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Abstract

The principle of no arbitrage says that identical assets should offer the same returns. However, experimental and anecdotal evidence suggests that people often rely on analogy making while valuing assets. The principle of analogy making says that similar assets should offer the same returns. I show that the principle of analogy making generates a closed-from alternative to the Black Scholes formula that does not require a complete market. The new formula differs from the Black Scholes formula only due to the appearance of a parameter in the formula that captures the risk premium on the underlying. The new formula, called the analogy option pricing formula, provides a new explanation for the implied volatility skew puzzle. The key empirical predictions of the analogy formula are discussed. Empirical evidence strongly supports these predictions.

Keywords: *Mental Accounting, Analogy Making, Incomplete Markets, Implied Volatility, Implied Volatility Skew, Option Prices, Risk Premium, Black Scholes Model*

JEL Classifications: G13; G12

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Mental Accounting: A Closed-Form Alternative to the Black Scholes Model

Many professional financial traders consider a call option to be a surrogate for the underlying stock and advise investors to consider replacing the underlying with the corresponding call option.[2] Such advice may result in investors placing a call option in the same mental account as the underlying. In a series of laboratory experiments, it has been found that mental accounting matters for pricing financial options. The first such experiment in a binomial setting is Rockenbach (2004) who tests for the mental accounting of a call option and the underlying and finds that participants demand the same expected return from a call option as available on the underlying. Experiments reported in Siddiqi (2012) and Siddiqi (2011) explore this further and find that the mental accounting of a call option with its underlying is due to the similarity in payoffs between the two assets as adding a third risky asset with dissimilar payoffs has no effect. It appears that participants in laboratory markets consider a call option to be a surrogate for the underlying without receiving any coaching to this effect due to the similarity in their payoffs. Arguably, investors in financial markets are even more likely to consider a call a surrogate for the underlying as they receive such advice from professional traders.

In this article, we investigate the theoretical implications of mental accounting of a call option with the underlying. We find that a new option pricing formula (the analogy formula) is obtained which differs only slightly from the Black Scholes formula due to the appearance of the risk premium on the underlying in the analogy formula. The analogy formula provides a new explanation for the implied volatility skew puzzle. Specifically, if the market prices are determined by the analogy formula and the Black Scholes formula is used to infer implied volatility, the skew is observed.

The analogy formula does not require the assumption of a dynamically complete market. It remains the same irrespective of whether we assume a complete market or an incomplete market. We do not take a position regarding market completeness in this article. However, as is well known, dynamically complete markets require non-trivial restrictions on market structure and price

[2] As illustrative examples, see the following:
http://ezinearticles.com/?Call-Options-As-an-Alternative-to-Buying-the-Underlying-Security&id=4274772,
http://www.investingblog.org/archives/194/deep-in-the-money-options/,
http://www.triplescreenmethod.com/TradersCorner/TC052705.asp,
http://daytrading.about.com/od/stocks/a/OptionsInvest.htm

processes. See Duffie and Huang (1985) as one example. Hence, perfect replication is unlikely to hold in practice.

If markets are incomplete, then the analogy formula almost always picks out a price within the arbitrage-free interval. If markets are complete, then transaction costs may prevent rational arbitrageurs from making money at the expense of analogy makers especially when the difference between the formulas is small and is due to only one parameter. We also provide testable predictions of the analogy formula. Existing empirical evidence strongly supports these predictions.

Thaler (1999) defines mental accounting as the set of cognitive operations used by individuals and households to organize, evaluate, and keep track of financial activities. Mental accounting of a call with its underlying implies that individuals mentally organize and evaluate investment in a call by comparing it with investment in a similar asset (the underlying). Specifically, they consider a call to be a surrogate for the underlying and if the call offers at least the same return as the underlying, it is considered a good investment.

Leading cognitive scientists argue that similarity spotting or analogy making forms the core of cognition and it is the fuel and fire of thinking (see Hofstadter and Sander (2013)). According to Hoftsadter and Sander (2013), we engage in analogy making when we spot a link between different situations/objects. They define analogy making as the act of placing objects in the same mental category due to perceived similarity between them. For example, if we see a pencil, we are able to recognize it as such because we have seen similar objects before and such objects are called pencils. Furthermore, spotting of this link is not only useful in assigning correct labels, it also gives us access to a whole repertoire of stored information regarding pencils including how they are used and how much a typical pencil costs. Hofstadter and Sander (2013) argue that analogy making not only allows us to carry out mundane tasks such as using a pencil, toothbrush or an elevator in a hotel but is also the spark behind all of the major discoveries in mathematics and the sciences. They argue that analogy making is responsible for all our thinking, from the most trivial to the most profound. Of course, there is always a danger of making wrong analogies. When a small private plane flew into a building in New York on October 11, 2006, the analogy with the events of September 11, 2001 was irrepressible and the Dow Jones Index fell sharply in response.

The recognition of analogy making as an important decision principle is not new. Hume wrote in 1748, "*From causes which appear similar, we expect similar effects. This is the sum of all our experimental conclusions*". (Hume 1748, Section IV). Similar ideas have been expressed in economic literature by Keynes (1921), Selten (1978), and Cross (1983) among others. To our knowledge, two formal approaches have been proposed to incorporate analogy making into economics: 1) case based decision theory of Gilboa and Schmeidler (2001) in which preferences are determined by the cases in a decision maker's memory and their similarity with the decision problem being considered, and 2) coarse thinking/analogy making model of Mullainathan, Schwartzstein, and Shleifer (2008) in which expectations about an attribute are formed by co-categorizing a situation with analogous situations and transferring the information content of the attribute across co-categorized situations. The approach in this paper relates to the model of Mullainathan et al (2008). The attribute of concern here is return on a call option, which is influenced by the return on the underlying as investors co-co-categorize a call with the underlying stock.

This paper adds to the literature in several ways: 1) We put forward a closed-form alternative to the Black Scholes formula. The Black Scholes model is the first option pricing model that provides a closed-form solution. Having a closed-form is advantageous for a number of reasons. Most importantly, it greatly simplifies computation enabling one to develop intuition about the impact of various parameters. The analogy approach results in a closed-form with a formula that differs from the Black Scholes formula due to the appearance of only one additional parameter, which is the risk premium on the underlying. Hence, the analogy formula is as simple as the Black Scholes formula. 2) The analogy formula provides a new explanation for the implied volatility skew puzzle. Specifically, if the market prices are determined by the analogy formula, and the Black Scholes formula is used to back-out implied volatility, the skew is observed. 3) In an interesting paper, Derman (2002) writes, "*If options prices are generated by a Black–Scholes equation whose rate is greater than the true riskless rate, and if these options prices are then used to produce implied volatilities via the Black–Scholes equation with a truly riskless rate, it is not hard to check that the resultant implied volatilities will produce a negative volatility skew*." (Derman (2002) page 295).

This paper provides a reason for the above mentioned effect.[3] The analogy formula is exactly identical to the Black Scholes formula apart from replacing the risk free rate with the return on the underlying stock (that is, the risk free is supplemented with the risk premium). Our approach is also broadly consistent with Shefrin (2008) who provides a systematic treatment of how behavioral assumptions impact the pricing kernel at the heart of modern asset pricing theory. 4) One limitation of the Black Scholes model is that it requires a complete market. In contrast, the analogy formula does not require a complete market. In an incomplete market there is no unique no-arbitrage price; rather a wide interval of arbitrage-free prices is obtained as the martingale measure is not unique.

[3] I am grateful to Emanuel Derman for pointing this out.

