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Abstract

The main idea of this paper is to introduce Tradeable Measures of Risk as an objective and model independent way of measuring risk. The present methods of risk measurement, such as the standard Value-at-Risk supported by BASEL II, are based on subjective assumptions of future returns. Therefore two different models applied to the same portfolio can lead to different values of a risk measure. In order to achieve an objective measurement of risk, we introduce a concept of *Realized Risk* which we define as a directly observable function of realized returns. Predictive assessment of the future risk is given by *Tradeable Measure of Risk* – the price of a financial contract which pays its holder the Realized Risk for a certain period. Our definition of the Realized Risk payoff involves a Weighted Average of Ordered Returns, with the following special cases: the worst return, the empirical Value-at-Risk, and the empirical mean shortfall. When Tradeable Measures of Risk of this type are priced and quoted by the market (even of an experimental type), one does not need a model to calculate values of a risk measure since it will be observed directly from the market. We use an option pricing approach to obtain dynamic pricing formulas for these contracts, where we make an assumption about the distribution of the returns. We also discuss the connection between Tradeable Measures of Risk and the axiomatic definition of Coherent Measures of Risk.

1 Introduction

The most popular risk measures, such as Value-at-Risk (VaR), expected shortfall, and standard deviation, are based on a distribution of future returns of an asset or a portfolio. These measures are inherently model dependent, and thus two different approaches may assign different risk values to an otherwise identical portfolio. The idea of this paper is to define a risk measure that would not depend on subjective assumptions about future returns, but instead would be determined by a market. In order to achieve an objective measurement of risk, we first introduce a concept of *Realized Risk*, which is a directly observable function of realized returns. A tradeable contract with the Realized Risk payoff is called *Tradeable Measure of Risk*.

We show that most of the currently traded contracts are in fact Tradeable Measures of Risk, but unfortunately they typically do not satisfy axioms which are desirable for a Measure of Risk. In particular, they are not good estimators of the popular Measures of Risk (such as VaR), and/or they do not satisfy properties for a Coherent Measure of Risk. Thus we focus on a contract on Weighted Average of Ordered Returns, which serves both as an estimator of a weighted average of quantiles from the distribution of returns (Theorem 4.1), and under certain conditions satisfies properties for Coherent Measure of Risk (Theorem 5.2).

Even a small or experimental market for Weighted Average of Ordered Returns would indicate the implied distribution and structure of future risk. In order to achieve objectivity of risk measurement, the market for Weighted Average of Ordered Returns can be organized by a central bank or similar regulatory entity. Let us consider greenhouse gas emissions as an analogy for financial risk. For a company, it is desirable to have low financial risk just as for society it is desirable to produce a low volume of emissions. However, it is frequently the case that in order to achieve economic success, a company needs to take financial risks. Analogously, to achieve success, a society might engage in industrial production which leads to emissions. Interestingly

enough, there is a market for trading emissions quota (such as the one set up by Kyoto protocol), but a market for trading coherent financial risk which would transfer an excess or unused financial risk quota does not exist.

There are two advantages of having a market where a contract on Realized Risk (especially on Weighted Average of Ordered Returns) would be traded: its market price can be viewed as a risk measure and the contract itself can serve as a way of financial risk insurance. In Section 5 we will show that under certain conditions, a Weighted Average of Ordered Returns satisfies generalized axioms for Coherent Measures of Risk.

Within the areas of mathematical finance and mathematical insurance, there has been almost simultaneous development in an axiomatic approach to measuring risk. Arztner et al. [2] and [3] established the representation theorem of a risk measure as a supremum of expectations under the axioms of monotonicity, subadditivity, positive homogeneity and translation invariance in a finite probability space. Wang et al. [21] deduced the Choquet integral representation of the distributional property of risk measures based on the work of Yaari [23] with additional assumptions of law invariance and comonotonicity. Kusuoka [16] developed equivalent representations to Wang et al. [21]. Recent research focused on extending the space where the representation theorem applies (Delbaen [9], Cherny [7]), attempted to develop a dynamic version of coherent risks (Artzner et al. [4], Riedel [18], Cheridito et al. [6], Frittelli and Scandolo [12], Kloppel and Schweizer [15], Weber [22]), or relaxed the axioms to convex risk measures (Föllmer and Schied [10]). The industry approach culminates with the latest BASEL II framework which adopts Value-at-Risk (VaR) as a universal minimal capital requirement. Though the fact that VaR is not a Coherent Measure of Risk motivated the original work of Artzner et al. [2] and [3], it has nevertheless remained as the industry standard up to date. On the transfer side, there already exists significant volume in trading non-coherent based risks in today's market. For example, volatility swaps provide a way to trade and hedge realized volatility. For options on realized variance, see Carr et al. [5]. Jarrow [14] also studied put option premium as a risk measure.

This paper is organized as follows: in Section 2, we define a payoff based on realized returns, which allows us to introduce a Tradeable Measure of Risk – a contract with a Realized Risk payoff. Section 3 provides formulas for a special case of Tradeable Measure of Risk, Weighted Average of Ordered Returns, if one makes an assumption about the distribution of the returns. Miura [17] set up a similar pricing problem of a lookback option on order statistics. Since it was applied directly to the asset price instead of the returns, a closed-form solution was not obtained. Section 4 shows that the Weighted Average of Ordered Returns converges to a weighted average of quantiles from the distribution of returns, and thus it can serve as an estimator to popular risk measures such as VaR or the expected shortfall. A consequence of obtaining the price process of the contract is that it can serve as a dynamic risk measure itself. This aspect will be detailed in Section 5. Section 6 concludes the paper.

2 Realized Risk and Tradeable Measures of Risk

Suppose the financial return on an asset of a bank over a finite time horizon $[0, t]$, $0 \leq t \leq T$, is an adapted stochastic process X_t (which could be either log return, percentage return, or absolute return) with values in $\mathbb{R}$ on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ that satisfies the usual conditions. BASEL II requires banks to put aside capital equal to $VaR_\lambda(X_T)$ which corresponds to the negative value of the λ-quantile from the distribution of the total return X_T. At time T, the real loss will be known and will almost certainly be different than the capital set aside initially. Therefore, the capital requirement targets the loss and provides a cushion under financial stress, but will not be an exact match. What we provide here is insurance against the Realized Risk.

Definition 2.1 *Suppose we have a collection of sampling times $0 = t_0 \leq t_1 \leq \ldots \leq t_N = T$. The return for period $[t_{i-1}, t_i]$ is denoted by X_i, for $1 \leq i \leq N$. The* Realized Risk *g is a function of realized returns*

$g : \mathbb{R}^N \to \mathbb{R}$. Tradeable Measure of Risk *is the market value* $\rho(t, T, (X_i)_{t_i \leq t})$ *of the contract with the Realized Risk payoff at any given time* t, $0 \leq t \leq T$.

Realistically speaking, for some of these contracts there will be no good hedging possibility for the Realized Risk, therefore this is an incomplete market pricing problem. Let us assume there is a risk neutral pricing measure $\mathbb{Q}$ and a constant interest rate r. The forward price of the Realized Risk (Tradeable Measure of Risk) is given by

$$(1) \qquad \rho(0, T) = e^{-rT} \mathbb{E}^{\mathbb{Q}}[g((X_i)_{1 \leq i \leq N})],$$

while the call and put option prices on the Realized Risk are given by

$$(2) \qquad c(0, T) = e^{-rT} \mathbb{E}^{\mathbb{Q}}[(g((X_i)_{1 \leq i \leq N}) - K)^+],$$

$$(3) \qquad p(0, T) = e^{-rT} \mathbb{E}^{\mathbb{Q}}[(K - g((X_i)_{1 \leq i \leq N}))^+].$$

Notice that both the call and the put options are Tradeable Measures of Risk directly if we use $(g((X_i)_{1 \leq i \leq N}) - K)^+$ and $(K - g((X_i)_{1 \leq i \leq N}))^+$ respectively in the definition of the Realized Risk.

Examples of Realized Risk:

1. Asset Itself. The most trivial example of Realized Risk is the underlying asset S_T itself. Assume that $\{X_1, \ldots, X_N\}$ are percentage returns: $X_i = \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}$. If we set

$$g((X_i)_{1 \leq i \leq N}) = S_0 \prod_{i=1}^{N} (1 + X_i) = S_T,$$

then $\rho(0, T)$ coincides with a forward on the underlying asset and $c(0, T)$ and $p(0, T)$ are respectively European call and put options on that asset.

