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ABSTRACT

The main aim of this paper is to derive the mathematical formulation to device an optimal purchasing policy for the service providing agency. An attempt has been made to maximize the output function of an agency subject to a nonlinear budget constraint by assuming that the agency gets price discounts for purchasing larger quantities of other inputs. Such quantity discounts alter the linear budget constraint and result in a nonlinear (convex type) budget constraint. We use the method of Lagrange multipliers and apply the first-order necessary conditions as well as the second-order sufficient conditions for maximization. We also use comparative static analysis and study the behavior of the agency when prices of inputs undergo change, besides providing useful interpretation of the Lagrange multipliers in this specific case. Illustrating an explicit example, we show that the optimization problems play an important role in the real world.

JEL. Classification: C51; C65; C61; D24

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1. INTRODUCTION

It is quite common to receive a discount on the price of each unit when ordering larger quantities of commodities, or in some markets the prices vary depending on quantities of commodity purchased. Also firms offer a lower per unit price if a consumer is willing to purchase larger quantities of a commodity. Such quantity discounts alter the linear budget constraint, and result in a nonlinear (convex type) budget constraint. This is an extension of the problem considered by Moolio *et al* (2009) by assuming that a government agency obtains price discounts by purchasing larger quantities of other inputs R. We assume that the government agency is allocated an annual budget B and required to maximize some sort of services to the community. If the agency uses factors $K, L,$ and R in the same sense as used by Moolio *et al* (2009) to produce and provide services to the community, then its objective is to maximize the output function subject to a nonlinear budget constraint.

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Baxley and Moorhouse (1984) suggested this problem. One can read the relevant articles to optimization problems by Islam (1997), Moolio (2002), Moolio and Islam (2008), Moolio *et al* (2009), and Islam *et al* (2010). All of the studies cited above are derived under the assumption of linear constraints. Moreover, fundamental relationships between mathematical economics, social choice and welfare theory by introducing utility functions, preference relations and Arrow's Impossibility Theorem are given in detail by Islam *et al* (2009). A detailed discussion on algebraic production functions and their uses is considered by Humphery (1997). An introduction to the Lagrange multipliers method and its application in the field of power systems economic operation is given by Li (2008). Kalvelagen (2003) discusses the well known optimization problem of utility maximization under a budget constraint, constructing interesting non-trivial variants by assuming some non-linear pricing structures actually observed in daily life. DeSalvo and Huq (2002) show that under some forms of nonlinear pricing, after a price rise people may buy more of a commodity than would have been bought under linear pricing.

To get intuitive ideas and understanding of the problem at hand, we consider here, explicitly, a simple algebraic function in three variables, and examine the behaviour of the agency, that is, how a change in the costs of input will affect the situation, or if the demand of the services undergoes some changes. We also give suitable interpretation of the Lagrange multiplier in the context of this specific situation, besides using it as a device for transforming a constrained problem into a higher dimensional unconstrained problem.

Organization of remaining paper is as below: section 2 details the building of model, in section 3 we consider an explicit example and find optimal output, section 4 explains the interpretation of Lagrange multipliers, in section 5 second-order sufficient conditions are applied for maximization, section 6 enlightens the comparative static analysis, and concluding remarks are given in section 7.

2. THE MODEL

We consider, for the fixed annual budget, a government agency is charged to produce and provide to the community a quantity Q of the services during a year, with the use of K quantity of capital, L quantity of labour, and R quantity of other inputs into its service oriented production process. The agency uses factors $K, L, \text{and } R$ to produce and provide services. Its objective is to maximize the output function: $Q = g\left(K, L, R\right)$ subject to a nonlinear budget constraint: $B = rK + wL + \rho(R)R$; where r is the rate of interest or services of capital per unit of capital K, w is the wage rate per unit of labour L, and ρ is the cost per unit of other inputs R, while g is a suitable production function. The government agency takes these and all other factor prices as given. We assume that second order partial derivatives of the function g with respect to the independent variables (factors) $K, L, \text{and } R$ exist.

3. AN EXPLICIT EXAMPLE

In order to get intuitive ideas and an intrinsic understanding of the problem, we consider explicitly a simple algebraic form of the output function in three variables:

$$Q = g\left(K, L, R\right) = KLR \tag{1}$$

subject to particular nonlinear budget constraint:

$B = rK + wL + \rho(R)R$, where $\rho\left(R\right) = \rho_0 R - \rho_0$, with ρ_0 being the discounted price of the inputs R. Therefore, the budget constraint takes the form:

$$B = rK + wL + \rho_0 R^2 - \rho_0 R. \quad (2)$$

We now formulate the maximization problem for the output function (1) in terms of the single Lagrange multiplier λ by defining the Lagrangian function Z:

$$Z(K, L, R, \lambda) = KLR + \lambda \{B - rK - wL - \rho_0 R^2 + \rho_0 R\}. \quad (3)$$

