 18*, 443–458 (1968).

[17] G. Herden, A. Pallack. On the continuous analogue of the Szpilrajn Theorem I. *Mathematical Social Sciences, 43*, 115–134 (2002).

[18] U. Höhle, N. Blanchard. Partial ordering in L-underdeterminate sets. *Information Sciences, 35*, 133–144 (1985).

[19] J.-Y. Jaffray. Semicontinuous extension of a partial order. *Journal of Mathematical Economics, 2*, 395–406 (1975).

[20] E. P. Klement, R. Mesiar and E. Pap, *Triangular Norms*, 386 pages, Kluwer Academic Publishers, Boston/London/Dordrecht, (2000).

[21] S. A. Orlovsky, Decision-making with a fuzzy preference relation. *Fuzzy Sets and Systems, 1*, 155–167 (1978).

[22] S. V. Ovchinnikov, On the transitivity property. *Fuzzy Sets and Systems, 20*, 241–243 (1986).

[23] K. Suzumura. Remarks on the theory of collective choice. *Economica, 43*, 381–390 (1976).

[24] K. Suzumura. Rational Choice, Collective Decisions, and Social Welfare. Cambridge University Press, 1983.

[25] E. Szpilrajn. Sur l'éxtension de l'ordre partiel. *Fundamenta Mathematicae, 16*, 386–389 (1930).

[26] G. Yi. Continuous extension of preferences. *Journal of Mathematical Economics, 22*, 547–555 (1993).

[27] L. Zadeh. Fuzzy sets. *Information and Control, 3*, 338–353 (1965).

[28] L. Zadeh. Similarity relations and fuzzy orderings. *Information Sciences, 3*, 177–200 (1971).