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Borda Voting is Non-manipulable but Cloning Manipulation is Possible

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Abstract: *This paper deals with Borda count which is sincere voting system and originally proposed by French mathematician and philosopher Jeans-Charles Borda. In Borda count a defeated candidate can manipulate the election result in his favor in sincere way by introducing a candidate which is a clone of him and voters ranked this clone candidate immediately below him. In this situation Borda rule is strictly follows but manipulation is possible. The paper shows that this type of manipulation is vulnerable. Both single and simultaneous vulnerabilities of cloning manipulations are discussed with detail calculations and easier ways.*

Keywords: Borda scores, cloning manipulation, vulnerabilities in Borda cloning manipulations.

Introduction

French mathematicians and philosophers Marquis de Condorcet (1743-1794) and Jeans-Charles Borda (1733-1799) introduced modern voting system. The voting aggregation rules proposed by these two eighteenth-century scholars are very different in the political history. Borda developed a voting method named "*method of marks*" in 1770 (Borda 1781). The Borda rule is a scoring method that yields a unique ranking, although not necessarily strict. Each elector ranks the alternatives according to his order of preference (ties disallowed). The Borda rule (Borda 1781) belongs to the class of point ranking rules where points are given to each candidate according to his rank in the preference of the voters. Once all votes have been counted and the candidate with the most points is the winner. It is currently used for the election of two ethnic minority members of the National Assembly of Slovenia, and in modified forms, to select presidential election candidates in Kiribati and to elect members of the Parliament of Nauru. It is also used throughout the world by various private organizations and competitions. In this method if there are n alternatives, an elector's first choice is assigned $(m-1)$ points, his second $(m-2)$ points and so on down to his last choice, which is assigned 0 point. One property of the Borda rule is that each of the voters of each type gives $m(m-1)$ marks to the candidates. Gibbard (1973) and Satterthwaite (1975) explained that all reasonable voting procedures are sensitive to manipulate by strategic voters. Dummett (1998) suggested that in election agenda manipulation is possible. In this paper we will call it cloning manipulation. A cloning candidate y of a candidate *x* is similar to him and all voters place *y* immediately below *x* in order rank. In this paper we follow the rules of Dummett (1998), Huang and Chua (2000) and Serais (2002). The Borda majority count is discussed in detail with many examples by Zahid and Swart (2010). A rule of voting is strategy-proof when every voter's best strategy is his true preference-order, otherwise it is manipulable.

We have included two propositions with proof to clarify the concept of Borda voting and its cloning manipulations.

Borda Voting

Let $N = \{1,2,...,n\}$ be the set of individual voters, and let $Y = \{x, y, z,...\}$ be the finite set of alternatives where $\#(N) = n$ and $\#(Y) = m$. If *x* is strictly preferred to *y* we write *xPy* and so on. If *x* is related to *y* we can write the binary relations as follows (Arrow 1963, Islam et al. 2009):

i) Reflexivity: $\forall\, x \in Y; xRx$.

ii) Completeness: $\forall\, x, y \in Y\ \&\ x \neq y \Rightarrow xRy$ or yRx.

iii) Transitivity: $\forall\, x, y, z \in Y$, if $xRy\ \&\ yRz \Rightarrow xRz$.

iv) Anti-symmetry: $\forall\, x, y \in Y$, if $xRy\ \&\ yRx \Rightarrow x = y$.

v) Asymmetry: $\forall\, x, y \in Y$, such that $xRy \Rightarrow\ \sim (yRx)$.

An order or ranking is weak if it is reflexive and transitive and an order is linear if it is reflexive, transitive and asymmetric. Let a vector $r = (r_1, r_2,...)$ is a rank form where r_1 the rank of alternative is in 1 and so on. We can also represent a complete weak order in a sequence form, that is, by a sequence $s_1 s_2...$, where s_1 and s_2 are the alternatives with ranks 1 and 2 respectively and so on. If each voter is asked to compare the alternatives pair by pair, as in the Condorcet procedure, her vote is summarized in a matrix $X^i = \left[x_{st}^i\right]_{s,t\in Y}$. For any pair of alternatives $(s,t) \in Y^2$, $x_{st}^i = 1$ if voter *i* chooses *s* over *t* and $x_{st}^i = 0$ otherwise, and $x_{st}^i = 0$ if *s* = *t*. Now aggregate votes for *n* voters is a poll and is denoted by $X = \sum_{i=1}^{n} X^i$. Once voters have expressed their opinions in a poll, the problem is to aggregate these opinions in order to select a final ranking. The Borda rule and the maximum likelihood rule are the two methods considered here to accomplish this aggregation.

