The Problem of Points

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25 March 2011

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According to Katz [6] some of the basic notions of probability existed in ancient civilizations. In The Talmud and in Roman calculations of annuities there is some evidence of this. However, no record of numerical probability calculations exists.

Hald [5], Bernstein [1] and Katz [6] agree that this numerical development was intimately linked to the study of gambling, contracts and profits. These authors also coincide in that the formulation of **"The Problem of Points"** is a crucial event.

The paper is centered on evaluating and explaining the history of the formulation of **"The Problem of Points"**. The solution to this problem originated the deductive notions of probability [5]. I will center my attention on the formulation and attempt of solution by Pacioli [4], Cardano [2], Tartaglia [14] and Forestani [4]. In this process Cardano began to unveil some principles that are coherent with a modern theory of probability.

Pacioli's Formulation and Attempted Solution of the Problem

Let us begin by taking a look at the original formulation in two problems by Pacioli [8] of **"The Problem of Points"** (or division of stakes).The translation to English of this text is from [9]. The text in smaller font and inside brackets corresponds to the explanation in modern terms of the original texts.

The following two problems are found in [8, F. 197 R. AND 198 V.]

Problem 1

> A company [two players] plays a ball game to 60 [until one of the players has reached 60 points] and each goal is 10 [the winner has to make six goals]. They stake 10 ducats in all [if they finish the game, the winner gets 10 ducats and the loser 0]. It happens by certain incidents [something unexpected not under the control of the players or caused by an external force] that they are not able to finish; and one party [player] has 50 [points] and the other 20 [points] One asks what portion of the stake is due to each party. [The game did not finish. The 10 ducats must be divided. It is interesting to see that there is a quest for a relationship between the points obtained by each player (up to the point when the game suddenly stops), a criterion of fairness, and the mathematical notion of dividing the stakes. The latter seems to be the 'rational' approach to the problem and hence the fair way. The problem is how to evaluate (measure) the outcomes of the game up to the mentioned point so we can relate it with the stakes to divide under a rational (fair) criteria and taking into account that there is no way to know how the game was going to end had not been interrupted. The problem is how to replace that lack of knowledge.]. For this problem I have found different opinions, going on the one direction to the other; all appear to me incoherent in their arguments. But the truth is what I will say, together with the right way [A bold assertion].
>
> I say then pursue this in 3 ways.
>
> [First way]

First consider how many goals at most are able to be made between the one and the other party [this is the maximum hypothetical score achievable by the two players taken together]; this will be 11, that is, when they are at 50 p(oints) each [The number 11 corresponds to 5 goals (50 points) made by each player, and the final winning goal made by the winner, who then has 60 points and wins]. Now you see what this part with 50 has of all these goals; this gives $\frac{5}{11}$ [5 is the total of goals achieved by the player that reaches 50 points when the game is interrupted and 11 is the maximum number of goals obtainable by both players together]; and 20 gives $\frac{2}{11}$. Therefore from this [observation] one p(ar)ty must take $\frac{5}{11}$ p(arts) and the other party $\frac{2}{11}$ p(arts). The sum makes $\frac{7}{11}$ [This is a measure of the portion of the game that has been played (7 goals) in relation to the maximum (11 goals)]. Then $\frac{7}{11}$ is worth 10 [ducats, because that is what is at stake and is what needs to be divided between the players]. What is due with $\frac{5}{11}$ [5 of 11 possible goals] and what with $\frac{2}{11}$ [2 of 11 possible goals]? Thus to the one with 50 [to the one who scored 5 goals] will come $7\frac{1}{7}$ and $2\frac{6}{7}$ to [the one with] 20 [Since "$\frac{7}{11}$ is worth 10 (ducats)", note that $\frac{5}{11}=\frac{5}{7}\cdot\frac{7}{11}$, so one player takes $\frac{5}{7}\cdot 10\ ducats$=$7\frac{1}{7}\ ducats$. Likewise, since $\frac{2}{11}=\frac{2}{7}\cdot\frac{7}{11}$, the other player takes $\frac{2}{7}\cdot 10\ ducats = 2\frac{6}{7}$ducats. A different way to look at it would be to notice that Pacioli defines the total as the sum of the goals achieved upon unexpected termination (Partial number of goals) 5+2=7. Then he finds the share that each of those scores has in that total: $\frac{5}{7}\ and\ \frac{2}{7}$. That is the way in which 1 ducat would be divided. $\frac{7}{11}$ is to the whole 10 ducats like $\frac{5}{11}$ is to the part x. Hence $\frac{7}{11}x=\frac{5}{11}\cdot 10$.This gives $x=\frac{50}{7}=7\frac{1}{7}$ ducats . Following the same procedure, $\frac{7}{11}$is to the whole 10 ducats like $\frac{2}{11}$ is to the part y. Hence $\frac{7}{11}y=\frac{2}{11}\cdot 10$.This gives $y=\frac{20}{7}=2\frac{6}{7}\ ducats$]. Finished.

