Preference of Social Choice in Mathematical Economics

Islam, Jamal and Mohajan, Haradhan and Moolio, Pahlaj

Indus Journal of Management Social Sciences

11 March 2008

Preference of Social Choice in Mathematical Economics

Jamal Nazrul Islam*, Haradhan Kumar Mohajan**, and Pahlaj Moolio***

ABSTRACT

Mathematical Economics is closely related with Social Choice Theory. In this paper, an attempt has been made to show this relation by introducing utility functions, preference relations and Arrow's impossibility theorem with easier mathematical calculations. The paper begins with some definitions which are easy but will be helpful to those who are new in this field. The preference relations will give idea in individual's and social choices according to their budget. Economists want to create maximum utility in society and the paper indicates how the maximum utility can be obtained. Arrow's theorem indicates that the aggregate of individuals' preferences will not satisfy transitivity, indifference to irrelevant alternatives and non-dictatorship simultaneously so that one of the individuals becomes a dictator. The Combinatorial and Geometrical approach facilitate understanding of Arrow's theorem in an elegant manner.

JEL. Classification: C51; D11; D21; D78; D92

Key words: Utility Function, Preference Relation, Indifference Hypersurface, Social Choice, Arrow's Theorem.

1. INTRODUCTION

This paper is related to Welfare of Economics and Sociology, in particular Social Choice Theory. Here we have tried to give various aspects of economics and sociology in mathematical terms. The presentation here is essentially a review of other's works, but we have tried to give the definitions and mathematical calculations more clearly, so that one may find the paper naive and simple. We

---

* Emeritus Professor, Research Centre for Mathematical and Physical Sciences, University of Chittagong, Chittagong, Bangladesh. Phone: +880-31-616780.

** Lecturer, Premier University, Chittagong, Bangladesh , E mail: haradhan_km@yahoo.com

*** Ph.D. Fellow, Research Centre for Mathematical and Physical Sciences, University of Chittagong, Chittagong, Bangladesh; and Professor and Academic Advisor, Pannasastra University of Cambodia, Phnom Penh, Cambodia. Corresponding Author: Pahlaj Moolio. E-mail: pahlajmoolio@gmail.com

Acknowledgements: Authors would like to thank the editors and anonymous referees for their comments and insight in improving the draft copy of this article. Authors furthur would like to declare that this manuscript is Original and has not previously been published, and that it is not currently on offer to another publisher; and also transfer copy rights to the publisher of this journal.

Recieved: 03-11- 2008; Revised: 20-01-2009; Accepted: 15-04-2009; Published:20-04-2009

hope that here the mathematicians will find the economics useful, and vice-versa. We have also included "Social Choice Theory" which is regarded as a part of Mathematical Economics.

In section 2, we give some definitions, which are very simple, but will be very helpful for those who are new in this field. Preference relations and utility functions are included in section 3 which are based on Arrow (1959, 1963), Cassels (1981), Myerson (1996), Islam (1997, 2008) and Pahlaj (2002). Arrow's impossibility theorem, its combinatorial and geometrical interpretation is given more clearly in section 4 which are based on Arrow (1963), Sen (1970), Barbera (1980), Cassels (1981), Islam (1997, 2008), Ubeda (2003), Geanokoplos (2005), Feldman , Serrano (2006) and Breton, Weymark (2006) , Feldman and Serrano (2007,2008), Suzumuro (2007), Miller (2009), Sato (2009).

2. A BRIEF DISCUSSION ON SETS, FUNCTIONS, VECTORS AND OPTIMIZATION

A set is any well defined collection of objects. Let A and B be two sets .The Cartesian product $A \times B$ of A and B is the set of pair (x, y) where $x \in A$ and $y \in B$. A function f from A to B is a rule which assigns to each $x \in A$, a unique element $f(x) \in B$. A more formal definition is as follows:

A function $f : A \to B$ is a subset of $A \times B$, such that
i) if $x \in A$, there is a set $y \in B$ such that $(x, y) \in f$ ii) such an element y is unique , that is , if $x \in A$, $y, z \in B$ such that $(x, y) \in f$, $(x, z) \in f$ then $y = z$.
If $f : A \to B$ is a function, then the image of f, $f(A)$, is the subset of B defined as follows: $f(A) = \{f(x) / x \in A\}$, that is, $f(A)$ consists of elements of B of the form $f(x)$, where x is some element of A. Here A is the domain and B is the co-domain. A function f: $A \to B$ is surjective if each element of B is the image of some element of A. The function f is an injective if for all x, $y \in A$, $f(x) = f(y)$ implies $x = y$. The function is bijective if it is both injective and surjective.

A correspondence ϕ from a set A to a set B is a relation which associates with each element x of A a non-empty subset $\phi(x)$ of B. Generally, if f is a function from a set A onto a set B then for every y $\in$ B, $f^{-1}(y)$ is a non-empty subset of A, and so f^{-1} is a correspondence from B to A.

Let us consider the set of all n -tuples of real numbers which is denoted by R^n and is called n-dimensional Euclidean space. A typical element or a vector in this space is denoted by $\mathbf{x} = (x_1, x_2, ..., x_n)$, where $x_i (i = 1,2,...,n)$ are real numbers. We will use the words 'points' and 'vectors' interchangeably; the point $\mathbf{x}$ can be associated with the directed line segment from the origin $\mathbf{0}$= (0,0,...,0) to the point $\mathbf{x}$. A convex set is defined as follows: Consider a set C which is such that , if $\mathbf{x}$ and $\mathbf{x}'$ are in C, so are all the vectors of the form $t\mathbf{x} + (1 - t)\ \mathbf{x}'$ with $0 \le t \le 1$, in other words, if the set C contains points $\mathbf{x}$ and $\mathbf{x}'$, it also contains all the points lying in the straight line joining $\mathbf{x}$ and $\mathbf{x}'$. For example in R^2 the interior of a circle is convex, in R^3 the interior of a sphere is convex, etc. Let us consider a function $f(\mathbf{x})$ where $\mathbf{x} = (x_1, x_2, ..., x_n)$, then by a hypersurface we mean the set of points in R^n for which f($\mathbf{x}$)= constant. For different values of the constant, we find corresponding different hypersurfaces. For n = 3 we have different surfaces, on the other hand for n = 2 we have simply curves. The indifference hypersurfaces do not intersect each other in the finite region. Since all the components of the vectors are non-negative so we will deal here only with non-negative coordinates. For $n = 2$ the curves lie in the first quadrant and for $n = 3$ the surfaces lie in the first octant.

For a function $f(x)$ to be optimum (maximum or minimum) $\frac{df}{dx} = f'(x) = 0$. If $\frac{d^2f}{dx^2} < 0$ at $x = x_0$ the function is maximum at a point $x = x_0$ and if $\frac{d^2f}{dx^2} > 0$ at $x = x_0$ the function is minimum at a point $x = x_0$. If $f(x, y)$ be a function of two variables x and y then for optimum $\frac{\partial f}{\partial x}(i.e. f_x) = 0 = \frac{\partial f}{\partial y}(i.e. f_y)$, and $f_{xx}f_{yy} - f_{xy}^2 > 0$. If $f_{xx} > 0$ (and $f_{yy} > 0$), then the function has a minimum point, if $f_{xx} < 0$ (and $f_{yy} < 0$) then the function has a maximum point. For $f_{xx}f_{yy} - f_{xy}^2 < 0$, there is neither a maximum nor a minimum, but a saddle point. In all cases, the tangent plane at the extremum (maximum, minimum or a saddle point) to the surface $z = f(x, y)$, is parallel to the z-plane. If $f_{xx}f_{yy} - f_{xy}^2 = 0$, one has to apply other considerations to determine the nature of the extremum.