Which price to pick then? Two approaches have been developed to search for solutions in an incomplete market. One is to pick a specific martingale measure according to some optimal criterion. See Follmer and Schweizer (1991), Miyahara (2001), Fritelli (2002), Bellini and Fritelli (2002), and Goll and Ruschendorf (2001) among others. The other approach is utility based option pricing. See Hodges and Neuberger (1989), Davis (1997), and Henderson (2002) for early treatment. Our approach relates to the former as it effectively specifies analogy making as a mechanism for picking a specific martingale measure. 5) We provide a number of testable predictions of the model and summarize existing evidence. The existing evidence strongly supports the analogy approach. 6) Duan and Wei (2009) use daily option quotes on the S&P 100 index and its 30 largest component stocks, to show that, after controlling for the underlying asset's total volatility, a higher amount of systematic risk leads to a higher level of implied volatility and a steeper slope of the implied volatility curve. In the analogy option pricing model, higher risk premium on the underlying for a given level of total volatility generates this result. As risk premium is related to systematic risk, this prediction of the analogy model is quite intriguing. 7) Our approach is also an example of behavioralization of finance. Shefrin (2010) argues that finance is in the midst of a paradigm shift, from a neoclassical based framework to a psychologically based framework. Behavioralizing finance is the process of replacing neoclassical assumptions with behavioral counterparts while maintaining mathematical rigor.

In general, pricing models that have been proposed to explain the implied volatility skew can be classified into three broad categories: 1) Stochastic volatility and GARCH models (Heston and Nandi (2000), Duan (1995), Heston (1993), Melino and Turnbull (1990), Wiggins (1987), and Hull and White (1987)). 2) Models with jumps in the underlying price process (Amin (1993), Ball and Torous (1985)). 3) Models with stochastic volatility as well as random jumps. See Bakshi, Cao, and Chen (1997) for a discussion of their empirical performance (mixed). Most of these models modify the price process of the underlying. Hence, the focus of these models is on finding the right distributional assumptions that could explain the implied volatility puzzles. Our approach differs from them fundamentally. In our approach, it is not the underlying's price process that leads to the skew. Rather, it is the mental accounting of a call with its underlying. We assume geometric Brownian motion for simplicity. More complicated versions of the results are obtained with other price processes.

Section 2 explains the difference between the principle of no arbitrage and analogy making through a simple example in a complete market context. Section 3 illustrates analogy making in an incomplete market and shows that analogy making picks out a specific martingale measure from the set of allowable martingale measures. Section 4 explores the implications of analogy making in a one period binomial model, which is the simplest case of a complete market. Section 5 does the same for the simplest case of an incomplete market: the trinomial model. Section 6 considers the general incomplete market case with N states. Section 7 puts forward the analogy option pricing formula that does not require the assumption of a complete market. Section 8 shows if prices are determined in accordance with the analogy formula and the Black Scholes formula is used to infer implied volatilities then the implied volatility skew is observed. Section 9 puts forward the key empirical predictions of the model. Section 10 concludes.

2. Analogy Making: A Complete Market Example

Consider an investor in a two state-two asset complete market world. The investor has initially put his money in the two assets: A stock (S) and a risk free bond (B). The stock has a price of $140 today. In the next period, the stock could either go up to $200 (the red state) or go down to $90 (the blue state). Each state has a 50% chance of occurring. The bond costs $100 today and it also pays $100 in the next period implying a risk free rate of zero. Suppose a new asset "A" is introduced to him. The asset "A" pays $140 in the red state and $30 in the blue state. How much should he be willing to pay for it?

Finance theory provides an answer by appealing to the principle of no-arbitrage: *identical assets should offer the same returns*. Consider a portfolio consisting of a long position in S and a short position in 0.60 of B. In the red state, S pays $200 and one has to pay $60 due to shorting 0.60 of B resulting in a net payoff of $140. In the blue state, S pays $90 and one has to pay $60 on account of shorting 0.60 of B resulting in a net payoff of $30. That is, payoffs from S-0.60B are identical to payoffs from "A". Hence, according to the no-arbitrage principle, "A" should be priced in such a way that its expected return is equal to the expected return from (S-0.60B). It follows that the no-arbitrage price for "A" is $80.

In practice, constructing a portfolio that replicates "A" is no easy task. When simple tasks such as the one described above are presented to participants in a series of experiments, they seem to rely on analogy-making to figure out their willingness to pay. See Rockenbach (2004), Siddiqi (2011), and Siddiqi (2012). So, instead of trying to construct a replicating portfolio which is identical to asset "A", people find an actual asset similar to "A" and price "A" in analogy with that asset. That is, they rely on the principle of analogy: *similar assets should offer the same returns* rather than on the principle of no-arbitrage: *identical assets should offer the same returns.*

Asset "A" is similar to asset S. It pays more when asset S pays more and it pays less when asset S pays less. Expected return from S is 1.0357 $\left(\frac{0.5\times200+0.5\times90}{140}\right)$. According to the principle of analogy, A's price should be such that it offers the same expected return as S. That is, the right price for A is $82.07.

In the above example, there is a gap of $2.07 between the no-arbitrage price and the analogy price. Rational investors should short "A" and buy "S-0.60B". However, if we introduce a small transaction cost of 1%, then the total transaction cost of the proposed scheme exceeds $2.07, preventing arbitrage. The transaction cost of shorting "A" is $0.8207 whereas the transaction cost of buying "S-0.60B" is $1.6 so the total transaction cost is $2.4207. Hence, in principle, the deviation between the no-arbitrage price and the analogy price may not be corrected due to transaction costs even if we assume perfect replication. It is also interesting to note that asset "A" is equivalent to a call option on "S" with a strike price of 60.

Even though the example discussed above is a complete market example in which asset "A" can be replicated by using other assets, the idea of analogy making does not require complete markets. For example, one can easily add another state, say Green, with a payoff that makes replication impossible. Consequently, we do not get a unique no-arbitrage price; however, a unique analogy price is still obtained. Market incompleteness determines fairly wide bounds for the rational price so, in principle, analogy making may pick out one price within the allowed arbitrage-free bounds.

Of course, "similarity spotting" is inherently subjective. What constitutes "similar" varies from person to person and depends on prior experiences. However, sometimes the similarity is pointed out in the decision context. This is the case with call options. A call option is a right to buy the underlying, so its payoff is related to the underlying's payoff by definition. Also, professional financial traders emphasize this similarity when they generate advice for investors. In fact, they use a call option's *moneyness* as a measure of similarity. Hence, the similarity between a call and its underlying is a salient feature of the decision context, and anybody who trades in call options is very likely to take this into account.

Inspired by market professionals, for call options, one can use *moneyness* as a measure of *similarity*. In the example discussed above, *moneyness* $= {}^{K}/_{S} = {}^{60}/_{140} = 0.43$ where K is the strike price which is 60 in this case and S is the stock price which is 140. A lower value of *moneyness* implies a higher degree of similarity. Lower the value of K for a given value of S, stronger is the similarity.

2.1 A Two Period Binomial Model Example

Consider a two period binomial model. The parameters are: Up factor=2, Down factor=0.5, Current stock price=100, Risk free interest rate per binomial period=0, Strike price=30, and the probability of up movement=0.5. It follows that the expected return from the stock per binomial period is 1.25 $(0.5 \times 2 + 0.5 \times 0.5)$.

The call option can be priced both via analogy as well as via no-arbitrage argument. The no-arbitrage price is denoted by C_R whereas the analogy price is denoted by C_A. Define $x_R = \frac{\Delta C_R}{\Delta S}$ and $x_A = \frac{\Delta C_A}{\Delta S}$.

Figure 1 shows the binomial tree and the corresponding no-arbitrage and analogy prices. Two things should be noted. Firstly, in the binomial case considered, before expiry, the analogy price is always larger than the no-arbitrage price. Secondly, the delta hedging portfolios in the two cases $Sx_R - C_R$ and $Sx_A - C_A$ grow at different rates. The portfolio $Sx_A - C_A$ grows at the rate equal to the expected return (gross) on stock per binomial period (which is 1.25 in this case). In the analogy case, the value of delta-hedging portfolio when the stock price is 100 is 17.06667 $(100 \times 0.98667 - 81.6)$. In the next period, if the stock price goes up to 200, the value becomes 21.33333 $(200 \times 0.98667 - 176)$. If the stock price goes down to 50, the value also ends up being equal to 21.33333 $(50 \times 0.98667 - 28)$. That is, either way, the rate of growth is the same and is equal to 1.25 as$17.06667 \times 1.25 = 21.33333$. It is easy to verify that the portfolio $Sx_R - C_R$ grows at the risk free rate per binomial period (which is 0 in this case).