This is a four dimensional unconstrained problem obtained from (1) and (2) by the use of Lagrange multiplier λ, as a device. Assuming that the government agency maximizes it's output, the optimal quantities K^*, L^*, R^*, λ^* of K, L, R, λ that necessarily satisfy the first-order conditions, which we obtained by partial differentiation of the Lagrangian function (3) with respect to four variables $\lambda, K, L,$ and R and setting them equal to zero:

$$Z_\lambda = B - rK - wL - \rho_0 R^2 + \rho_0 R = 0, \quad (4a)$$

$$Z_K = LR - \lambda r = 0, \quad (4b)$$

$$Z_L = KR - \lambda w = 0, \quad (4c)$$

$$Z_R = KL - \lambda\{2\rho_0 R - \rho_0\} = 0, \quad (4d)$$

where, $Z_K = \frac{\partial Z}{\partial K}, Z_L = \frac{\partial Z}{\partial L}, Z_R = \frac{\partial Z}{\partial R}, Z_\lambda = \frac{\partial Z}{\partial \lambda}$, and $Q_K = \frac{\partial g}{\partial K}, Q_L = \frac{\partial g}{\partial L}, Q_R = \frac{\partial g}{\partial R}$.

From (4a), we get $B = rK + wL + \rho_0 R^2 - \rho_0 R$, while from (4b-d), we get the Lagrange multiplier:

$$\lambda = (LR)/r = (KR)/w = (KL)/(2\rho_0 R - \rho_0). \quad (5)$$

If we consider the infinitesimal changes dK, dL, dR in K, L, R respectively, the corresponding changes dQ and dB in Q and B are:

$$dQ = Q_K dK + Q_L dL + Q_R dR, \quad (6)$$

$$dB = rdK + wdL + (\rho_0 R - \rho_0) dR. \quad (7)$$

With the use of (4a-d) or (5), we obtain the equation for Lagrange multiplier:

$$\frac{dQ}{dB} = \frac{Q_K dK + Q_L dL + Q_R dR}{rdK + wdL + (\rho_0 R - \rho_0) dR} = \lambda. \quad (8)$$

Thus, here the Lagrange multiplier gives the change in output consequent to change in the inputs. For example, if one of the inputs, say K, is held constant, means $dK = 0$, then (8) represents the partial derivative: $\left(\partial Q / \partial B \right)_K$ (with $dK = 0$), and so.

We now solve the set of four simultaneous equations in (4a-d) produced by the first-order conditions for the optimum values of $\lambda, K, L,$ and R:

$$B = rK + wL + \rho_0 R^2 - \rho_0 R, \quad (9a)$$

$$\lambda = (LR)/r, \quad (9b)$$

$$\lambda = (KR)/w, \quad (9c)$$

$$\lambda = (KL)/(2\rho_0 R - \rho_0). \quad (9d)$$

Equations (9b-d) verify the equation (5), and after solving them, we get:

$$\frac{1}{rK}=\frac{1}{wL}=\frac{1}{(2\rho_0 R-\rho_0)R} \tag{10}$$

$$K=\frac{wL}{r}=\frac{(2\rho_0 R-\rho_0)R}{r},\ L=\frac{rK}{w}=\frac{(2\rho_0 R-\rho_0)R}{w},\ R=\frac{wL}{2\rho_0 R-\rho_0}=\frac{rK}{2\rho_0 R-\rho_0}. \tag{11}$$

From (9a) we get:

$$K=\left(B-wL+\rho_0 R-\rho_0 R^2\right)/r, \tag{12a}$$

$$L=\left(B-rK+\rho_0 R-\rho_0 R^2\right)/w, \tag{12b}$$

$$R=(B-rK-wL)/(\rho_0 R-\rho_0). \tag{12c}$$

By substituting the value of K from (12a) into (11) and after simplification we get:

$$L=\left(B+\rho_0 R-\rho_0 R^2\right)/(2w). \tag{13}$$

Similarly, putting the value of L from (13) into (11) and after simplification, we get:

$$5\rho_0 R^2-3\rho_0 R-B=0.$$

Using the quadratic formula, we solve the above equation and get:

$$R=\left(3\rho_0\pm\sqrt{9\rho_0{}^2+20\rho_0 B}\right)\Big/(10\rho_0).$$

In order to get the results to analyze the comparative statics, we manipulate the above equation. Let us suppose that $B=2\rho_0$, therefore $B/\rho_0=2$, and hence $\rho_0=B/2$. Thus, following the simple steps of calculation, we get one positive and other negative value of R. We consider the positive value to be the optimal value, which is:

$$R^*=B/(2\rho_0). \tag{14a}$$

Now putting the optimal value of R from (14a) into (11), we get the optimal value of K:

$$K^*=B/(2r). \tag{14b}$$

And similarly by putting the optimal value of K from (14b) into (11), we get the optimal value of L:

$$L^*=B/(2w). \tag{14c}$$

And, finally by substituting the optimal values K^*, L^*, and R^* from (14a-c) into (9b-d) and after a simple calculation, we get two values of the Lagrange multiplier:

