MPRA
Munich Personal RePEc Archive

Utility Maximization Subject to Multiple Constraints

Jamal Islam and Haradhan Mohajan and Pahlaj Moolio

Indus Journal of Management & Social Sciences

15. February 2010

Online at http://mpra.ub.uni-muenchen.de/50667/
MPRA Paper No. 50667, posted 16. October 2013 07:49 UTC

Utility Maximization Subject to Multiple Constraints

Jamal Nazrul Islam[1], Haradhan Kumar Mohajan[2], and Pahlaj Moolio[3]

ABSTRACT

Applying method of Lagrange multipliers, an attempt has been made to derive mathematical formulation to workout optimal purchasing policy in order to maximize utility of an individual consumer subject to multiple constraints; in this particular illustration, two constraints: 1) budget constraint, and 2) coupon constraint. An explicit example is given in order to examine the behaviour of an individual consumer and to support the analytical arguments, using comparative static analysis.

JEL. Classification: C51; C70; H6; O12;

Keywords: Lagrange multipliers, Utility function, Multiple constraints, Comparative static analysis, Necessary and sufficient conditions, Consumer behaviour.

1. INTRODUCTION

The method of Lagrange multipliers is a very useful and powerful technique in multivariable calculus and has been used to facilitate the determination of necessary conditions; normally, this method was considered as a device for transforming a constrained problem to a higher dimensional unconstrained problem (Islam 1997). Using this method, Baxley and Moorhouse (1984) analyzed an example of utility maximization subject to a budget constraint, and provided a mathematical formulation for nontrivial constrained optimization problem with special reference to application in economics. They considered implicit functions with assumed characteristic qualitative features and

---

The material presented by the authors does not necessarily represent the viewpoint of editors and the management of the Indus Institute of higher education as well as the author' institute.

[1] *Emeritus Professor, Research Centre for Mathematical and Physical Sciences, University of Chittagong, Chittagong, Bangladesh. Phone: +880-31-616780.*

[2] *Assistant Professor, Premier University, Chittagong, Bangladesh. Email: haradhan_km@yahoo.com*

[3] *PhD Fellow, Research Centre for Mathematical and Physical Sciences, University of Chittagong, Bangladesh; and Professor, Paññāsāstra University of Cambodia, Phnom Penh, Cambodia.*

Corresponding Author: Pahlaj Moolio. Email: pahlajmoolio@gmail.com

Acknowledgements: Authors would like to thank the editors and anonymous referees for their comments and insight in improving the draft copy of this article. Author further would like to declare that this manuscript is original and has not previously been published, and that it is not currently on offer to another publisher; and also transfer copy rights to the publisher of this journal.

Received: 15-02-2010; Revised: 09-04-2010; Accepted: 15-06-2010; Published: 30-06-2010

provided illustration of an example, generating meaningful economic behaviour. This approach and formulation may enable one to view optimization problems in economics from a somewhat wider perspective.

Islam (1997) considered a problem of maximizing utility in two commodities subject to a budget constraint and studied the behaviour of an individual consumer, providing preference relations. Moolio (2002) extended the work of Islam (1997) to n commodities and applied necessary and sufficient conditions; as well as taking into account Cobb-Douglas production function into two variables (factors: capital, and labour) and using Lagrange multipliers method, he studied the behaviour of a competitive firm by considering a problem of cost minimization subject to an output constraint. Moolio and Islam (2008) considering the function in three variables (factors: capital, labour, and other inputs) provided the formulation of the problem, applying necessary and sufficient conditions, thus extended the work of Moolio (2002). They gave reasonable interpretation of the Lagrange multipliers and examined the behaviour of the firm by analyzing comparative static results. Moolio, Islam and Mohajan (2009) considered theoretically a variation of the problem studied by Moolio and Islam (2008), assuming that a government agency is allocated an annual budget and charged to maximize as well as make available some sort of services to the community; hence they maximized the output function subject to a budget constraint.

Moreover, fundamental relationship between mathematical economics, social choice and welfare theory by introducing utility functions, preference relations and Arrow's Impossibility Theorem is given in detail by Islam, Mohajan and Moolio. (2009).

Typically, during some emergencies, such as in the times of war or a natural disaster, and as there is inadequate supply, and the civilian populations are subject to some form of rationing, the governments intervene and legislate a maximum ceiling prices of certain basic consumer goods. This occurred in Pakistan in 1970s because of political disturbances and the results were quite satisfactory. Initially, this is done through a "first come first served" approach, with or without limiting sales to each consumer. Lines are formed, and much time is spent foraging for those rationed goods. Eventually, some kind of non-price rationing mechanism is evolved (Samuelson and Nordhaus 2001).