[Second way]

Another way is similar: That is, in all they are able to make 110 [Where 110 points is 11 possible goals for the two players added together multiplied by 10 which is the number of points for each goal.]. See what part of this is 50 [divide 50 by 110]; you will find as above, $\frac{5}{11}$ and thus 20 will be $\frac{2}{11}$ [50 points is $\frac{5}{11}$ of 110 and 20 points is $\frac{2}{11}$ of 110]. And do further as above.

[Third way]

The third is a very short way, that you add what both parties have together: that is, 50 and 20, makes 70. And this is the divisor, by which 70 is worth 10[ducats]. What is due with 50 and what with 20? Etc [50 is $\frac{50}{70}$=$\frac{5}{7}$of the whole, meaning $\frac{5}{7}\cdot(10\ ducats)=7\frac{1}{7}ducats$. And, 20 is $\frac{20}{70}=\frac{2}{7}$of the whole, meaning $\frac{2}{7}\cdot(10\ ducats)=2\frac{6}{7}\ ducats$]

Problem 2:

Three [players] compete at crossbow; who first makes 6 best [goals] wins. They stake altogether 10 ducats. When the first has 4 goals, the second 3, and the third 2 [a total of 9 goals], they intend to continue no longer and agree to share the stake [In this case the termination of the game is agreed rather than caused by external factors]. I ask, to how much is each entitled?

Do thus: First of all look how many best goals at most all three together are able to make. You will find that they are able to make no more than 16; because it could be that all three each have 5 goals and one next will be made in order to have 6 which wins the stake. Thus they make at most 16 [best goals]; of these 16 the first has 4, which makes $\frac{1}{4}$ [$=\frac{4}{16}$]; thus he has $\frac{1}{4}$ of the stake: that is of 10 Denari [ducats; this seems to be an error.], which is $2\frac{1}{2}$[$\frac{1}{4}\cdot(10\ ducats)=\frac{10}{4}=2\frac{1}{2}$ ducats.This approach[1] is different from the one taken in the first problem although numerically equivalent (see Appendix)]. The second has 3 goals, which amounts to $\frac{3}{16}$ of what he could make [If they where to achieve the highest number of goals]. Therefore he will have $\frac{3}{16}$ of the stake, which is $1\frac{7}{8}$ ducats [$\frac{3}{16}\ (10)=\frac{30}{16}=\frac{15}{8}=1\frac{7}{5}$]. The third has two [goals], which is $\frac{2}{16}$. Therefore to him is $\frac{2}{16}$, that is $\frac{1}{8}$ of the stake, which is $1\frac{1}{4}$ [$\frac{1}{8}\ (10)=\frac{10}{8}=\frac{5}{4}=1\frac{1}{4}$ ducats]. All these you add up: that is $2\frac{1}{2}$, $1\frac{7}{8}$, $1\frac{1}{4}$. They make $5\frac{5}{8}$, and you subtract this from 10, that is from the whole stake [$10-5\frac{5}{8}=4\frac{3}{8}$][this is the new approach]

One must now divide this as in a trading company and say: one has 4 and the other 3 and the other 2 goals [on total of goals], indeed $\frac{1}{4}$, $\frac{3}{16}$ and $\frac{1}{8}$ [of the stake respectively] , and they have to divide [the reminder of the stake] $4\frac{3}{8}$. How much for each? Work! You will find, that the first will amount to $1\frac{17}{18}$ [Since 4 is $\frac{4}{9}$ of 9, then $\frac{4}{9}$ of $4\frac{3}{8}$ is $\frac{4}{9}\cdot\frac{35}{8}=\frac{35}{18}$] so, the second to $1\frac{11}{24}$[$\frac{3}{9}\cdot 4\frac{3}{8}=\frac{1}{3}\cdot\frac{35}{8}=\frac{35}{24}=1\frac{11}{24}$] and the third to $\frac{35}{36}$ [$\frac{2}{9}\cdot 4\frac{3}{8}=\frac{2}{9}\cdot\frac{35}{8}=\frac{35}{36}$]. Finished

