Social welfare and social choice in different individuals' preferences

Mohajan, Haradhan

International Journal of Human Development and Sustainability

24 January 2011

Mohajan, H.K. (2012n), Social Welfare and Social Choice in Different Individuals' Preferences, *International Journal of Human Development and Sustainability*, 5(1): 11–22.

Social Welfare and Social Choice in Different Individuals' Preferences

Haradhan Kumar Mohajan
Assistant Professor, Faculty of Business Studies, Premier University, Chittagong, Bangladesh
E-mail: haradhan_km@yahoo.com

Abstract: *This paper discusses both social welfare and social choice using Arrow's impossibility theorem for multi-profile preference case and two versions of it for single-profile preference case. Between these two versions first one assumes a two-individual society and the second one, which is similar to a theorem of Pollak, assuming two or more individuals. In single-profile version decisiveness, simple and complex diversity must occur. This paper considers a special case of Arrow's theorem, namely single-profile preference. Diversity and decisiveness of preferences are discussed for two individuals and more than two individuals in a society.*

Keywords: Single-profile preference, Social welfare, Pollak diversity, Dictator.

Introduction

This paper is related to welfare economics and sociology, in particular social choice theory. Here we have tried to give various aspects of economics and sociology in mathematical terms and simple ways. The social welfare function (SWF) first studied by Arrow, are the rules for transforming preference profile into social preference orderings. Arrow's theorem indicates that the aggregate of individuals' preferences will not satisfy the five conditions namely i) completeness and transitivity, ii) universality, iii) Weak Pareto optimality, iv) independence of irrelevant alternatives and v) non-dictatorship simultaneously. So that in Arrow's theorem one of the individuals becomes a dictator.

Arrow in his impossibility theorem used multi-profile preference. But here we will describe both multi-profile and two versions of single-profile preference. Single-profile Arrow theorems were first proved in the late 1970s and early 1980s by Parks (1976), Hammond (1976), Kemp and Ng (1976), Pollak (1979), Roberts (1980) and Rubinstein (1984). Single-profile theorems were developed in response to an argument of Samuelson (1967) against Arrow. Single-profile Arrow theorem established that bad results (dictatorship, or illogic of social preferences, or, more generally, impossibility of aggregation) could be proved with one fixed preference profile (or set of ordinal utility functions), provided the profile is "diverse" enough (Parks 1976, Hammond 1976, Kemp and Ng 1976, Pollak 1979, Roberts 1980 and Rubinstein 1984).

In multi-profile Arrow theorem we consider *N*-individuals as like Arrow (1963). But in single-profile case we consider two-individuals for theorem-1 and two or more individuals for theorem-2. In single-profile models, neutrality is the natural assumption to substitute for Arrow's independence of irrelevant alternatives (IIA) assumption. The second version is close to Pollak's theorem, where there are two or more individuals and strengthens neutrality to neutrality/monotonicity as like Pollak (1979) and others. In both multi-profile and single-profile cases of course there is a dictator. But dictators in single profile models are sometimes innocuous than in multi-profile case.

Notations

Let $N = \{1,2,\dots, n\}$ be the set of individuals or voters which are finite and N is finite subset of the non-negative real line R_+ and $\#(N) = n \geq 2$. The set of alternatives or social options is denoted by $X=\{x,y,z,...\}$. For any $x,y\in X$ one prefers *x* to *y* or he prefers *y* to *x*, or he is indifferent to the choice between *x* and *y*. We can write these possibilities, respectively, as follows: *xPy*, *yPx*, *xIy*. Sometimes we use the notation *xRy* to mean that either *x* is preferred to *y* or *x* is indifferent to *y*, so that *y* is not preferred to *x*. Person *i*'s preference relation is R_i. xR_iy means person *i* prefers *x* to *y* or is indifferent between them; xP_iy means *i* prefers (strictly) *x* to *y*; xI_iy means *i* is indifferent between them. The utility function (Islam, Mohajan and Moolio 2009 a,b, 2011) as follows: $u(x)=u(x_1,x_2,...,x_n)$. In preference relation we can write $u(x) > u(y)$ then *xPy*.

Multi-Profile Arrow's Impossibility Theorem

Arrow's impossibility theorem (Arrow 1963) is very subtle but delicate. Arrow showed that the preferences of many individuals be aggregated into social preference but there is a flaw in this aggregation. Because a social welfare function cannot be derived by democratic vote to reflect the preferences of all the individuals in the society. Here we will try to give a very simple version of Arrow's impossibility theorem following Feldman 1974, Barbera 1980, 2010, Bossert and Weymark 2003, Geanokoplos 2005, Breton and Weymark 2006, Islam1997, 2008, Islam, Mohajan and Moolio 2009 a ,b, 2011 and Miller 2009.

For simplicity let us consider there are two individuals in the society and three social alternatives *x*, *y*, *z*. For the preference orderings for individual 1 or 2 there are exactly 6×6=36 different constellations of individual preferences possible in the society (figure-1) where alternatives are ordered from top to bottom (Feldman 1974, Islam, Mohajan and Moolio 2009 a, b).