The points of Borda count are aggregated across voters to give the Borda score $b_s(X) = \sum_{t=1}^{m} x_{st}$ of *s* and the ranking $s_1 s_2...$ such that $b_{s_1}(X) \geq b_{s_2}(X) \geq b_{s_3}(X) \geq \ldots$ is the Borda ranking. It is unique but it admits ties. Again the Borda ranking is the weak order *B* (*X*) such that:

$$\forall s,t \in Y\ ;\ B_s(X) \leq B_t(X) \Leftrightarrow b_s(X) \geq b_t(X).$$

where $B_s(X)$ is the rank of alternative *s* and *B* (·) is the Borda rule (Truchon 2006).

Let v_i be the number of voters, then for the alternative '*a*' the Borda count (*B*) can be defined by;

$$B(a) = v_1 \times 0 + v_2 \times 1 + ... + v_k \times (k-1)$$

$$=\sum_{i=1}^{k} v_i \times (i-1).$$

Borda rule is reflexive and transitive and also obey neutrality that is all candidates are treated equally. It is also unanimous that is if every voter gives to a candidate C a higher rank than to all other candidates, then C wins in the election and Borda count avoid dictatorship.

Arithmetical Calculations

Let us assume that there are 17 voters of three types and three alternatives x, y, z and the preference profile be as follows (Islam et al. 2011):

Type 1: $xPyPz$ by 8 voters,
Type 2: $yPzPx$ by 5 voters,
Type 3: $zPxPy$ by 4 voters.

Borda count in this profile be as follows:

For x: 8×2+5×0+4×1 = 20 marks,
For y: 8×1+5×2+4×0 = 18 marks,
For z: 8×0+5×1+4×2 = 13 marks.

Here x gets highest marks 20, so x wins. We observed that Borda method is for sincere voters and has no voter paradox but it has some problems. Black (1948, 1958) and Satterthwaite (1975) modified the Borda method by misrepresentation of their preferences by the electors. Now we modify the above example by adding two alternatives u and v. The preference profile would be as follows:

Type 1: $xPyPzPuPv$ by 8 voters,
Type 2: $yPzPxPuPv$ by 5 voters,
Type 3: $zPxPyPuPv$ by 4 voters.

Now Borda counts would be as follows:

For x: 8×4+5×2+4×3 = 54 marks,
For y: 8×3+5×4+4×2 = 52 marks,
For z: 8×2+5×3+4×4 = 47 marks,
For u: 8×1+5×1+4×1 = 17 marks,
For v: 8×0+5×0+4×0 = 0 mark.

So that in this case x wins again. Type-3 voters have realized that x would win in the election then they would have change their preference profile as;

Type 3: $zPyPuPvPx$ by 4 voters,

so that the Borda counts would be,

For x: 8×4+5×2+4×0 = 42 marks,
For y: 8×3+5×4+4×3 = 56 marks,
For z: 8×2+5×3+4×4 = 47 marks,
For u: 8×1+5×1+4×2 = 21 marks,
For v: 8×0+5×0+4×1 = 4 marks.

In this case y would have won. The voters of type-3 would have been better off than when they voted honestly; the method provides a temptation for misrepresentation of preferences. The possibility of manipulation of the result of an election through the misrepresentation of preferences as described above was considered neither by Borda nor by Condorcet.

Cloning Manipulation in Borda Voting

We have seen that Borda did not use manipulation in his voting method. But we can manipulate the Borda rule by introducing a cloning candidate (Serais 2002). Suppose x would be defeated in an election following Borda count (Islam et al. 2011). The candidate x can manipulate the election outcome in his favor by introducing his clone y (say) in the choice set, the clone y being defined as an alternative which is ranked immediately below x in the individual preferences.