The proof: Join together [add], what they are due first with what is due to them later [The proportion on the reminder for each player]. It must make 10 [ducats]. Consequently you will say, that the first received altogether of it $2\frac{1}{2}$ and $1\frac{17}{18}$, which is $4\frac{4}{9}$; the second $1\frac{7}{8}$ and $1\frac{11}{24}$, which is $3\frac{1}{3}$; the third $1\frac{1}{4}$ and $\frac{35}{36}$, which is $2\frac{2}{9}$; which is well

[$\left(2\frac{1}{2}+1\frac{17}{18}\right)+\left(1\frac{7}{8}+1\frac{11}{24}\right)+\left(1\frac{1}{4}+\frac{35}{36}\right)=\left(\frac{5}{2}+\frac{35}{18}\right)+\left(\frac{15}{8}+\frac{35}{24}\right)+\left(\frac{5}{4}+\frac{35}{36}\right)=\frac{45+35}{18}+\frac{45+35}{24}+\frac{45+35}{36}=\frac{80}{18}+\frac{80}{24}+\frac{80}{36}=\frac{40}{9}+\frac{10}{3}+\frac{20}{9}$.

This is Pacioli's

$4\frac{4}{9}+3\frac{1}{3}+2\frac{2}{9}$.

Continuing:

$\frac{40}{9}+\frac{10}{3}+\frac{20}{9}=\frac{40+30+20}{9}=\frac{90}{9}=10$ ducats.]

[Dividing the stakes in proportion to the points achieved with respect to the total achieved upon premature termination is equivalent to dividing the stakes in proportion to the maximum points possible and then dividing the reminder accordingly.]

Now it will give this same thing immediately according to the way of a trading company, as we say [above] in that of the ball game [Problem 1]. What you make there in two steps, here you make in one. There is however to say: 3 establish a trading company; the one puts 4 shares, the second 3 and the

> third 2, and they have to divide 10. What is due to each? Work! You find it, has been said already etc.

The procedure above can be generalized to the case of n players. What is interesting about this second example is that is shows more clearly the analogy of Pacioli's reasoning with the division of shares in a trade company. This highlights the relationship of this problem with the economic arena. This relationship involves the connection between logic, fairness and the occurrence of the unexpected.

Observe that the main difference in procedure between this approach and the one taken in the first problem is that in the former the whole stake was divided in proportion to the share of each player in the total of goals achieved by the group upon premature termination; there is no reminder involved. In the second problem on the other hand, the calculations are done first of the share of the best wins scored upon premature termination with respect to the total of maximum possible points, hence leaving a reminder that will be distributed in the same proportions with that total. Both methods render exactly the same results. Thus the two options presented here are calculating the proportions using the points achieved upon premature termination and the total of points achieved at that stage, or using those points/goals as a proportion of the maximum total achievable and calculating the share of each player in the reminder and add it to the calculated proportions of the share. These alternatives are good in order to visualize that different perspectives shed the same light over the problem. It is also interesting that Pacioli (as Tartaglia, Forestani and a little less Cardano) assumes that the only solution to the problem is arithmetic. In that sense he limited the solution of the problem to the tools available. Those tools constrained their possibilities of analysis of the problem. It was also difficult for him to determine whether those tools were enough or not.

It is also worth noting that uncertainty to Pacioli, Cardano, Tartaglia and Forestani is not an entity or a well defined concept that should be subject of inquiry. The need to know about the future that comes from astrology and other esoteric practices had no mathematical meaning. It begins to appear with the proliferation of the games of chance under the renaissance movement that questioned the church's prohibition. Therefore it is interesting that tackling this issue gave birth to the hope in acquiring knowledge about what we don't know with certainty. The result will eventually be known as Probability. There is a continuum between knowing and not knowing, and with probability we seem to identify how far the observer is from one extreme or the other. Until that time uncertainty did not seem to be a matter susceptible to mathematics. It was not also an issue on scientific grounds. The human interaction in gambling and in business made it an issue. And as those issues gained relevance, so did the need for a solution. But the answer had to wait until more mathematical tools were available (see conclusions at the end). Once developed, the notion of probability opened the door for analyzing scientific questions not graspable by previous methods. And the solution of those problems by means of probability gave birth to further developments in science. In this way human interaction, logic and technology interact in order to transform themselves. It is also worth noting that this type of approximation (to knowledge in this case) was also used in the development of calculus.