3. UTILITY FUNCTION

We consider vectors

$\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$ in R^n, then $\mathbf{x} \geq \mathbf{y} \Rightarrow x_i \geq y_i$ for all i (1a)

$\mathbf{x} > \mathbf{y} \Rightarrow \mathbf{x} \geq \mathbf{y}$ but $\mathbf{x} \neq \mathbf{y}$, that is x_i is different from y_i for at least one i (1b)

$\mathbf{x} >> \mathbf{y} \Rightarrow x_i > y_i$ for all i (1c)

Now we introduce the notion of *preference* (Arrow 1959, 1963, Islam 1997, Myerson 1996, Breton and Weymark 2006 ,Feldman and Serrano 2007,2008 , Suzumuro 2007, Miller 2009, Sato 2009).

Suppose two bundles of commodities are represented by the vectors **x** and **y**. The components represent amounts of different commodities in some unit, such as kilogram. We assume that one prefers the bundle **x** to the bundle **y** or he prefers **y** to **x**, or he is indifferent to the choice between **x** and **y**. We can write these possibilities, respectively, as follows:

$$\mathbf{x}P\mathbf{y},\ \mathbf{y}P\mathbf{x},\ \mathbf{x}I\mathbf{y}. \tag{2}$$

Sometimes we use the notation

$$\mathbf{x}\,R\mathbf{y} \tag{3}$$

to mean that either **x** is preferred to **y** or **x** is indifferent to **y**, so that **y** is not preferred to **x**. If **x***P***y** then it is not necessary that all the commodities of **x** are greater than all the corresponding components of **y**. We can write that it is not necessary that **x** >> **y** or even $\mathbf{x} \geq \mathbf{y}$.

We now define the utility function (Islam, 1997, 2008) as,

$$u(\mathbf{x}) = u(x_1, x_2, ..., x_n). \tag{4}$$

In preference relation we can write

$$u(\mathbf{x}) > u(\mathbf{y}) \Leftrightarrow \mathbf{x}P\mathbf{y}. \tag{5}$$

Let us consider a fixed vector $\mathbf{x}_0$, and consider the set of all the vectors $\mathbf{x}$ which are preferred to $\mathbf{x}_0$. If we denote this set by $V(\mathbf{x_0})$, we can write (Cassels, 1981)

$$V(\mathbf{x_0}) = \{\mathbf{x} : \mathbf{x}P\mathbf{x_0}\}. \tag{6}$$

For the utility function it can be written as,

$$V(\mathbf{x_0}) = \{\mathbf{x} : u(\mathbf{x}) > u(\mathbf{x}_0)\} \tag{7}$$

where $V(\mathbf{x_0})$ is a convex set.

We now introduce the idea of a budget constraint. For bundle $\mathbf{x}$ with a price vector $\mathbf{p}$ let us consider one has maximum c amount of taka or dollars to spend, then we can write,

$$\mathbf{p}.\mathbf{x} \le c \text{ ; } (\mathbf{p}.\mathbf{x} \text{ is the price of the bundle } \mathbf{x}) \tag{8}$$

which is referred to as budget constraint. Let us consider the hypersurfaces

$$u(\mathbf{x}) = \text{constant}, \tag{9}$$

for various values of the constant. According to (5) the individual concerned is indifferent to the bundles represented by all these vectors i.e., all these bundles for him are ‘equally good’ (or ‘equally bad’). That is why (9) are indifferent hypersurfaces. For simplicity we consider $n = 2$, so,

$$u(\mathbf{x}) = x_{\mathbf{1}}x_2. \tag{10}$$

The indifference curves are given by rectangular hyperbolae,

$$x_1x_2 = k \tag{11}$$

where, k = constant > 0.

Let the fixed price vector be $\mathbf{p} = (p_1, p_2)$ then by (8) the budget constraint is

$$p_1x_1 + p_2x_2 \le c \tag{12}$$

with fixed c.

If we draw a straight line (AB),

$$p_1x_1 + p_2x_2 = c \tag{13}$$

then there is only one member of family of indifference curves (11) that touches the straight line (13). Let it touch at the point $(\bar{x}_1, \bar{x}_2)$ which is a vector and it maximizes the utility (see figure – 1).

The inequality (12) restricts $(\bar{x}_1, \bar{x}_2)$ to the interior or boundary of the triangle OAB,

where, $ON = \left| \dfrac{c}{\sqrt{(p_1^2 + p_2^2)}} \right|$, which is parallel to the vector $\mathbf{p}$.

The maximum of the utility function must occur on the line AB but not in the interior of triangle OAB. The indifference curve which gives the maximum is (Islam, 1997)

$$x_1x_2 = \bar{x}_2\bar{x}_{\mathbf{1}} \equiv \frac{c^2}{4p_1p_2}. \qquad \text{(See Appendix – I)} \tag{14}$$

From (14), we get $x_2 = \dfrac{c^2}{4p_1p_2x_1}$ and substituting in (13) yields,

$$p_1x_1 + \frac{c^2}{4p_1x_1} = c$$

whose discriminant is zero, so (13) has two common roots $x_1 = x_2$ and the curve and the line touch at a point $(\bar{x}_1, \bar{x}_2)$. We will show maximality of indifference hypersurface in Appendix – I.

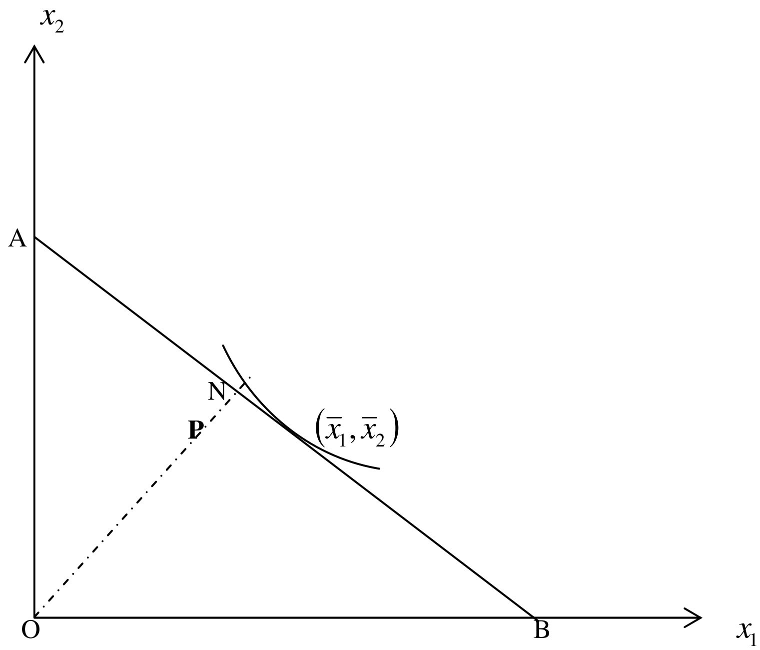

Figure-1

Figure-1: The point $(\bar{x}_1, \bar{x}_2)$ maximizes the utility. ON is parallel to price vector **p** which is perpendicular to AB.