Exp. Ret	1.25
Up Prob.	0.5
Up	2
Down	0.5
Risk-Free r	0
Strike	30

Stock Price	400
$Call_R$	370
$Call_A$	370

Stock Price	200
x_R	1
B	-30
$Call_R$	170
x_A	1
$Call_A$	176

Stock Price	100
x_R	0.977778
B	-25.5556
$Call_R$	72.22222
x_A	0.986667
$Call_A$	81.6

Stock Price	100
$Call_R$	70
$Call_A$	70

Stock Price	50
x_R	0.933333
B	-23.3333
$Call_R$	23.33333
x_A	0.933333
$Call_A$	28

Stock Price	25
$Call_R$	0
$Call_A$	0

Figure 1

3. Analogy Making: An Incomplete Market Example

Consider a simple incomplete market in which there are two assets and three states. Each state is equally likely to occur. Asset "S" has a price of 100 today and the risk free asset "B" also has a price of 100 today. The state-wise payoffs are summarized in table 1.

Table 1				
Asset Type	**Price**	**Red**	**Blue**	**Green**
S	100	200	90	50
B	100	110	110	110

Suppose a new asset, "A" is introduced with the following payoffs: Red state payoff is 140; Blue state payoff is 30; and Green state payoff is 0. This claim is equivalent to a call option on "S" with a strike price of 60. "A" cannot be replicated with S and B. Hence, there is no unique no-arbitrage price. However, an arbitrage free interval can be specified: $45.5 < arbitrage\ free\ price < 54.5$.

Applying the principle of analogy making picks out the following price from the arbitrage free interval: 50. It can be shown that this price corresponds to the following martingale measure: (0.363633, 0.136364, 0.5). Hence, in this example, analogy making picks out a specific martingale measure from the set of allowable martingale measures. Consequently, it can be considered a selection mechanism.

The problem of pricing "A" in an incomplete market can be stated as: Given the actual probability measure $\boldsymbol{P}$, there is a set $\boldsymbol{Q}$ of *equivalent martingale measures* such that the price of "A" is in an arbitrage-free interval:

$$\left(e^{r(T-t)} inf_{Q\in\boldsymbol{Q}} E_Q[max(S_T - K, 0)], e^{r(T-t)} sup_{Q\in\boldsymbol{Q}} E_Q[max(S_T - K, 0)]\right).$$

Analogy making selects a Q.

3.1 A Two Period Trinomial Model Example

As in the binomial example discussed earlier, consider a two period trinomial situation. The parameters are: Up factor=2, Down factor=0.5, Middle factor=1, Risk free interest rate per binomial period=0, Strike price=30, Probabilities of up, down, and middle movements are equal to 1/3. It follows that the expected return on the stock per period is 1.166666. Of course, in this case, a unique no-arbitrage price cannot be calculated, however, a unique analogy price can be found.

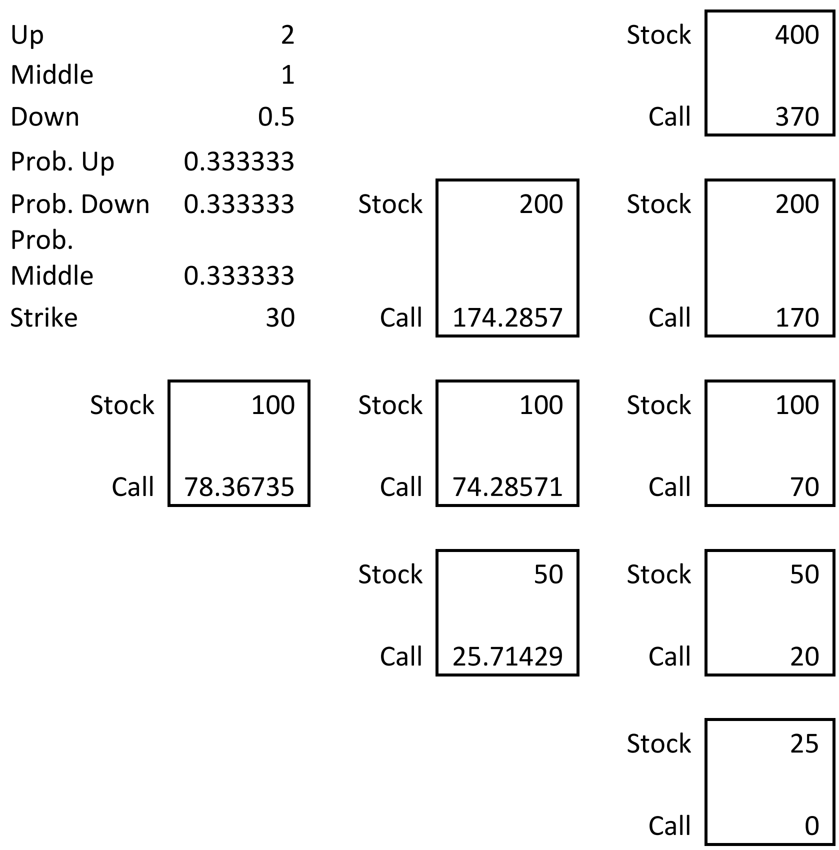


Figure 2

Figure 2 shows the trinomial tree and analogy prices in each node. Define $x = \frac{\Delta C}{\Delta S}$. However, x is not unique. At each node, it can take two possible values depending on whether the next period difference is taken between the top and the middle node or between the middle and the bottom node. Hence, it is useful to work with the expected value of x, denoted by $\bar{x}$. An interesting result is that the expected growth rate of the delta-hedging portfolio $S \cdot \bar{x} - C$ per period is equal to the expected growth rate of the underlying stock. To see this, note that $\bar{x} = 0.985714$ at time 0 when the stock price is 100. So, the value of delta hedging portfolio at that time is 20.20408. Depending on whether Up, Middle, or Down state is realized, the delta-hedging portfolio is worth 22.85714, 24.28571, or 23.57143. Hence, the expected value of the delta-hedging portfolio in the next period is 23.57143. That is, the expected growth rate is $\frac{23.57143}{20.20408} = 1.1666666$, which is equal to the expected return on the stock. The expected growth rate of the delta-hedging portfolio is equal to the expected growth rate of the stock throughout the trinomial tree.

4. Analogy Making: The Binomial Case

Consider a simple two state world as in a one period binomial model. The equally likely states are Red, and Blue. There is a stock with payoffs $X_1, and\ X_2$ corresponding to states Red, and Blue respectively. The state realization takes place at time T. The current time is time t. We denote the risk free discount rate by r. The current price of the stock is S. There is another asset, which is a European call option on the stock. By definition, the payoffs from the call option in the two states are:

$$C_1 = max\{(X_1 - K), 0\}, C_2 = max\{(X_2 - K), 0\} \quad (1)$$

Where K is the striking price, and $C_1, and\ C_2,$ are the payoffs from the call option corresponding to Red, and Blue states respectively.

As can be seen, the payoffs in the two states depend on the payoffs from the stock in corresponding states. Furthermore, by appropriately changing the striking price, the call option can be made more or less similar to the underlying stock with the similarity becoming exact as K approaches zero (all payoffs are constrained to be non-negative). Without loss of generality, we assume:

$X_1 - K > 0, and\ X_2 - K > 0.$ [4]

[4] There is no loss of generality involved as versions of all the propositions in this section are obtained if the call option is also allowed to expire out-of-the-money.

How much is an analogy maker willing to pay for this call option?

An analogy maker co-categorizes this call option with the underlying and values it in *transference* with the underlying stock. In other words, an analogy maker relies on the principle of analogy: *similar assets should offer the same return*. In contrast, a rational investor relies on the principle of no-arbitrage: *identical assets should offer the same return.*

We denote the return on an asset by $q \in Q$, where Q is some subset of $\Re$ (the set of real numbers). In calculating the return of the call option, an analogy maker faces two similar, but not identical, observable situations, $s \in \{0,1\}$. In $s = 0$, "return demanded on the call option" is the attribute of interest and in $s = 1$, "actual return available on the underlying stock" is the attribute of interest. The analogy maker has access to all the information described above. We denote this public information by I.