2. Weighted Average of Ordered Returns. The main focus of our paper is the case when Realized Risk $g((X_i)_{1 \leq i \leq N})$ is a weighted average of order statistics:

$$(4) \qquad g((X_i)_{1 \leq i \leq N}) = -\sum_{i=1}^{N} w_i X_{(i)}, \quad \text{where } w_i \geq 0 \text{ and } \sum_{i=1}^{N} w_i = 1.$$

Order statistics is a collection of ordered returns $\{X_{(1)}, X_{(2)}, \ldots, X_{(N)}\}$ with $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(N)}$ from the sample $\{X_1, X_2, \ldots, X_N\}$. The Tradeable Measure of Risk at time t, $\rho(t, T, (X_i)_{t_i \leq t})$, with partial observations up to time t, is given by the conditional expectation:

$$(5) \qquad \rho(t, T, (X_i)_{t_i \leq t}) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[g((X_i)_{1 \leq i \leq N}) | \mathcal{F}_t].$$

If the weights in (4) are decreasing, $w_1 \geq \ldots \geq w_N$, then $g((X_i)_{1 \leq i \leq N})$ is a statistical approximation of the class of law invariant convex comonotonic risk measures, called Weighted VaR, that is based on probability distortion of Conditional VaR and is equivalent to the Choquet integral representation (see Kusuoka [16] and Wang et al. [21]). The details are given in Appendix A, in order not to deviate from the current presentation. For additional justification from an axiomatic approach in finite probability space, see Heyde et al. [13]. Important special cases of Weighted Average of Ordered Returns include: the worst return, the empirical VaR, and the empirical mean shortfall.

2a. Worst Return. The Worst Return and its corresponding Tradeable Measure of Risk are defined as:

$$g((X_i)_{1 \leq i \leq N}) = -X_{(1)}, \quad \rho(t, T, (X_i)_{t_i \leq t}) = -e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[X_{(1)} | \mathcal{F}_t],$$

i. e.,

$$w_i = \begin{cases} 1, & i = 1; \\ 0, & i > 1. \end{cases}$$

2b. Empirical VaR. The empirical $VaR_\lambda(X)$ is given by weights:

$$w_i = \begin{cases} 1, & i = \lfloor N\lambda \rfloor; \\ 0, & i \neq \lfloor N\lambda \rfloor, \end{cases}$$

where $\lfloor N\lambda \rfloor$ denotes the largest integer less than or equal to $N\lambda$, $0 < \lambda < 1$.

2c. Empirical Mean Shortfall. The empirical shortfall is defined by weights:

$$w_i = \begin{cases} \frac{1}{\lfloor N\lambda \rfloor}, & i \leq \lfloor N\lambda \rfloor; \\ 0, & i > \lfloor N\lambda \rfloor. \end{cases}$$

The corresponding Realized Risk and the Tradeable Measure of Risk are:

$$g((X_i)_{1 \leq i \leq N}) = -\frac{1}{\lfloor N\lambda \rfloor} \sum_{i=1}^{\lfloor N\lambda \rfloor} X_{(i)}, \quad \rho(t, T, (X_i)_{t_i \leq t}) = -e^{-r(T-t)} \frac{1}{\lfloor N\lambda \rfloor} \sum_{i=1}^{\lfloor N\lambda \rfloor} \mathbb{E}^{\mathbb{Q}}[X_{(i)}|\mathcal{F}_t].$$

3. Maximum Drawdown. Another example of Realized Risk is the discretely monitored maximum drawdown of the price process S_t, which we obtain if we define the payoff as

$$g((X_i)_{1 \leq i \leq N}) = -\min_{1 \leq k < l \leq N} \sum_{i=k+1}^{l} X_i = \max_{0 \leq k < l \leq N} (S_{t_k} - S_{t_l}),$$

where the returns are absolute changes in S : $X_i = S_{t_i} - S_{t_{i-1}}$ for $i = 1, \ldots, N$. For a more systematic treatment of Maximum Drawdown, see Vecer [20].

4. Realized Variance. As mentioned earlier, contracts on realized variance are already traded. In fact, such contracts are Tradeable Measures of Risk with payoff:

$$g((X_i)_{1 \leq i \leq N}) = \frac{1}{N} \sum_{i=1}^{N} (X_i - \bar{X})^2.$$

See Carr et al. [5] for details.

A particular feature of Tradeable Measures of Risk is that the past is seamlessly connected to the future in a non-parametric way. If the current time is t and the time of maturity T, the realized returns $(X_i)_{t_i \leq t}$ have impact on $\rho(t, T, (X_i)_{t_i \leq t})$ through its reflection in the payoff function $g((X_i)_{1 \leq i \leq N})$.

3 Pricing Contracts on Weighted Average of Ordered Returns

In this section, we will limit our focus to a special case of Realized Risk, namely on the Weighted Average of Ordered Returns. Assume that the financial returns $\{X_1, X_2, ..., X_N\}$ are independent and identically distributed with cumulative distribution function $F_X(x)$. The cumulative distribution function of the ith order statistic $X_{(i)}$ is:

(6) $$F_{X_{(i)}}(x) = \sum_{k=i}^{N} \binom{N}{k} [F_X(x)]^k [1 - F_X(x)]^{N-k}.$$

When the returns have a continuous distribution with probability density function $f_X(x)$, the density of $X_{(i)}$ is given by:

$$(7) \qquad f_{X_{(i)}}(x) = \frac{N!}{(i-1)!(N-i)!} f_X(x)[F_X(x)]^{i-1}[1 - F_X(x)]^{N-i}.$$

If we assume the information available at any given time comprises the observed returns X_i's up to that time, meaning $\mathcal{F}_t = \sigma\left((X_i)_{t_i \leq t}\right)$, then we can rewrite the dynamic forward price (5) as:

$$\rho(t, T, (X_i)_{t_i \leq t}) = -e^{-r(T-t)} \sum_{i=1}^{N} w_i \mathbb{E}^{\mathbb{Q}}[X_{(i)}|\mathcal{F}_t]$$

$$(8) \qquad = -e^{-r(T-t)} \sum_{i=1}^{N} w_i \mathbb{E}^{\mathbb{Q}}[X_{(i)}|X_1, ..., X_n], \quad \text{where } t_n \leq t < t_{n+1}.$$

In general, we need to find the conditional distributions of order statistics $X_{(i)}$'s based on the first n observations $X_1, X_2, ..., X_n$. Let us formulate the question in the following way: suppose we have an ordered set of real numbers $-\infty = x_0 < x_1 < x_2 < ... < x_n < x_{n+1} = \infty$ and a set of random variables $Y_1, Y_2, ..., Y_m$ which are independent draws from a common distribution with cumulative distribution function $F(x)$. Let us mix the samples $x_1, x_2, ..., x_n, Y_1, Y_2, ..., Y_m$ and call them $Z_1, Z_2, ..., Z_{n+m}$. We need to compute the conditional distribution of $Z_{(i)}$ as a function of $x_1, x_2, ..., x_n$: $F_{Z_{(i)}}(z|x_1, ..., x_n)$.

Lemma 3.1 *The conditional cumulative distribution function has the following representation:*

$$(9) \qquad F_{Z_{(i)}}(z|x_1, ..., x_k) = \sum_{k=0 \vee (i-m)}^{n \wedge i} F_{Y_{(i-k)}}(z)\mathbb{I}_{[x_k, x_{k+1})}(z), \quad \text{for } 1 \leq i \leq n+m,$$

where $\mathbb{I}_A(x)$ is the indicator function of set A.

PROOF. The order statistics of Y_i's are written as $Y_{(0)}, Y_{(1)}, ..., Y_{(m)}$, where we have added an extra observation $Y_{(0)} = -\infty$. The x_i's divide the real line into $n+1$ intervals, and we need to keep track of which interval each $Y_{(i)}$ falls in. When $z \in [x_k, x_{k+1})$, the event $Z_{(i)} \leq z$ is equivalent to $Y_{(i-k)} \leq z$. Therefore we can write:

$$\{Z_{(i)} \leq z\} = \sum_{k=0 \vee (i-m)}^{n \wedge i} \{Y_{(i-k)} \leq z\} \mathbb{I}_{[x_k, x_{k+1})}(z), \quad \text{for } 1 \leq i \leq n+m.$$

Note that the constraints $0 \leq k \leq n$ and $0 \leq i - k \leq m$ give the range of summation $0 \vee (i-m) \leq k \leq n \wedge i$ in the above equation. The cumulative distribution function is therefore given by the following expression:

$$F_{Z_{(i)}}(z|x_1, ..., x_n) = \mathbb{Q}(Z_{(i)} \leq z|x_1, ..., x_n)$$

$$= \sum_{k=0 \vee (i-m)}^{n \wedge i} \mathbb{Q}(Y_{(i-k)} \leq z)\mathbb{I}_{[x_k, x_{k+1})}(z)$$

$$(10) \qquad = \sum_{k=0 \vee (i-m)}^{n \wedge i} F_{Y_{(i-k)}}(z)\mathbb{I}_{[x_k, x_{k+1})}(z), \quad \text{for } 1 \leq i \leq n+m.$$