Usually, the method of rationing is applied through the use of redeemable coupons used by government agencies. The government agencies give each consumer an allocated number of coupons each month. Under rationing system, each consumer must have money as well as enough number of coupons in order to purchase the required goods. In turn, each consumer pays money and converts a certain number of coupons at the time of purchase of rationed goods. This effectively means that the consumer pays two prices at the time of purchase of rationed goods. He or she pays both the coupon price and the monetary price of rationed goods. This requires the consumer to have both sufficient funds as well as coupons to buy a unit of rationed goods. In result, there are two types of money involved for purchasing rationed goods. Hence, hence consumer faces two constraints: budgetary constraint, and the rationing coupon constraint. This situation develops a problem of utility maximization subject to multiple constraints, in this particular case of double constraints.

Therefore, in this paper, we consider a problem of maximization of utility of an individual consumer subject to two constraints. Baxley and Moorhouse (1984) suggested this problem in their paper entitled "Lagrange Multiplier Problems in Economics". In section 2, following Moolio and Islam

(2008) and Moolio, Islam and Mohajan (2009), we formulate the mathematical model for the problem. Considering an explicit example in section 3, we apply necessary conditions to find optimal values of the commodities in order to maximize utility of an individual consumer. In section 4, we give a reasonable interpretation of the Lagrange multipliers in the context of this particular illustration; and sufficient conditions, involving bordered Hessian determinants, are applied in section 5. In section 6, following Cassels (1981), Chiang (1984), Islam , Mohajan and Moolio (2009), and Samuelson and Nordhaus (2001), we analyze the comparative static results, examining the behaviour of an individual consumer. In the final section 7, concluding remarks are given.

2. THE MODEL

In order to get intrinsic understanding of the problem and to keep it manageable, we confine ourselves to two-commodity world, assuming that an individual consumer obtains his utility (i.e. satisfaction) from the consumption of two types of goods x and y, which are purchased in the marketplace in the quantities of X and Y, respectively. We assume that individual consumer spends all his income and surrenders all his coupons on the purchase of these two goods. Then, individual consumer's utility function $U(X,Y)$ must be maximized subject to budget constraint,

$$B = P_X X + P_Y Y \tag{1}$$

and coupon constraint,

$$R = r_X X + r_Y Y \tag{2}$$

where P_X, P_Y are the prices and r_X, r_Y are the ration coupons required in order to purchase a unit of commodity x, y, respectively.

We introduce two Lagrange multipliers λ_1, λ_2 to define the Lagrangian function L as below:

$$L(X,Y,\lambda_1,\lambda_2) = U(X,Y) + \lambda_1(B - P_X X - P_Y Y) + \lambda_2(R - r_X X - r_Y Y) \tag{3a}$$

Setting up partial derivatives of (3) equal to zero, we get following first order necessary conditions for maximization:

$$L_{\lambda_1} = B - P_X X - P_Y Y = 0, \tag{4a}$$

$$L_{\lambda_2} = R - r_X X - r_Y Y = 0, \tag{4b}$$

$$L_X = U_X - \lambda_1 P_X - \lambda_2 r_X = 0, \tag{4c}$$

$$L_Y = U_Y - \lambda_1 P_Y - \lambda_2 r_Y = 0 \tag{4d}$$

In principle, (4a-d) lead to the optimal solutions $X^*, Y^*, \lambda_1^*, \lambda_2^*$, each quantity being a function of the parameters P_X, P_Y, r_X, r_Y. Following usual procedure, we ignore λ_1^*, λ_2^* and regard X^*, Y^*

as the necessary solutions for the extrema. If we consider infinitesimal changes dX, dY, we get corresponding changes in U, B, R as below:

$$dU = U_X dX + U_Y dY \tag{5a}$$

$$dB = P_X dX + P_Y dY \tag{5b}$$

$$dR = r_X dX + r_Y dY \tag{5c}$$

If, for instance, we consider the money constraint to remain constant (not to change); that is, if $dB = 0$, then we get:

$$\frac{dU}{dR} = \frac{U_X dX + U_Y dY}{r_X dX + r_Y dY} = \lambda_2 \tag{6}$$

where (4a-d) have been used with $\lambda_1 = 0$. The Lagrange multiplier λ_2 may then be interpreted as $\left(\frac{\partial U}{\partial R}\right)_R$. Similarly, λ_1 is equated to $\left(\frac{\partial U}{\partial B}\right)_R$. Therefore, the Lagrange multipliers, in this specific illustration, give the changes in the utility consequent to one of the constraints being operative, but not the other.