The actual expected return available on the underlying stock is given by,

$$E[q|I, s = 1] = \frac{(X_1 - S) + (X_2 - S)}{2S} \tag{2}$$

For the analogy maker, the expected return demanded on the call option is:

$$\begin{aligned} E[q|I, s = 0] &= E[q|I, s = 1] \\ &= \frac{\{X_1 - S\} + \{X_2 - S\}}{2 \times S} \end{aligned} \tag{3}$$

So, the analogy maker infers the price of the call option, P_c , from:

$$\frac{\{C_1 - P_c\} + \{C_2 - P_c\}}{2 \times P_c} = \frac{\{X_1 - S\} + \{X_2 - S\}}{2 \times S} \tag{4}$$

It follows,

$$P_c = \frac{C_1 + C_2}{X_1 + X_2} \times S$$

$$=> P_c = \left(1 - \frac{2K}{X_1 + X_2}\right) S \tag{5}$$

We know[5],

$$S = e^{-(r+\delta)(T-t)} \times \frac{X_1 + X_2}{2} \tag{6}$$

where δ is the risk premium on the underlying.

Substituting (6) into (5):

$$P_c = S - Ke^{-(r+\delta)(T-t)} \tag{7}$$

The above equation is the one period analogy option pricing formula for in-the-money binomial case.

The rational price P_r is (from the principle of no-arbitrage):

$$P_r = S - Ke^{-r(T-t)} \tag{8}$$

If limits to arbitrage prevent rational arbitrageurs from making riskless profits at the expense of analogy makers, both types will survive in the market. If α is the weight of rational investors in the market price and $(1-\alpha)$ is the weight of analogy makers, then one may choose to write:

$$P_c^M = \alpha\left(S - Ke^{-r(T-t)}\right) + (1-\alpha)\left(S - Ke^{-(r+\delta)(T-t)}\right) \tag{9}$$

Proposition 1 ***The price of a European call option in the presence of analogy makers*** **($\alpha < 1$)** ***is always larger than the price in the absence of analogy makers*** **($\alpha = 1$)** ***as long as the underlying stock price reflects a positive risk premium. Specifically, the difference between the two prices is*** $\boldsymbol{(1-\alpha)\left(Ke^{-r(T-t)} - Ke^{-(r+\delta)(T-t)}\right)}$ ***where*** $\boldsymbol{\delta}$ ***is the risk premium reflected in the price of the underlying stock.***

[5] Stock price can be written as a product of a discount factor and expected payoff if the underlying follows a binomial process as assumed here, or any other discrete approximation of geometric Brownian motion such as trinomial or multinomial, or the continuous geometric Brownian motion.

Proof.

Subtracting equation (8) from equation (9) yields the desired expression which is greater than zero as long as $\delta > 0$.

■

Corollary 1.1 ***If the European call option is out-of-the-money in one state, the analogy price is still larger than the rational price as long as the underlying reflects a positive risk premium.***

Proof. Without loss of generality assume that the European call option is out-of-the-money in the blue state. By a similar logic to the earlier case, the analogy price follows:

$$Call = S \cdot \frac{X_1}{X_1 + X_2} - \frac{1}{2} K e^{-(r+\delta)(T-t)}$$

Comparing the above with the corresponding rational price leads to the stated result. ■

Corollary 1.2 ***If there is more than one binomial period, then using backward induction and applying the relevant analogy formula at each node leads to the price of the European call option.***

Proposition 2 shows the condition under which rational arbitrageurs cannot make arbitrage profits at the expense of analogy makers. Consequently, both types may co-exist in the market.

Proposition 2 ***Analogy makers cannot be arbitraged out of the market if***

$\boldsymbol{(1-\alpha)\left\{Ke^{-r(T-t)} - Ke^{-(r+\delta)(T-t)}\right\}} < c$ ***where c is the transaction cost involved in the arbitrage scheme and δ>0***

Proof.

The presence of analogy makers increases the price of an in-the-money call option beyond its rational price. A rational arbitrageur interested in profiting from this situation should do the following: Write a call option and create a replicating portfolio. If there are no transaction costs involved then he would pocket the difference between the rational price and the market price without creating any liability for him when the option expires. As proposition 1 shows, the difference is $(1-\alpha)\{\mathrm{K}e^{-r(T-t)} - \mathrm{K}e^{-(r+\delta)(T-t)}\}$. However, if there are transactions costs involved then he would follow the strategy only if the benefit is greater than the cost. Otherwise, arbitrage profits cannot be made.

■

In the binomial case, analogy makers overprice a call option (if the risk premium on the underlying is positive). When such overpriced calls are added to portfolios then the dynamics of such portfolios would be different from the dynamics without overpricing. Proposition 3 considers the case of covered call writing and shows that the two portfolios grow with different rates with time.

Proposition 3 ***If analogy makers set the price of an in-the-money European call option then the covered call writing position (long stock+short call) grows in value at the rate of $r+\delta$. If rational investors set the price of an in-the-money European call option then the covered call writing position grows in value at the risk free rate r.***

Proof.

Re-arranging equation (8):

$$Ke^{-r(T-t)} = S - P_r$$

The right hand side of the above equation is the covered call writing position with rational pricing. Hence, it follows that the covered call portfolio grows in value at the risk free rate with time if investors are rational.

For analogy makers, re-arrange equation (7):

$$Ke^{-(r+\delta)(T-t)} = S - P_c$$

The right hand side of the above equation is the covered call writing position when analogy makers price the call option. As the left hand side shows, this portfolio grows at a rate of $r + \delta$ with time.

■

Corollary 3.1 ***Proposition 3 extends to the case when the portfolio is*** $\boldsymbol{S \cdot \frac{\Delta C}{\Delta S} - C}$ ***instead of covered call writing where C is the price of a European call option.***

Proof. Follows from realizing that for in-the-money binomial case, $\frac{\Delta C}{\Delta S} = 1$.

Corollary 3.2 ***Suppose the call option is out of the money in one state. The delta hedging portfolio*** $\boldsymbol{S \cdot \frac{\Delta C}{\Delta S} - C}$ ***grows at the rate of*** $\boldsymbol{r + \delta}$ ***in the analogy case whereas it grows at the risk free rate*** $\boldsymbol{r}$ ***in the no-arbitrage case. C denotes the price of a European call option.***

Proof. Analogy Case: First consider the case where the option is out of the money in the blue state. It follows that $\frac{\Delta C}{\Delta S} = \frac{X_1 - K}{X_1 - X_2}$. So, $S \cdot \frac{\Delta C}{\Delta S} - C$ can be simplified to $e^{-(r+\delta)(T-t)} \left\{ \frac{X_1 X_2 - K\frac{(X_1 + X_2)}{2}}{(X_1 - X_2)} + \frac{K}{2} \right\}$.

Hence, it grows at the rate $r + \delta$ with time. If the option is out-of-the-money in the red state, then $\frac{\Delta C}{\Delta S} = \frac{X_2 - K}{X_2 - X_1}$, so $S \cdot \frac{\Delta C}{\Delta S} - C$ can be simplified to $e^{-(r+\delta)(T-t)} \left\{ \frac{X_1 X_2 - K\frac{(X_1 + X_2)}{2}}{(X_2 - X_1)} + \frac{K}{2} \right\}$.

If there are multiple binomial periods, it is easy to verify that the rate of growth is still $r + \delta$.

No-arbitrage Case: If the option is out-of-the-money in the blue state, then $\frac{\Delta C}{\Delta S} = \frac{X_1 - K}{X_1 - X_2}$. It follows that $S \cdot \frac{\Delta C}{\Delta S} - C = X_2 \frac{X_1 - K}{X_1 - X_2} e^{-(r)(T-t)}$ which grows at the rate r. Similarly, if the option is out-of-the money in the red state then $\frac{\Delta C}{\Delta S} = \frac{X_2 - K}{X_2 - X_1}$. Hence, $S \cdot \frac{\Delta C}{\Delta S} - C = X_1 \frac{X_2 - K}{X_2 - X_1} e^{-(r)(T-t)}$ which grows at the rate r.■

Proposition 3 and its corollaries show that the portfolio $\mathrm{S} \cdot \frac{\Delta \mathrm{C}}{\Delta \mathrm{S}} - \mathrm{C}$ grows at different rates under analogy vs. no arbitrage pricing. This result provides the intuitive foundation of the analogy option pricing formula when the underlying follows geometric Brownian motion.