Thus, the proof is complete. $\diamond$

Note that the conditional cumulative distribution function, $F_{Z_{(i)}}(z|x_1, ..., x_n)$, is a piecewise function. On the

interval $z \in [x_{k-1}, x_k)$, it is equal to $F_{Y_{(i-k+1)}}(z)$, and on the interval $z \in [x_k, x_{k+1})$, it is equal to $F_{Y_{(i-k)}}(z)$. Therefore, there is a discrete probability mass at every $z = x_k$, whenever $1 \leq k \leq i$:

$$
\begin{aligned}
\mathbb{Q}(Z_{(i)} = x_k | x_1, ..., x_n) &= F_{Z_{(i)}}(x_k | x_1, ..., x_n) - F_{Z_{(i)}}(x_k - | x_1, ..., x_n) \\
&= F_{Y_{(i-k)}}(x_k) - F_{Y_{(i-k+1)}}(x_k) \\
&= \sum_{j=i-k}^{m} \binom{m}{j} [F(x_k)]^j [1 - F(x_k)]^{m-j} - \sum_{j=i-k+1}^{m} \binom{m}{j} [F(x_k)]^j [1 - F(x_k)]^{m-j}
\end{aligned}
$$

$$
(11) \qquad = \binom{m}{i-k} [F(x_k)]^{i-k} [1 - F(x_k)]^{m-i+k}.
$$

If the Y_i's have a continuous distribution with density function $f(x)$, we can write the conditional probability density (mass) function of $Z_{(i)}$, with the help of the Dirac delta function $\delta_a(x)$:

$$
\begin{aligned}
f_{Z_{(i)}}(z | x_1, ..., x_n) &= \sum_{k=0 \vee (i-m)}^{n \wedge (i-1)} f_{Y_{(i-k)}}(z) \mathbb{I}_{(x_k, x_{k+1})}(z) + \sum_{k=1}^{i} \mathbb{Q}(Z_{(i)} = x_k | x_1, ..., x_n) \delta_{x_k}(z) \\
&= \sum_{k=0 \vee (i-m)}^{n \wedge (i-1)} \frac{m!}{(i-k-1)!(m-i+k)!} f(z) [F(z)]^{i-k-1} [1 - F(z)]^{m-i+k} \mathbb{I}_{(x_k, x_{k+1})}(z)
\end{aligned}
$$

$$
(12) \qquad + \sum_{k=1}^{i} \binom{m}{i-k} [F(x_k)]^{i-k} [1 - F(x_k)]^{m-i+k} \delta_{x_k}(z),
$$

for $1 \leq i \leq n + m$.

Theorem 3.2 *Suppose the returns $\{X_1, X_2, ..., X_N\}$ are independent and identically distributed with cumulative distribution function $F_X(x)$ under $\mathbb{Q}$, and $t \in [t_n, t_{n+1})$. Let $\hat{X}_{(1)}, ..., \hat{X}_{(n)}$ be the order statistics of observed returns $X_1, ..., X_n$, and $\tilde{X}_{(1)}, ..., \tilde{X}_{(N-n)}$ the order statistics of future returns $X_{n+1}, ..., X_N$. The dynamic forward price process defined in (8) is:*

$$
(13) \qquad \rho(t, T, (X_i)_{t_i \leq t}) = -e^{-r(T-t)} \sum_{i=1}^{N} w_i \mathbb{E}^{\mathbb{Q}}[X_{(i)} | X_1, ..., X_n]
$$

$$
(14) \qquad = -e^{-r(T-t)} \sum_{i=1}^{N} w_i \sum_{k=0 \vee (i-N+n)}^{n \wedge i} \int_{(\hat{X}_{(k)}, \hat{X}_{(k+1)})} x F_{\tilde{X}_{(i-k)}}(dx)
$$

$$
- e^{-r(T-t)} \sum_{i=1}^{N} w_i \sum_{k=1}^{i} \hat{X}_{(k)} \binom{N-n}{i-k} [F_X(\hat{X}_{(k)})]^{i-k} [1 - F_X(\hat{X}_{(k)})]^{N-n-i+k},
$$

where

$$
F_{\tilde{X}_{(i-k)}}(x) = \sum_{j=i-k}^{N-n} \binom{N-n}{j} [F_X(x)]^j [1 - F_X(x)]^{N-n-j}.
$$

Furthermore, when the distribution of X_i is continuous with probability density function $f_X(x)$, we can write:

$$
(15) \qquad \rho(t, T, (X_i)_{t_i \leq t}) = -e^{-r(T-t)} \sum_{i=1}^{N} w_i \sum_{k=0 \vee (i-N+n)}^{n \wedge (i-1)} \int_{(\hat{X}_{(k)}, \hat{X}_{(k+1)})} x f_{\tilde{X}_{(i-k)}}(x) dx
$$

$$
- e^{-r(T-t)} \sum_{i=1}^{N} w_i \sum_{k=1}^{i} \hat{X}_{(k)} \binom{N-n}{i-k} [F_X(\hat{X}_{(k)})]^{i-k} [1 - F_X(\hat{X}_{(k)})]^{N-n-i+k},
$$

where:

$$f_{\tilde{X}_{(i-k)}}(x) = \frac{(N-n)!}{(i-k-1)!(N-n-i+k)!} f_X(x)[F_X(x)]^{i-k-1}[1-F_X(x)]^{N-n-i+k}.$$

PROOF. The results follow directly from Lemma 3.1 and the comments afterwards, where we replace $x_1, ..., x_n$ with $X_1, ..., X_n$, and $Y_1, ..., Y_m$ with $X_{n+1}, ..., X_N$. $\diamond$

The pricing formulas given in Theorem 3.2 are based on the distribution and density functions (6) and (7) defined at the beginning of this section. Here, we present a lemma that associates the distribution of order statistics to a Value-at-Risk transformation of Beta distribution in a general continuous distribution case and provide an alternative pricing formula. As before, $X_{(1)}, X_{(2)}, ..., X_{(N)}$ are the order statistics of independent and identically distributed random variables $X_1, X_2, ..., X_N$ with cumulative distribution function $F_X(x)$ under $\mathbb{Q}$. Since $VaR_\lambda(X)$ is the negative value of the $\lambda-$quantile function of X, it is the negative value of an inverse function of $F_X(x)$. Therefore, we write it out with two variables

$$VaR(X; \lambda) = VaR_\lambda(X).$$

Note that a Beta(α, β) random variable Y has probability density function:

(16) $$f_{B(\alpha,\beta)}(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, 0 < x < 1,$$

where $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$ and $\alpha > 0, \beta > 0$. Its expectation and variance are simply given by $\mathbb{E}Y = \frac{\alpha}{\alpha+\beta}$ and $Var(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

Lemma 3.3 *Suppose $\{X_1, X_2, ..., X_N\}$ are independent and identically distributed random variables with continuous distribution. Then the j-th order statistics $X_{(j)}$ and $-VaR(X; Y)$ have the same law, where Y is a random variable with Beta(j, N-j+1) distribution.*

PROOF. It is well-known that the j-th order statistics from an independent identically distributed Uniform(0,1) random sample of size N has a Beta(j, N-j+1) distribution. Since $F_X(X_j) \sim$ Uniform(0,1) and F_X is an increasing function and therefore preserve the order of the statistics, $F_X(X_{(j)}) \sim$ Beta(j, N-j+1) and the result follows easily. A direct proof using the probability density function in (7) is also straightforward. Recall that:

$$f_{X_{(j)}}(x) = \frac{N!}{(j-1)!(N-j)!} f_X(x)[F_X(x)]^{j-1}[1-F_X(x)]^{N-j}.$$

We have:

$$\begin{aligned}
\mathbb{Q}(X_{(j)} \le z) &= \int_{-\infty}^z \frac{N!}{(j-1)!(N-j)!} f_X(x)[F_X(x)]^{j-1}[1-F_X(x)]^{N-j}dx \\
&= \int_{-\infty}^z \frac{N!}{(j-1)!(N-j)!} [F_X(x)]^{j-1}[1-F_X(x)]^{N-j}dF_X(x) \\
&= \int_0^{F_X(z)} \frac{1}{B(j, N-j+1)} y^{j-1}(1-y)^{N-j}dy \\
&= \mathbb{Q}(Y \le F_X(z)) = \mathbb{Q}(-VaR(X; Y) \le z),
\end{aligned}$$

where we have denoted $F_X^{-1}(\lambda) = -VaR(X; \lambda)$. $\diamond$

Lemma 3.3 allows us to think of any order statistics as a transform of a Beta random variable with parameters depending only on the order of the statistics and the sample size. From a computational perspective, it also makes the expectation formula simpler to evaluate. Theorem 3.2 is based on the direct formula (7):

$$\mathbb{E}^{\mathbb{Q}}X_{(j)} = \int_{-\infty}^\infty x \frac{1}{B(j, N-j+1)} f_X(x)[F_X(x)]^{j-1}[1-F_X(x)]^{N-j}dx.$$

7

If we use Lemma 3.3 and (16) instead, we arrive to an alternative formula:

$$(17) \quad \mathbb{E}^{\mathbb{Q}} X_{(j)} = -\int_0^1 VaR_\lambda(X) f_{Beta(j,N-j+1)}(\lambda) d\lambda = -\int_0^1 VaR_\lambda(X) \frac{1}{B(j, N-j+1)} \lambda^{j-1}(1-\lambda)^{N-j} d\lambda.$$

The second approach is much more systematic for evaluation or simulation, and formula (15) in Theorem 3.2 can be simplified correspondingly in the unconditional case which we will state in the following Corollary.