Tartaglia's attempt at a solution

The following excerpt is taken from (Tartaglia N. , 1556) Prima parte del General Tratatto Book 16, Section 206 translated by (Pulskamp, 2009 b)

After setting up an example in which two players play a ball game to 60 points and stake 22 ducats each, Tartaglia asserts the following in regard to Pacioli's proposed solution to the first problem:

> This rule of his seems to me to be neither beautiful nor good [correct], because if per chance [for example(First Example:)] one of the parties [players] had 10 [points] and the other none [This is the extreme example in which proportions are 0], one would proceed according to his rule, that that party which had the said 10 should take all, and the other should take nothing [Applying Pacioli's procedure, 0 points would make the proportion over the total of points achieved by both players taken together upon premature termination also 0 and so the proportion that that player will be able to take from the stake.], which is completely beyond reason, that one with 10 could take the whole.
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> And Therefore I say, that the resolution of such a problem is rather judicial [involving judgment calls in the solution] than [purely] through computation, because in whatever manner through which it is resolved to you, there will be found arguing. Nevertheless men argue, it seems to me, this or that [people propose solutions without reason].
>
> First one must see what part each [player] has of the complete game [of the points necessary for winning the game], that is, if one perchance had 10 [points] and the other 0, thus the [one] who has 10, will have the sixth of the complete game [of the points necessary to win the game (60)]; and therefore I say, that in this case, he should receive the sixth part [$\frac{10}{60} = \frac{1}{6}$] of the money that each [player] have put ; that is, if they stake 22 ducats per party, he should have the sixth part of the said 22 ducats, [the opponent's stake], which makes 3 and two-thirds[$\frac{22}{6} = \frac{11}{3} = 3\frac{2}{3}$], the total with his 22 ducats makes 25 and two- thirds[22+$3\frac{2}{3} = 25\frac{2}{3}$], and the other party [his opponent] may take the rest, this remainder is 18 and one third ducats [$44 - 25\frac{2}{3} = 18\frac{1}{3}$]. [Second Example:] And if [instead of having 10 points and 0 points] one party had 50 [points] and the other 30 [points], one should deduct [subtract] 30 from 50. There will remain 20, and these 20 come to be one-third of the whole game [$50 - 30 = 20$. And $20 = \frac{1}{3}$ of 60 which is 'the whole game']. And however one should take (besides his own [22 ducats]) one-third part of the money of the other party [Therefore the player with 50 should take $\frac{1}{3}$ of his opponent's stake: $\frac{1}{3} \cdot 22$ ducats], which third part makes 7 and one third ducats [$\frac{22}{3} = 7\frac{1}{3}\ ducats$], that with his own [this amount added to his stake (22 ducats)] will make [will be equal to] 29 and one third ducats [$7\frac{1}{3} + 22 = 29\frac{1}{3}$ducats]. And the other party [the player with 30 points] may take the rest, which will make 14 and two thirds ducats [$44 - 29\frac{1}{3}$=$14\frac{2}{3}\ ducats$]. And proceeding so, nothing inconvenient will be found to follow, as make in the solution of brother Luca.

Tartaglia's main concern is that although one of the players has the whole of the points made upon premature termination, there is a factor that is not taken into account in Pacioli's rule which has to do with the lack of knowledge about the resolution of the game had it reached normal termination. This uncertainty implies that both players had the possibility of winning the game and thus getting the whole stake. This is what Pacioli is not taking into account from Tartaglia's perspective.

Unlike Pacioli, Tartaglia is calculating the proportions using the points required for winning (60) as the whole, of which the points achieved (10) are the part. Then he is using those proportions to divide the corresponding player's opponent stake which he will claim. In Pacioli's Problem 1, on the other hand, the whole is in the 'first way' the maximum total of goals, in the 'second way' the maximum total of points and in the 'third way' the sum of the points achieved by the players. In the second of the Pacioli's problems the proportions calculated with the sum of points achieved as the whole are used on the total of the stake leaving a reminder that is distributed with the calculated proportions. Tartaglia, on the other hand is taking the points necessary to win as the total. Then he multiplies these proportions by each player's stake. Since the reminder that is left is not divided according to the proportions but equally (as we will see) amongst the players, Pacioli's problem of assigning 0 to the player with 0 points/goals seems solved. The fact that the whole in Tartaglia is represented by the points necessary to win seems to look for a more impartial criteria with the minimum set of assumptions. Instead of taking as hypothesis that all the players score their maximum in order to define the whole, Tartaglia assumes that one of the players obtains this score and that is why his whole is 6 goals (or 60 points).