5. Analogy Making: The Trinomial Case

A simple example of an incomplete market is a one period trinomial model with three states and two assets. In such a market, in general, it is not possible to replicate a given claim by using existing assets. The equally likely states are Red, Blue, and Green. There is a stock with payoffs $X_1, X_2, and\ X_3$ corresponding to states Red, Blue, and Green respectively. The state realization takes place at time T. The current time is time t. The second asset is a risk free asset with the risk free rate of return given by r. The current price of the stock is S.

Suppose, a new asset is introduced, which is a European call option on the stock. By definition, the payoffs from the call option in the three states are:

$$C_1 = max\{(X_1 - K), 0\}, C_2 = max\{(X_2 - K), 0\},$$

$$and\ C_3 = max\{(X_3 - K), 0\} \tag{10}$$

Where K is the striking price, and $C_1, C_2, and\ C_3$ are the payoffs from the call option corresponding to Red, Blue, and Green states respectively. To ensure that this claim is non-replicable, we need one state payoff from the call option to be 0. Without loss of generality, we assume the following:

$$C_1 = X_1 - K, C_2 = X_2 - K, and\ C_3 = 0$$

Proposition 4 ***If analogy makers set the price of the call option, it is given by***

$$\boldsymbol{C = S \cdot \frac{X_1+X_2}{X_1+X_2+X_3} - \frac{2}{3} K e^{-(r+\delta)(T-t)}}$$

Proof.

Follows from equating the expected return demanded on a call option with the expected return available on the underlying. ■

Corollary 4.1 ***If there is more than one trinomial period, the analogy price is calculated by using backward induction and applying the relevant analogy formula at each node.***

The incomplete market case does not have counterparts to proposition 1 and 2 of the binomial case discussed in the last section, as there is no unique rational price in the trinomial case. However, counterpart to proposition 3 exists.

Proposition 5 shows the rate at which the portfolio $\mathrm{S} \cdot \mathrm{E}\left[\frac{\Delta C}{\Delta S}\right] - \mathrm{C}$ grows under analogy making.

Proposition 5 ***If analogy makers set the price of a call option then the expected growth rate of the portfolio $S \cdot E\left[\frac{\Delta C}{\Delta S}\right] - C$ is $r + \delta$ where C is the price of the call option.***

Proof. Note that $E\left[\frac{\Delta C}{\Delta S}\right] = \left\{0.5 + 0.5 \cdot \frac{X_2 - K}{X_2 - X_3}\right\}$. Plugging-in this expression in $S \cdot E\left[\frac{\Delta C}{\Delta S}\right] - C$ and simplifying leads to the desired result■

Proposition 5 is important because it provides an intuitive foundation to the analogy formula in the incomplete market case. Both the binomial process considered earlier and the trinomial process considered here, converge to the geometric Brownian motion in the limit of small time steps. As the rate of growth of the delta hedging portfolio in the binomial case is equal to the expected growth rate of delta hedging portfolio in the trinomial case, one can see the intuition behind the analogy formula taking the same form irrespective of market completeness.

It is pertinent to ask under what conditions the analogy formula picks out a price from within the arbitrage-free interval. Proposition 6 provides an answer for the trinomial case considered in this section.

Proposition 6 ***Suppose the risk premium on the underlying is such that the following holds $-ln\left(\frac{3K}{X_3+2K}\right) \leq \delta(T-t) \leq ln\left(\frac{X_3+2K}{3X_3}\right)$. It follows that the analogy price will be in the arbitrage-free interval.***

Proof. Starting from $S - Ke^{-(r+\delta)(T-t)} \leq S\left(\frac{X_1+X_2}{X_1+X_2+X_3}\right) - \frac{2K}{3}e^{-(r+\delta)(T-t)} \leq S - X_3 e^{-(r+\delta)(T-t)}$ and simplifying leads to the desired expression. ■

It is clear from proposition 6 that for reasonable parameter values, the analogy price is within the arbitrage-free bounds.

6. Analogy Making: The General Discrete Case with an Incomplete Market

Consider a one period multinomial model. Suppose there are only two assets in a world with N states. As before the assets are a stock and a risk free bond. The stock (S) pays a state dependent payoff X_i, where i is the index for state that ranges from 1 to N. The respective state probabilities are π_i. The current time is t. Next period is T. The risk free asset (B) pays $Be^{r(T-t)}$in the next period regardless of which state is realized. Suppose a European call option (C) on S is introduced with a strike price of K. The option is in-the-money in some states and out-of-the-money in others. Proposition 7 gives the price of the option:

***Proposition* 7 If analogy makers set the price, then the price of the call option is given by:**

$$\boldsymbol{C = S\frac{\sum_{X_i>K}\pi_i X_i}{\sum_{i=1}^{N}\pi_i X_i} - \sum_{X_i>K}\pi_i K e^{-(r+\delta)(T-t)}}$$

Proof. By induction

■

Corollary 7.1 ***If there is more than one multinomial period, the analogy price is obtained by using backward induction and applying the relevant analogy formula at each node.***

Proposition 8 shows the rate of growth of the delta hedging portfolio.

Proposition 8 ***If analogy makers set the price of the call option then the expected growth rate of the portfolio*** $\boldsymbol{S\cdot E\left\{\frac{\Delta C}{\Delta S}\right\} - C}$ ***is*** $\boldsymbol{r+\delta}$ ***where C is the price of the call option.***

Unsurprisingly, the delta hedging portfolio grows at the expected rate of $r + \delta$, just like we saw in the trinomial case. Whichever of the three processes (binomial, trinomial, or multinomial) we assume, it converges to the geometric Brownian motion in the limit of large number of periods and small time steps (the continuous case).

Proposition 9 ***Suppose the lowest value the underlying can take in the next multinomial period is X_L and the corresponding state probability is π_L. The analogy price is in the arbitrage-free interval if***

$$-\ln\left\{\frac{K}{\sum_{X_i \geq K} \pi_i K + \sum_{X_i < K} \pi_i X_i}\right\} \leq \delta(T-t) \leq \ln\left\{\frac{\sum_{X_i \geq K} \pi_i K + \sum_{X_i < K} \pi_i X_i}{X_L}\right\}$$

Proof

The arbitrage-free interval is:

$$S - Ke^{-r(T-t)} \leq S\left(\frac{\sum_{X_i > K} X_i \pi_i}{\sum_{i=1}^{N} \pi_i X_i}\right) - \sum_{X_i > K} \pi_i K e^{-(r+\delta)(T-t)} \leq S - X_L e^{-r(T-t)}$$

Re-arranging the above and simplifying leads to the desired result.

■

It is clear from proposition 9 that the analogy price is with-in the arbitrage-free interval for reasonable parameter values. In the next section, we consider the continuous case, and derive the analogy option pricing formula.

7. The Option Pricing Formula

In this section, we derive a new option pricing formula by allowing the underlying to follow a geometric Brownian motion instead of the discrete processes assumed earlier (binomial, trinomial, or N state discrete incomplete market). It is well known that the Brownian motion is the limiting case of these processes. By exploring the implications of analogy making in the discrete processes, the previous sections develop intuition which carries over to the continuous case discussed here. Two key results from the previous sections stand out in this respect. Firstly, a new parameter, δ, which is the risk premium on the underlying stock, appears in the analogy price with the analogy price being larger than the rational price as long as the risk premium is positive in the binomial case. Of course, there is no unique rational price in the incomplete market cases and the analogy price is with-in the arbitrage-free interval for reasonable parameter values. Secondly, in the three discrete cases, the (expected) rate of growth of the delta hedging portfolio ($S \cdot \frac{\Delta C}{\Delta S} - C$) is equal to $r + \delta$. Specifically, in the binomial case, the rate of growth in every state is $r + \delta$ (that is, the state-wise growth as well as the expected growth rate is equal to $r + \delta$), whereas in the other two discrete cases, the expected rate of growth is $r + \delta$.