$$\lambda^*{}_1=B^2/(4rw\rho_0), \tag{14d}$$

$$\lambda^*{}_2=B^2/4rw(B-\rho_0). \tag{14e}$$

Thus, we get the stationary point:

$$\left(K^*,L^*,R^*\right)=\left(B/2r,B/2w,B/2\rho_0\right). \tag{15}$$

Moreover, substituting the values of K^*, L^*, R^* from (14a-c) into (1), and after a straightforward calculation, we get:

$$Q^*=g(K,L,R)=\left(\frac{B}{2r}\right)\left(\frac{B}{2w}\right)\left(\frac{B}{2\rho_0}\right)=\frac{B^3}{8rw\rho_0}. \tag{16}$$

This is the optimal output of the agency in terms of r, w, ρ_0, and B.

4. Interpretation of Lagrange Multiplier

In order to provide an interpretation of the Lagrange multiplier, specifically in this case, with the aid of the chain rule, we get from (16):

$$\frac{\partial Q^*}{\partial B} = Q_K \frac{\partial K}{\partial B} + Q_L \frac{\partial L}{\partial B} + Q_R \frac{\partial R}{\partial B}. \tag{17}$$

From (1) we get:

$Q_K = LR$, $Q_L = KR$, $Q_R = KL$.

And from (4b-d), we get:

$r\lambda = LR$, $w\lambda = KR$, $(2\rho_0 R - \rho_0)\lambda = KL$.

Therefore, we write (17):

$$\frac{\partial Q^*}{\partial B} = \lambda^* \left[r\frac{\partial K}{\partial B} + w\frac{\partial L}{\partial B} + (2\rho_0 R - \rho_0)\frac{\partial R}{\partial B} \right]. \tag{18}$$

From (4a) we get:

$B = rK + wL + \rho_0 R^2 - \rho_0 R$.

Differentiation of the above equation, keeping $K, L,$ and R constant, yields:

$1 = r\frac{\partial K}{\partial B} + w\frac{\partial L}{\partial B} + (2\rho_0 R - \rho_0)\frac{\partial R}{\partial B}$, which allows us to re-write (18):

$$\frac{\partial Q^*}{\partial B} = \lambda^* . \tag{19}$$

Equation (19) verifies the equation (8). Thus the Lagrange multiplier λ^* is the same as obtained by Moolio *et al* (2009); that it may be interpreted as the marginal output; that is, the change in total output incurred from an additional unit of budget B. In other words, if the agency wants to increase (decrease) a 1-unit of its output, it would cause the total budget to increase (decrease) by approximately λ^* units.

5. SECOND-ORDER SUFFICIENT CONDITIONS

Now, in order to be sure that the optimal solution obtained in (16) is the maximum, we check it against the second-order sufficient conditions, which imply that for the solution K^*, L^*, R^* and λ^* of (4a-d) to be a relative maximum, the bordered principal minors of the bordered Hessian:

$$|\overline{H}| = \begin{vmatrix} 0 & -B_K & -B_L & -B_R \\ -B_K & Z_{KK} & Z_{KL} & Z_{KR} \\ -B_L & Z_{LK} & Z_{LL} & Z_{LR} \\ -B_R & Z_{RK} & Z_{RL} & Z_{RR} \end{vmatrix}$$

should alternate in sign; the sign of $|\overline{H}_{m+1}|$ being that of $(-1)^{m+1}$, where m is the number of constraints, in this case $m = 1$. In other words, particularly in this specific case, if

$$\left|\overline{H}_2\right| = \begin{vmatrix} 0 & -B_K & -B_L \\ -B_K & Z_{KK} & Z_{KL} \\ -B_L & Z_{LK} & Z_{LL} \end{vmatrix} > 0 \tag{20a}$$

and, $$\left|\overline{H}_3\right| = \left|\overline{H}\right| = \begin{vmatrix} 0 & -B_K & -B_L & -B_R \\ -B_K & Z_{KK} & Z_{KL} & Z_{KR} \\ -B_L & Z_{LK} & Z_{LL} & Z_{LR} \\ -B_R & Z_{RK} & Z_{RL} & Z_{RR} \end{vmatrix} < 0 \tag{20b}$$

with all the derivatives evaluated at the critical values $K^*, L^*, R^*, \text{and } \lambda^*$, then the stationary value of output Q will assuredly be the maximum. We check this condition. From (20a) we expand the determinant to get:

$$\left|\overline{H}_2\right| = -B_K B_K Z_{LL} + 2B_K B_L Z_{KL} - B_L B_L Z_{KK}. \tag{21}$$