Choose a set $A \subseteq Y$ be a finite set where $\#(A) \leq m$. Now for $A = \{x, y, z\}$ the six possible preference orderings over A will be numbered as in following table-1:

Table 1: Possible preference orderings over the set *A*.

n_1	n_2	n_3	n_4	n_5	n_6
x	x	y	y	z	z
y	z	x	z	x	y
z	y	z	x	y	x

A voting situation is a vector, $s = (n_1, ..., n_6)$, where n_j $(j = 1, ..., 6)$ be the number of type j voters and $\sum_{j=1}^{6} n_j = n$. Here n is the total number of voters in an election. Let $S^n = \{s^1, ..., s^n\}$ be the set of all possible voting situations. A social choice function $f : S^n \to A$, assigns to each voting situation a non-empty subset of *A*. Let N_{xy} be the number of voters who prefer x to y, $S^x_{B,s}$ be the Borda score of x, and $S^{xy}_{B,s}$ be the difference of Borda score between x and y for the voting situations i.e., $S^{xy}_{B,s} = S^x_{B,s} - S^y_{B,s}$.

Now we can introduce the mathematical definition of a clone as follows: A candidate y is a clone of x for a voting situations if and only if $\forall z \in X \setminus \{x, y\}$, $\forall i \in N$ $xP_iz \Leftrightarrow yP_iz$ and $\forall i \in N$, xP_iy.

This type of manipulation was introduced by Dummett (1998) where he called it agenda manipulation. Dummett observed that the Borda rule may suffer from this manipulation and explained by a series of examples.

Arithmetical Calculations

First we set an example where there are 12 voters and they have to choose preference relation among four alternatives x, y, z and u. Let the preference profile be as follows:

Type 1: $yPuPzPx$ by 2 voters,
Type 2: $uPzPxPy$ by 2 voters,
Type 3: $zPuPyPx$ by 2 voters,
Type 4: $xPuPyPz$ by 3 voters,
Type 5: $xPyPuPz$ by 1 voter,
Type 6: $zPyPxPu$ by 2 voters.

Borda votes for this profile be as follows:

For x: 2×0+2×1+2×0+3×3+1×3+2×1 = 16 marks,
For y: 2×3+2×0+2×1+3×1+1×2+2×2 = 17 marks,

For z: 2×1+2×2+2×3+3×0+1×0+2×3 = 18 marks,
For u: 2×2+2×3+2×2+3×2+1×1+2×0 = 21 marks.

Here u gets highest 21 marks, so u wins and y gets second of the lowest 17 marks. Dummett assumes that before the election, a fifth candidate, v is introduced by y whom every voter ranks immediately below y. Then the preference profile would be as follows:

Type 1: $yPvPuPzPx$ by 2 voters,
Type 2: $uPzPxPyPv$ by 2 voters,
Type 3: $zPuPyPvPx$ by 2 voters,
Type 4: $xPuPyPvPz$ by 3 voters,
Type 5: $xPyPvPuPz$ by 1 voter,
Type 6: $zPyPvPxPu$ by 2 voters.

Then the Borda votes would be as follows:

For x: 2×0+2×2+2×0+3×4+1×4+2×1 = 22 marks,
For y: 2×4+2×1+2×2+3×2+1×3+2×3 = 29 marks,
For z: 2×1+2×3+2×4+3×0+1×0+2×4 = 24 marks,
For u: 2×2+2×4+2×3+3×3+1×1+2×0 = 28 marks,
For v: 2×3+2×0+2×1+3×1+1×2+2×2 = 17 marks.

Now y gets highest score of 29 marks and wins in the election. Here we observed that in initial voting situation y scored second of the lowest 17 marks but after cloning a candidate v as fifth candidate by y placed him in first position. So that cloning manipulation is sufficiently powerful to win in an election by a losing candidate.

Single Cloning Manipulation

Let there are two losing candidates in an election. Now we will discuss the cloning manipulation by a single loser. Let $A=\{x, y, z\}$ then the voting situation s^1 be as follows:

Table 2: The Voting Situation of s^1.

n_1	n_2	n_3	n_4	n_5	n_6	Scores
x	x	z	z	y	y	2
y	z	x	y	z	x	1
z	y	y	x	x	z	0

From table-2 we find;

$S_{B,s^1}^{x} = 2(n_1+n_2)+n_3+n_6$, $S_{B,s^1}^{y} = n_1+n_4+2(n_5+n_6)$ and

$S_{B,s^1}^{z} = n_2+2(n_3+n_4)+n_5$

$N_{xy} = n_1+n_2+n_3$, $N_{yx} = n_4+n_5+n_6$, $N_{xz} = n_1+n_2+n_6$, $N_{zx} = n_3+n_4+n_5$

$N_{yz} = n_1+n_5+n_6$ and $N_{zy} = n_2+n_3+n_4$

$$N_{yz}+N_{zy} = n_1+n_2+n_3+n_4+n_5+n_6 = n$$

Similarly, $N_{yx}+N_{yx} = n$ and $N_{xz}+N_{zx} = n$.