3. AN EXPLICIT EXAMPLE

We now consider an explicit form of utility function U in two commodities

$$U = U(X, Y) = XY \tag{7}$$

and provide a detailed discussion. Using (7), the (3a) takes the following form:

$$L(X, Y, \lambda_1, \lambda_2) = XY + \lambda_1 (B - P_X X - P_Y Y) + \lambda_2 (R - r_X X - r_Y Y) \tag{3b}$$

Therefore, the set of four equations (4a-d) also takes the following form:

$$L_{\lambda_1} = B - P_X X - P_Y Y = 0 \tag{8a}$$

$$L_{\lambda_2} = R - r_X X - r_Y Y = 0 \tag{8b}$$

$$L_X = Y - \lambda_1 P_X - \lambda_2 r_X = 0 \tag{8c}$$

$$L_Y = X - \lambda_1 P_Y - \lambda_2 r_Y = 0 \tag{8d}$$

Solution of the set of four simultaneous equations (8a-d) produced by the first order conditions for the optimum values of λ_1, λ_2, X and Y gives the following optimal values:

$$X^* = \frac{r_Y B - P_Y R}{P_X r_Y - P_Y r_X}, \text{ with } P_X r_Y \neq P_Y r_X \tag{9a}$$

$$Y^* = \frac{P_X R - r_X B}{P_X r_Y - P_Y r_X}, \text{ with } P_X r_Y \neq P_Y r_X \tag{9b}$$

$$\lambda_1^* = \frac{\left(P_X r_Y + P_Y r_X\right)R - 2 r_X r_Y B}{\left(P_X r_Y - P_Y r_X\right)^2}, \text{ with } P_X r_Y \neq P_Y r_X \tag{10a}$$

$$\lambda_2^* = \frac{\left(P_X r_Y + P_Y r_X\right)B - 2 P_X P_Y R}{\left(P_X r_Y - P_Y r_X\right)^2}, \text{ with } P_X r_Y \neq P_Y r_X \tag{10b}$$

Thus, following is the stationary point.

$$\left(X^*, Y^*\right) = \left(\frac{r_Y B - P_Y R}{\left(P_X r_Y - P_Y r_X\right)}, \frac{P_X R - r_X B}{\left(P_X r_Y - P_Y r_X\right)}\right) \tag{11}$$

Moreover, by substituting the values of X^* and Y^* from (9a-b) into (7), we get optimal value of the utility of an individual consumer in terms of P_X , r_X , P_Y , r_Y , B , and R :

$$U^* = \frac{\left(P_X r_Y + P_Y r_X\right)BR - P_X P_Y R^2 - r_X r_Y B^2}{\left(P_X r_Y - P_Y r_X\right)^2}, \text{ with } P_X r_Y \neq P_Y r_X \tag{12}$$

4. INTERPRETATION OF LAGRANGE MULTIPLIERS

Now, in order to provide a useful interpretation of Lagrange multipliers, in this specific case, with the aid of chain rule, assuming first the money constraint not to change; that is, if $dB = 0$, then $\lambda_1 = 0$; from (12) we get:

$$\left(\frac{\partial U^*}{\partial R}\right)_R = U_X \frac{\partial X}{\partial R} + U_Y \frac{\partial Y}{\partial R} \tag{13a}$$

From (7), we get:
$U_X = Y$; $U_Y = X$. Then, (13a) becomes:

$$\left(\frac{\partial U^*}{\partial R}\right)_R = Y \frac{\partial X}{\partial R} + X \frac{\partial Y}{\partial R} \tag{13b}$$

And from (8c-d), assuming that $\lambda_1 = 0$, we get:
$Y = \lambda_2 r_X$, and $X = \lambda_2 r_Y$.

Therefore, we re-write (13b) as below:

$$\left(\frac{\partial U^*}{\partial R}\right)_R = \lambda_2^* \left(r_X \frac{\partial X}{\partial R} + r_Y \frac{\partial Y}{\partial R} \right) \tag{14}$$

Differentiation of (2), keeping X and Y constant, yields:
$1 = r_X \frac{\partial X}{\partial R} + r_Y \frac{\partial Y}{\partial R}$, which allows us to re-write (14) as below:

$$\left(\frac{\partial U^*}{\partial R}\right)_R = \lambda_2^* \tag{15}$$

Equation (15) verifies (6). Thus, the Lagrange multiplier λ_2^* obtained in (15) may be interpreted as the marginal utility; that is, the change in total utility incurred from an additional unit of coupon R. In other words, if an individual wants to increase (decrease) 1 unit of his utility, it would cause the total coupon quantity to increase (decrease) by approximately λ_2^* units; here we assume that budget constraint remains unchanged.

Next, we assume that the coupon is not constant; that is, $dR = 0$, then $\lambda_2 = 0$; and following straightforward steps as mentioned above, we get:

$$\left(\frac{\partial U^*}{\partial B}\right)_B = \lambda_1^* \tag{16}$$

Equation (16) also verifies (6). Thus, the Lagrange multiplier λ_1^* obtained in (16) may be interpreted as the marginal utility; that is, the change in total utility incurred from an additional unit of budget B. In other words, if an individual wants to increase (decrease) 1 unit of his utility, it would cause the total budget to increase (decrease) by approximately λ_1^* units; here we assume that coupon constraint remains unchanged.