For clarity, we list all the assumptions applicable to the continuous case below:

1) The underlying follows constant coefficient geometric Brownian motion

2) The risk free rate of borrowing and lending is r

3) There are no dividends.

4) Assets are infinitely divisible

5) There are no transaction costs

6) There are no taxes

7) All options are European style

Note that we do not assume that options are perfectly replicable by some combination of the underlying and the risk-free asset. That is, we do not take a position regarding market completeness.

Proposition 10 ***If analogy makers set the price of a European call option, the analogy option pricing partial differential Equation (PDE) is***

$$(r + \delta)C = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}(r + \delta)S + \frac{\partial^2 C}{\partial S^2}\frac{\sigma^2 S^2}{2}$$

Proof.

See Appendix A ■

The analogy option pricing PDE can be solved by transforming it into the heat equation. Proposition 11 shows the resulting call option pricing formula for European options.

Proposition 11 ***The formula for the price of a European call is obtained by solving the analogy based PDE. The formula is*** $\boldsymbol{C = SN(d_1) - Ke^{-(r+\delta)(T-t)}N(d_2)}$ ***where*** $\boldsymbol{d_1 = \frac{ln(S/K)+(r+\delta+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}}$ ***and*** $\boldsymbol{d_2 = \frac{ln\left(\frac{S}{K}\right)+\left(r+\delta-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}$

Proof.

See Appendix B.

■

Corollary 11.1 ***The formula for the analogy based price of a European put option is*** $\boldsymbol{Ke^{-(r+\delta)(T-t)}N(-d_2) - SN(-d_1)}$

Proof. Follows from put-call parity.

■

Note that put-call parity does not require a complete market. That is, corresponding European call and put options, even if not perfectly replicable with the underlying and the risk-free asset, should satisfy put-call parity.

The analogy option pricing formula is different from the Black-Scholes formula due to the appearance of risk premium on the underlying in the analogy formula. It suggests that the risk premium on the underlying stock does matter for option pricing. The analogy formula is derived by keeping all the assumptions behind the Black-Scholes formula except one: in the case of complete markets, the assumption of no-arbitrage pricing of call in replication with the underlying and the risk-free asset is dropped, and in the case of incomplete markets, of course, the assumption that a replicating portfolio exists which perfectly replicates a call option is dropped.

8. The Implied Volatility Skew

All the variables in the Black Scholes formula are directly observable except for the standard deviation of the underlying's returns. So, by plugging in the values of observables, the value of standard deviation can be inferred from market prices. This is called implied volatility. If the Black Scholes formula is correct, then the implied volatility values from options that are equivalent except for the strike prices should be equal. However, in practice, for equity index options, a skew is observed in which in-the-money call options' (out-of-the money puts) implied volatilities are higher than the implied volatilities from at-the-money and out-of-the-money call options (in-the-money puts).

The analogy approach developed here provides an explanation for the skew. If the analogy formula is correct, and the Black Scholes model is used to infer implied volatility then skew arises as table 1 shows.

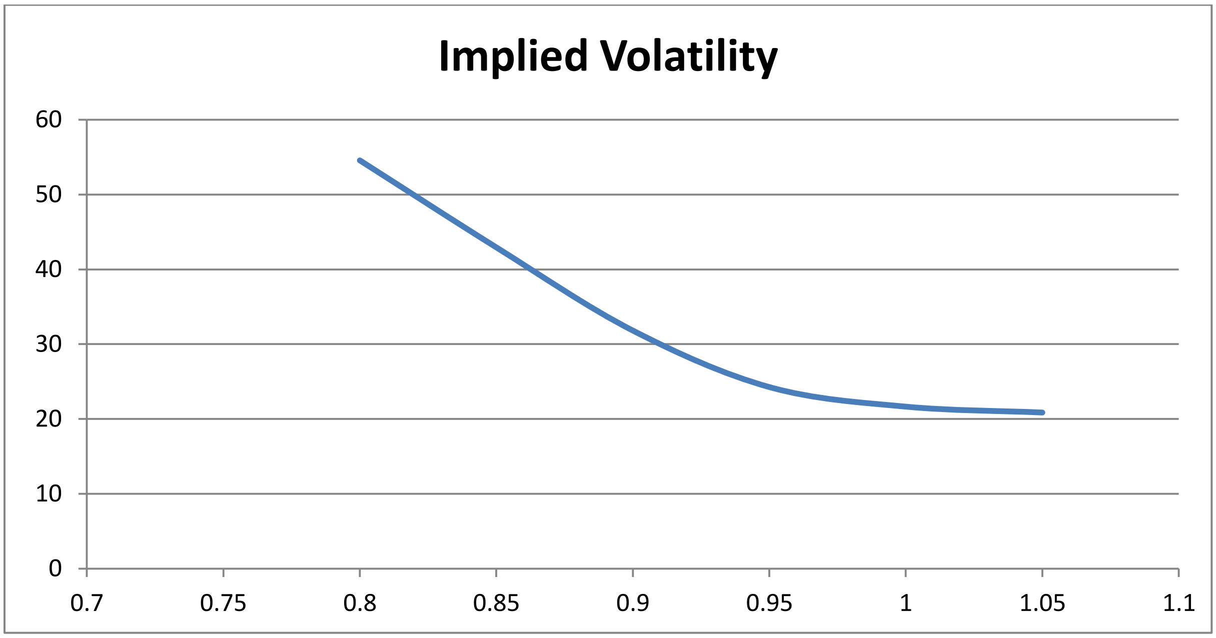
Table 1

Implied Volatility Skew

Underlying's Price=100, Volatility=20%, Risk Premium on the Underlying=5%, Time to Expiry=0.06 year

K	**Black Scholes Price**	**Analogy Price**	**Difference**	**Implied Volatility**
105	0.5072	0.5672	0.06	20.87
100	2.160753	2.326171	0.165417	21.6570
95	5.644475	5.901344	0.25687	24.2740
90	10.30903	10.58699	0.277961	31.8250
85	15.26798	15.53439	0.266419	42.9400
80	20.25166	20.50253	0.250866	54.5700

As table 1 shows, implied volatility skew is seen if the analogy formula is correct, and the Black Scholes formula is used to infer implied volatility. Notice that in the example considered, difference between the Black Scholes price and the analogy price is quite small even when implied volatility gets more than double the value of actual volatility.



K/S

Figure 1

Figure 1 is the graphical illustration of table 1. It is striking to observe from table 1 and figure 1 that the implied volatility skew is quite steep even when the price difference between the Black Scholes price and the analogy price is small. In the next section, we outline a number of key empirical predictions that follow from the analogy making model.

9. Key Predictions of the Analogy Model

Prediction#1 ***After controlling for the underlying asset's total volatility, a higher amount of risk premium on the underlying leads to a higher level of implied volatility and a steeper slope of the implied volatility curve.***

Risk premium on the underlying plays a key role in analogy option pricing formula. Figure 2 illustrates this. In the figure, implied volatility skews for two different values of risk premia are plotted. Other parameters are the same as in table 1.