Now apply what was done for the player with 10 points in the first example, to both players in this second example. According to the rule that Tartaglia introduced in the first example $\frac{50}{60}$ is the share that the player with 50 should get from his opponent's stake; $\frac{30}{60}$ is the share that the player with 30 should get from his opponent's stake. Both stakes are 22 ducats, totaling 44 ducats. If we subtract what each player should get from his counterpart we arrive to a net gain ($= gain - loss$) for each player. Let us begin with the net gain for the player with 50 points: Gain= $\frac{50}{60} \cdot 22$. Loss=$\frac{30}{60} \cdot 22$. $gain - loss = \frac{50}{60} \cdot 22 - \frac{30}{60} \cdot 22$=22 $\cdot \left(\frac{50-30}{60}\right) = 22 \cdot (\frac{1}{3})$; where 60 is the 'whole game'.

Forestani's proposed solution

The following passage is taken from [4, Libro Quinto, pp. 364-367], translated by Pulskamp [11].

> An already old Gentleman, finding himself again at his villa, and taking a great delight in the game of ball, called two young Laborers, and said, "Here you have 4 ducats to play in my presence with the ball here. And who of you first gains 8 games, I wish that he has won the 4 ducats."[In this case the money is put by a third party as opposed to being put by the players] And therefore they began to play. And when one

of them had gained 6 games, and the other 3 they lost the ball, and could not end, & the gentleman said, "Here you have the money, divide it between you". I ask how much of it each one receives

In resolving various similar propositions the opinions are diverse, but this to us seems most correct, and most common. And first we will ask for 4 ducats [we will assume that the stake is 4 ducats], that he [the player] will have need to win 8 games [If he is to win]; and [hence] the other [player] is not [would not be] able to win any more than 7 [games]. Therefore between them [adding up the games won by both players] it cannot run more than [the maximum number of games that they would be able to win is] 15 games; so that the first one [the player], winning 5 games, comes to gain [should gain] $\frac{5}{15}$ [which is the games won by the player divided by the maximum number of points achievable by both players taken together], that is $\frac{1}{3}$ of the 4 ducats & the second, who wins 3 games, comes to gain $\frac{3}{15}$, that is $\frac{1}{5}$ of the said 4 ducats in a way that between the first one, and the second, they come to gain $\frac{8}{15}$ of the 4 ducats [$\frac{1}{3}+\frac{1}{5}=\frac{8}{15}$]. For which thing clearly it is known, that $\frac{7}{15}$ remain to you which is not exhausted , neither by games, nor by any of them winning [$\frac{15}{15}-\frac{8}{15}=\frac{7}{15}$]. And therefore [in the solution that I propose] you must divide it in a half [divide the reminder in half $\frac{7}{15}\cdot\frac{1}{2}=\frac{7}{30}$]. Then seize the half from $\frac{7}{15}$, that is $\frac{7}{30}$; & you add it to $\frac{1}{3}$. It makes $\frac{17}{30}$ [$\frac{7}{30}+\frac{1}{3}=\frac{17}{30}$] and such part [of the stake (4 ducats)] the first one [player] receives; and the other half [of the remainder $\frac{7}{15}$] that is $\frac{7}{30}$, you add to $\frac{1}{5}$. It makes $\frac{13}{30}$ [$\frac{7}{30}+\frac{1}{5}=\frac{13}{30}$] and such part [of the stake] the second [player] receives. Now divide 4 ducats [which is the stake] in the societal way [between the players in the aforementioned way], saying therefore, the first one must receive for 17 [$\frac{17}{30}$] & the second for 13 [$\frac{13}{30}$] of the said 4 ducats. Work. You will find that the first one must have $2\frac{4}{15}$ [$\frac{17}{30}\cdot 4=\frac{68}{30}=2\frac{4}{15}$ goes to the one who won 5 games] ducats & the second $1\frac{11}{15}$ [$\frac{13}{30}\cdot 4=\frac{52}{30}=1\frac{11}{15}$] ducats and this is true solution of similar proposals.