5. SECOND ORDER SUFFICIENT CONDITIONS

Now, in order to be sure that the optimal solution obtained in (12) is maximum, we check it against the second order sufficient conditions, which implies that for a solution X^* Y^* λ_1^* and λ_2^* of (8a-

d) to be a critical point for the maximum problem, the bordered principal minors of the bordered Hessian,

$$\left|\overline{H}\right| = \begin{vmatrix} 0 & 0 & -B_X & -B_Y \\ 0 & 0 & -R_X & -R_Y \\ -B_X & -R_X & U_{XX} & U_{XY} \\ -B_Y & -R_Y & U_{YX} & U_{YY} \end{vmatrix} \tag{17}$$

should alternate in sign; namely, the sign of $\left|\overline{H}_{m+1}\right|$ being that of $(-1)^{m+1}$, where m is number of constraints, in this case $m = 2$, with all the derivatives evaluated at critical values X^*, Y^*, λ_1^* and λ_2^*; then the stationary value of utility U obtained in (12) will assuredly be the maximum. We check this condition by expanding the determinant in (17):

$$\left|\overline{H}_2\right| = \left|\overline{H}\right| = B_X{}^2 R_Y{}^2 - 2B_X B_Y R_X R_Y + B_Y{}^2 R_X{}^2 \tag{18}$$

Now, since from (8a-b) and (7), we have
$P_X X + P_Y Y = B$; $r_X X + r_Y Y = R$, and $U = U(X, Y) = XY$

Therefore, various partial differentiations yield:

$$B_X = P_X,\ B_Y = P_Y;\ R_X = r_X,\ R_Y = r_Y \tag{19}$$

$$U_X = Y,\ U_Y = X,\ U_{XX} = 0,\ U_{YY} = 0,\ U_{XY} = U_{YX} = 1 \tag{20}$$

Putting the values from (19) and (20) into (18), and after simplifying, we get:

$$\left|\overline{H}_2\right| = \left|\overline{H}\right| = \left(P_X r_Y - P_Y r_X\right)^2 \tag{21}$$

From (21), it seems that two possible situations might arise: I) if $P_X r_Y = P_Y r_X$, then the determinant is zero; and II) if $P_X r_Y \neq P_Y r_X$ (i.e., any one of the values is different than the other three) then the determinant is non zero and is a positive number. Economists can provide better interpretation of this situation. However, in practical life it is hardly to have situation (I); however, situation (II) seems more practical, which we consider to be the case.

6. COMPARATIVE STATIC ANALYSIS

In order to derive results of economic interest, we mathematically solve the four equations in (8a-d) for X, Y, λ_1 and λ_2 in terms of P_X, P_Y, r_X, r_Y, B and R, and calculate the twenty-four partial derivatives: $\frac{\partial \lambda_1}{\partial P_X}, \Lambda, \frac{\partial \lambda_2}{\partial P_X}, \Lambda, \frac{\partial X}{\partial P_X}, \Lambda, \frac{\partial Y}{\partial P_X}, \Lambda$, etc. These partial derivatives are called the comparative static of the model. The model's usefulness is to determine how accurately it predicts adjustment in the consumer's behavior, that is, how a consumer reacts to the changes in the price of goods or to the changes in quantities of coupons to be surrendered while buying respective goods.

Since we have assumed that left side of each in (8a-b) is continuously differentiable and solutions exist, then by the implicit-function theorem X, Y, λ_1 and λ_2 will each be continuously differentiable functions of P_X, P_Y, r_X, r_Y, B, and R; if following Jacobian matrix,

$$J = \begin{bmatrix} 0 & 0 & -B_X & -B_Y \\ 0 & 0 & -R_X & -R_Y \\ -B_X & -R_X & U_{XX} & U_{XY} \\ -B_Y & -R_Y & U_{YX} & U_{YY} \end{bmatrix} \tag{22}$$

is non-singular (inverse exists) at the optimum point $\left(X^*, Y^*, \lambda_1^*, \lambda_2^*\right)$.

As second order conditions have been satisfied, so the determinant of (22) does not vanish at the optimum (situation II), that is, $|J| = |\overline{H}|$; accordingly we apply the implicit-function theorem. We denote left hand sides of (8a-d) by four components of a vector $\underline{F}$, which all depend on λ_1^*, λ_2^*, X^*, Y^*, P_X, P_Y, r_X, r_Y, B, and R, which may be regarded as points in a ten dimensional Euclidian space, E^{10}. Thus, $\underline{F} = \left(F_1, F_2, F_3, F_4\right)$,

$$F_i = F_i\left(\lambda_1^*, \lambda_2^*, X^*, Y^*, P_X, P_Y, r_X, r_Y, B, R\right) = 0\,;\; i = 1, 2, 3, 4 \tag{23}$$

the latter representing four equations in (8a-d). Thus, $\underline{F}$ is a four vector-valued function taking values in E^4 and defined for points in E^{10}. By the implicit function theorem, we solve (23) for the functions $\lambda_1^*, \lambda_2^*, X^*, Y^*$ in terms of P_X, P_Y, r_X, r_Y, B and R:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ X \\ Y \end{bmatrix} = \underline{G}\left(P_X, P_Y, r_X, r_Y, B, R\right) \tag{24}$$

where $\underline{G} = \left(G_1, G_2, G_3, G_4\right)$, being a four-vector valued functions of P_X, P_Y, r_X, r_Y, B and R.