Let $S_{B,s^1}^{xy} \geq 0$ and $S_{B,s^1}^{xz} \geq 0$, so that x wins by Borda counts. Now suppose y is in the lowest position by Borda counts. The candidate y could introduce u whom every voter ranks immediately below y. Then the voting situation s^2 would be as follows:

Table 3: The Voting Situation of s^2.

n_1	n_2	n_3	n_4	n_5	n_6	Scores
x	x	z	z	y	y	3
y	z	x	y	u	u	2
u	y	y	u	z	x	1
z	u	u	x	x	z	0

From table-3 we find:

$$\begin{aligned} S^{x}_{B,s^2} &= 3(n_1+n_2)+2n_3+n_6 \\ &= (2(n_1+n_2)+n_3+n_6)+(n_1+n_2+n_3) \\ &= S^{x}_{B,s^1}+N_{xy}, \end{aligned}$$

$$\begin{aligned} S^{y}_{B,s^2} &= 2n_1+n_2+n_3+2n_4+3(n_5+n_6) \\ &= (n_1+n_4+2(n_5+n_6))+(n_1+n_2+n_3+n_4+n_5+n_6) \\ &= S^{y}_{B,s^1}+n, \end{aligned}$$

$$\begin{aligned} S^{z}_{B,s^2} &= 2n_2+3(n_3+n_4)+n_5 \\ &= (n_2+2(n_3+n_4)+n_5)+(n_2+n_3+n_4) \\ &= S^{z}_{B,s^1}+N_{zy}, \end{aligned}$$

$$S^{u}_{B,s^2} = n_1+n_4+2(n_5+n_6)=S^{y}_{B,s^1}.$$

Since u is cloned by y, so u is always beaten by y. Now y beats x if,

$$N_{yx} > S^{xy}_{B,s^1} \text{ i.e., } S^{y}_{B,s^1}+n > S^{x}_{B,s^1}+N_{xy}$$

$$\text{i.e., } 2(n_1+n_4)+3(n_5+n_6)+n_2+n_3 > 3(n_1+n_4)+2n_3+n_6 . \qquad (1)$$

Now y beats z if,

$$N_{yz} \geq S^{zy}_{B,s^1} \text{ i.e., } S^{y}_{B,s^1}+N_{yz}+N_{zy} \geq S^{z}_{B,s^1}+N_{zy} \text{ i.e., } S^{y}_{B,s^1}+n \geq S^{z}_{B,s^1}+N_{zy}$$

$$\text{i.e., } 2(n_1+n_4)+3(n_5+n_6)+n_2+n_3 > 3(n_3+n_4)+2n_2+n_5 . \qquad (2)$$

Here we have 6 preference types and 4 candidates but u is the clone of y. The new candidate u is always ranked after y in the individual preferences. Inequalities (1) and (2) satisfy all the properties of Borda rule, so that y wins in the election.

In this section we have shown by calculations that Borda voting is for sincere voters and manipulation is impossible but a defeated candidate can manipulate Borda voting in his favor by introducing his clone (Islam et al. 2011).

Simultaneous Cloning Manipulation

Now we describe the cloning manipulation by both of the losing candidates. In table-2 we have considered that x wins but y and z have defeated. Now both y and z could introduce cloning candidates. Let u be the clone of y and v be the clone of z and let only y would be benefited by cloning. The voting situation s^3 would be as follows:

From table-4 we find:

$$\begin{aligned}S^{x}_{B,s^3} &= 4(n_1+n_2)+2(n_3+n_6)\\ &= (2(n_1+n_2)+n_3+n_6)+(n_1+n_2+n_3)+(n_1+n_2+n_6)\\ &= S^{x}_{B,s^1}+N_{xy}+N_{xz},\end{aligned}$$

Table 4: The voting situation of s^3.