From (2) we get the following derivatives:

$$B_K = r\,;\; B_L = w\,;\; B_R = 2\rho_0 R - \rho_0. \tag{22a}$$

And from (4b-d) we get the following derivatives:

$$Z_{KK} = 0\,;\; Z_{LL} = 0\,;\; Z_{RR} = -2\rho_0\lambda. \tag{22b}$$

$$Z_{KL} = Z_{LK} = R\,;\; Z_{KR} = Z_{RK} = L\,;\; Z_{LR} = Z_{RL} = K. \tag{22c}$$

By substituting the values of $B_K, B_L, Z_{KK}, Z_{LL}, Z_{KL}$ from (22a-c) into (21), and after simplification we get:

$$\left|\overline{H}_2\right| = 2rwR.$$

By substituting the critical value R^* from (14a) into the above equation, we get:

$$\left|\overline{H}_2\right| = 2rw\left(\frac{B}{2\rho_0}\right) = \frac{rwB}{\rho_0}. \tag{23}$$

Since r, w, ρ_0 are the costs of inputs and hence are positive, while B is budget, which will never be negative, therefore, $\left|\overline{H}_2\right| > 0$, as required.

And from (20b) we expand the determinant (since the second partial derivatives of $Z_{KL} = Z_{LK}, Z_{KR} = Z_{RK}, \text{and } Z_{LR} = Z_{RL}$, therefore, for the ease of writing, we use only one notation), and substituting the values of $B_K, B_L, B_R, Z_{KK}, Z_{LL}, Z_{RR}, Z_{KL}, Z_{KR}, Z_{LR}$ from (22a-c) we get:

$$\left|\overline{H}\right| = r^2K^2 + 2rwR(-2\rho_0\lambda) - 2r(2\rho_0R - \rho_0)RK - 2rwLK + w^2L^2 - 2w(2\rho_0R - \rho_0)LR + (2\rho_0R - \rho_0)^2R^2.$$

Simplifying and substituting the critical values K^*, L^*, R^* from (14a-c) we get:

$$\left|\overline{H}\right| = \frac{B^2}{4} - 2rw\lambda B - \frac{B^3}{2\rho_0} + \frac{B^2}{2} - \frac{B^2}{2} + \frac{B^2}{4} - \frac{B^3}{2\rho_0} + \frac{B^2}{2} + \frac{B^4}{4\rho_0{}^2} + \frac{B^2}{4} - \frac{B^3}{2\rho_0}.$$

Though we found two values of the Lagrange multiplier; however, we consider here only one. Hence, substituting the value of $\lambda = \lambda_1^* = B^2/(4rw\rho_0)$ from (14d) and simplifying we get:

$$\left|\overline{H}\right| = \frac{5B^2}{4} - \frac{2B^3}{\rho_0} + \frac{B^4}{4\rho_0^{\ 2}}.$$

For this particular case, we have taken the value of $B = 2\rho_0$, so by re-substituting it into the above expression, we get:

$$\left|\overline{H}\right| = -7\rho_0^{\ 2}. \qquad (24)$$

Although ρ_0 is the discounted cost of other inputs, it cannot be negative. Therefore $\left|\overline{H}\right| < 0$, as required. Equations (23) and (24) are second-order sufficient conditions satisfied to state that the stationary point obtained in (15) is the relative maximum point. Thus, the value of the output function (16) is indeed a relative maximum value.

6. COMPARATIVE STATIC ANALYSIS

Now, since the sufficient conditions are satisfied, we drive further results of economic interest. Mathematically, we solve the four equations (4a-d) for $K, L, R, \text{and } \lambda$ in terms of $r, w, \rho_0, \text{and } B$, and compute sixteen partial derivatives: $\frac{\partial K}{\partial r}, \frac{\partial K}{\partial w}, \frac{\partial K}{\partial \rho_0}, \frac{\partial K}{\partial B}$, etc. These partial derivatives are referred to as the comparative static of the model (Chiang 1984). The model's usefulness is to determine how accurately it predicts adjustments in the agency's input behaviour; that is, how the agency will react to the changes in the costs of capital, labour, and other inputs.

Since we have assumed that the left side of each in (4a-d) is continuously differentiable and that the solution exists, then by the implicit function theorem, $K, L, R, \text{and } \lambda$ will each be continuously differentiable functions of $r, w, \rho_0, \text{and } B$, if the following Jacobian matrix:

$$J = \begin{bmatrix} 0 & -B_K & -B_L & -B_R \\ -B_K & Z_{KK} & Z_{KL} & Z_{KR} \\ -B_L & Z_{LK} & Z_{LL} & Z_{LR} \\ -B_R & Z_{RK} & Z_{RL} & Z_{RR} \end{bmatrix}$$

is non-singular at the optimum point $\left(K^*, L^*, R^*, \lambda^*_1\right)$. As the second-order conditions are met, so the determinant of J does not vanish at the optimum; that is, $\left|J\right| = \left|\overline{H}\right|$; consequently, we apply the implicit function theorem. Let F be the vector-valued function defined for the point $\left(\lambda^*_1, K^*, L^*, R^*, r, w, \rho_0, B\right) \in R^8$, and taking the values in R^4, whose components are given by the left side of the equations in (4a-d). By the implicit function theorem, the equation $F\left(\lambda^*_1, K^*, L^*, R^*, r, w, \rho_0, B\right) = 0$ may be solved in the form of

$$\begin{bmatrix} \lambda^*_1 \\ K^* \\ L^* \\ R^* \end{bmatrix} = G\left(r, w, \rho_0, B\right).$$