In R^n we consider a single indifference hypersurface,

$$u(\mathbf{x}) = k_0, \tag{15}$$

for some fixed k_0. For every price vector **p**>0, there is a particular vector **x** which minimizes the cost $\mathbf{p}.\mathbf{x}$ for all the vectors **x** on this hypersurface. Since the vector depends on **p** we write it as $\bar{\mathbf{x}}(\mathbf{p})$, for all **x** lying on (15). If there are two price vectors $\mathbf{p}'$ and $\mathbf{p}''$ and write $\bar{\mathbf{x}}' = \bar{\mathbf{x}}(\mathbf{p}')$, $\bar{\mathbf{x}}'' = \bar{\mathbf{x}}(\mathbf{p}'')$ that is, the vectors $\bar{\mathbf{x}}'$ and $\bar{\mathbf{x}}''$ minimize the total cost $\mathbf{p}'.\mathbf{x}$ and $\mathbf{p}''.\mathbf{x}$ respectively on the hypersurface (15), so that we have

$$\mathbf{p}'.\bar{\mathbf{x}}'' \geq \mathbf{p}'.\bar{\mathbf{x}}' \text{ and } \mathbf{p}''.\bar{\mathbf{x}}' \geq \mathbf{p}''.\bar{\mathbf{x}}''$$

$$\Rightarrow \mathbf{p}'.(\bar{\mathbf{x}}' - \bar{\mathbf{x}}'') \leq 0 \text{ and } \mathbf{p}''.(\bar{\mathbf{x}}'' - \bar{\mathbf{x}}') \leq 0. \tag{16}$$

Adding these two inequalities we get,

$$\Rightarrow (\mathbf{p}' - \mathbf{p}'').(\bar{\mathbf{x}}' - \bar{\mathbf{x}}'') \leq 0. \tag{17}$$

This is known as the *substitution theorem*.

Now let the two vectors $\mathbf{p}'$ and $\mathbf{p}''$ differ only in their ith components that is $p_i' \neq p_i''$, $p_j' = p_j''$, $\mathrm{j} \neq \mathbf{i}$, then

$$\Rightarrow (p_i' - p_i'').(\bar{x}_i' - \bar{x}_i'') \leq 0. \tag{18}$$

Since $\mathbf{p}'.\mathbf{x}$ minimized by $\overline{\mathbf{x}}(\mathbf{p}') = \overline{\mathbf{x}}'$ and $\overline{\mathbf{x}}(\mathbf{p}'') = \overline{\mathbf{x}}''$ then substituting $\mathbf{p}_i'' = \mathbf{p}' + \delta\mathbf{p}_i'$ in (18) we get

$$-\delta\mathbf{p}_i'(\overline{\mathbf{x}}(\mathbf{p}') - \overline{\mathbf{x}}(\mathbf{p}_i' + \delta\mathbf{p}_i')) \leq 0.$$

Now $\overline{\mathbf{x}}_i(\mathbf{p}_i' + \delta\mathbf{p}_i') = \overline{\mathbf{x}}(\mathbf{p}_i')\dfrac{\partial \bar{x}_i}{\partial p_i}\delta\mathbf{p}_i'$ so that we can write (assuming $\delta\mathbf{p}_i'$>0)

$$\frac{\partial \bar{x}_i}{\partial p_i} \leq 0. \tag{19}$$

The Reciprocity theorem is given as follows. (For proof see Appendix -II);

$$\frac{\partial \bar{x}_i}{\partial p_j} = \frac{\partial \bar{x}_j}{\partial p_i}, \tag{20}$$

where $i \neq j$.

To examine the significance of the Reciprocity theorem (20) we let n =3 and consider x_1, x_2, x_3 to refer to the three commodities tea, coffee and sugar respectively. For i=1 , j=2 we get from (20),

$$\frac{\partial \bar{x}_1}{\partial p_2} = \frac{\partial \bar{x}_2}{\partial p_1}. \tag{21}$$

If the common value of (21) is positive, the function x_1 increases with p_2 and same as for x_2 and p_1. This can be explained as, if the price of tea goes up, we drink more coffee, and vice versa. In this case the commodities are said to be *substitutes*.

For i=1, j=3 we get from (20),

$$\frac{\partial \bar{x}_1}{\partial p_3} = \frac{\partial \bar{x}_3}{\partial p_1}. \tag{22}$$

If the common value is negative, so that the function x_1 decrease as p_3 increase and x_3 decreases as p_1 increases, the rate of decrease being the same, which we can interpret as saying that as the price of sugar goes up we drink less tea, and if the price of tea goes up we buy less sugar to minimize the cost and keep the total utility the same. In this situation the commodities are said to be *complements*.

4. ARROW'S THEOREM

4.1 Pre-Requisites

Arrow's original form of his theorem appeared in his book (1963). The form given here is based as Sen (1970), Cassels (1981), Islam (1997, 2008), Ubeda (2005), Geanokoplos (2006) and Breton , Weymark (2006), Sato (2009). Arrow's theorem deals with the manner in which the preferences of a group of individuals are combined to yield the preferences of a group. We can explain it by a simple example known as paradox of the voter. Suppose we have a community consisting of three individuals *A*, *B* and *C*. Assume that they have three alternatives *x*, *y*, *z* from which to choose. Let *x*, y, and z stands respectively for hot war, cold war or peace with another group of individuals. If *A* prefers *x* to y, and y to z then we write

$$x_A P y_A P z_A \text{ etc.} \quad (23)$$

Here we omit indifference between two alternatives; that is for *x* and *y* we have xPy or yPx .We assume that choices *x*, *y* and *z* are transitive, that is,

$$xPy \text{ and } yPz \Rightarrow xPz. \quad (24)$$

For voter paradox, suppose the preference relation for *A*, *B* and *C* are as follows;

$$x_A P y_A P z_A \quad (25a)$$

$$y_B P z_B P x_B \quad (25b)$$

$$z_C P x_C P y_C . \quad (25c)$$

Now we impose two conditions on the group preference of *x, y, z* as follows;

i) it must be transitive

ii) it should satisfy the majority rule, that is, if out of three people two prefer *x* to *y*, then the group prefers x to *y*.

Now we want to impose two conditions which are (i) the relation should be transitive and (ii) the relation should satisfy the majority rule. From (25) we see that *x* is preferred to *y* by *A* and *C,* so that, by the majority rule, *x* is preferred to *y* by the group. Again, we see that *y* is preferred to *z* by *A* and *B*, again by the majority rule *y* is preferred to z by the group. Since we claim that the group choice be transitive, so that *x* will be preferred to z by the group. If we now require that the group choice be transitive, we deduce that *x* is preferred to *z* by the group. However, from (25 b, c) we see that in fact *z* is preferred to *x* by *B* and *C*, so that by the majority rule z should be preferred to *x*. Thus we see that in the situation that the individual choice is given by (25a-c) it is not possible to impose the requirements of transitivity and majority rule simultaneously, although these conditions are fairly reasonable.

The above problem expresses the fact that certain difficulties arise when we try to work out the preference of a group from those of the individuals in it, even when one wants reasonable requirements to be satisfied. Arrow's theorem deals with such impossibility of finding group preference.