n_1	n_2	n_3	n_4	n_5	n_6	Scores
x	x	z	z	y	y	4
y	z	v	v	u	u	3
u	v	x	y	z	x	2
z	y	y	u	v	z	1
v	u	u	x	x	v	0

$$\begin{aligned}S^{y}_{B,s^3} &= 3n_1+n_2+n_3+2n_4+4(n_5+n_6)\\ &= (n_1+n_4+2(n_5+n_6))+(n_1+n_2+n_3+n_4+n_5+n_6)+(n_1+n_5+n_6)\\ &= S^{y}_{B,s^1}+n+N_{yz},\end{aligned}$$

$$\begin{aligned}S^{z}_{B,s^3} &= n_1+3n_2+4(n_3+n_4)+2n_5+n_6\\ &= (n_2+2(n_3+n_4)+n_5)+(n_1+n_2+n_3+n_4+n_5+n_6)+(n_2+n_3+n_4)\\ &= S^{z}_{B,s^1}+n+N_{zy},\end{aligned}$$

$$\begin{aligned}S^{u}_{B,s^3} &= 2n_1+n_4+3(n_5+n_6)\\ &= (n_1+n_4+2(n_5+n_6))+(n_1+n_5+n_6)\\ &= S^{y}_{B,s^1}+N_{yz},\end{aligned}$$

$$\begin{aligned}S^{v}_{B,s^3} &= 2n_2+3(n_3+n_4)+n_5\\ &= (n_2+2(n_3+n_4)+n_5)+(n_2+n_3+n_4)\\ &= S^{z}_{B,s^1}+N_{zy}.\end{aligned}$$

Since *u* is the clone of *y*, so that *u* is beaten by *y* always. Hence *y* beats *x* if, $S^{yx}_{B,s^3}>0$

$\Rightarrow S^{y}_{B,s^1}>S^{xy}_{B,s^1}+N_{xz}$ i.e., $2S^{y}_{B,s^1}>S^{x}_{B,s^1}+N_{xz}$ i.e., $S^{y}_{B,s^1}+N_{yx}+N_{yz}>S^{x}_{B,s^1}+N_{xz}$

i.e., $S^{y}_{B,s^1}+N_{yx}+N_{yz}+N_{xy}>S^{x}_{B,s^1}+N_{xz}+N_{xy}$

i.e., $S^{y}_{B,s^1}+n+N_{yz}>S^{x}_{B,s^1}+N_{xz}+N_{xy}$,

$$n_1+n_4+2(n_5+n_6)+n+n_1+n_5+n_6>2(n_1+n_2)+n_3+n_6+(n_1+n_2+n_6)+(n_1+n_2+n_3)$$

i.e., $4(n_5+n_6)+3n_1+n_2+n_3+2n_4>4(n_1+n_2)+2(n_3+n_6)$. (3)

Again *v* is the clone of z so that $S^{yz}_{B,s^3}\geq 0$ gives $S^{zv}_{B,s^3}\geq 0$ which implies $S^{yv}_{B,s^3}\geq 0$.

Again y beats z if $S^{yz}_{B,s^3}\geq 0$ $\Rightarrow S^{yz}_{B,s^3}\geq N_{zy}-N_{yz}$ $\Rightarrow S^{y}_{B,s^1}+n+N_{yz}\geq S^{z}_{B,s^1}+n+N_{zy}$

i.e., $4(n_5+n_6)+3n_1+n_2+n_3+2n_4>4(n_3+n_4)+3n_2+2n_5+n_1+n_6$. (4)

Here we have 6 preference types and 5 candidates but we find that there are two new candidates *u* and *v* are always ranked respectively after *y* and *z* in the individual preferences. Hence u is the clone of y and v is the clone z. Inequalities (3) and (4) satisfy all the properties of Borda voting, so that y wins in the election.

In this section we have shown by calculations that Borda voting is for sincere voters and manipulation is impossible but two defeated candidates can manipulate Borda voting in their favor by introducing their clones (Islam et al. 2011).

Proposition 1: *Borda voting is manipulation free but it is not free from cloning manipulation.*

Proof: From the above discussion we have observed that Borda voting is for sincere voters. In this method voting manipulation is impossible said by Jeans-Charles Borda. Also we have shown by arithmetic calculations that cloning manipulation of Borda voting is possible in sincere ways. Again we have discussed by algebraic calculations of the single cloning manipulation of Borda voting in detail. The simultaneous cloning manipulation of Borda voting is also elucidated clearly by algebraic calculations. Hence Borda voting is manipulation free but it is not free from cloning manipulation. Q.E.D.