Corollary 3.4 *Suppose $\{X_1, X_2, ..., X_N\}$ are independent and identically distributed random variables with continuous distribution. The initial forward price, a special case of Theorem 3.2, can be calculated as*

$$\rho(0, T) = -e^{-rT} \sum_{i=1}^N w_i \mathbb{E}^{\mathbb{Q}}[X_{(i)}]$$

$$= -e^{-rT} \int_0^1 VaR_\lambda(X) \left(\sum_{i=1}^N w_i f_{B(i,N-i+1)}(\lambda) \right) d\lambda.$$

Another interesting perspective of the above formula (17) becomes clear when we compare it to equation (32) in Appendix A. The negative value of the expectation of a particular order statistic is not going to be a good risk measure that satisfies the usual axioms because the Beta density does not have the monotonicity property which function ψ possess in the representation (32) for Weighted VaR. On the other side, the negative value of the expectation of Weighted Average of Ordered Returns when the weights are decreasing will serve as a good risk measure. Its convergence to Weighted VaR is proved in Section 4, and its properties as a dynamic risk measure are discussed in Section 5.

Now let us turn to the question how to apply Theorem 3.2 to a portfolio process W_t, $0 \leq t < \infty$, and give some numerical examples. The value of this process will be recorded at times $0 = t_0 < t_1 < \ldots < t_n = T$ to compute the log returns:

$$X_i = \ln\left(\frac{W_i}{W_{i-1}}\right), \quad i = 1, 2, ..., N,$$

where we denote $W_i = W_{t_i}$. Without loss of generality, we will choose uniform time intervals with length $\Delta t = t_i - t_{i-1}$. Therefore, $N+1$ observations of the portfolio value $W_0, W_1, ..., W_{N+1}$ correspond to N returns $X_1, X_2, ..., X_N$ on time interval $[0, T] = [0, N\Delta t]$. We allow both t and T to vary, that is if $t = M\Delta t$ and $T = N\Delta t$, then the number of observed returns M and the number of future observations N can both vary. The cumulative continuous time return process is defined as

$$X_t = \ln\left(\frac{W_t}{W_0}\right), \quad \text{for} \quad t \geq 0.$$

The exact link between the discrete and the continuous returns is easy to find: $X_i = X_{t_i} - X_{t_{i-1}}$. If W_t follows the Black-Scholes model, then:

$$dW_t = W_t(\alpha dt + \sigma dB_t),$$

where α and σ are positive constants, and B_t a standard Brownian motion. Discrete returns in the Black-Scholes model,

$$(18) \qquad\qquad X_i = (\alpha - \tfrac{\sigma^2}{2})\Delta t + \sigma(B_{t_i} - B_{t_{i-1}}),$$

are independent and identically distributed with $N\left((\alpha - \frac{\sigma^2}{2})\Delta t, \sigma^2 \Delta t\right)$ distribution. Thus, Theorem 3.2 applies, and the dynamic forward price can be explicitly calculated as:

$$\rho(t, T, (X_i)_{t_i \leq t}) = -e^{-r(T-t)} \sum_{i=1}^N w_i \sum_{k=0\vee(i-N+n)}^{n\wedge(i-1)} \int_{(\hat{X}_{(k)}, \hat{X}_{(k+1)})} x f_{\tilde{X}_{(i-k)}}(x) dx$$

$$- e^{-r(T-t)} \sum_{i=1}^N w_i \sum_{k=1}^i \hat{X}_{(k)} \binom{N-n}{i-k} \left[\Phi\left(\frac{\hat{X}_{(k)} - m\Delta t}{\sigma\sqrt{\Delta t}}\right)\right]^{i-k} \left[1 - \Phi\left(\frac{\hat{X}_{(k)} - m\Delta t}{\sigma\sqrt{\Delta t}}\right)\right]^{N-n-i+k},$$

where $m = \alpha - \frac{\sigma^2}{2}$, Φ is the cumulative distribution function of the standard normal distribution, and:

$$f_{\tilde{X}_{(i-k)}}(x) = \frac{(N-n)!}{(i-k-1)!(N-n-i+k)!} \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} e^{-\frac{(x-m)^2}{2\sigma^2\Delta t}} \left[\Phi\left(\frac{x-m\Delta t}{\sigma\sqrt{\Delta t}}\right)\right]^{i-k-1} \left[1 - \Phi\left(\frac{x-m\Delta t}{\sigma\sqrt{\Delta t}}\right)\right]^{N-n-i+k}.$$

We mentioned in Section 1 that VaR is not coherent and the assigned weights for empirical VaR does not satisfy the conditions in (4). However, VaR is widely used in practice and adopted by BASEL II, therefore it is useful to give a benchmark numerical example for the forward price based on the empirical VaR. We will denote the dynamic forward price of the empirical VaR as ρ_λ:

$$\rho_\lambda(t, T, (X_i)_{t_i \leq t}) = -e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \left[X_{(\lfloor N\lambda \rfloor)} | X_1, \ldots, X_n\right],$$

where $t_N \leq T < t_{N+1}$, and $t_n \leq t < t_{n+1}$. Figure 1 shows a path of process $-\rho_{5\%}(t, T, (X_i)_{t_i \leq t})$, where $T = 3$ months and $\Delta t = 1$ day. In this case, $\lfloor N\lambda \rfloor = 3$, therefore the last element of the process, $-\rho_{5\%}(T, T, (X_i)_{1 \leq i \leq N})$, is equal to the third worst return, $X_{(3)}$.

In Figure 2, we present the term structure of forward prices: $\rho_{5\%}(0, T)$ for different values of T (from 6 months to 12 years). We can see that the prices with longer time to maturity tend to be closer to the theoretical value of quantile $q_{5\%}(X)$.
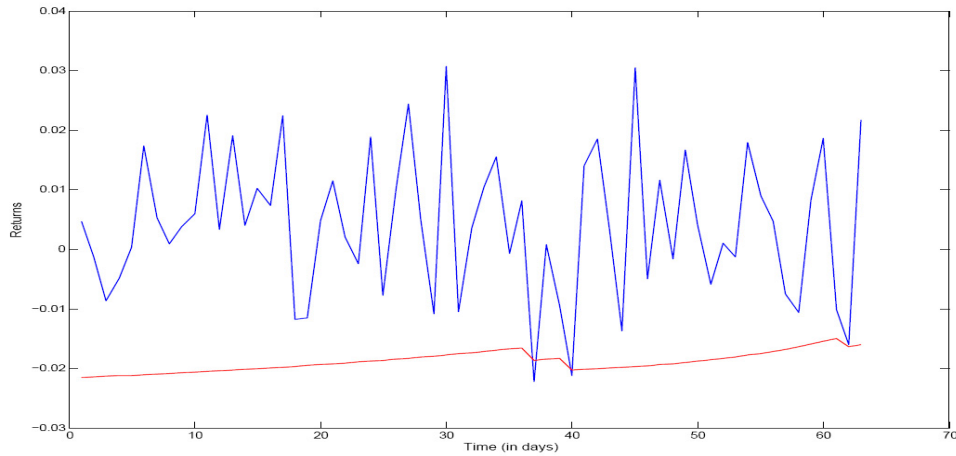

Figure 1: Evolution of $-\rho_{5\%}(t, T, (X_i)_{t_i \leq t})$, where $T = 3$ months and $\Delta t = 1$ day. The daily returns, X, are assumed to follow the Black-Scholes model with interest rate $r = 4\%$ and annual volatility $\sigma = 20\%$.