Moreover, the Jacobian matrix for $\underline{G}$, regarded as J_G is given by

$$\begin{bmatrix} \frac{\partial \lambda_1^*}{\partial P_X} & \frac{\partial \lambda_1^*}{\partial P_Y} & \frac{\partial \lambda_1^*}{\partial r_X} & \frac{\partial \lambda_1^*}{\partial r_Y} & \frac{\partial \lambda_1^*}{\partial B} & \frac{\partial \lambda_1^*}{\partial R} \\ \frac{\partial \lambda_2^*}{\partial P_X} & \frac{\partial \lambda_2^*}{\partial P_Y} & \frac{\partial \lambda_2^*}{\partial r_X} & \frac{\partial \lambda_2^*}{\partial r_Y} & \frac{\partial \lambda_2^*}{\partial B} & \frac{\partial \lambda_2^*}{\partial R} \\ \frac{\partial X^*}{\partial P_X} & \frac{\partial X^*}{\partial P_Y} & \frac{\partial X^*}{\partial r_X} & \frac{\partial X^*}{\partial r_Y} & \frac{\partial X^*}{\partial B} & \frac{\partial X^*}{\partial R} \\ \frac{\partial Y^*}{\partial P_X} & \frac{\partial Y^*}{\partial P_Y} & \frac{\partial Y^*}{\partial r_X} & \frac{\partial Y^*}{\partial r_Y} & \frac{\partial Y^*}{\partial B} & \frac{\partial Y^*}{\partial R} \end{bmatrix} = -J^{-1} \begin{bmatrix} -X^* & -Y^* & 0 & 0 & 1 & 0 \\ 0 & 0 & -X^* & -Y^* & 0 & 1 \\ -\lambda_1 & 0 & -\lambda_2 & 0 & 0 & 0 \\ 0 & -\lambda_1 & 0 & -\lambda_2 & 0 & 0 \end{bmatrix}. \quad (25)$$

where the ith row in the last matrix on the right is obtained by differentiating the ith left hand side in (8a-d) with respect to P_X, then P_Y, then r_X, then r_Y, then B, and then R. Let C_{ij} be the cofactor of the element in the ith row and jth column of Jacobian matrix J, and then inverting J using the method of cofactor gives:

$J^{-1} = \frac{1}{|J|} C^T$, where $C = \left(C_{ij}\right)$, the matrix of cofactors of J, and T means transpose.

Thus, we express (25) as follows:

$$\begin{bmatrix} \frac{\partial \lambda_1^*}{\partial P_X} & \frac{\partial \lambda_1^*}{\partial P_Y} & \frac{\partial \lambda_1^*}{\partial r_X} & \frac{\partial \lambda_1^*}{\partial r_Y} & \frac{\partial \lambda_1^*}{\partial B} & \frac{\partial \lambda_1^*}{\partial R} \\ \frac{\partial \lambda_2^*}{\partial P_X} & \frac{\partial \lambda_2^*}{\partial P_Y} & \frac{\partial \lambda_2^*}{\partial r_X} & \frac{\partial \lambda_2^*}{\partial r_Y} & \frac{\partial \lambda_2^*}{\partial B} & \frac{\partial \lambda_2^*}{\partial R} \\ \frac{\partial X^*}{\partial P_X} & \frac{\partial X^*}{\partial P_Y} & \frac{\partial X^*}{\partial r_X} & \frac{\partial X^*}{\partial r_Y} & \frac{\partial X^*}{\partial B} & \frac{\partial X^*}{\partial R} \\ \frac{\partial Y^*}{\partial P_X} & \frac{\partial Y^*}{\partial P_Y} & \frac{\partial Y^*}{\partial r_X} & \frac{\partial Y^*}{\partial r_Y} & \frac{\partial Y^*}{\partial B} & \frac{\partial Y^*}{\partial R} \end{bmatrix} = \frac{-1}{|J|} C^T \begin{bmatrix} -X^* & -Y^* & 0 & 0 & 1 & 0 \\ 0 & 0 & -X^* & -Y^* & 0 & 1 \\ -\lambda_1 & 0 & -\lambda_2 & 0 & 0 & 0 \\ 0 & -\lambda_1 & 0 & -\lambda_2 & 0 & 0 \end{bmatrix}$$