We consider a finite set *U* of *n* individuals and we denote a typical individual by u_i (*i*=1, 2,…, *n*). In the above example *n* = 3 and a society *U* = {*A, B, C*}.We consider a finite set *S* consisting of '*a*' alternatives or social choices which we denote by *x, y, z,…*. Every member of the set *U* has a preference ordering on the set *S* in the sense that if $x, y \in S$ we have one of the following three possibilities for member u_i ;

$$x_i P y_i ,\ y_i P x_i ,\ x_i I y_i . \quad (26)$$

For the individual u_i , we shall denote by W_i any given ordering of the set S. Similarly, we shall denote by *W* the preference ordering of the whole group *U*. If the individual u_i prefers *x* to *y* we shall write,

$$x_i P y_i \left(w_i\right). \qquad (27)$$

We now want to determine *W* if we are given w_i for all *i*. If all the individuals prefer *x* to y, then the group should prefer *x* to y, that is,

$$x_i P y_i \quad \text{for all } i \Rightarrow x\,P\,y\,(W). \qquad (28)$$

Arrow's theorem is concerned with attempting to find a group or social ordering W from the individual orderings w_i,

$$W\left(w_1, w_2, ..., w_n\right). \qquad (29)$$

The followings are the conditions of the theorem;

I) *W* is defined when each of the w_i runs independently through all orderings of the set *S*.
II) The condition (28) is satisfied.
III) This condition is referred to as *indifference to irrelevant alternatives* and is given as follows:

Let *T* be a subset of *S*. For each *i*, let w_i' and w_i'' induce the same ordering on *T*. In this case, $W\left(w_1', w_2', ..., w_n'\right)$ and $W\left(w_1'', w_2'', ..., w_n''\right)$ induces the same ordering on *T*. We denote these two conditions by W', W'' respectively.

The condition (III) may be slightly difficult. Let *S*={*x, y, z*}, and *T*={*x, y*}$\subset$ S. Consider the orderings of *A, B, C* given by w_A', w_B', w_C' and w_A'', w_B'', w_C'' which induce the same ordering on {*x, y*}. For example, this might be (Islam, 1997)

$$x_A P y_A\left(w_A'\right),\ x_A P y_A\left(w_A''\right) \qquad (30a)$$

$$x_B P y_B\left(w_B'\right),\ x_B P y_B\left(w_B''\right) \qquad (30b)$$

$$y_C P x_C\left(w_C'\right),\ y_C P x_C\left(w_C''\right). \qquad (30c)$$

In this case $W' = W\left(w_A', w_B', w_C'\right),\ \ W'' = W\left(w_A'', w_B'', w_C''\right)$ induce the same ordering on *x, y*; that is, either

$$xPy\left(W'\right) \text{ and } xPy\left(W''\right) \text{ or } yPx\left(W'\right) \text{ and } yPx\left(W''\right). \qquad (31)$$

Similar conditions hold if *T* is the subset {y, z} or {*z, x*}.
We are now in a position to state Arrow's Theorem;

Arrow's theorem: Suppose that S has at least three elements and the conditions I, II and III are satisfied. Then there exists an individual $u_k \in U$, such that

$$W\left(w_1, w_2, ..., w_n\right) = w_k, \text{ some } k, \qquad 1 \le k \le n \qquad (32)$$

that is, the group preference coincides with that of some one (single) individual.

4.2 A COMBINATORIAL APPROACH TO ARROW'S THEOREM

Let us consider the sets *U* and *S* to have three elements each (Islam, 1997). As before we denote by *x, y, z* the group choices, and by x_A, y_A, z_A etc., the individual choices. Now there are six possibilities for the group preference ordering, as follows:

$$xPyPz(W_1) \tag{33a}$$
$$xPzPy(W_2) \tag{33b}$$
$$yPzPx(W_3) \tag{33c}$$
$$yPxPz(W_4) \tag{33d}$$
$$zPxPy(W_5) \tag{33e}$$
$$zPyPx(W_6). \tag{33f}$$

Corresponding to (33 a-f), we have the individual preferences, six of each individual which we denote by $w_{A_1}, w_{A_2}, w_{A_3}, w_{A_4}, w_{A_5}, w_{A_6}, w_{B_1},...$ etc. The possibilities for the arguments of the function *W* are as follows;

$$W\left(w_{A_i}, w_{B_j}, w_{C_k}\right);\ i, j, k = 1,2,3,...,6. \tag{34}$$

Thus there are 6^3=216 possibilities for the arguments of *W*, and there are six possible values (33a-f); so, the function *W* represents a map from a set consisting of 216 elements to a set consisting of six elements. Arrow's theorem guarantees that one of the following three possibilities must necessarily hold;

$$\begin{aligned} W\left(w_{A_i}, w_{B_j}, w_{C_k}\right) &= W_i \\ W\left(w_{A_i}, w_{B_j}, w_{C_k}\right) &= W_j \\ W\left(w_{A_i}, w_{B_j}, w_{C_k}\right) &= W_k. \end{aligned} \tag{35}$$

That is, the group preference coincides with one of the individual preferences, so that there has to be a 'dictator' if conditions I, II, III of Arrow's theorem are to be satisfied.

Now we state briefly how Arrow's theorem is to be considered in the combinational approach. In this case (34) can be introduced as follows;

$$W\left(w_{A_i}, w_{B_j}, w_{C_k}\right) = W_a \tag{36}$$

where $a \in \{1,2,3,...,6\}$ and *i, j, k* runs independently the values over the same set. The six values of '*a*' give six possibilities (33a-f) for the group preference.

$$\therefore\ a = a(i, j, k) = a(i\ j\ k). \tag{37}$$

Arrow's theorem implies that if conditions I, II, III are satisfied, this map must reduce to one of the following three

$$a\ (i\ j\ k) = i\ ;\quad a\ (i\ j\ k) = j\ ;\quad a\ (i\ j\ k) = k. \tag{38}$$

First we consider condition II for $\{x, y\}$;

$$x_A P y_A, x_B P y_B, x_C P y_C \Rightarrow xPy. \tag{39}$$

We see from (33) that *xPy* obtains for W_1, W_2, W_5. If we denote the set of integers {1, 2, 5}, then $i, j, k \in \{1,2,5\} \Rightarrow a(ijk) \in \{1,2,5\}$.

Now we consider the condition III. Let $(i', j', k'), (i'', j'', k'')$ be two possible set of values of the indices *i, j, k* and let $T=\{x, y\}$. Condition III asserts that if these two sets of values corresponds to the same ordering for *x, y*; then $a(i', j', k')$ and $a(i'', j'', k'')$ must induce the same ordering on *x, y*. So that

$$i', i'' \in \{1, 2, 5\} \text{ or } \{3, 4, 6\} \tag{40a}$$

$j', j'' \in \{1, 2, 5\}$ or $\{3, 4, 6\}$ (40b)

$k', k'' \in \{1, 2, 5\}$ or $\{3, 4, 6\}$ (40c)

then,

$a(i', j', k')$ and $a(i'', j'', k'')$ are both from the set {1, 2, 5} or both from {3, 4, 6}.

4.3 A GEOMETRICAL APPROACH TO THE COMBINATIONAL FORMALISM

Here we introduce equations (33a-f) in the new notation:

$0 : xPy\,\mathrm{P}z$ (41a)
$1 : xPzPy$ (41b)
$2 : yPzPx$ (41c)
$3 : yPx\,Pz$ (41d)
$4 : zPxPy$ (41e)
$5 : zPy\,Px.$ (41f)

Thus (000), for example, gives the group decision or preference (41a) denoted by the integer 0. In this case, from the rules I, II and III, it is clear that (000) =0. There are $6^3 = 216$ such possibilities, which can be grouped into 6 groups, for convenience, as follows, in notation which should be clear from the above remarks (Islam, 1997).

The above six groups corresponds to *A*'s choice. In the first group *A*'s choice is uniformly '0' in the second group *A*'s choice is '1', and so on.
A more symmetric way of representing these 216 values of the function *a*(*i j k*) in which choices of *A, B, C* are represented symmetrically, is through a cubic lattice in a three-dimensional Euclidean space containing $6 \times 6 \times 6 = 216$ points. This is displayed in the figure-2.