Unlike Pacioli who would divide the remainder according to the calculated proportions, Forestani is proposing to divide it in halves. Forestani is presumably trying to take into account the possibilities that both players still had of winning, up to the point when the game finished prematurely. This is so because the history of the game (the number of won games) is not used to calculate the proportion that each player gets from the reminder. Thus the reminder is divided on equal parts because unlike the part of the whole game that has been played (which takes into account the games won by each player according to the procedure just explained) the part that was left without resolution is not known and cannot be inferred. That is why Forestani seems to distribute the benefit of the doubt amongst both players equally as opposed to Pacioli's proposed solution according to which the reminder should be divided in a pro-rate basis with the games won by each player. Pacioli's solution assumes that the history of the game (games won) is as fair in terms of deciding who gets the part of the stake attribuible to the games played, as it is to deciding the part of the stake corresponding to the games that have not been played.

Cardano's attempt at a solution

The following excerpt is taken form Cardano [2, Chapter LXI, De Extraorinariis, §§13–17 (f. 143 r. - f. 144 r.) and Last Chapter On the Error of Fra Luca, §5 (f. 289 v. – f. 290 r.)] translated by Pulskamp [12].

> 13. What should be known about the reckoning in games is that one takes into consideration with regard to games only the end to which [the difference between the points made by each player and the points necessary to win the game] and this [difference] in progression [According to Pulskamp [12, p.5] "Cardano means by *progressio* the summing of the natural numbers up to a certain point. The *progressio* of n is thus $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$"] dividing the whole [of the stake] by the same parts [in the same way the total progression is divided]. [First Example] For example two [players] gamble to ten [the first to make 10 goals wins]. One has 7 [goals], the other 9 [goals]. One now asks how much each should have in the case of division [in the case the players have to divide the stakes] if the game is not finished. Subtract 7 [the number of goals obtained by one of the players] from 10 [the goals necessary to win the game]; there remains 3. Subtract 9 from 10; there remains 1. The progression of 3 is 6 [1+2+3=6]. The progression of 1 is 1. Therefore by dividing the total [stake] into 7 parts [which is the sum of the aforementioned progressions:$6 + 1 = 7$] you will give 6 parts to the one having 9 [goals], and one part to the one having 7 [goals]. Let us assume that they have staked 7 gold pieces each, then the total stake would be 14 [gold pieces], out of which 12 falls to the one [player] having 9 [goals] [the total stake (14 gold pieces) divided in 7 parts (which is the sum of the two progressions) is: $\frac{14}{7}$ = 2 gold pieces for each part. Of those 7 parts, 6 correspond to the player with 9 goals. Thus $6 \cdot 2 = 12$ gold pieces], and 2 [gold pieces] to the one having 7 games [Accordingly, $1 \cdot 2 = 2$]. Hence who has 7 [goals], loses $\frac{5}{7}$ of the capital. [The player with 9 goals gets $\frac{6}{7}$ of the stake while the player with 7 gets $\frac{1}{7}$ and hence 'loses' $\frac{5}{7} = \frac{6}{7} - \frac{1}{7}$ parts][of the whole stake]
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> [Second Example] Another example: let us assume that the game is to 10 [the first to make 10 goals wins] and one [player] has 3 [goals], [and] the other [player has] 6 [goals]. Subtract [find the differences between the goals made and the goals necessary to win for each player like in the first example]. The residuals 7 [10-3=7] and 4 [10-6=4] are made. The progression of 7 is 28 [$1 + 2 + \cdots + 7 = 28$]. The progression of 4 is 10 [$1 + 2 + 3 + 4 = 10$] . Therefore of the entire sum [$28 + 10 = 38$], I shall give 28 parts to the one having 6 games [corresponding to his opponents difference in progression], and to the one having 3, I shall give 10 parts; & therefore I divide the total stake into 38 parts, and whoever has 3, loses $\frac{9}{19}$ [If the 38 parts were divided equally, each player would get $\frac{38}{2} = 19$ parts. But Cardano assigns 28 parts to the player with 6 and 10 to the player with 6, so the latter player instead losses 9 parts out of the 19 he could have gotten with an equal division] of his capital.
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> 14. But the demonstrating rule concerning this: If a game must be started again after division happened, the parties would have to be put the same [part of the stake] as what they have received under the existing condition [if the proposed division rule is to work]. And it is in the first example, that [the player] [,] to which one [more goal is necessary for winning (the player with 9][,] says, "I wish to play with the condition, that you [the player with 7] cannot win except by winning 3 times [making three goals[2]] without