Moreover, the Jacobian matrix for G is given by

$$\begin{bmatrix} \frac{\partial \lambda^*_1}{\partial r} & \frac{\partial \lambda^*_1}{\partial w} & \frac{\partial \lambda^*_1}{\partial \rho_0} & \frac{\partial \lambda^*_1}{\partial B} \\ \frac{\partial K^*}{\partial r} & \frac{\partial K^*}{\partial w} & \frac{\partial K^*}{\partial \rho_0} & \frac{\partial K^*}{\partial B} \\ \frac{\partial L^*}{\partial r} & \frac{\partial L^*}{\partial w} & \frac{\partial L^*}{\partial \rho_0} & \frac{\partial L^*}{\partial B} \\ \frac{\partial R^*}{\partial r} & \frac{\partial R^*}{\partial w} & \frac{\partial R^*}{\partial \rho_0} & \frac{\partial R^*}{\partial B} \end{bmatrix} = -J^{-1} \begin{bmatrix} -K^* & -L^* & R^* - R^{*2} & 1 \\ -\lambda^*_1 & 0 & 0 & 0 \\ 0 & -\lambda^*_1 & 0 & 0 \\ 0 & 0 & \lambda^*_1 - 2\lambda^*_1 R^* & 0 \end{bmatrix} \tag{25}$$

where the ith row in the last matrix on the right is obtained by differentiating the ith left side in (4) with respect to r, then w, then ρ_0, and then B. Let C_{ij} be the cofactor of the element in the ith row and jth column of J, and then inverting J using the method of cofactor gives:

$J^{-1} = \frac{1}{|J|} C^T$, where $C = (C_{ij})$.

Thus, equation (25) can further be expressed in the form:

$$\begin{bmatrix} \frac{\partial \lambda^*_1}{\partial r} & \frac{\partial \lambda_1^*}{\partial w} & \frac{\partial \lambda_1^*}{\partial \rho_0} & \frac{\partial \lambda_1^*}{\partial B} \\ \frac{\partial K^*}{\partial r} & \frac{\partial K^*}{\partial w} & \frac{\partial K^*}{\partial \rho_0} & \frac{\partial K^*}{\partial B} \\ \frac{\partial L^*}{\partial r} & \frac{\partial L^*}{\partial w} & \frac{\partial L^*}{\partial \rho_0} & \frac{\partial L^*}{\partial B} \\ \frac{\partial R^*}{\partial r} & \frac{\partial R^*}{\partial w} & \frac{\partial R^*}{\partial \rho_0} & \frac{\partial R^*}{\partial B} \end{bmatrix} = -\frac{1}{|J|} \begin{bmatrix} C_{11} & C_{21} & C_{31} & C_{41} \\ C_{12} & C_{22} & C_{32} & C_{42} \\ C_{13} & C_{23} & C_{33} & C_{43} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{bmatrix} \begin{bmatrix} -K^* & -L^* & R^* - R^{*2} & 1 \\ -\lambda^*_1 & 0 & 0 & 0 \\ 0 & -\lambda^*_1 & 0 & 0 \\ 0 & 0 & \lambda^*_1 - 2\lambda^*_1 R^* & 0 \end{bmatrix}$$

Or,

$$\Rightarrow \; = -\frac{1}{|J|} \begin{bmatrix} -K^*C_{11} - \lambda^*_1 C_{21} & -L^*C_{11} - \lambda^*_1 C_{31} & R^*C_{11} - R^{*2}C_{11} + \lambda^*_1 C_{41} - 2\lambda^*_1 R^* C_{41} & C_{11} \\ -K^*C_{12} - \lambda^*_1 C_{22} & -L^*C_{12} - \lambda^*_1 C_{32} & R^*C_{12} - R^{*2}C_{12} + \lambda^*_1 C_{42} - 2\lambda^*_1 R^* C_{42} & C_{12} \\ -K^*C_{13} - \lambda^*_1 C_{23} & -L^*C_{13} - \lambda^*_1 C_{33} & R^*C_{13} - R^{*2}C_{13} + \lambda^*_1 C_{43} - 2\lambda^*_1 R^* C_{43} & C_{13} \\ -K^*C_{14} - \lambda^*_1 C_{24} & -L^*C_{14} - \lambda^*_1 C_{34} & R^*C_{14} - R^{*2}C_{14} + \lambda^*_1 C_{44} - 2\lambda^*_1 R^* C_{44} & C_{14} \end{bmatrix}. \tag{26}$$