The points can be grouped into six lattice planes (each containing 36 points) which are parallel to the (*i j*) plane, to the (*j k*) plane, or to the (*i k*) plane. These correspond to the grouping according to *C*'s choice, to *B*'s choice and to *A*'s choice respectively.
By Arrow's theorem if *A*'s choice prevails then all the points on any one lattice plane parallel to the (*j k*) plane must have the same value, the value given by the *i* entry in (*i j k*), for B all the points in any lattice plane parallel to the (*i k*) plane has the same value, corresponding to the entry *j* in (*i j k*); similar condition holds for *C*.

(000)	(001)	(002)	(003)	(004)	(005)	
(010)	(011)	(012)	(013)	(014)	(015)	
(020)	(021)	(022)	(023)	(024)	(025)	
(030)	(031)	(032)	(033)	(034)	(035)	
(040)	(041)	(042)	(043)	(044)	(045)	
(050)	(051)	(052)	(053)	(054)	(055)	
(100)	(101)	(102)	(103)	(104)	(105)	
(110)	(111)	(112)	(113)	(114)	(115)	
(120)	(121)	(122)	(123)	(124)	(125)	
(130)	(131)	(132)	(133)	(134)	(135)	
(140)	(141)	(142)	(143)	(144)	(145)	
(150)	(151)	(152)	(153)	(154)	(155)	
(200)	(201)	(202)	(203)	(204)	(205)	
(210)	(211)	(212)	(213)	(214)	(215)	
(220)	(221)	(222)	(223)	(224)	(225)	
(230)	(231)	(232)	(233)	(234)	(235)	
(240)	(241)	(242)	(243)	(244)	(245)	
(250)	(251)	(252)	(253)	(254)	(255)	
(300)	(301)	(302)	(303)	(304)	(305)	
(310)	(311)	(312)	(313)	(314)	(315)	
(320)	(321)	(322)	(323)	(324)	(325)	
(330)	(331)	(332)	(333)	(334)	(335)	
(340)	(341)	(342)	(343)	(344)	(345)	
(350)	(351)	(352)	(353)	(354)	(355)	
(400)	(401)	(402)	(403)	(404)	(405)	
(410)	(411)	(412)	(413)	(414)	(415)	
(420)	(421)	(422)	(423)	(424)	(425)	
(430)	(431)	(432)	(433)	(434)	(435)	
(440)	(441)	(442)	(443)	(444)	(445)	
(450)	(451)	(452)	(453)	(454)	(455)	
(500)	(501)	(502)	(503)	(504)	(505)	
(510)	(511)	(512)	(513)	(514)	(515)	
(520)	(521)	(522)	(523)	(524)	(525)	
(530)	(531)	(532)	(533)	(534)	(535)	
(540)	(541)	(542)	(543)	(544)	(545)	
(550)	(551)	(552)	(553)	(554)	(555).	(42)

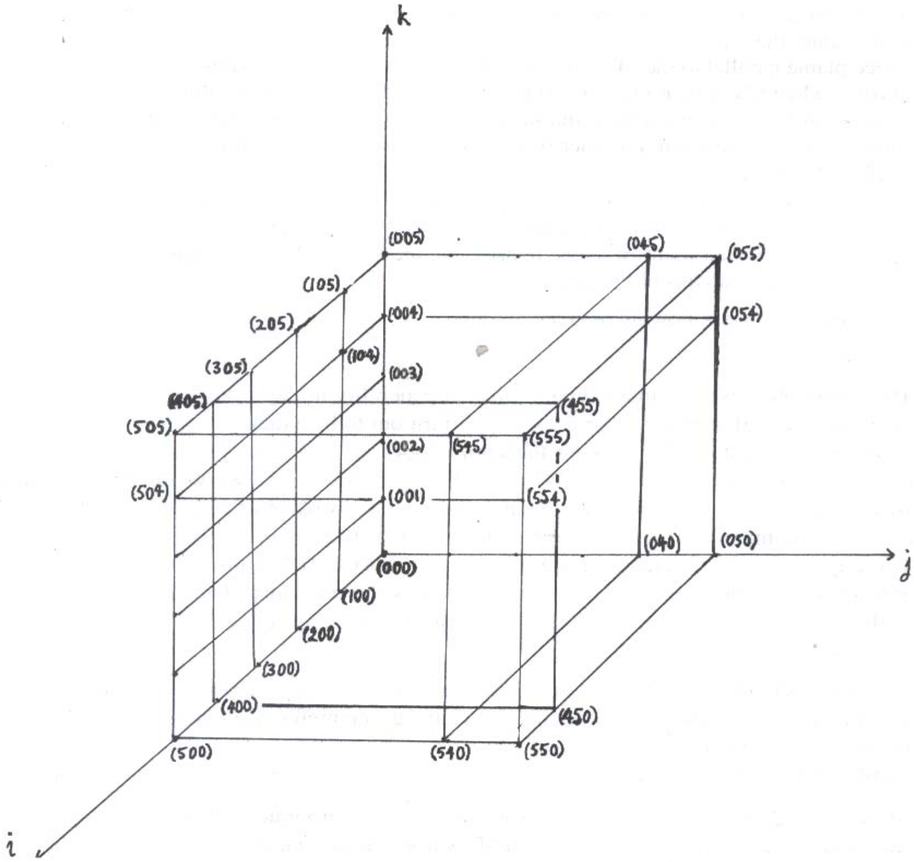

Figure-2

Figure-2: There are 216 points in the lattice cube where some of the points are displayed. The points are grouped into six lattice planes, each containing 36 points.

We now explain how one can use the above formalism to give a 'combinatorial' proof of Arrow's theorem for the particular case of three individuals and three choices.
Let us consider the basic assumption:

$$(0\ 1\ 2)=0. \tag{43}$$

Here we are simply fixing on *A* as the dictator. If instead of (43) we had chosen (0 1 2) =1,2 we would have chosen *B, C* respectively as the possible dictator.
Again we consider (0 1 0). From (41 a-f) we see that in this case all three individuals prefer x to *y* and prefer *x* to *z*. The value of (0 1 0), that is, the group preference must also reflect this. So that it is clear from (41a-f) that

$$(010) \in \{0,1\}. \tag{44}$$

We now introduce an example in the support of condition III. Let us consider the choices (010), (012) and the subset $\{y, z\}$; then (010) and (012) are both in the set {0,2,3} or both in the set
{1,4,5}. (45)

From (43) it follows that (012) is in the set {0,2,3} and so (010) must also be in this set. But from (44), (010) is also in the set {0,1}. The only common value between the sets {0,1} and {0, 2, 3} is 0, and so we must have (0 10)=0.

CONCLUDING REMARKS

The paper has been particularly concerned with the role of preference in mathematical economics. The paper is difficult and we have tried to give a basic concept how an economist depends on a mathematician and vice-versa. Most of the material in this paper has been taken from References: Islam (1997, 2008) and Pahlaj (2002).

REFERENCES

Arrow, K. J. 1959. Rational Choice Functions and Orderings: *Economica,* 26(102):121-127

Arrow, K. J. 1963. *Social Choice and Individual Values* (2nd ed.) New York: *John Wiley & Sons*.

Barbera, S. 1980. Pivotal voters: A New Proof of arrow's Theorem. *Economics letter*, 6(1): 13-16

Breton Le M., and J.Weymark. 2006. Arrovian Social Choice Theory on Economics Domains. In: Arrow K. J., A. K. Sen and K. Suzumura (Eds.). *A Handbook of social Choice and Welfare,* Volume-2. North Holland: Amsterdam.