We give a numerical example to compare the forward prices when the returns are derived from the Black-Scholes model and the Merton model with jumps while the total volatilities in both models are kept the same. The Merton model is defined as:

$$dW_t = W_{t-}(\tilde{\alpha}dt + \tilde{\sigma}dB_t + Y_t dN_t),$$

where B_t is a standard Brownian motion, N_t is a standard Poisson process with intensity λ_{Po}, and Y_t are independent and identically distributed normal random variables with mean μ and standard deviation ν, $Y_t \sim N(\mu, \nu^2)$. The discrete return can be calculated from Doléans-Dade exponential formula:

(19)
$$X_i = (\tilde{\alpha} - \frac{\tilde{\sigma}^2}{2})\Delta t + \tilde{\sigma}(B_{t_i} - B_{t_{i-1}}) + \sum_{k=1}^{N_{t_i} - N_{t_{i-1}}} Y_k.$$
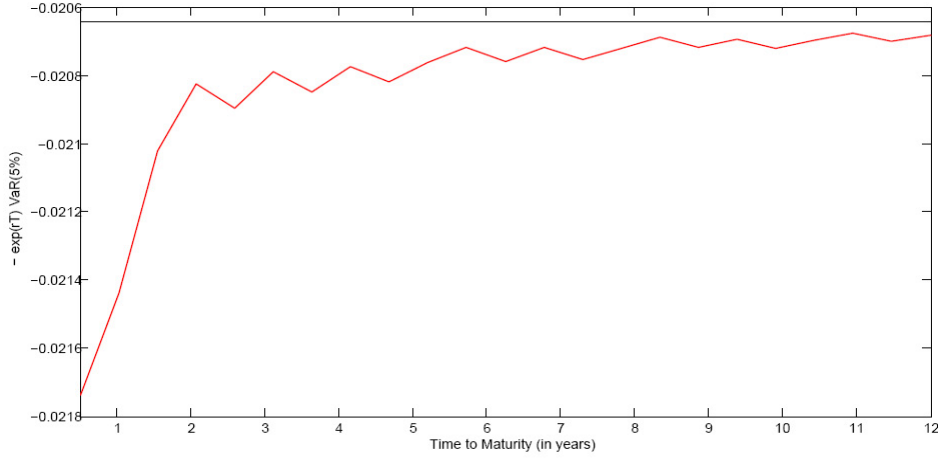
Figure 2: Term structure of $-e^{rT}\rho_{5\%}(0,T)$, where $T \in [0.5, 12]$ years and $\Delta t=$ one day. Daily returns X are assumed to follow the Black-Scholes model with interest rate $r = 4\%$ and annual volatility $\sigma = 20\%$. The horizontal line represents the theoretical 5% quantile, $q_{5\%}(X) = F_X^{-1}(0.05)$.

The density function of X_i has a series expansion:

$$f_{X_i} = e^{-\lambda_{Po}\Delta t} \sum_{k=0}^{\infty} \frac{(\lambda_{Po}\Delta t)^k \exp\left\{-\frac{\left(x-(\tilde{\alpha}-\tilde{\sigma}^2/2)\Delta t-k\mu\right)^2}{2(\tilde{\sigma}^2\Delta t+k\nu^2)}\right\}}{k!\sqrt{2\pi(\tilde{\sigma}^2\Delta t + k\nu^2)}}.$$

In Figure 3, we plot values of $\rho_{5\%,M}(0,T)$, where $T = 6$ months and $\Delta t = 1$ day, obtained from the Merton model with different intensities λ_{Po}. The total volatility of the returns is kept at 30% by reducing $\tilde{\sigma}$ along with increasing λ_{Po}. We compared these values to $\rho_{5\%,BS}(0,T)$ from the Black-Scholes model, which is represented by the horizontal line in the figure. For intensity $\lambda_{Po} = 0$, the two models are identical. If we increase λ_{Po}, $\rho_{5\%,M}(0,T)$ decreases until it reaches its minimum, and then it grows, as the effects of more frequent jumps begin to outweigh lower values of $\tilde{\sigma}$.

Now we will give formulas for the dynamic version of the call and put prices defined in (2) and (3). These formulas are derived only for the empirical VaR and the worst return because we have calculated the conditional distribution of one order statistic $X_{(i)}$ in Lemma 3.1, not of a linear combination $\sum_{i=1}^{N} w_i X_{(i)}$.

Theorem 3.5 *Suppose that returns $\{X_1, X_2, ..., X_N\}$ are independent and identically distributed with cumulative distribution function $F_X(x)$ under $\mathbb{Q}$. Let $\hat{X}_{(1)}, \ldots, \hat{X}_{(n)}$ be the order statistics of observed returns $X_1, \ldots, X_n$, and $\tilde{X}_{(1)}, \ldots, \tilde{X}_{(N-n)}$ the order statistics of future returns $X_{n+1}, \ldots, X_N$. If the Realized Risk is defined as a quantile, $g_\lambda((X_i)_{1 \leq i \leq N}) = -X_{(\lfloor N\lambda \rfloor)}$, then the dynamic call and put option price processes are equal to:*

$$(20) \qquad c_\lambda(t, T, (X_i)_{t_i \leq t}) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\left(-X_{(\lfloor N\lambda \rfloor)} - K\right)^+ | X_1, ..., X_n\right]$$

$$= e^{-r(T-t)} \sum_{k=0\vee(\lfloor N\lambda \rfloor-N+n)}^{n\wedge\lfloor N\lambda \rfloor} \int_{[\hat{X}_{(k)},\hat{X}_{(k+1)})} (-x-K)^+ F_{\tilde{X}_{(\lfloor N\lambda \rfloor-k)}}(dx)$$

$$+ e^{-r(T-t)} \sum_{k=1}^{\lfloor N\lambda \rfloor} (-\hat{X}_{(k)} - K)^+ \binom{N-n}{\lfloor N\lambda \rfloor - k} [F_X(\hat{X}_{(k)})]^{\lfloor N\lambda \rfloor-k}[1 - F_X(\hat{X}_{(k)})]^{N-n-\lfloor N\lambda \rfloor+k},$$

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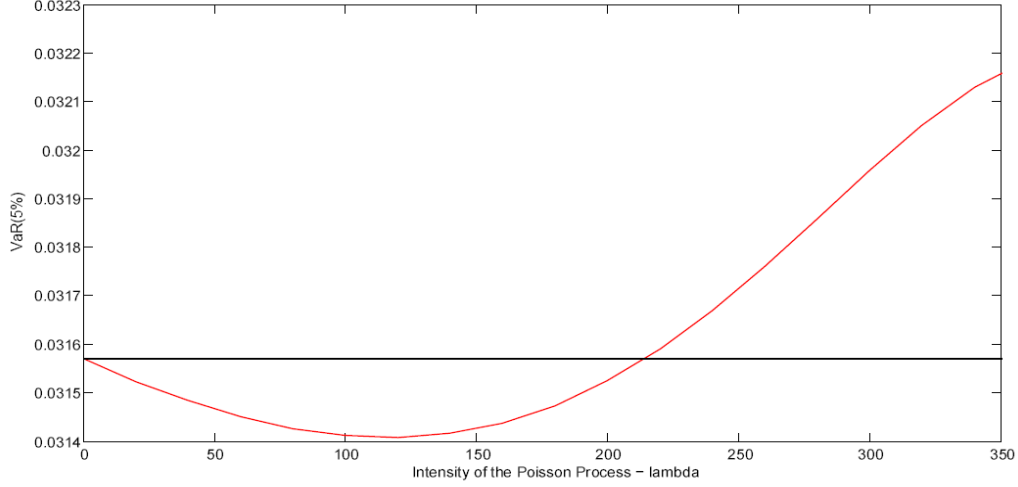Figure 3: Comparison of $\rho_{5\%}(0,T)$ for the Black-Scholes and the Merton model of the daily returns, where $T = 6$ months and $\Delta t = 1$ day. The horizontal line represents the price of the empirical VaR in the Black-Scholes model with annual volatility $\sigma = 30\%$ and interest rate $r = 4\%$. The curve displays the dependence of $\rho_{5\%}(0,T)$ on λ_{Po} in the Merton model. Other parameters are set to be: $\mu = 0$, $\nu = 0.015$, $r = 4\%$. Volatility of the diffusion component, $\tilde{\sigma}$, changed along with λ in order to preserve the total volatility of 30% : $\sqrt{\tilde{\sigma}^2 + \lambda_{Po}(\nu^2 + \mu^2)} = 30\%$.

$$(21) \qquad p_\lambda(t,T,(X_i)_{t_i \leq t}) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[(K + X_{(\lfloor N\lambda \rfloor)})^+ | X_1, ..., X_n]$$

$$= e^{-r(T-t)} \sum_{k=0 \vee (\lfloor N\lambda \rfloor - N + n)}^{n \wedge \lfloor N\lambda \rfloor} \int_{[\hat{X}_{(k)}, \hat{X}_{(k+1)})} (K + x)^+ F_{\tilde{X}_{(\lfloor N\lambda \rfloor - k)}}(dx)$$

$$+ e^{-r(T-t)} \sum_{k=1}^{\lfloor N\lambda \rfloor} (K + \hat{X}_{(k)})^+ \left(\begin{array}{c} N - n \\ \lfloor N\lambda \rfloor - k \end{array} \right) [F_X(\hat{X}_{(k)})]^{\lfloor N\lambda \rfloor - k} [1 - F_X(\hat{X}_{(k)})]^{N - n - \lfloor N\lambda \rfloor + k},$$

where $t_N \leq T < t_{N+1}$, $t_n \leq t < t_{n+1}$, and

$$F_{\tilde{X}_{(\lfloor N\lambda \rfloor - k)}}(x) = \sum_{j=\lfloor N\lambda \rfloor - k}^{N-n} \left(\begin{array}{c} N - n \\ j \end{array} \right) [F_X(x)]^j [1 - F_X(x)]^{N-n-j}.$$