Vulnerabilities in Borda Cloning Manipulation

Although Borda voting is cloning manipulable in sincere ways but it is vulnerable. Huang and Chua (2000) have discussed the vulnerability in Borda cloning manipulation. Let $Z(n)$ be the set of the cloning manipulation of the Borda voting situations which can be write as a polynomial of n as follows:

$$Z(n) = x_5 n^5 + x_4 n^4 + x_3 n^3 + x_2 n^2 + x_1 n + x_0 \tag{5}$$

where $n = a + bi$, i = 0,1,...,5 and b is the productivity of the integer sequence for which the representation is valid, and a is such that $n = a\,(\mathrm{mod}\,b)$. From (5) the six equations can be written as;

$$x_5 n^5 + x_4 n^4 + x_3 n^3 + x_2 n^2 + x_1 n + x_0 = K_1, \tag{6a}$$

$$x_5 n^5 + x_4 n^4 + x_3 n^3 + x_2 n^2 + x_1 n + x_0 = K_2, \tag{6b}$$

$$x_5 n^5 + x_4 n^4 + x_3 n^3 + x_2 n^2 + x_1 n + x_0 = K_3, \tag{6c}$$

$$x_5 n^5 + x_4 n^4 + x_3 n^3 + x_2 n^2 + x_1 n + x_0 = K_4, \tag{6d}$$

$$x_5 n^5 + x_4 n^4 + x_3 n^3 + x_2 n^2 + x_1 n + x_0 = K_5, \tag{6e}$$

$$x_5 n^5 + x_4 n^4 + x_3 n^3 + x_2 n^2 + x_1 n + x_0 = K_6, \tag{6f}$$

where $Z = \{K_1, K_2, ..., K_6\}$ are the set of the cloning manipulation of the Borda voting situations. The Huang and Chua algorithm (Huang and Chua 2000) obtain the periodicities are b = 24 and b = 15 for single and simultaneous cloning manipulation respectively. They have set 6 equations with 6 unknowns $(x_0, x_1, ..., x_5)$ as like (6) and have solved these 6 simultaneous equations to obtain the value of these 6 unknowns. Huang and Chua (2000) solved the values of 6 unknowns for vulnerability in single cloning manipulation as follows:

$x_0 = -440880$, $x_1 = 94221$, $x_2 = 208580$, $x_3 = 111750$, $x_4 = 24420$, and $x_5 = 1909$.

Huang and Chua (2000) also solved the values of 6 unknowns for vulnerability in simultaneous cloning manipulation as follows:

$x_0 = -214208$, $x_1 = 32280$, $x_2 = 109260$, $x_3 = 58745$, $x_4 = 12900$, and $x_5 = 1023$.

Hence the polynomials of vulnerability of in single cloning manipulation are as;

$$Z(n) = 1909n^5 + 24420n^4 + 111750n^3 + 208580n^2 + 94221n - 440880 \quad (7)$$

and the polynomials of vulnerability of in single cloning manipulation are as;

$$Z(n) = 1023n^5 + 12900n^4 + 58745n^3 + 109260n^2 + 32280n - 214208 . \quad (8)$$

Let $V(B^{cl}, a(b))$ and $V(B^{simcl}, a(b))$ be the vulnerability of Borda rule to single and simultaneous cloning manipulation respectively. We divide the cardinality $Z(n)$ by the total number of situations. From (7) and (8) we obtain the following vulnerabilities of Borda rule to single and simultaneous cloning manipulation respectively for $a = 1$ (Huang and Chua 2000, Serais 2002) as follows:

$$V(B^{cl}, 1(24)) = \frac{1909n^5 + 24420n^4 + 111750n^3 + 208580n^2 + 94221n - 440880}{3072(n+1)(n+2)(n+3)(n+4)(n+5)}, \quad (9)$$

$$V(B^{simcl}, 1(15)) = \frac{1023n^5 + 12900n^4 + 58745n^3 + 109260n^2 + 32280n - 214208}{1875(n+1)(n+2)(n+3)(n+4)(n+5)}. \quad (10)$$

Now we calculate the vulnerabilities of single and simultaneous cloning manipulations for different values of n.