Cassels, J.W.S. 1981. *Economics for the Mathematicians*. Cambridge, UK: Cambridge University Press.

Feldman M.A. and R. Serrano. 2006. *Welfare Economics and Social Choice Theory*, (2nd Ed.). New York: Springer.

Feldman M.A. and R. Serrano. 2007. *Arrow's Impossibility Theorem: Preference Diversity in a Single-Profile World.* Working Paper No. 2007-12. Brown University Department of Economics.

Feldman M.A. and R. Serrano. 2008. *Arrow's Impossibility Theorem: Two Simple Single-Profile Version*. Working Paper. Brown University Department of Economics.

Geonokoplos John. 2005. The Three Brief Proof of Arrow's Impossibility Theorem. *Economic Theory,* 26(1):211-215.

Islam, J.N. 1997. Aspects of Mathematical Economics and Social Choice Theory. *Proceedings of the Second Chittagong Conference on Mathematical Economics and its Relevance for Development (*Ed.) J.N. Islam. Chittagong, Bangladesh: Chittagong. University of Chittagong.

Islam, J.N. 2008. An *Introduction to Mathematical Economics and Social Choice Theory.* Book to appear.

Miller M.K. 2009. Social Choice Theory Without Pareto: The pivotal voter approach. *Mathematical Social Sciences,* (Accepted 20 Feb.2009).

Myerson, R.B. 1996. *Fundamentals of Social Choice Theory*. Discussion Paper-1162. Northwestern: Centre for Mathematical Studies in Economics and Management Science, Northwestern University.

Pahlaj, M. 2002. *Theory and Applications of Classical Optimization to Economic Problems.* M. Phil. Thesis. Chittagong: University of Chittagong, Bangladesh.

Sato, S. 2009. Strategy-Proof Social Choice with exogenous Indifference classes. *Mathematical Social Sciences,* 57:48-57.

Sen, A. K. 1970. *Collective Choice and Social Welfare.* Holden Day: Oliver and Boyd.

Suzumura K. 2007. Choice, Opportunities, and Procedures. *Collected Papers of Kotaro Suzumura.* Tokyo, Japan:Institute of Economic Research, Hitotsubashi University, Kunitatchi.

Ubeda, Luis. 2003. Neutrality in Arrow and Other Impossibility Theorems. *Economic Theory*, 23(10):195-204

APPENDIX-I

Here we will show that the bundle with maximum utility must lie on the line

$$p_1x_1 + p_2x_2 = c \qquad \text{(AI-1)}$$

but not inside the triangle. For suppose one chooses the bundles $\mathbf{x}'$ lying within the triangle, as is figure AI-1. If we join the origin to $\mathbf{x}''$ and continue the straight line until it makes the line (AI-1) at $\mathbf{x}''$, then clearly

$$\mathbf{x}'' > \mathbf{x}', \text{ so } u(\mathbf{x}'') > u(\mathbf{x}').$$

Therefore, we can always find a bundle on the line (AI-1) whose utility is higher than any given bundle within the triangle. However, on the line there are many possible bundles (x_1, x_2) each satisfying (AI-1); that is, each satisfying the budget constraint,

$$p_1x_1 + p_2x_2 \le c \qquad \text{(AI-2)}$$

and which bundle should we choose to maximize his utility

$$u(x_1, x_2) = x_1x_2 = k\,. \qquad \text{(AI-3)}$$

Now we will show this by geometrically. We choose the bundle $\mathbf{x}' = (x_1', x_2')$ on (AI-1). Consider the indifference curve passing through $\mathbf{x}'$. Let this meet the line (AI-1) again at $\mathbf{x}''$ (Figure AI-1). Choose any bundle $\hat{\mathbf{x}}'$ lying between $\mathbf{x}'$ and $\mathbf{x}''$ on the line (AI-1). Consider the indifference curve passing through $\hat{\mathbf{x}}' = \hat{\mathbf{x}}_1'\hat{\mathbf{x}}_2'$ (dotted curve in Fig. AI-1). Here the utility of all the points on this curve will be higher than k. Similarly if we chose a point between the points $\hat{\mathbf{x}}'$ and $\hat{\mathbf{x}}''$ whose utility will be higher. Clearly this process can be continued until we come to the point **y** at which on indifference curve is tangent to the line (AI-1). This point or bundle will clearly maximize the utility, whose amount will be k_0.

The same result can be obtained algebraically as follows:
From (AI-1) we get

$x_2 = \frac{1}{p_2}(c - p_1 x_1)$, and utility is $u(\mathbf{x}) = x_1 x_2 = \frac{1}{p_2} x_1 (c - p_1 x_1) = f(x_1)$ say,

$$\frac{df}{dx_1} = \frac{1}{p_2}(c - 2p_1 x_1) = 0 \Rightarrow x_1 = \frac{c}{2p_1}$$

$$\frac{d^2 f}{dx_1^2} = -\frac{2p_1}{p_2} < 0,$$

so that $\mathbf{y} = (x_1, x_2) = \left(\frac{c}{2p_1}, \frac{c}{2p_2} \right)$ is a maximum point on the line (AI-1) and hyperbolic curve (AI-3) but not inside the triangle.

Let us now consider n=3. Here we will show that the maximum utility must lie on the plane. Let us consider the utility function

$$u(x_1, x_2, x_3) = x_1 x_2 x_3 \tag{AI-4}$$

and the budget constraint;

$$p_1 x_1 + p_2 x_2 + p_3 x_3 \le c \tag{AI-5}$$

and the plane,

$$p_1 x_1 + p_2 x_2 + p_3 x_3 = c \tag{AI-6}$$

$$x_3 = \frac{1}{p_3}(c - p_1 x_1 - p_2 x_2) \text{ and } f(x_1, x_2) = \frac{x_1 x_2}{p_3}(c - p_1 x_1 - p_2 x_2)$$

$$f_{x_1} = 0 \Rightarrow \frac{x_2}{p_3}(c - 2p_1 x_1 - p_2 x_2) = 0$$

$$\Rightarrow 2p_1 x_1 + p_2 x_2 - c = 0 \tag{AI-7}$$

$$f_{x_2} = 0 \Rightarrow p_1 x_1 + 2p_2 x_2 - c = 0. \tag{AI-8}$$

Solving (AI-7) & (AI-8) we get,

$$x_1 = \frac{c}{3p_1}, \ x_2 = \frac{c}{3p_2};$$

$$f_{x_1 x_1} = -\frac{2p_1 x_2}{p_3} = -\frac{2p_1 c}{3p_2 p_3} \text{ and } f_{x_2 x_2} = -\frac{2p_2 c}{3p_1 p_3},$$

$$f_{x_1 x_2} = \frac{1}{p_3}\left(c - 2p_1 x_1 - 2p_2 x_2\right) = -\frac{c}{3p_3},$$

$$D = \frac{4c^2}{9p_3^2} - \frac{c^2}{9p_3^2} = \frac{c^2}{3p_3^2} > 0 \quad \text{and} \quad f_{x_1 x_2} = -\frac{2p_1 c}{3p_2 p_3} < 0.$$

So, the utility function is maximum on the plane and on the indifference hypersurface.