Furthermore, when the distribution of X_i is continuous with probability density function $f_X(x)$, we can write:

$$c_\lambda(t,T,(X_i)_{t_i \leq t}) = e^{-r(T-t)} \sum_{k=0 \vee (\lfloor N\lambda \rfloor - N + n)}^{n \wedge (\lfloor N\lambda \rfloor - 1)} \int_{(\hat{X}_{(k)}, \hat{X}_{(k+1)})} (-x - K)^+ f_{\tilde{X}_{(\lfloor N\lambda \rfloor - k)}}(x) dx$$

$$+ e^{-r(T-t)} \sum_{k=1}^{\lfloor N\lambda \rfloor} (-\hat{X}_{(k)} - K)^+ \left(\begin{array}{c} N - n \\ \lfloor N\lambda \rfloor - k \end{array} \right) [F_X(\hat{X}_{(k)})]^{\lfloor N\lambda \rfloor - k} [1 - F_X(\hat{X}_{(k)})]^{N - n - \lfloor N\lambda \rfloor + k},$$

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$$p_\lambda(t, T, (X_i)_{t_i \leq t}) = e^{-r(T-t)} \sum_{k=0 \vee (\lfloor N\lambda \rfloor - N + n)}^{n \wedge (\lfloor N\lambda \rfloor - 1)} \int_{(\hat{X}_{(k)}, \hat{X}_{(k+1)})} (K + x)^+ f_{\tilde{X}_{(\lfloor N\lambda \rfloor - k)}}(x) dx$$

$$+ e^{-r(T-t)} \sum_{k=1}^{\lfloor N\lambda \rfloor} (K + \hat{X}_{(k)})^+ \left(\begin{array}{c} N - n \\ \lfloor N\lambda \rfloor - k \end{array} \right) [F_X(\hat{X}_{(k)})]^{\lfloor N\lambda \rfloor - k} [1 - F_X(\hat{X}_{(k)})]^{N - n - \lfloor N\lambda \rfloor + k},$$

where:

$$f_{\tilde{X}_{(\lfloor N\lambda \rfloor - k)}}(x) = \frac{(N - n)!}{(\lfloor N\lambda \rfloor - k - 1)!(N - n - \lfloor N\lambda \rfloor + k)!} f_X(x)[F_X(x)]^{\lfloor N\lambda \rfloor - k - 1}[1 - F_X(x)]^{N - n - \lfloor N\lambda \rfloor + k}.$$

A path of the process $c_{5\%}(t, T, (X_i)_{t_i \leq t})$, where $T = 6$ months and $\Delta t = 1$ day, is shown in Figure 4. In this example, $\lfloor N\lambda \rfloor = 6$, therefore $c_{5\%}(t, T, (X_i)_{t_i \leq t})$ is the price of the sixth worst excess loss $(-X_{(6)} - K)^+$.
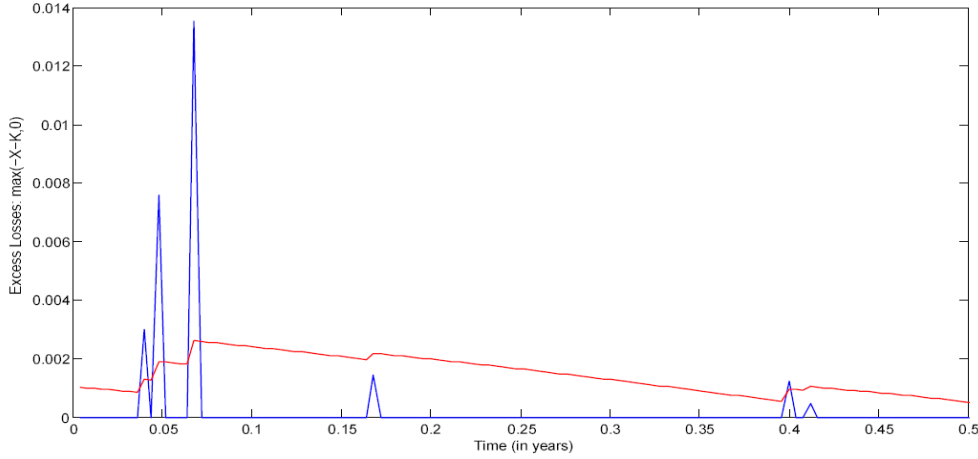

Figure 4: Evolution of $c_{5\%}(t, T, (X_i)_{t_i \leq t})$, where $T = 6$ months, $\Delta t =$ one day, and $K = 0.0155$. The pikes represent excess losses: $(-X_i - K)^+$. The daily returns, X, are assumed to follow the Black-Scholes model with interest rate $r = 4\%$, and annual volatility $\sigma = 15\%$.

4   Convergence Theorems

Let us discuss the convergence of a Tradeable Measure of Risk, $\rho(0, T)$, if we let the time of maturity $T \to \infty$. The number of future returns increases and the expected discrete payoff, $\mathbb{E}^{\mathbb{Q}}[g((X_i)_{0 < i \leq N})]$, approaches the continuous payoff of which $g((X_i)_{0 < i \leq N})$ is an estimate. The continuous payoff is in fact the Weighted VaR (see Appendix A). We define the $\lambda-$quantile of a distribution with cumulative distribution function $F_X(x)$ as:

(22) $q_\lambda(X) = \sup\{x : F_X(x) < \lambda\}, \quad 0 < \lambda < 1.$

We see from (31) in Appendix A that the VaR is simply defined as the negative value of the above $\lambda-$quantile function:

$$VaR_\lambda(X) = -q_\lambda(X) = \inf\{m \mid \mathbb{P}(X + m \leq 0) < \lambda\},$$

where X has distribution $F_X(x)$.

Theorem 4.1 *Suppose that returns $\{X_1, \ldots, X_N\}$ have a continuous and increasing cumulative distribution function $F_X(x)$.*

(i) *If $g((X_i)_{0<i\leq N}) = -X_{(\lfloor N\lambda \rfloor)}$, for some $\lambda \in (0,1)$, and $\int_0^1 VaR_\lambda(X)d\lambda$ is finite, then:*

$$(23) \qquad e^{rT}\rho(0,T) \longrightarrow VaR_\lambda(X) \quad as \quad T \to \infty.$$

(ii) *Let $\psi(\lambda)$ be a nonnegative and continuous function on $[0,1]$, such that $\int_0^1 \psi(\lambda)d\lambda = 1$ and $\int_0^1 VaR_\lambda(X)\psi(\lambda)d\lambda$ is finite. If $g((X_i)_{0<i\leq N}) = -\sum_{i=1}^N w_i X_{(i)}$ and the weights are given as:*

$$w_i = \frac{1}{s_N N}\psi\left(\frac{i-1}{N-1}\right) \quad for \quad i = 1, \ldots, N, \quad where \quad s_N = \frac{1}{N}\sum_{i=1}^N \psi\left(\frac{i-1}{N-1}\right),$$

then:

$$(24) \qquad e^{rT}\rho(0,T) \longrightarrow \int_0^1 VaR_\lambda(X)\psi(\lambda)d\lambda \quad as \quad T \to \infty.$$

PROOF.

(i) If g is defined as the empirical VaR, then (see (17)):

$$
\begin{aligned}
e^{rT}\rho(0,T) &= -\mathbb{E}^{\mathbb{Q}}\left[X_{(\lfloor N\lambda \rfloor)}\right] \\
&= -\int_0^1 F_X^{-1}(y)\frac{N!}{(\lfloor N\lambda \rfloor - 1)!(N - \lfloor N\lambda \rfloor)!}y^{\lfloor N\lambda \rfloor - 1}(1-y)^{N-\lfloor N\lambda \rfloor}dy \\
&= \int_0^1 VaR_y(X)\mathbb{I}_{\{y\in(\varepsilon,1-\varepsilon)\}}dF_{B(\lfloor N\lambda \rfloor, N-\lfloor N\lambda \rfloor + 1)}(y) \\
&\quad + \int_0^1 VaR_y(X)\mathbb{I}_{\{y\notin(\varepsilon,1-\varepsilon)\}}\frac{1}{B(\lfloor N\lambda \rfloor, N - \lfloor N\lambda \rfloor + 1)}y^{\lfloor N\lambda \rfloor - 1}(1-y)^{N-\lfloor N\lambda \rfloor}dy,
\end{aligned}
$$

where $t_N \leq T < t_{N+1}$ (thus, $T \to \infty$ is equivalent to $N \to \infty$). In the above formula, we choose $\varepsilon \in (0,1)$, so that $\lambda \in (\varepsilon, 1-\varepsilon)$. In this proof, $F_{B(\lfloor N\lambda \rfloor, N-\lfloor N\lambda \rfloor + 1)}(y)$ denotes the cumulative distribution function of a Beta distribution with parameters $\lfloor N\lambda \rfloor$ and $N - \lfloor N\lambda \rfloor + 1$.
A sequence of random variables, which have Beta distributions with parameters $\lfloor N\lambda \rfloor$ and $N-\lfloor N\lambda \rfloor + 1$, converges in distribution to λ, as $N \to \infty$. Hence, $F_{B(\lfloor N\lambda \rfloor, N-\lfloor N\lambda \rfloor + 1)}(y)$ converges weakly to $\mathbb{I}_{(0,\lambda)}(y)$. Since $VaR_y(X)\mathbb{I}_{\{y\in(\varepsilon,1-\varepsilon)\}}$ is bounded and $\lambda \in (\varepsilon, 1-\varepsilon)$, we have:

$$
\begin{aligned}
\lim_{N\to\infty}\int_0^1 VaR_y(X)\mathbb{I}_{\{y\in(\varepsilon,1-\varepsilon)\}}dF_{B(\lfloor N\lambda \rfloor, N-\lfloor N\lambda \rfloor + 1)}(y) &= \int_0^1 VaR_y(X)\mathbb{I}_{\{y\in(\varepsilon,1-\varepsilon)\}}d\mathbb{I}_{(0,\lambda)}(y) \\
&= VaR_\lambda(X).
\end{aligned}
$$