For $n = 1$ from (9) we get,

$$V(B^{cl}, 1(24)) = \frac{1909 + 24420 + 111750 + 208580 + 94221 - 440880}{3072 \times 2 \times 3 \times 4 \times 5 \times 6} = 0,$$

For $n = 1$ from (10) we get,

$$V(B^{simcl}, 1(15)) = \frac{1023 + 12900 + 58745 + 109260 + 32280 - 214208}{1875 \times 2 \times 3 \times 4 \times 5 \times 6} = 0.$$

For $n = 2$ from (9) we get,

$$V(B^{cl}, 1(24)) = \frac{1909 \times 32 + 24420 \times 16 + 111750 \times 8 + 208580 \times 4 + 94221 \times 2 - 440880}{3072 \times 3 \times 4 \times 5 \times 6 \times 7}$$

$$= 0.4286.$$

For $n = 2$ from (10) we get,

$$V(B^{simcl}, 1(15)) = \frac{1023 \times 32 + 12900 \times 16 + 58745 \times 8 + 109260 \times 4 + 32280 \times 2 - 214208}{1875 \times 3 \times 4 \times 5 \times 6 \times 7}$$

$$= 0.2857.$$

For $n = 3$ from (9) we get,

$$V(B^{cl}, 1(24)) = \frac{1909 \times 243 + 24420 \times 81 + 111750 \times 27 + 208580 \times 9 + 94221 \times 3 - 440880}{3072 \times 4 \times 5 \times 6 \times 7 \times 8}$$

$$= 0.3571.$$

For $n = 3$ from (10) we get,

$$V(B^{simcl}, 1(15)) = \frac{1023 \times 243 + 12900 \times 81 + 58745 \times 27 + 109260 \times 9 + 32280 \times 3 - 214208}{1875 \times 4 \times 5 \times 6 \times 7 \times 8}$$

$$=0.2973.$$

For $n = 4$ from (9) we get,

$$V(B^{cl}, 1(24)) = \frac{1909 \times 1024 + 24420 \times 256 + 111750 \times 64 + 208580 \times 16 + 94221 \times 4 - 440880}{3072 \times 5 \times 6 \times 7 \times 8 \times 9}$$

$$= 0.4011.$$

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For n = 4 from (10) we get,

$$V\left(B^{simcl},1(15)\right)=\frac{1023\times1024+12900\times256+58745\times64+109260\times16+32280\times4-214208}{1875\times5\times6\times7\times8\times9}$$

=0.3447.

Similarly we proceed for n = 5, 6, … and finally we calculate for $n=\infty$ as follows:

$$V\left(B^{cl},1(24)\right)=\frac{1909}{3072}=0.6214\,,$$

$$V\left(B^{cl},1(15)\right)=\frac{1023}{1875}=0.5456\,.$$

The aggregated calculating values of the vulnerabilities of the Borda rule are given in table-5 below.

Table 5: Vulnerabilities of the Borda rule in cloning manipulations.

n	Single manipulation	Simultaneous manipulation
1	0	0
2	0.4286	0.2857
3	0.3571	0.2973
4	0.4011	0.3447
...	...	...
...	...	...
...	...	...
∞	0.6214	0.5456

Proposition 2: *Borda voting suffers from vulnerabilities of cloning manipulations.*

Proof: In proposition-1 we have seen that the cloning manipulations of Borda voting in single and simultaneous ways are possible. Again we have discussed that Borda cloning manipulation is not strong and it is vulnerable. The vulnerabilities in cloning manipulations of Borda voting have shown with arithmetic and algebraic calculations and aggregated the vulnerabilities in table-5. For n = 1 there is no vulnerability but vulnerabilities varies for both single and simultaneous cloning manipulations from n = 2,3,...,∞. Hence Borda voting suffers from vulnerabilities of cloning manipulations. Q.E.D .

Concluding Remarks

In this paper we have discussed Borda voting and cloning manipulation of Borda count. Dummett (1998), Huang and Chua (2000) have analyzed the cloning manipulation of Borda count. In this paper the Borda voting and its cloning manipulation is given in simple way and with some detail calculations. Finally we have shown both single and simultaneous vulnerabilities of the Borda rule in cloning manipulation. The calculations, tables and propositions with proof are given in more clear ways.

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