Now the same result can be generalized algebraically as follows:

Let us consider the parabola $x_2^2 = 4b^2\left(a^2 - x_1\right)$ and the utility is,

$$u(\mathbf{x}) = x_1 x_2 = \frac{2ab}{p_2} x_1 \left(a^2 - x_1\right)^{1/2} = f(x_1), \text{ (say)}$$

$$\frac{df}{dx_1} = \frac{ab\left(2a^2 - 3x_1\right)}{p_2\left(a^2 - x_1\right)^{1/2}} \quad \Rightarrow x_1 = \frac{2a^2}{3} \quad \text{and } x_2 = \sqrt{2}ab$$

$\dfrac{d^2 f}{dx_1^2} = -\dfrac{9\sqrt{3}b}{p_2} < 0.$ So that $\mathbf{y} = \left(\dfrac{2a^2}{3}, \sqrt{2}ab\right)$ is a maximum point on the parabola and hyperbolic curve (AI-3).

.

Now $u = x_1 x_2 = \dfrac{2\sqrt{2}}{3} a^3 b$ = constant

Point of intersection;

$$\frac{2\sqrt{2}}{3} a^3 b = \frac{2ab}{p_2} x_1 \left(a^2 - x_1\right)^{1/2}$$

$$9x_1^3 - 9a^2 x_1^2 + 2a^4 p_2^2 = 0.$$

Let us choose $p_2 = \dfrac{3}{2} a^2$, we have

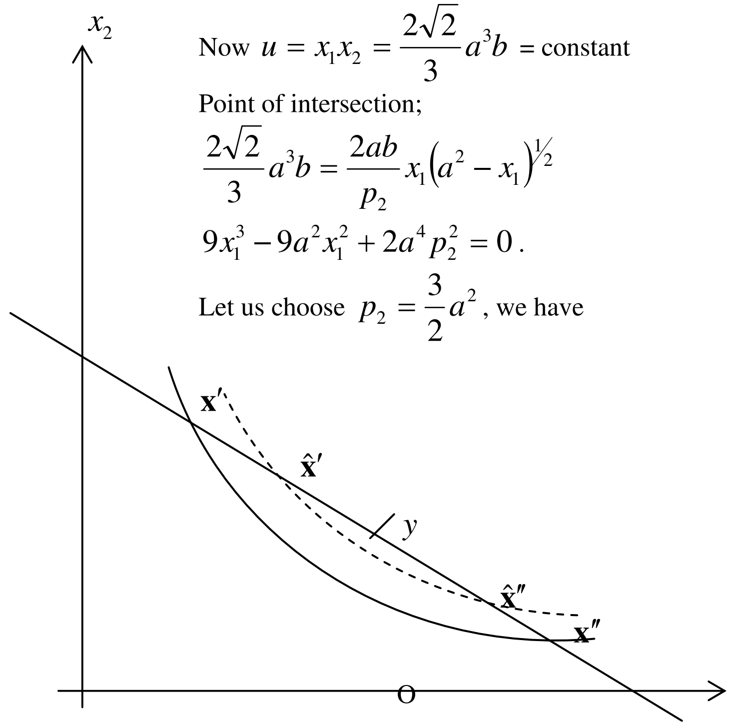

Figure-AI-1

Figure-AI-1: The $\mathbf{x}'$ - $\mathbf{x}''$ and $\hat{\mathbf{x}}'$ - $\hat{\mathbf{x}}''$ curves attain a maximum utility until they touch at the point '*y*' . So that maximum utility occur on the line but not in the interior of the triangle.

$$9x_1^3 - 9a^2x_1^2 + 3a^6 = 0 \Rightarrow \left(x_1 - \frac{2a^2}{3}\right)^2 \left(3x_1 + a^2\right) = 0 .$$

$\therefore\ x_1 = \dfrac{2}{3}a^2$ is a double root. So, the parabola and hyperbolic curve are tangent only for the special value $p_2 = \dfrac{3}{2}a^2$ and the maximum point lies on the parabola and hyperbolic curve (AI-3) but not inside the parabola.

BY THE METHOD OF LAGRANGIAN MULTIPLIER

Maximize $f(x) = u(x_1, x_2, ..., x_n) = x_1 x_2 ... x_n$,

subject to $g(x) = p_1x_1 + p_2x_2 + ... + p_nx_n = k$. (AI-9)

Let us introduce Lagrangian multiplier λ ,

$$F(\mathbf{x}) = f(x) - \lambda(g(x) - c) = x_1x_2...x_n - \lambda(p_1x_1 + p_2x_2 + \cdots + p_nx_n - c).$$

Now taking $\dfrac{\partial F}{\partial x_1} = 0 = \dfrac{\partial F}{\partial x_2} = \cdots = \dfrac{\partial F}{\partial x_n}$, (AI-10)

we get,

$$\begin{aligned} x_2 x_3 \cdots x_n - \lambda p_1 &= 0 \\ x_1 x_3 \cdots x_n - \lambda p_2 &= 0 \\ \cdots \quad \cdots \quad & \cdots \\ x_1 x_2 \cdots x_{n-1} - \lambda p_n &= 0 \end{aligned}$$

so that

$$\left(x_2x_3 \cdots x_n, x_1x_3 \cdots x_n, x_1x_2 \cdots x_{n-1}\right) = \lambda\left(p_1, p_2, \cdots, p_n\right) = \lambda \mathbf{p}. \qquad \text{(AI-11)}$$

Consider the indifference hypersurface $x_1x_2...x_n = c'$ .The normal to this at the point

$(x_1, x_2, ..., x_n)$ is given by

$$\nabla(x_1 \cdots x_n - c') = (x_2 x_3 \cdots x_n, x_1 x_3 \cdots x_n, x_1 x_2 \cdots x_{n-1}).$$

According to (AI-11) this vector is proportional to **p**, which is normal to the plane (AI-9). Thus the normal to the indifference hypersurface is parallel to the normal to the plane. This is consistent with the plane being tangent to the hypersurface.

Maximize, $u = A x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ $\quad (\mathrm{AI}-9) \Rightarrow x^n = \dfrac{1}{p_n}(c - p_1 x_1 - \cdots - p_{n-1} x_{n-1})$

$$\therefore f(x_1 \cdots, x_{n-1}) = A x_1^{a_1} x_2^{a_2} \cdots x_{n-1}^{a_{n-1}} (c - p_1 x_1 - \cdots - p_{n-1} x_{n-1})^{a_n} \div p_n^{a_n}.$$

$$\frac{\partial f}{\partial x_i} = A \left\{ \begin{array}{l} a_i x_1^{a_1} \cdots \dfrac{x_{n-1}^{a_{n-1}}}{x_i} (c - p_1 x_1 - \cdots - p_{n-1} x_{n-1})^{a_n} - p_i x_1^{a_1} \cdots \\ \qquad\qquad x_{n-1}^{a_{n-1}} (c - p_1 x_1 - \cdots - p_{n-1} x_{n-1})^{a_n - 1} \end{array} \right\} \div p_n^{a_n} = 0$$

i=1, 2,…,n-1.