Before going to define Arrow's impossibility theorem we have to study the following requirements (Arrow 1963, Sen 1970, Feldman 1974, Feldman and Serrano 2006, Islam 1997, 2008, and Islam, Mohajan and Moolio 2009 a, b, 2011):

Completeness and Transitivity

Completeness in social choice implies for any pair of alternatives *x* and *y* then either *xRy* or *yRx* must hold, and transitivity in social choice implies for any triple *x*, *y*, *z* we have for *xRy* and *yRz* must imply *xRz*. The social preference relations generated by a collective choice rule must be complete and transitive. The requirement says that a collective choice

Figure-1 The preference orderings for individual 1 or 2 there are exactly 36 different constellations of individual preferences possible in the society.

	1	2	1	2	1	2	1	2	1	2	1	2
1st	x	x	x	x	x	y	x	y	x	z	x	z
2nd	y	y	y	z	y	z	y	x	y	x	y	y
3rd	z	z	z	y	z	x	z	z	z	y	z	x
1st	x	x	x	x	x	y	x	y	x	z	x	z
2nd	z	y	z	z	z	z	z	x	z	x	z	y
3rd	y	z	y	y	y	x	y	z	y	y	y	x
1st	y	x	y	x	y	y	y	y	y	z	y	z
2nd	x	y	x	z	x	z	x	x	x	x	x	y
3rd	z	z	z	y	z	x	z	z	z	y	z	x

1st	y	x	y	x	y	y	y	y	y	z	y	z
2nd	z	y	z	z	z	z	z	x	z	x	z	y
3rd	x	z	x	y	x	x	x	z	x	y	x	x
1st	z	x	z	x	z	y	z	y	z	z	z	z
2nd	x	y	x	z	x	z	x	x	x	x	x	y
3rd	y	z	y	y	y	x	y	z	y	y	y	x
1st	z	x	z	x	z	y	z	y	z	z	z	z
2nd	y	y	y	z	y	z	y	x	y	x	y	y
3rd	x	z	x	y	x	x	x	z	x	y	x	x

rule must always permit social choices between alternatives, and that social choices must be consistent, or not inherently self-contradictory. The axiom of completeness seems sufficient reasonable but we explain some objections with it as follows (Feldman and Serrano 2006):

- If x and y are two complicated parts of a new type computer, an individual Mr. X does not know about them and may be unwilling to choose.
- If Mr. X is given the choice between shooting his two intimate friends he must prevent for this.
- If Mr. X depends on his parents, his friends, his religious books, his government etc., he may be incapable of making choices himself.

We have seen that transitivity is fairly reasonable but in a real society there are objections to the transitivity assumption. In some circumstances transitivity may not be true. Suppose Mr. X prefers apple (A) to banana (B) and banana to cherry (C). Transitivity relation implies that Mr. X must prefer A to C but sometimes he may prefer C to A. If xIy, yIz then by transitivity xIz which is least realistic, since it can be applied repeatedly to get nonsense results: Let t_1 be a cup of tea with one grain of sugar in it; let t_2 be a cup of tea with two grains of sugar in it; and so on. Hence Mr. X can not taste the difference between t_n and t_{n+1}, for any $n \in N$ he must be indifferent between them. But he must be indifferent between t_0 and $t_{1,000,000,000,000}$ which is probably false. We will assume that every individual in a society is rational that is every member in the society choose best alternative for himself.

Universality or Unrestricted Domain (U)

In social choice theory U is a property of SWF's in which all preferences of all the voters are factored into the final ordering of societal choices. Hence U is a common requirement for all social choice functions. U indicates that the SWF accounts for all preferences among all voters to yield a unique and complete ranking of social choices. The SWF must be wide enough in scope to work from any logical set of individual orderings. The Pareto principal gives a perfectly fine social orderings if the preferences of individuals are unanimous. But in the case of incomplete preference relations it will not have a social orderings so that it fails to satisfy this requirement of Arrow. Similarly, the method of majority decision (MMD) may yield intransitives in some cases so the MMD also fails for U. According to this restriction in figure-1, the rule should give us a social preference ordering for every cell not just for the easy ones, like those where there is unanimous agreement that is the diagonal cells in figure-1.

Pareto Principal

Weak Pareto Principal: The Social choice function must satisfy the Pareto principle in the weak form that is if everyone prefers x to y, then the society must also prefer x to y. Mathematically, for any $x, y \in X$, $\forall i : xP_i y \Rightarrow xPy$ i.e., if alternative x is ranked above y for all orderings $(R_1,..., R_N)$ then x is ranked higher than y by $F(R_1,..., R_N)$. Pareto consistency is a very mild requirement for a collective choice rule (CCR). If the societies are ruled by external forces then one can not expect it to hold in societies. For example, (Feldman 1974) everyone prefers lust and gambling, on the one hand, to chastity and frugality on the other, but where, according to a Holy Book, the society state of chastity and frugality is preferable to the society state of lust and gambling. But if the external forces, thinking for their economy, naturally would recommend lust and gambling.