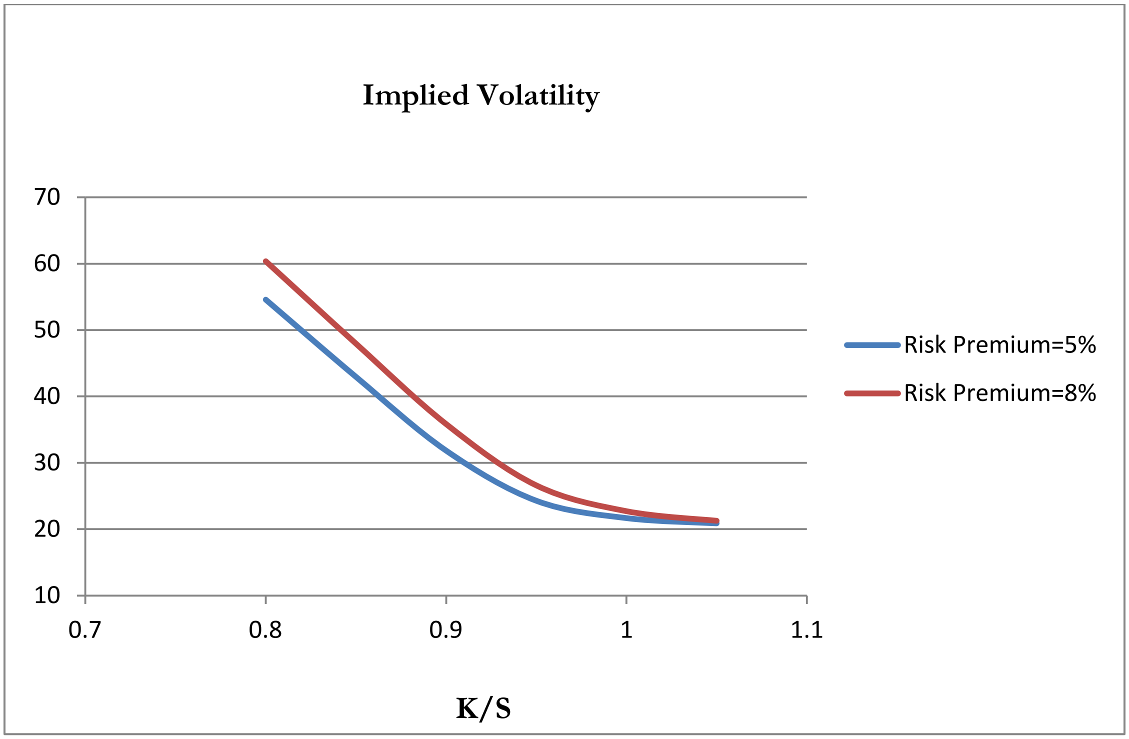


Figure 2

Duan and Wei (2009) use daily option quotes on the S&P 100 index and its 30 largest component stocks, to show that, after controlling for the underlying asset's total risk, a higher amount of systematic risk leads to a higher level of implied volatility and a steeper slope of the implied volatility curve. As risk premium is related to systematic risk, the prediction of the analogy model is quite intriguing.

Prediction#2 ***Implied volatility should typically be higher than realized/historical volatility***

It follows directly from the analogy formula that as long as the risk premium on the underlying is positive, implied volatility should be higher than actual volatility. Anecdotal evidence is strongly in favor of this prediction. Rennison and Pederson (2012) calculate implied volatilities from at-the-money options in 14 different options markets over a period ranging from 1994 to 2012. They show that implied volatilities are typically higher than realized volatilities.

Prediction#3 ***Implied volatility curve should flatten out with expiry***

Figure 3 plots implied volatility curves for two different expiries. All other parameters are the same as in table 1. It is clear from the figure that as expiry increases, the implied volatility curve flattens out.

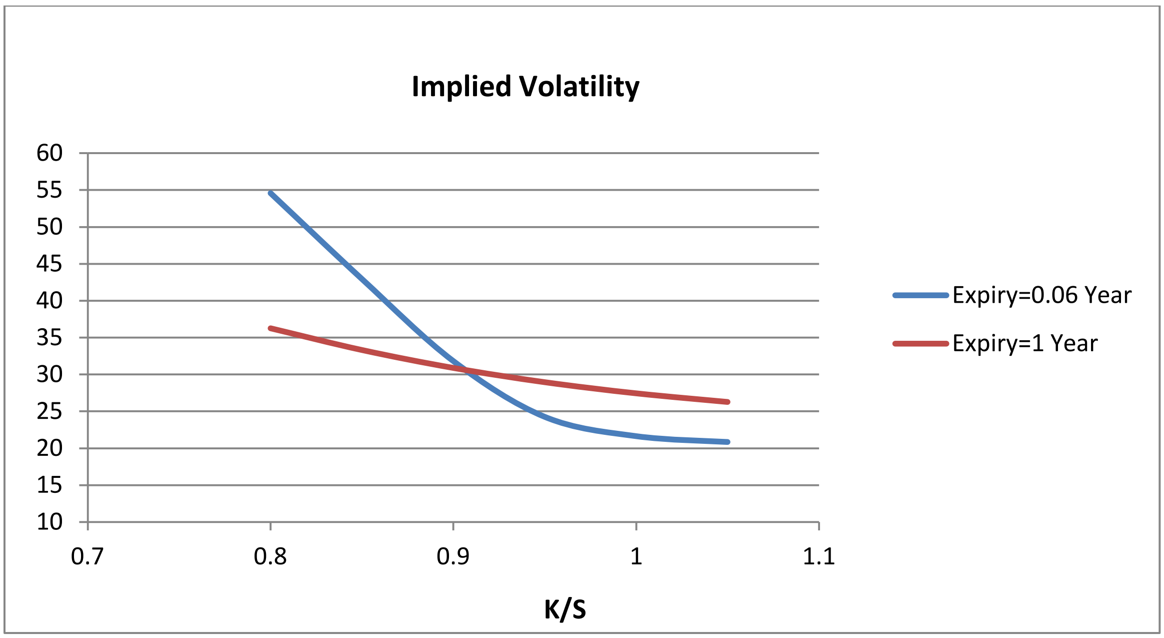


Figure 3

Empirically, implied volatility curve typically flattens out with expiry (see Greiner (2013) as one example). Hence, this match between a key prediction of the analogy model and empirical evidence is quite intriguing.

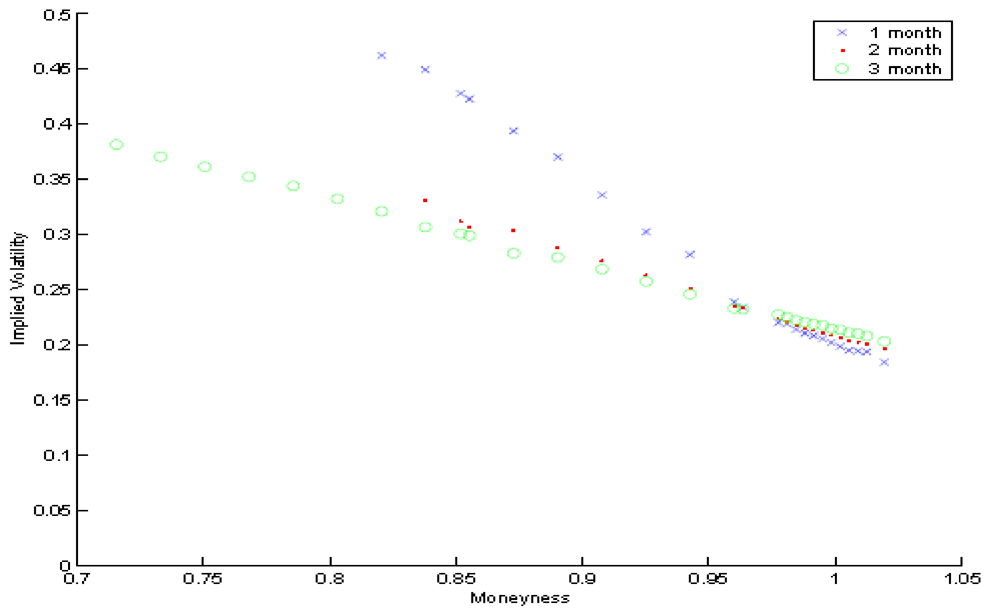


Figure 4 *Implied volatility as a function of moneyness on January 12, 2000, for options with at least two days and at most three months to expiry.*

As an illustration of the fact that implied volatility curve flattens with expiry, figure 4 is a reproduction of a chart from Fouque, Papanicolaou, Sircar, and Solna (2004) (figure 2 from their paper). It plots implied volatilities from options with at least two days and at most three months to expiry. The flattening is clearly seen.

10. Conclusions

Even though this paper only considers the case of a call and its underlying asset, it is interesting to note that the idea of analogy making is potentially extendable to a general class of assets. In this regard, the following two approaches may be taken. Firstly, any equity claim can be considered a call option on the underlying firm's assets with the face value of debt as the striking price. This line of inquiry may open up new ways of exploring the relationship between the economic decisions by a firm and their impact on share prices. It is not hard to see that decisions that would matter in one way without similarity based co-categorizations may impact the share prices differently with similarity based co-categorizations. Secondly, similarity based reasoning, when extended to a general class of assets, typically, either leads to an underestimation or overestimation of risk. Exploring the consequences of such misperceptions for investor behavior is another interesting line on inquiry.