$$= \frac{-1}{|J|} \begin{bmatrix} -X^*C_{11} - \lambda_1^* C_{31} & -Y^*C_{11} - \lambda_1^* C_{41} & -X^*C_{21} - \lambda_2^* C_{31} & -Y^*C_{21} - \lambda_2^* C_{41} & C_{11} & C_{21} \\ -X^*C_{12} - \lambda_1^* C_{32} & -Y^*C_{12} - \lambda_1^* C_{42} & -X^*C_{22} - \lambda_2^* C_{32} & -Y^*C_{22} - \lambda_2^* C_{42} & C_{12} & C_{22} \\ -X^*C_{13} - \lambda_1^* C_{33} & -Y^*C_{13} - \lambda_1^* C_{43} & -X^*C_{23} - \lambda_2^* C_{33} & -Y^*C_{23} - \lambda_2^* C_{43} & C_{13} & C_{23} \\ -X^*C_{14} - \lambda_1^* C_{34} & -Y^*C_{14} - \lambda_1^* C_{44} & -X^*C_{24} - \lambda_2^* C_{34} & -Y^*C_{24} - \lambda_2^* C_{44} & C_{14} & C_{24} \end{bmatrix}. \quad (26)$$

Now, we are in position to derive comparative static results. Firstly, how does the level of consumption of commodity x changes when its' price increases? From (26), we get:

$$\frac{\partial X^*}{\partial P_X} = \frac{-1}{|J|}\left[-X^* C_{13} - \lambda_1^* C_{33}\right] = \frac{X^*}{|J|}\left[C_{13}\right] + \frac{\lambda_1^*}{|J|}\left[C_{33}\right].$$

$$\frac{\partial X^*}{\partial P_X} = \frac{X^*}{|J|}\begin{vmatrix} 0 & 0 & -R_Y \\ -B_X & -R_X & U_{XY} \\ -B_Y & -R_Y & U_{YY} \end{vmatrix} + \frac{\lambda_1^*}{|J|}\begin{vmatrix} 0 & 0 & -B_X \\ 0 & 0 & -R_X \\ -B_Y & -R_Y & U_{YY} \end{vmatrix}.$$

$$\frac{\partial X^*}{\partial P_X} = \frac{X^*}{|J|}\left\{-R_Y\left(B_X R_Y - B_Y R_X\right)\right\} + \frac{\lambda_1^*}{|J|}(0).$$

By substituting the values of X^* and $|J| = |\overline{H}|$ from (9a) and (21) respectively into above equation and also using (19) and after simplifying, we get:

$$\frac{\partial X^*}{\partial P_X} = -\frac{r_Y\left(r_Y B - P_Y R\right)}{\left(P_X r_Y - P_Y r_X\right)^2} \qquad (27)$$

Since P_X, P_Y, r_X, r_Y are prices and ration coupons for goods x and y, and so are greater than zero, as well as B, R are budget and ration coupons, so are positive. Therefore, the sign of $\frac{\partial X^*}{\partial P_X}$ depends on the term $\left(r_Y B - P_Y R\right)$, assuming that $P_X r_Y \neq P_Y r_X$. Then, there seems to be three situations:

a) If $r_Y B > P_Y R$, then $\frac{\partial X^*}{\partial P_X} < 0$, which indicates that if the price of commodity x increases, the level of consumption of x will decrease. This situation seems a reasonable result in the sense that commodity x has many substitute goods; and hence consumers switch to substitutes when price of commodity x goes up.

b) If $r_Y B < P_Y R$, then $\frac{\partial X^*}{\partial P_X} > 0$, which indicates that even if the price of commodity x increases, the level of consumption of x will also increase. It seems that commodity x is superior goods in this situation and it has no other substitutes.

c) And finally if $r_Y B = P_Y R$, then $\frac{\partial X^*}{\partial P_X} = 0$, which indicates that if the price of commodity x increases, there seems no effect on the level of consumption of goods x. It looks as if commodity x is a necessity and it has neither complementary nor substitutes goods.

Secondly, in order to see how does the level of consumption of commodity x changes when it's quantity of surrendering ration coupons increase? From (26), following the steps mentioned above and by substituting the values of X^* and $|J| = |\overline{H}|$ respectively from (9a) and (21), also using values from (19), we get:

$$\frac{\partial X^*}{\partial r_X} = -\frac{P_Y\left(P_Y R - r_Y B\right)}{\left(P_X r_Y - P_Y r_X\right)^2} \tag{28}$$

Since P_X, P_Y, r_X, r_Y are prices of commodities and quantities of coupons, so can never be negative, as well as B, R are budget and ration coupon, so are also positive, therefore, the sign of $\frac{\partial X^*}{\partial r_X}$ depends on the term $\left(P_Y R - r_Y B\right)$, assuming that $P_X r_Y \neq P_Y r_X$. Then, again there seems to be three situations:

a) If $P_Y R > r_Y B$, then $\frac{\partial X^*}{\partial r_X} < 0$, which indicates that if the quantity of surrendering ration coupon to purchase the commodity x increases, the level of consumption of x will decrease. This situation seems reasonable result in the sense that commodity x has many substitute goods; and so consumers switch to substitutes when its quantity of surrendering ration coupons to purchase the commodity increases.

b) If $P_Y R < r_Y B$, then $\frac{\partial X^*}{\partial r_X} > 0$, which indicates that even if the quantity of surrendering ration coupon to purchase the commodity x increases, the level of consumption of x will also increase. It seems that commodity x is a superior good in this situation, and it has no other substitute goods.

c) And finally if $r_Y B = P_Y R$, then $\frac{\partial X^*}{\partial r_X} = 0$, which indicates that if the quantity of surrendering ration coupon for purchasing the commodity x increases, there seems no effect on the level of the consumption of goods x. It looks as if commodity x is a necessity and it has neither complementary nor supplementary goods.