To complete the proof, we need to show:

$$(25) \quad \lim_{N\to\infty}\int_0^1 VaR_y(X)\mathbb{I}_{\{y\notin(\varepsilon,1-\varepsilon)\}}\frac{1}{B(\lfloor N\lambda \rfloor, N - \lfloor N\lambda \rfloor + 1)}y^{\lfloor N\lambda \rfloor - 1}(1-y)^{N-\lfloor N\lambda \rfloor}dy = 0.$$

This follows from the fact that

$$\mathbb{I}_{\{y\notin(\varepsilon,1-\varepsilon)\}}\frac{1}{B(\lfloor N\lambda \rfloor, N - \lfloor N\lambda \rfloor + 1)}y^{\lfloor N\lambda \rfloor - 1}(1-y)^{N-\lfloor N\lambda \rfloor}$$

are tails of Beta density functions with parameters $\lfloor N\lambda \rfloor$ and $N-\lfloor N\lambda \rfloor + 1$. As $N \to \infty$, these functions converge to zero uniformly on $\{y \notin (\varepsilon, 1-\varepsilon)\}$. Moreover, $\int_0^1 VaR_y(X)dy$ is assumed to be finite. Hence, the bounded convergence theorem implies result (25).

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(ii) We can write the initial price of a forward on Weighted Average of Ordered Returns as follows (see Lemma 3.3):

$$
\begin{aligned}
e^{rT}\rho(0,T) &= -\mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{N} w_i X_{(i)}\right] = -\sum_{i=1}^{N} w_i E^{\mathbb{Q}}\left[X_{(i)}\right] \\
&= -\sum_{i=1}^{N} w_i \int_0^1 F_X^{-1}(\lambda) \frac{N!}{(i-1)!(N-i)!}\lambda^{i-1}(1-\lambda)^{N-i}d\lambda \\
&= \int_0^1 \left[-F_X^{-1}(\lambda)\right]\sum_{i=1}^{N} w_i \frac{N!}{(i-1)!(N-i)!}\lambda^{i-1}(1-\lambda)^{N-i}d\lambda \\
&= \int_0^1 VaR_\lambda(X)\sum_{i=1}^{N}\frac{1}{s_N N}\psi\left(\frac{i-1}{N-1}\right)\frac{N!}{(i-1)!(N-i)!}\lambda^{i-1}(1-\lambda)^{N-i}d\lambda \\
&= \int_0^1 VaR_\lambda(X)\frac{1}{s_{M+1}}\sum_{k=0}^{M}\psi\left(\frac{k}{M}\right)\left(\begin{array}{c}M\\k\end{array}\right)\lambda^k(1-\lambda)^{M-k}d\lambda,
\end{aligned}
$$

where $t_N \leq T < t_{N+1}$, $M = N-1$, and $k = i-1$. Term

$$
\sum_{k=0}^{M}\psi\left(\frac{k}{M}\right)\left(\begin{array}{c}M\\k\end{array}\right)\lambda^k(1-\lambda)^{M-k}
$$

is a Bernstein polynomial of λ, which converges uniformly to the function ψ as $M \to \infty$. $\{s_{M+1}\}_{M=0}^{\infty}$ is a sequence of Riemann sums which converges to $\int_0^1 \psi(\lambda)d\lambda = 1$ as $M \to \infty$. Thus, the entire integrand converges uniformly to $VaR_\lambda(X)\psi(\lambda)$, which is assumed to be an integrable function. As a result:

$$
\begin{aligned}
\lim_{T\to\infty} e^{rT}\rho(0,T) &= \int_0^1 VaR_\lambda(X)\lim_{M\to\infty}\frac{1}{s_{M+1}}\sum_{k=0}^{M}\psi\left(\frac{k}{M}\right)\left(\begin{array}{c}M\\k\end{array}\right)\lambda^k(1-\lambda)^{M-k}d\lambda \\
&= \int_0^1 VaR_\lambda(X)\psi(\lambda)d\lambda.
\end{aligned}
$$

$\diamond$

Another important feature of a Tradeable Measure of Risk is that $e^{r(T-t)}\rho(t,T,(X_i)_{t_i\leq t})$ becomes a better estimate of payoff $g((X_i)_{0<i\leq N})$ as t converges to the time of maturity T. This observation is stated and proved in the following theorem.

Theorem 4.2 *Suppose that $\{X_1, X_2, ..., X_N\}$ are independent and identically distributed random variables with finite variance. Then:*

(26)
$$
\mathbb{E}^{\mathbb{Q}}\left[e^{r(T-t)}\rho(t,T,(X_i)_{t_i\leq t}) - g((X_i)_{0<i\leq N})\right]^2,
$$

where $g((X_i)_{0<i\leq N}) = -\sum_{i=1}^{N} w_i X_{(i)}$, is a nonincreasing function of t.

PROOF. Conditional expectation $\mathbb{E}^{\mathbb{Q}}[.|X_1,\ldots,X_n]$ will be denoted by $\mathbb{E}_n^{\mathbb{Q}}[.]$ in this proof. Note that if $t_n \leq t < t_{n+1}$, then $e^{r(T-t)}\rho(t,T,(X_i)_{t_i\leq t}) = -\mathbb{E}_n^{\mathbb{Q}}\left[\sum_{i=1}^{N} w_i X_{(i)}\right]$. The following inequality proves claim

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(26):

$$\mathbb{E}^{\mathbb{Q}}\left[-\mathbb{E}_n^{\mathbb{Q}}\left[\sum_{i=1}^N w_i X_{(i)}\right] + \sum_{i=1}^N w_i X_{(i)}\right]^2 = \mathbb{E}^{\mathbb{Q}}\left[-\mathbb{E}_n^{\mathbb{Q}}\left[\sum_{i=1}^N w_i X_{(i)}\right] + \mathbb{E}_{n+1}^{\mathbb{Q}}\left[\sum_{i=1}^N w_i X_{(i)}\right]\right]^2$$

$$+\mathbb{E}^{\mathbb{Q}}\left[-\mathbb{E}_{n+1}^{\mathbb{Q}}\left[\sum_{i=1}^N w_i X_{(i)}\right] + \sum_{i=1}^N w_i X_{(i)}\right]^2$$

$$\geq \mathbb{E}^{\mathbb{Q}}\left[-\mathbb{E}_{n+1}^{\mathbb{Q}}\left[\sum_{i=1}^N w_i X_{(i)}\right] + \sum_{i=1}^N w_i X_{(i)}\right]^2.$$

$\diamond$

5   Tradeable Measure of Risk as a Dynamic Risk Measure

In Section 3, we studied the properties of Tradeable Measure of Risk $\rho(t, T, (X_i)_{t_i \leq t})$ as the forward price process on Weighted Average of Ordered Returns. That section addressed the "tradeable" part of the title of this paper. In this section, we will discuss the "measure of risk" part. In fact, as t changes, the definition of the conditional expectation in (5) makes Tradeable Measure of Risk, $\rho(t, T, (X_i)_{t_i \leq t})$, particularly suited to serve as a dynamic measure of risk. In fact, the adoption of the option pricing approach implies its discounted value is naturally a martingale. Representation theorems for Dynamic Risk Measures that satisfy either coherent or convex principles have been proven in Artzner et al. [4], Riedel [18], Cheridito et al. [6], Frittelli and Scandolo [12], Klöppel and Schweizer [15], and Weber [22]. We will focus on listing some usual desirable properties of a Dynamic Risk Measure in Definition 5.1 and stating the properties which $\rho(t, T, (X_i)_{t_i \leq t})$ satisfies in Theorem 5.2. For some of the properties, we will need to assume that the weights are decreasing: $w_1 \geq \ldots \geq w_N$.