[2] From here on I exchanged Cardano's word games for goals, as it is less confusing insofar we identify the word 'game' with the activity that the players are carrying on. 'Goals' on the other hand are the partial victories. In short, for us, game is the war, whereas goals are each one of the battles that are part of the game. Since Cardano is referring to those partial battles as games, it is better for the sake of clarity to exchange it for 'goals'.

pause [because the pause implies that I win], and, if I win 1 [1 goal], I will win [the game]." And [Therefore] he who wishes to win 3 games [the player with 7] must wager 2 gold pieces. How much must the other [player's] stake be? I say that he will stake 2 [even though the player with 7 has to make more goals]. For the [following] reasoning: If [for instance] they must play to one game [the first to make one goal wins], it is sufficient to stake 2 [gold pieces: 2 gold pieces *times* 1 goal], and if [if they must make] two [goals in order to win], he [the player with 9] would have to stake triple [6 gold pieces]. The reason [is] because by winning simply 1 game [to the one who first makes a goal], he would win 4 [gold pieces; the two that he staked plus the two that his opponent staked], but [if they play to 2 goals] here he [the player with 9] stands with danger of losing the second [goal] with the first win [thus losing what he won in the first goal and having to make one more goal to win the game.]; therefore he must win the triple [One goal made by his opponent plus 2 necessary to win the game: 1+2=3 'the triple'] , and if to 3 [and if they play to the one who first makes 3 goals,] six times, because the difficulty is duplicated [the player with 9 has to make 3 goals in order to win. His opponent might make 0, 1, or 2 goals. Thus the number of times that the ball might be in play would be 1+2+3=6 times]; therefore he [the player with 9] would have to stake 12 [gold pieces in this example: 6 times 2 gold pieces]. Now also he already has received 12 [gold pieces] and that other [player] 2; therefore the division has been accomplished with the agreement made with the consent of the parties. Otherwise, if it is caused by whoever has the more, it is divided into equals; if it was caused by whoever has the fewer, he loses all.

The logic behind Cardano's reasoning is the following: When the game starts the two players are in the same conditions and have to make the same number of goals whichever is to win. At some point before resolution the game stops. One of the players is ahead. Thus the player that is ahead must receive a compensation proportional to the effort necessary for the other player necessary to win the game. This effort is measured by Cardano with the "progression" of the difference between the number of points reached by the other player and the points necessary to win the game. The "progression" is the hypothetical scores with which the other player might have ended the game had the player, that is ahead, won the game. The rest, corresponds to the procedure applied to the other player symmetrically.

Conclusions:

Only Cardano realized that the distribution of the stake should be done based on the points that each player had to achieve in order to win the game when it suddenly stopped. This is a step ahead to Pascal who finally solved the problem with a combinatorics technique that was not available to these four authors.

Pascal's solution is illustrated as follows:

Pascal Solution to "The Problem of Points":

The following statement of Pascal's theorem is taken from [15].

<u>Pascal's two principles (c. 1654): </u>(6, p.451)

1- If the position of a given player is such that a certain sum belongs to him whether he wins or loses, [then] he should receive that sum even if the game is halted.
2- If the position of the two players is such that if one wins, a certain sum belongs to him, and if he loses, it belongs to the other, and if both players have equally good chances of winning, the they should divide the sum equally, if they are unable to play.

Pascal's Note: What determines the split of the stakes is the number of games remaining, and the total number that the rules say either player must win to obtain the entire stake.

We can see how Pascal's note is very similar to the way Cardano approached the problem. However, although Cardano is coherent with the note, his procedure for calculating the stakes is different from Pascal's. Following [15] we have:

Pascal's Theorem:

Suppose that the first player lacks r games of winning the set, while the second player lacks s games, where both r and s are at least 1.