Next, how does the level of consumption of commodity y change when the price of commodity x increases? Similarly, from (26), we get:

$$\frac{\partial Y^*}{\partial P_X} = \frac{-1}{|J|}\left[-X^* C_{14} - \lambda_1^* C_{34}\right] = \frac{-X^*}{|J|}\left[C_{14}\right] - \frac{\lambda_1^*}{|J|}\left[C_{34}\right].$$

$$\frac{\partial Y^*}{\partial P_X} = \frac{-X^*}{|J|}\begin{vmatrix} 0 & 0 & -R_X \\ -B_X & -R_X & U_{XX} \\ -B_Y & -R_Y & U_{YX} \end{vmatrix} - \frac{\lambda_1^*}{|J|}\begin{vmatrix} 0 & 0 & -B_X \\ 0 & 0 & -R_X \\ -B_Y & -R_Y & U_{YX} \end{vmatrix}.$$

$$\frac{\partial Y^*}{\partial P_X} = \frac{-X^*}{|J|}\{-R_X(B_X R_Y - B_Y R_X)\} - \frac{\lambda_1^*}{|J|}(0).$$

Similarly, by substituting the values of X^* and $|J| = |\overline{H}|$ respectively from (9a) and (21) into the above equation, also using values from (19), we get:

$$\frac{\partial Y^*}{\partial P_X} = \frac{r_X(r_Y B - P_Y R)}{(P_X r_Y - P_Y r_X)^2} \tag{29}$$

Since P_X, P_Y, r_X, r_Y are prices of commodities and quantities of ration coupons, so can never be negative, as well as B, R are budget and ration coupon, so are also positive. Therefore, the sign of $\frac{\partial Y^*}{\partial P_X}$ depends on $(r_Y B - P_Y R)$, assuming that $P_X r_Y \neq P_Y r_X$. Again, there seems to be three situations:

a) If $r_Y B > P_Y R$, then $\frac{\partial Y^*}{\partial P_X} > 0$, which indicates that if the price of the commodity x increases, the level of consumption of y will also increase. This situation shows that goods x and y are substitute goods to each other; that is, when price of x goes up people switch to its substitute goods y; for instance, tea and coffee.

b) If $r_Y B < P_Y R$, then $\frac{\partial Y^*}{\partial P_X} < 0$, which indicates that if the price of the commodity x increases, the level of consumption of commodity y will decrease. This situation shows that goods x and y are complementary goods; that is, when price of x goes up people buy less of it, consequently level of consumption of y also decreases, as because complementary goods are used together; for instance, gasoline and engine oil.

c) And finally if $r_Y B = P_Y R$, then $\frac{\partial Y^*}{\partial P_X} = 0$, which indicates that if the price of the commodity x increases, there seems no effect on the level of consumption of goods y. This is reasonable result in the sense that commodities x and y are unrelated goods; for instance, “jelly beans and mathematics textbook.”

Secondly, in order to see how does the level of consumption of commodity y changes when the quantity of surrendering ration coupons for purchasing of the commodity x increases? Again, from (26) following the steps as mentioned above and by using the values of X^* and $|J| = |\overline{H}|$ respectively from (9a) and (21), and also using values from (19), we get:

$$\frac{\partial Y^*}{\partial r_X} = -\frac{P_X(r_Y B - P_Y R)}{(P_X r_Y - P_Y r_X)^2} \tag{30}$$

Since P_X, P_Y, r_X, r_Y are prices of commodities and quantities of coupons, so can never be negative, as well as B, R are budget and total ration coupon, so are also positive. Therefore, assuming $P_X r_Y \neq P_Y r_X$, again three similar situations and discussion as mentioned above can be worked out.

The above analysis relates to the effects of an increase in price and surrendering quantity of ration coupons for commodity x; our results are readily adaptable to the case of a change in the price and quantity of ration coupons are required to be surrendered for buying commodity y.