There is also an interesting link between research in the growing area of unawareness (agents are unaware of the full state space) and the principle of analogy making. Analogy making is an inductive principle and the intuitive appeal of inductive reasoning when faced with unawareness is undeniable. Exploration of this connection is the subject of future research.

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Appendix A

In the binomial analogy case, the portfolio $S\frac{\partial C}{\partial S} - C$ grows at the rate $r + \delta$. Divide $[0, T - t]$ in n time periods, and with $n \to \infty$, the binomial process converges to the geometric Brownian motion. To deduce the analogy based PDE consider:

$$V = S\frac{\partial C}{\partial S} - C$$

$$\Rightarrow dV = dS\frac{\partial C}{\partial S} - dC$$

Where $dS = uSdt + \sigma SdW$ and by Ito's Lemma $dC = \left(uS\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt + \sigma S\frac{\partial C}{\partial S}dW$

$$\Rightarrow (r + \delta)Vdt = (uSdt + \sigma SdW)\frac{\partial C}{\partial S} - \left(uS\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt - \sigma S\frac{\partial C}{\partial S}dW$$

$$(r + \delta)Vdt = -\left(\frac{\partial C}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt$$

$$\Rightarrow (r + \delta)\left(S\frac{\partial C}{\partial S} - C\right) = -\left(\frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right)$$

$$\Rightarrow (r + \delta)C = (r + \delta)S\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} \qquad (A1)$$

The above is the analogy based PDE.

In the trinomial and multinomial analogy cases, the expected growth rate of the delta-hedging portfolio is $r + \delta$. Divide $[0, T - t]$ in n time periods, and with $n \to \infty$, the trinomial and and multinomial processes converge to the geometric Brownian motion. To deduce the analogy based PDE consider:

$$E[dV] = E[dS]\frac{\partial C}{\partial S} - E[dC]$$

$$=> (r + \delta)Vdt = -\left(\frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt$$

$$\Rightarrow (r+\delta)\left(S\frac{\partial C}{\partial S} - C\right) = -\left(\frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right)$$

$$=> (r+\delta)C = (r+\delta)S\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}$$

The above equation is identical to A1.

Appendix B

The analogy based PDE derived in Appendix A can be solved by converting to heat equation and exploiting its solution.

Start by making the following transformation:

$$\tau = \frac{\sigma^2}{2}(T-t)$$

$$x = ln\frac{S}{K} => S = Ke^x$$

$$C(S,t) = K \cdot\ c(x,\tau) = K \cdot c\left(ln\left(\frac{S}{K}\right), \frac{\sigma^2}{2}(T-t)\right)$$

It follows,

$$\frac{\partial C}{\partial t} = K \cdot \frac{\partial c}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = K \cdot \frac{\partial c}{\partial \tau} \cdot \left(-\frac{\sigma^2}{2}\right)$$

$$\frac{\partial C}{\partial S} = K \cdot \frac{\partial c}{\partial x} \cdot \frac{\partial x}{\partial S} = K \cdot \frac{\partial c}{\partial x} \cdot \frac{1}{S}$$

$$\frac{\partial^2 C}{\partial S^2} = K \cdot \frac{1}{S^2} \cdot \frac{\partial^2 C}{\partial x^2} - \ K \cdot \frac{1}{S^2}\frac{\partial C}{\partial x}$$

Plugging the above transformations into (A1) and writing $\tilde{r} = \frac{2(r+\delta)}{\sigma^2}$, we get:

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial x^2} + (\tilde{r}-1)\frac{\partial c}{\partial x} - \tilde{r}c \qquad (B1)$$

With the boundary condition/initial condition:

$$C(S,T) = max\{S-K,0\}\ becomes\ c(x,0) = max\{e^x - 1,0\}$$

To eliminate the last two terms in (B1), an additional transformation is made:

$$c(x,\tau) = e^{\alpha x+\beta\tau}u(x,\tau)$$

It follows,

$$\frac{\partial c}{\partial x} = \alpha e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 c}{\partial x^2} = \alpha^2 e^{\alpha x+\beta\tau}u + 2\alpha e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} + e^{\alpha x+\beta\tau}\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial c}{\partial \tau} = \beta e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial \tau}$$

Substituting the above transformations in (B1), we get:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (\alpha^2 + \alpha(\tilde{r}-1) - \tilde{r} - \beta)u + \big(2\alpha + (\tilde{r}-1)\big)\frac{\partial u}{\partial x} \qquad (B2)$$

Choose $\alpha = -\frac{(\tilde{r}-1)}{2}$ and $\beta = -\frac{(\tilde{r}+1)^2}{4}$. (B2) simplifies to the Heat equation:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad (B3)$$

With the initial condition:

$$u(x_0,0) = max\{\left(e^{(1-\alpha)x_0} - e^{-\alpha x_0}\right),0\} = max\left\{\left(e^{\left(\frac{\tilde{r}+1}{2}\right)x_0} - e^{\left(\frac{\tilde{r}-1}{2}\right)x_0}\right),0\right\}$$

The solution to the Heat equation in our case is:

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{4\tau}}u(x_0,0)dx_0$$

Change variables: $= \frac{x_0 - x}{\sqrt{2\tau}}$, which means: $dz = \frac{dx_0}{\sqrt{2\tau}}$. Also, from the boundary condition, we know that $u > 0 \; iff \; x_0 > 0$. Hence, we can restrict the integration range to $z > \frac{-x}{\sqrt{2\tau}}$

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\pi}}}^{\infty} e^{-\frac{z^2}{2}} \cdot e^{\left(\frac{\tilde{r}+1}{2}\right)(x+z\sqrt{2\tau})} dz - \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{z^2}{2}} \cdot e^{\left(\frac{\tilde{r}-1}{2}\right)(x+z\sqrt{2\tau})} dz$$

$$=: H_1 - H_2$$

Complete the squares for the exponent in H_1:

$$\frac{\tilde{r}+1}{2}\left(x + z\sqrt{2\tau}\right) - \frac{z^2}{2} = -\frac{1}{2}\left(z - \frac{\sqrt{2\tau}(\tilde{r}+1)}{2}\right)^2 + \frac{\tilde{r}+1}{2}x + \tau\frac{(\tilde{r}+1)^2}{4}$$

$$=: -\frac{1}{2}y^2 + c$$

We can see that $dy = dz$ and c does not depend on z. Hence, we can write:

$$H_1 = \frac{e^c}{\sqrt{2\pi}} \int_{-x/\sqrt{2\pi} - \sqrt{\tau/2}(\tilde{r}+1)}^{\infty} e^{-\frac{y^2}{2}} dy$$

A normally distributed random variable has the following cumulative distribution function:

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{y^2}{2}} dy$$

Hence, $H_1 = e^c N(d_1)$ where $d_1 = {}^{x}/_{\sqrt{2\pi}} + \sqrt{{}^{\tau}/_{2}}\,(\tilde{r}+1)$

Similarly, $H_2 = e^f N(d_2)$ where $d_2 = {}^{x}/_{\sqrt{2\pi}} + \sqrt{{}^{\tau}/_{2}}\,(\tilde{r}-1)$ and $f = \frac{\tilde{r}-1}{2}x + \tau\frac{(\tilde{r}-1)^2}{4}$

The analogy based European call pricing formula is obtained by recovering original variables:

$Call = SN(d_1) - Ke^{-(r+\delta)(T-t)}N(d_2)$

Where$\boldsymbol{d_1} = \frac{\boldsymbol{ln(S/K)+(r+\delta+\frac{\sigma^2}{2})(T-t)}}{\boldsymbol{\sigma\sqrt{T-t}}}$ and $\boldsymbol{d_2} = \frac{\boldsymbol{ln\left(\frac{S}{K}\right)+\left(r+\delta-\frac{\sigma^2}{2}\right)(T-t)}}{\boldsymbol{\sigma\sqrt{T-t}}}$