By (AI-10) we get,

$$a_i (c - p_1 x_1 - \cdots - p_{n-1} x_{n-1}) - a_n p_i x_i = 0, \quad i = 1, 2, \ldots, n-1. \qquad \text{(AI-12)}$$

For n = 3 we have i =1, 2 then (AI-12) becomes

$$a_1 (c - p_1 x_1 - p_2 x_2) - a_3 p_1 x_1 = 0 \quad \text{and} \quad a_2 (c - p_1 x_1 - p_2 x_2) - a_3 p_2 x_2 = 0 \qquad \text{(AI-13)}$$

$$\Rightarrow a_1^{-1} p_1 (a_1 + a_3) x_1 + p_2 x_2 = c \quad \text{and} \quad p_1 x_1 + a_2^{-1} (a_2 + a_3) x_2 = c$$

$$\Rightarrow \frac{p_1 x_1}{a_1} = \frac{p_2 x_2}{a_2} = \frac{c}{a_1 + a_2}.$$

For n = 5, i = 1, 2, 3, 4 we get from (AI-12)

$$a_1 (c - p_1 x_1 - \ldots - p_4 x_4) - a_5 p_1 x_1 = 0$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$a_4 (c - p_1 x_1 - \ldots - p_4 x_4) - a_5 p_4 x_4 = 0.$$

Similarly, we get

$$\frac{p_1 x_1}{a_1} = \ldots = \frac{p_4 x_4}{a_4} = \frac{c}{a_1 + \ldots + a_4}.$$

Therefore, we get

$$\frac{p_1 x_1}{a_1} = \ldots = \frac{p_n x_n}{a_n} = \frac{c}{a_1 + \ldots + a_n}$$

$$x_n = \frac{1}{p_n}(c - p_1 x_1 - \ldots - p_n x_n)$$

$$= \frac{c a_n}{p_n}(a_1 + \ldots + a_n)^{-1}.$$

Indifference hypersurface is,

$Ax_1^{a_1}x_2^{a_2}...x_n^{a_n} = c'$ and the normal is ,

$$\nabla\left(Ax_1^{a_1}...x_n^{a_n} - c'\right) = \frac{cA}{a_1 + ... + a_n}\left(p_1,..., p_n\right) = \frac{cA}{a_1 + ... + a_n}\mathbf{P}.$$

Thus, the plane being tangent to the hypersurface. Therefore, that in every case has the same result.

Special case: 1

Let us consider the parabola,

$$y = 4\left(b^2 - a^2x^2\right) \qquad \text{(AI-14)}$$

and indifference hyperbolic curve

$$xy = k \Rightarrow y = \frac{k}{x} \qquad \text{(AI-15)}$$

$$4a^2x^3 - 4b^2x + k = 0\,.$$

Let coincident roots occur at $x = \alpha$

$$4a^2x^3 - 4b^2x + k = 4a^2(x-\beta)(x-\alpha)^2$$

$$= 4a^2\left[x^3 - (\beta + 2\alpha\ )x^2 + \left(\alpha^2 + 2\alpha\beta\right)x - \beta\alpha^2\right].$$

Equating the coefficients;

$$\beta + 2\alpha = 0,$$

$$4b^2 = 4\left(\alpha^2 + 2\alpha\beta\right)a^2,$$

$$4\alpha^2\beta a^2 = -k,$$

$$\Rightarrow \alpha^2 = \frac{b^2}{3a^2} \quad \text{and } k = \frac{8b^3}{3\sqrt{3}a}.$$

Again we have, $x = \dfrac{b}{\sqrt{3}a}$ and $y = \dfrac{k}{x} = \dfrac{8}{3}b^2$.

For hyperbolic curve, $\nabla(xy - k) = (y, x) = \left(\dfrac{8b^2}{3}, \dfrac{b}{\sqrt{3}a}\right) = \dfrac{b}{\sqrt{3}a}\left(\dfrac{8ab}{\sqrt{3}}, 1\right)$.

For parabola, $\nabla\left[y - 4\left(b^2 - a^2x^2\right)\right] = \left(8a^2x, 1\right) = \left(\dfrac{8ab}{\sqrt{3}}, 1\right)$.

So parabola and hyperbolic-curve are tangents i.e. maximum utility occurs on the parabola but not inside.

Special Case 2:

Let us consider the parabola and hyperbolic- curve

$y = ax^2 + bx + c$ and $x\,y = k$,

$$ax^3 + bx^2 + cx - k = 0\,.$$

Let coincident roots occur at $x = \alpha$

$$ax^3 + bx^2 + cx - k = a(x-\beta)(x-\alpha)^2$$
$$= a\left[x^3 - a(\beta + 2\alpha)x^2 + a\left(\alpha^2 + 2\alpha\beta\right)x - \beta\alpha^2\right].$$

Equating the coefficients we get,

$$\beta + 2\alpha = 0$$

$$2\alpha\beta + \alpha^2 = \frac{c}{a}$$

$$a\alpha^2\beta = k\,.$$

So, $3a\alpha^2 + 2b\alpha + c = 0 \qquad \Rightarrow \alpha = \dfrac{-b - \sqrt{b^2 - 3ac}}{3a}$ and $\beta = \dfrac{-b + 2\sqrt{b^2 - 3ac}}{3a}$

$$k = \frac{2b^3 - 9abc + 2\left(b^2 - 3ac\right)\sqrt{b^2 - 3ac}}{27a^2}$$

$$\frac{y}{x} = \frac{-b + 2\sqrt{b^2 - 3ac}}{3}\,.$$

For hyperbolic curve, $\nabla(xy - k) = (y, x) = \alpha\left(\dfrac{-b + 2\sqrt{b^2 - 3ac}}{3}, 1\right)$.

For parabola, $\nabla\left[y - ax^2 - bx - c\right] = (-2ax - b, 1) = \left(\dfrac{-b + 2\sqrt{b^2 - 3ac}}{3}, 1\right)$.

So, the parabola and hyperbolic-curve are tangents i.e. maximum utility occurs on the parabola but not inside.

APPENDIX-II

Here we prove the reciprocity theorem (20). We first write down the conditions for the minimization of $\mathbf{p}.\mathbf{x}$ subject to the constraint (16) using the Lagrangian multipliers. Writing $u_i = \partial u / \partial x_i$, these conditions can be written as,

$$\frac{\partial}{\partial x_i}\left[\mathbf{p}.\mathbf{x} - \lambda(u - k_0)\right] = 0\,, \qquad i = 1, 2, \ldots, n \qquad \text{(AII-1)}$$

where λ is the so called Lagrangian multiplier. Condition (AII-1) can be written as follows;

$$p_i = \lambda u_i(\mathbf{x}) \qquad i = 1, 2, \ldots, n. \tag{AII-2}$$

By (16) and (AII-2) constitute (n+1) equations for the n +1 unknowns $x_1, x_2, \ldots x_n, \lambda$.

The solution for this set of equations for **x** is what we have denoted by $\bar{\mathbf{x}}(\mathbf{p})$. Consider now a small variation in the price vector given by $d\mathbf{p}$, that is each component p_i changes to $p_i + dp_i$. Let the corresponding change in $\bar{\mathbf{x}}$ be denoted by $d\bar{\mathbf{x}}$, that is, the vector which minimizes (**p**+d**p**).**x** subject to (16) is $\bar{\mathbf{x}} + d\bar{\mathbf{x}}$.

$$du = \sum u_i dx_i = 0. \tag{AII-3}$$

By (AII-2) we can write

$$\sum p_i d\bar{x}_i = 0. \tag{AII-4}$$

Let P denote the total minimum price vector **p** subject to (16), that is,

$$P = \mathbf{p}.\bar{\mathbf{x}}(\mathbf{p}). \tag{AII-5}$$

Taking the differentiation we get

$$d\mathbf{P} = \mathbf{p}.d\bar{\mathbf{x}} + d\mathbf{p}.\bar{\mathbf{x}}(\mathbf{p}) = \bar{\mathbf{x}}(\mathbf{p}).d\mathbf{p}\ . \tag{AII-6}$$

The left side of (AII-6) is a perfect differential so the right side must be of the form $\sum\left({}^{\partial Q}\!/_{\partial p_i}\right)dp_i$ for some function Q of **p**. So that we can write

$$\frac{\partial \bar{x}_i}{\partial p_j} = \frac{\partial \bar{x}_j}{\partial p_i}.$$