If the set of games is interrupted at this point, [then] the stakes should be divided so that the first player gets that proportion of the total as

$$\sum_{k=0}^{s-1} \binom{n}{k} \ is\ to\ 2^n,$$

Where $n = r + s - 1$ (The maximum number of games left)

Observe:

$\sum_{k=0}^{s-1}\binom{n}{k}$ is the sum of the first s terms if the binomial expansion of $(1+1)^n$, while

$2^n$is the sum of all terms, where

$\binom{n}{0}$ = The number of chances for the first player to win n points,

$\binom{n}{1}$ = ,, ,, ,, ,, ,, ,, ,, ,, ,, ,, $(n-1)$ points

⋮

$\binom{n}{s-1}$ = ,, ,, ,, ,, ,, ,, ,, ,, ,, ,, $n-(s-1) = r$ points.

Thus, none of the solutions by Cardano, Tartaglia, Forestani and Pacioli managed to identify Pascal's principles as fundamental for the solution of the problem. Even more, none of them stated clearly which assumption they were making or tried to justify them. Had they tried to make explicit their assertions in those grounds, their way of reasoning the problem would have been more effective.

Like Pacioli, Forestani and Tartaglia, Cardano considers the problem of division of stakes as a problem of proportion. All of them define a whole composed by parts that can be attributed to the players. Then they find what part is each of those parts in the whole. Then they treat the defined whole as if it was the total stake (or to the total opponent's stake like in Tartaglia and Cardano). Hence, in the same way the component parts of the total are part of the total, so it is the part that each player takes from the stake. Thus, the difference lies in the definition of the whole. Cardano is defining the whole as the sum of the progression of the difference of each player's score with the minimum required to win; the parts are each player's difference between the minimum required to win and those in progression. Thus the total stake is divided into the defined whole, and the same part the players take in the defined whole, they also take in the stake. In Cardano each player participates in the whole with the other player's progression.

It is interesting to see that the problem of distribution of stakes is very similar to the situation of an economy that experiences a crisis. The stakes are represented by investment and in general expenditure decisions. The goals are profits, sales and incomes. The players are the families, investors and the government. Whenever there is a crisis, investors want to liquidate their investments (sell assets) without any desire to wait until the accorded pay-back period is reached. In the economy the families, investors and governments who make the higher number of 'right' choices in market terms are the ones who win a bigger portion of the stakes (incomes and profits) in a normal situation. However, a crisis leads to the unexpected liquidation of gains and incomes causing an unexpected termination of the economic game. Full employment is the maximum number of possible points and the level of employment is the goals upon unexpected termination. The problem of distribution gets solved by prices and wages; however it is not necessarily a fair distribution. In this sense the economy could be seen as a game in which the maximum possible points are determined by the capacity of production.

During a crisis the economy is in less than full employment. Therefore, it would be necessary to evaluate whether the market is dividing in a fair way the stakes of the economy and if there was an alternative way to divide them , which one should it be? The problem of the division of the stakes may shed some light on this issue.

Appendix: Totalities and Parts in Pacioli:

In the present appendix I aim to present the interplay of the concepts of parts and whole in Pcioli's formulation of the problem of points. This also highlights that Pacioli's assumption that the only way to solve the problem was with arithmetic.

Problem 1:

1)

$11 = 5 + 6$; 11 is the maximum number of goals. 5 and 6 are the parts.

$\frac{5}{11}$ *and* $\frac{6}{11}$ are the parts that 5 and 6 are of the whole 11

2)

$7 = 2 + 5$; 7 is the total of number of goals upon unexpected termination.

$\frac{2}{7}$ *and* $\frac{5}{7}$ are the parts that 2 and 5 are of the whole 7.

3)

$10 = x + y$; 10 is the total at stake. x and y are the part in which the stake will be divided.

4)

$2 + 5 + 4 = 11$ total of possible goals expressed as the sum of the goals (2 and 5) upon unexpected termination and a remainder (4)

$\frac{2}{11}, \frac{5}{11}, \frac{4}{11}$ are the parts that 2,5,4 are in the whole 11.

Or

$\frac{7}{11}, \frac{4}{11}$ are the parts that 7 and 4 are in the whole 11.

5)

$\frac{5}{11} + \frac{2}{11} = \frac{7}{11}$ Total of parts upon unexpected termination

Depending on what perspective is elected, wholes and parts are interrelated in order to calculate x and y. The whole (The stake: 10 ducats) will be divided as wholes ($\frac{7}{11}, \frac{11}{11}, \frac{7}{7}, \frac{110}{110}$...) are divided between one player and the other.

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