Now, we study and examine the effects of changes in $r, w, \rho_0,$ and B on $K, L,$ and R. Firstly, we find out the effect on capital K when its interest rate r increases. From (26), we get:

$$\frac{\partial K^*}{\partial r} = -\frac{1}{|J|}\left[-K^*C_{12} - \lambda^*_1 C_{22}\right] = -\frac{K^*}{|J|}\left[C_{12}\right] + \frac{\lambda^*_1}{|J|}\left[C_{22}\right].$$

$$\frac{\partial K^*}{\partial r} = -\frac{K^*}{|J|}\{-B_K Z_{LL} Z_{RR} + B_K Z_{LR} Z_{LR} + B_L Z_{KL} Z_{RR} - B_R Z_{KL} Z_{LR} - B_L Z_{KR} Z_{LR} + B_R Z_{KR} Z_{LL}\}$$
$$+\frac{\lambda^*_1}{|J|}\{-B_L B_L Z_{RR} + 2B_L B_R Z_{LR} - B_R B_R Z_{LL}\}.$$

Substituting the values of $B_K, B_L, B_R, Z_{LL}, Z_{RR}, Z_{KL}, Z_{KR}$, and Z_{LR} from (22a-c) into the above equation, we get:

$$\frac{\partial K^*}{\partial r} = -\frac{K^*}{|J|}\{rK^2 - 2w\rho_0 R\lambda - 2\rho_0 R^2 K + \rho_0 RK - wLK\} + \frac{\lambda^*_1}{|J|}\{2w^2\rho_0\lambda + 4w\rho_0 RK - 2w\rho_0 K\}.$$

Substituting the critical values $K^*, L^*, R^*, \lambda^*_1$ from (14a-d) into the above equation, and following straightforward steps of calculation, we get:

$$\frac{\partial K^*}{\partial r} = \frac{1}{|J|}\left\{\frac{5B^4}{8r^2\rho_0} - \frac{5B^3}{8r^2}\right\}.$$

Since for this specific case, we have taken the value of $B = 2\rho_0$. Therefore, by re-substituting it into the above equation, we get:

$$\frac{\partial K^*}{\partial r} = \frac{1}{|J|}\left\{\frac{10\rho_0{}^3}{r^2} - \frac{5\rho_0{}^3}{r^2}\right\} = \frac{1}{|J|}\left\{\frac{5\rho_0{}^3}{r^2}\right\}.$$

Since $|J| = |\overline{H}|$, therefore substituting the value of $|\overline{H}|$ from (24) into the above equation, we get:

$$\frac{\partial K^*}{\partial r} = -\frac{5\rho_0}{7r^2}. \tag{27}$$

Since $r > 0, \rho_0 > 0$, therefore, $\frac{\partial K^*}{\partial r} < 0$, which indicates that if the interest rate of the capital K increases, the agency has to consider to decrease the level of input K. This is a reasonable result.

Secondly, we study and examine the effects on labour L when the interest rate of capital K increases. Again from (26), we get:

$$\frac{\partial L^*}{\partial r} = -\frac{1}{|J|}\left[-K^* C_{13} - \lambda^*_1 C_{23}\right] = \frac{K^*}{|J|}[C_{13}] - \frac{\lambda^*_1}{|J|}[C_{23}].$$

$$\frac{\partial L^*}{\partial r} = \frac{K^*}{|J|}\{-B_K Z_{LK} Z_{RR} + B_K Z_{RK} Z_{LR} + B_L Z_{KK} Z_{RR} - B_R Z_{KK} Z_{LR} - B_L Z_{KR} Z_{RK} + B_R Z_{KR} Z_{LK}\}$$
$$-\frac{\lambda^*_1}{|J|}\{-B_K B_L Z_{RR} + B_K B_R Z_{LR} + B_L B_R Z_{RK} - B_R B_R Z_{LK}\}.$$

Substituting the values of $B_K, B_L, B_R, Z_{KK}, Z_{RR}, Z_{LK}, Z_{KR}, Z_{LR}$ from (22a-c) into the above equation, and following straightforward steps of calculation, we get:

$$\frac{\partial L^*}{\partial r} = \frac{K^*}{|J|}\left\{2r\rho_0 R\lambda_1 + rLK - wL^2 + 2\rho_0 R^2 L - \rho_0 LR\right\}$$
$$-\frac{\lambda^*_1}{|J|}\left\{2rw\rho_0\lambda_1 + 2r\rho_0 RK - r\rho_0 K + 2w\rho_0 RL - w\rho_0 L - 4{\rho_0}^2 R^3 - {\rho_0}^2 R + 4{\rho_0}^2 R^2\right\}.$$

Substituting the critical values $K^*, L^*, R^*, \lambda^*_1$ from (14a-d) into the above equation, and following straightforward steps of calculation, we get:

$$\frac{\partial L^*}{\partial r} = \frac{1}{|J|}\left\{\frac{B^5}{8rw{\rho_0}^2} - \frac{3B^4}{8rw\rho_0} + \frac{B^3}{4rw}\right\}.$$

For this specific case, we have taken the value of $B = 2\rho_0$. Therefore, re-substituting it into the above equation, we get:

$$\frac{\partial L^*}{\partial r} = 0. \tag{28}$$

This indicates that there will be no effect on the level of labour L, if the interest rate of capital K increases. This also indicates that labour and capital are complementary.