Finally, we analyze the effect of a change in budget B, and in turn a change in quantity of rationing coupons R. Suppose that the individual consumer gets additional budget, and so he wants to increase his utility. Naturally, we can expect that, because of additional money, the consumer would like to buy more amounts of commodity x and y; however, purchasing of commodities is also affected by availability of ration coupons. We examine and verify this mathematically as follows, first by a change in budget B. Again, from (26), we get:

$$\frac{\partial X^*}{\partial B} = \frac{-1}{|J|}[C_{13}] = \frac{-1}{|J|}\begin{vmatrix} 0 & 0 & -R_Y \\ -B_X & -R_X & U_{XY} \\ -B_Y & -R_Y & U_{YY} \end{vmatrix}.$$

$$\frac{\partial X^*}{\partial B} = \frac{R_Y}{|J|}(B_X R_Y - B_Y R_X).$$

Similarly, by substituting the value of $|J| = |\overline{H}|$ from (21) into the above equation, and also using values from (19), we get:

$$\frac{\partial X^*}{\partial B} = \frac{r_Y}{(P_X r_Y - P_Y r_X)} \tag{31}$$

Since P_X, P_Y, r_X, r_Y are prices of commodities and quantities of ration coupons, so can never be negative, and so are positive. Therefore, the sign of $\frac{\partial X^*}{\partial B}$ depends on the denominator $(P_X r_Y - P_Y r_X)$, assuming here that $P_X r_Y \neq P_Y r_X$. Then there can be two possible situations:

a) If $P_X r_Y > P_Y r_X$, then $\frac{\partial X^*}{\partial B} > 0$, which indicates that if the budget increases, the level of consumption of commodity x will also increase. This is a reasonable result in the sense that commodity x is not an inferior good; it may be a superior good.

b) However, if $P_X r_Y < P_Y r_X$, then $\frac{\partial X^*}{\partial B} < 0$, which indicates that even if the budget increases, but the level of consumption of commodity x can decrease. This seems to be valid if commodity x is an inferior good.

And then, in order to get results for the change in ration coupons, from (26) following steps as mentioned above, and by using the value of $|J| = |\overline{H}|$ from (21) as well as using (19), we get:

$$\frac{\partial X^*}{\partial R} = \frac{P_Y}{(P_Y r_X - P_X r_Y)} \tag{32}$$

Since P_X, P_Y, r_X, r_Y are the prices of the commodities and the quantities of the ration coupons, so can never be negative, and so are positive. Therefore, the sign of $\frac{\partial X^*}{\partial R}$ depends on the denominator $(P_Y r_X - P_X r_Y)$. Assuming that $P_Y r_X \neq P_X r_Y$, two possible situations and similar results as well as discussions as mentioned above can easily be worked out.

7. CONCLUDING REMARKS

We have applied the technique of Lagrange multipliers to maximize utility function subject to two constraints: 1) budget constraint, and 2) coupon constraint, and derived mathematical formulation to devise optimal purchasing policy for an individual consumer. With the help of an explicit example, we studied the behaviour of an individual consumer applying comparative static analysis; that is, if the price and / or coupon to purchase a certain commodity rise, how an individual consumer behaves; as well as it is also demonstrated that if individual consumer's budget and / or coupons increases how an individual consumer is going to behave. This is the third paper in the series of our papers published earlier in Indus Journal of Management & Social Sciences.

REFERENCES

Baxley, J.V. and Moorhouse, J.C. 1984. Lagrange Multiplier Problems in Economics. *The American Mathematical Monthly*, 91(7): 404-412 (Aug. – Sep).

Cassels, J.W.S. 1981. Economics for Mathematicians. Cambridge University Press.

Chiang, A.C. 1984. *Fundamental Methods of Mathematical Economics*. 3rd ed. Singapore: McGraw-Hill.

Humphery, T.M. 1997. Algebraic Production Functions and Their Uses Before Cobb- Douglas. *Federal Reserve Bank of Richmond Economic Quarterly,* 83(1) Winter.

Islam, J.N. 1997. *Aspects of Mathematical Economics and Social Choice Theory.* Proceedings of the Second Chittagong Conference on Mathematical Economics and its Relevance for Development. J.N. Islam (Ed.), Chittagong: University of Chittagong, Bangladesh.

Islam, J.N. 2008. *An Introduction to Mathematical Economics and Social Choice Theory,* (book to appear).

Islam, J.N., Mohajan, H.K., and Moolio, P. 2009. Preference of Social Choice in Mathematical Economics, *Indus Journal of Management and Social Sciences*, 3(1):17-37 (Spring 2009)

Moolio, P. 2002. *Theory and Applications of Classical Optimization to Economic Problems.* M. Phil. Thesis. Chittagong: University of Chittagong, Bangladesh.

Moolio, P. and Islam, J.N. 2008. Cost Minimization of a Competitive Firm. *Indus Journal of Management and Social Sciences*, 2(2): 184-160 (Fall 2008).

Moolio, P., Islam, J.N., and Mohajan, H.K. 2009. Output Maximization of an Agency. *Indus Journal of Management and Social Sciences,* 3(1): 39-51 (Spring 2009).

Samuelson, P.A. and Nordhaus, W.D. 2001. *Microeconomics,* 17th ed. New York: McGraw-Hill Irwin.