Definition 5.1 (Coherence, Relevance, and Time Consistency) *Suppose X_t and Y_t are stochastic processes for $t \in [0, T]$, and $\rho(t, T, X)$ and $\rho(t, T, Y)$ are dynamic risk measures on X_t and Y_t. We define the following axioms for ρ:*

1. *Monotonicity: $X_t \leq Y_t$ for all $t \in [0, T]$ implies $\rho(t, T, X) \geq \rho(t, T, Y)$ for all $t \in [0, T]$.*

2. *Subadditivity: $\rho(t, T, X + Y) \leq \rho(t, T, X) + \rho(t, T, Y)$ for all $t \in [0, T]$.*

3. *Positive Homogeneity: $\rho(t, T, \lambda X) = \lambda \rho(t, T, X)$ for any real number $\lambda \geq 0$ and for all $t \in [0, T]$.*

4. *Predictable Translation Invariance: fix $t \in [0, T]$, $\rho(t, T, X + Z) = \rho(t, T, X) - Z$ for any $\mathcal{F}_t$-measurable random variable Z.*

5. *Relevance: fix $u \in [0, T]$, if $X_s(\omega) = -\epsilon \mathbb{I}_A(\omega)\mathbb{I}_{[u,T]}(s)$ for some $A \in \mathcal{F}_u$ where $\mathbb{Q}(A) > 0$, and all $\epsilon > 0$, then $\rho(t, T, X) > 0$ for all $t \in [0, u)$.*

6. *Time Consistency: $e^{-rs}\rho(s, T, X) = \mathbb{E}^{\mathbb{Q}}\left[e^{-rt}\rho(t, T, X)|\mathcal{F}_s\right]$ for $0 \leq s \leq t \leq T$.*

Properties 1-4 are axioms of Coherent Measures of Risk; Relevance is related to the no-arbitrage principle; Time Consistency makes it convenient to implement dynamic programming methods.

Theorem 5.2 (Coherence, Relevance, and Time Consistency) *Let $\rho(t, T, (X_i)_{t_i \leq t})$ be a Tradeable Measure of Risk defined by*

(27) $$\rho(t, T, (X_i)_{t_i \leq t}) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[g((X_i)_{1 \leq i \leq N})|\mathcal{F}_t]$$

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with

$$(28) \qquad g((X_i)_{1 \leq i \leq N}) = -\sum_{i=1}^{N} w_i X_{(i)}, \quad w_1 \geq w_2 \geq ... \geq w_N \geq 0 \ and \ \sum_{i=1}^{N} w_i = 1.$$

Then $\rho(t, T, (X_i)_{t_i \leq t})$ satisfies the axioms of Monotonicity, Subadditivity, Positive homogeneity, Predictable Translation Invariance, Relevance, and Time Consistency.

PROOF. The results are straightforward consequences of the definition of $\rho(t, T, (X_i)_{t_i \leq t})$ as the conditional expectation of weighted order statistics. The assumption about decreasing weights is a sufficient condition for axioms of Relevance, and Subadditivity (the proof can be found in Heyde et al. [13]). We choose to show only Relevance here (according to item 5 in Definition 5.1, $X \leq 0$):

$$\rho(t, T, (X_i)_{t_i \leq t}) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[g((X_i)_{1 \leq i \leq N})|\mathcal{F}_t] = -e^{-r(T-t)}\sum_{i=1}^{N} w_i \mathbb{E}^{\mathbb{Q}}[X_{(i)}|\mathcal{F}_t]$$

$$\geq -e^{-r(T-t)}w_1\mathbb{E}^{\mathbb{Q}}[X_{(1)}|\mathcal{F}_t] = e^{-r(T-t)}w_1\epsilon\mathbb{Q}(A) > 0.$$

$\diamond$

Note that $\rho(t, T, (X_i)_{t_i \leq t})$ satisfies all the properties mentioned in Definition 5.1.

6 Conclusion

The idea of this paper is to introduce tradeable financial contracts which can serve both as risk measures, as well as a means to transfer financial risk. One can view such contracts as financial insurance. In particular, we introduce option contracts to be traded on specific measures of risk to provide market participants (banks, financial, and non-financial companies) a way to manage their risk through financial markets.

The call option on Weighted Average of Ordered Returns $g((X_i)_{1 \leq i \leq N})$ can be considered a risk management tool which reduces a holder's exposure to the market risk associated with a position in asset X. In practice, trading on behalf of an investor or a financial institution may be subject to limit L on the risk exposure:

$$(29) \qquad \rho(t, T, (X_i)_{t_i \leq t}) \leq L.$$

Rules of financial institutions contain measures which need to be taken whenever the risk exposure exceeds the limit. Increasing capital reserves or closing out the position are examples of such measure. However, the Tradable Measure of Risk allows for a reduction to exposure in a different way: by entering a long position in a call option c on the Realized Risk with strike price K, such that:

$$(30) \qquad \rho_X(t, T, (X_i)_{t_i \leq t}) - c(t, T, (X_i)_{t_i \leq t}) \leq L.$$

Finally, we would like to draw an analogy between our idea of trading contracts on Realized Risk and trading emissions. According to the cap and trade rules for emissions, each company has a limit on the amount of a pollutant, but can exceed it by buying "extra emissions" from another company which does not use its entire limit. As a result, emissions are priced as any other goods and services. One can think of risk in a similar way: risk is analogous to emissions, where high levels are considered bad, while low levels are considered good. There are financial institutions (such as banks and funds) which comply with certain limits on risk. If there was a market for risk, an institution could take a riskier position in an asset than it is allowed, buying the financial insurance contract on the excess risk, and let the seller manage the risk in a more efficient way, probably through diversification. The regulator entity would play a natural role in keeping the risk market liquid, and in turn the price of risk observed directly from the market could provide useful information about the health of the economy and serve as an input for monetary or regulatory policy purpose.

A   Realized Risk Measure Based on Weighted VaR

In this Section, we will assume that Realized Risk $g((X_i)_{1 \leq i \leq N})$ is given by Weighted Average of Ordered Returns with decreasing weights: $w_1 \geq \ldots \geq w_N$. Let us relate formula (4) for g to Weighted VaR, noting that the former is the discrete approximation of the latter. Weighted VaR (see Wang et al. [21], Föllmer and Schied [11], Cherny [7]) is in fact a probabilistically distorted Conditional VaR (see Acerbi et al. [1], Rockafellar and Uryasev [19]), and it is the only possible form of law-invariant, comonotonic, convex risk measures on L^∞ on an atomless probability space (see Kusuoka [16]). $VaR_\lambda(X)$ is by definition the negative value of the λ-quantile of a random variable X defined in (22):

$$(31) \qquad VaR_\lambda(X) = -q_X(\lambda) = \inf\{\, m \,|\, \mathbb{P}(X + m \leq 0) < \lambda \,\}.$$

Averaging VaR, we get Conditional VaR:

$$CVaR_\lambda(X) = \frac{1}{\lambda} \int_0^\lambda VaR_\gamma(X) d\gamma, \quad \text{for } \lambda \in (0, 1].$$

Suppose μ is a probability measure on $(0, 1]$ and the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless. Weighted VaR is a distortion of Conditional VaR by probability measure μ:

$$WVaR_\mu(X) = \int_{(0,1]} CVaR_\gamma(X) \mu(d\gamma).$$

We can define $CVaR_0(X) = -\text{essinf } X$ to extend the definition of μ on the closed interval $[0, 1]$. However, the essinf X will be captured if μ assigns positive probability on the point 0. In case X is unbounded below, this point will overweight the rest of the distribution of X. Therefore, we avoid this situation here by excluding point 0. Fubini's theorem gives:

$$(32) \qquad WVaR_\mu(X) = \int_0^1 VaR_\gamma(X) \psi(\gamma) d\gamma, \quad \text{where } \psi(\gamma) = \int_{(\gamma,1]} \frac{1}{\alpha} \mu(d\alpha).$$

It is obvious that $\psi(\gamma)$ is a decreasing, right-continuous function. If we define $\nu(\lambda) = \int_0^\lambda \psi(\gamma) d\gamma$, then $\nu(\lambda)$ is increasing and concave with $\nu(0) = 0$ and $\nu(1) = 1$. Thus $\nu(\lambda)$ can be viewed as a distribution function with density function $\psi(\gamma)$. The discrete approximation of the $WVaR_\mu(X)$ from N statistical observations becomes our payoff function (4) with decreasing weights:

$$g((X_i)_{1 \leq i \leq N}) = -\sum_{i=1}^N w_i X_{(i)}, \quad \text{where } w_1 \geq w_2 \geq \ldots \geq w_N \geq 0 \text{ and } \sum_{i=1}^N w_i = 1.$$

The convergence from the discrete to the continuous case is given in Theorem 4.1. For additional justification from an axiomatic approach in finite probability space see Heyde et al. [13].

Acknowledgements

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