The above analysis relates to the effects of a change in the interest rate of the capital K; our comparative static results are readily adaptable to the case of a change in the wage rate of labour L. However, the comparative static results are bit different in the case of a change in the cost of other inputs R; which we analyze here.

First, we find the effect on other inputs R when its' discounted cost ρ_0 increases. From (26), we get:

$$\frac{\partial R^*}{\partial \rho_0} = -\frac{1}{|J|}\left[RC_{14} - R^2 C_{14} + \lambda_1 C_{44} - 2R\lambda_1 C_{44}\right] = \frac{R - R^2}{|J|}\left[C_{14}\right] + \frac{2R\lambda_1 - \lambda_1}{|J|}\left[C_{44}\right].$$
$$\frac{\partial R^*}{\partial \rho_0} = \frac{R - R^2}{|J|}\left\{-B_K Z_{LK} Z_{RL} + B_K Z_{RK} Z_{LL} + B_L Z_{KK} Z_{RL} - B_R Z_{KK} Z_{LL} - B_L Z_{KL} Z_{RK} + B_R Z_{LK} Z_{KL}\right\}$$
$$+\frac{2R\lambda_1 - \lambda_1}{|J|}\left\{-B_K B_K Z_{LL} + B_K B_L Z_{KL} + B_L B_K Z_{LK} - B_L B_L Z_{KK}\right\}.$$

Substituting the values of $B_K, B_L, B_R, Z_{KK}, Z_{LL}, Z_{RR}, Z_{LK}, Z_{KR}, Z_{LR}$ from (22a-c) into the above equation, and following straightforward steps of calculation, we get:

$$\frac{\partial R^*}{\partial \rho_0} = \frac{1}{|J|}\left\{\begin{array}{l} -rR^2K - wR^2L + 2\rho_0 R^4 - \rho_0 R^3 + rR^3K + wR^3L - 2\rho_0 R^5 + \rho_0 R^4 \\ +4rwR^2\lambda_1 - 2rwR\lambda_1. \end{array}\right\}.$$

Substituting the critical values $K^*, L^*, R^*, \lambda^*_1$ from (14a-d) into the above equation, and following straightforward steps of calculation, we get:

$$\frac{\partial R^*}{\partial \rho_0} = \frac{1}{|J|}\left\{-\frac{B^5}{16{\rho_0}^4} + \frac{9B^4}{16{\rho_0}^3} - \frac{5B^3}{8{\rho_0}^2}\right\}.$$

Again, since for this specific case, we have taken the value of $B = 2\rho_0$. Therefore, re-substituting it into the above equation and after simplifying it, we get:

$$\frac{\partial R^*}{\partial \rho_0} = \frac{1}{|J|}\{2\rho_0\}.$$

Since $|J| = |\overline{H}|$, therefore substituting the value of $|\overline{H}|$ from (24) into the above equation, we get:

$$\frac{\partial R^*}{\partial \rho_0} = -\frac{2}{7\rho_0}. \tag{29}$$

Since $\rho_0 > 0$, therefore, $\frac{\partial R^*}{\partial \rho_0} < 0$, which indicates that if the discounted cost of the other inputs R increases, the agency has to consider decreasing the level of other inputs R. This is also a reasonable result.

Now, we study and examine the effects on labour L when the discounted cost of other inputs R increases. Again from (26), we get:

$$\frac{\partial L^*}{\partial \rho_0} = -\frac{1}{|J|}\left[RC_{13} - R^2C_{13} + \lambda_1 C_{43} - 2R\lambda_1 C_{43}\right] = \frac{R^2 - R}{|J|}\left[C_{13}\right] + \frac{\lambda_1 - 2R\lambda_1}{|J|}\left[C_{43}\right].$$

$$\frac{\partial L^*}{\partial \rho_0} = \frac{R^2 - R}{|J|}\{-B_K Z_{LK} Z_{RR} + B_K Z_{RK} Z_{LR} + B_L Z_{KK} Z_{RR} - B_R Z_{KK} Z_{LR} - B_L Z_{KR} Z_{RK} + B_R Z_{LK} Z_{KR}\}$$
$$+ \frac{\lambda_1 - 2R\lambda_1}{|J|}\{-B_K B_K Z_{LR} + B_K B_L Z_{KR} + B_K B_R Z_{LK} - B_L B_R Z_{KK}\}.$$

Substituting the values of $B_K, B_L, B_R, Z_{KK}, Z_{RR}, Z_{LK}, Z_{KR}, Z_{LR}$ from (22a-c) into the above equation, and following straightforward steps of calculation, we get:

$$\frac{\partial L^*}{\partial \rho_0} = \frac{1}{|J|}\begin{Bmatrix} 2r\rho_0 R^3\lambda_1 + rLKR^2 - wL^2R^2 + 2\rho_0 R^4 L - \rho_0 LR^3 - 2r\rho_0 R^2\lambda_1 - rKLR + wL^2R \\ -2\rho_0 R^3 L + \rho_0 LR^2 - r^2K\lambda_1 + rwL\lambda_1 + 2r\rho_0 R^2\lambda_1 - r\rho_0 R\lambda_1 + 2r^2KR\lambda_1 \\ -2rwLR\lambda_1 - 4r\rho_0 R^3\lambda_1 + 2r\rho_0 R^2\lambda_1. \end{Bmatrix}.$$

Substituting the critical values $K^*, L^*, R^*, \lambda^*_1$ from (14a-d) into the above equation, and following straightforward steps of calculation, we get:

$$\frac{\partial L^*}{\partial \rho_0} = \frac{1}{|J|}\left\{-\frac{B^4}{16w{\rho_0}^2}\right\}.$$

Again, since for this specific case, we have taken the value of $B = 2\rho_0$. Therefore, re-substituting it into the above equation and after simplifying it, we get:

$$\frac{\partial L^*}{\partial \rho_0} = \frac{1}{7w}. \tag{30}$$

Since $w > 0$, therefore, $\frac{\partial L^*}{\partial \rho_0} > 0$, which indicates that even if the discounted cost of the other inputs R increases, the agency still may like to consider increasing the level of labour L. This comparative static result obtained under nonlinear constraint (discounted price) is not the same result under linear constraint (linear price) obtained by Moolio *et al* (2009); however, it seems a reasonable result, as because if the discounted cost of other inputs increases, agency will get much more quantity of other inputs R, and hence it needs more labour force.

Next, we analyze the effect of a change in budget B. Let us suppose that the service-providing agency is provided with an additional budget, and asked to increase its output. Naturally, we can expect that it will increase in its inputs $K, L,$ and R. We examine and verify this mathematically. Again from (26), we get:

$$\frac{\partial K^*}{\partial B} = -\frac{1}{|J|}[C_{12}] = \frac{1}{|J|}\begin{vmatrix} -B_K & Z_{KL} & Z_{KR} \\ -B_L & Z_{LL} & Z_{LR} \\ -B_R & Z_{RL} & Z_{RR} \end{vmatrix}$$

$$\frac{\partial K^*}{\partial B} = \frac{1}{|J|}\left\{-B_K Z_{LL} Z_{RR} + B_K Z_{LR} Z_{RL} + B_L Z_{KL} Z_{RR} - B_R Z_{KL} Z_{LR} - B_L Z_{KR} Z_{RL} + B_R Z_{KR} Z_{LL}\right\}.$$

Substituting the values of $B_K, B_L, B_R, Z_{LL}, Z_{RR}, Z_{LK}, Z_{KR}, Z_{LR}$ from (22a-c) into the above equation, we get:

$$\frac{\partial K^*}{\partial B} = \frac{1}{|J|}\left\{rK^2 - 2w\rho_0 R\lambda_1 - 2\rho_0 R^2 K + \rho_0 RK - wLK\right\}.$$

Substituting the critical values K^*, L^*, R^* and $\lambda^*{}_1$ from (14a-d) into the above equation, we get:

$$\frac{\partial K^*}{\partial B} = \frac{1}{|J|}\left\{\frac{B^2}{4r} - \frac{B^3}{4r\rho_0} - \frac{B^3}{4r\rho_0}\right\}.$$

Again, since for this specific case, we have taken the value of $B = 2\rho_0$. Therefore, re-substituting it into the above equation, we get:

$$\frac{\partial K^*}{\partial B} = \frac{1}{|J|}\left\{\frac{-3\rho_0{}^2}{r}\right\}.$$

Since $|J| = |\overline{H}|$, therefore substituting the value of $|\overline{H}|$ from (24) into the above equation, we get:

$$\frac{\partial K^*}{\partial B} = \frac{3}{14} > 0. \qquad (31)$$

And accordingly, $\frac{\partial L^*}{\partial B} > 0$, $\frac{\partial R^*}{\partial B} > 0$, which indicate that when budget size increases the level of input of capital K, labour L and other inputs R also increase, in order to increase the output services provided by the agency.

7. CONCLUDING REMARKS

We have applied the method of Lagrange multipliers to maximize output function subject to a nonlinear constraint, and derived mathematical formulation to devise optimal purchasing policy for a service providing agency. With the help of an explicit example, we studied the behaviour of the agency applying comparative static analysis; that is, if the price of an input rises, how an agency behaves; as well as it is also demonstrated that if an agency's budget increases how the agency is going to behave. Illustrating an explicit example, we showed that the optimization problems play an important role in real world, as well as in the cases where objective function and constraint have specific meanings; the Lagrange multipliers often have an identifiable significance. This is the fourth paper in the series of our papers published earlier in Indus Journal of Management & Social Sciences.

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