Strong Pareto Principal: For all $x, y \in X$ if $xR_i y$ for all i, and $xP_i y$ for some i, then xPy.

Independent of Irrelevant Alternatives (IIA): IIA means that the social ranking of x vis-à-vis y must depend only on individual rankings of x vs. z or y vs. w or z vs. w or any other such irrelevancy. Let X be a set of alternatives, R be the set of social ordering and $C(Y, R)$ be the choice function, then for a single individual the choice made from any fixed environment Y should be independent of the very existence of alternatives outside of Y. Again consider, if R and R' be the relations determined by f corresponding respectively to two sets of individual preferences, $(R_1,..., R_n)$ and $(R'_1,..., R'_n)$. If $x, y \in X, xR_i y \Leftrightarrow xR'_i y \ \forall i$, then $C(Y, R)$ and $C(Y, R')$ are the same. Condition IIA is the most subtle of all the requirements. Suppose society chooses democracy (d) over communism (c) and fascism (f) is their third alternative. At one stage everyone of the society suddenly changes the desirability of f but no one changes his mind about d vs. c. The independence requirement says that, if society is faced with the choice between d and c, and only those two, it must still choose d over c. The standard example of a CCR that violates independence is the rule of weighted voting. Let the society be made up of two individuals $\underline{1}$ and $\underline{2}$. Suppose $\underline{1}$'s initial preferences are fP_1dP_1c, while $\underline{2}$'s initial preferences are fP_2dP_2c . Suppose a person's first choice gets a weight of 10 points, a second choice gets 7 points, and a third choice get 3 points. If the social choice is between c and d, d gets 7+7=14, and c gets 10+3=13 points; so d is socially preferred to c. Now let $\underline{1}$ become totally disillusioned with f; his ordering changes to dP_1cP_1f . Now if d vs. c vote is repeated, d gets, 10+7=17 points, and c gets, 10+7=17 points. Society has become indifferent between d and c; even though neither $\underline{1}$ nor $\underline{2}$ changed his mind about these two alternatives.

Non-dictatorship

It is required that the Social welfare function should not be dictatorial. That is, there should be no individual such that whenever he prefers x to y, society must prefers x to y, irrespective of the preferences else. This is called the condition of non-dictatorship. Mathematically, there is no individual i such that for every element in the domain of rule f,

$\forall x, y \in X$ such that $xP_i y \Rightarrow xPy$. Anonymous voting systems with at least two voters satisfy the non-dictatorship property. The dictatorship is undesirable in the society. First, it is undesirable because one's worst enemy might be dictator. Second, it is not a collective choice rule. So that dictatorship may cause the violation of human rights. No dictator assumption is different in a single-preference profile from what it is in the multi-profile case. For example, in the single-profile case if all the individuals in a society have the same preferences and if Pareto optimality holds then by definition everyone is a dictator. If individual *i* is indifferent among all the alternatives, she is by definition a dictator.

The Pareto consistency requirement says a CCR must respect unanimous opinion: if both $\underline{1}$ and $\underline{2}$ prefer one alternative to another, then society must also prefer the one to the other. For example, given the configuration of individual preferences of the 1st row, 3rd column cell of figure-1, the Pareto requirement says *x* and *y* must be socially preferred to *z*. Now consider the independence requirement. Suppose that, when applied to the constellation of preferences,

	$\underline{1}$	$\underline{2}$
1st	x	y
2nd	y	x
3rd	z	z

a collective choice rule gives *x* is socially preferred to y. Then xP_1y providing that xP_1y and yP_2x, no matter how $\underline{1}$ and $\underline{2}$ changes their feelings about the irrelevant alternative *z*. Similarly, we must have yPx or xIy whenever xP_1y and yP_2x.

We are now in a position to discuss Arrow's impossibility theorem as follows (for geometrical interpretation and combinational approach to Arrows theorem see Islam, Mohajan and Moolio 2009a:

Arrow's theorem: *Suppose that the set of alternatives X has at least three elements and the conditions* (*i*) *to* (*v*) *are satisfied. Then there exists an individual* $u_k \in U$, *such that*

$$W(w_1, w_2, ..., w_n) = w_k, \text{ some } k, \qquad 1 \le k \le n$$

that is, the group preference coincides with that of some one (single) individual.

Single-Profile Arrow's Impossibility Theorem

We have discussed multi-profile version of Arrow's theorem. In this section we will discuss two simple versions of the same theorem but for single-profile case only following Feldman and Serrano (2008). We will also introduce some illustrative examples and definitions to clarify the concepts that we attempt to discuss. We have defined some definitions above and here we add some definitions which are related to single-profile model. Some of them are discussed as follows:

Voting Paradox: Now we discuss the Condorcet voting paradox in which there is no Condorcet winner (an alternative that beats every other alternative in sequence of pair-wise majority contests), (Condorcet 1785). Let us assume that there are 17 voters of three types and three alternatives *x*, *y*, *z*. Let preference relations being as follows:

Type 1: $xPyPz$ by 8 voters,

Type 2: $yPzPx$ by 5 voters,

Type 3: $zPxPy$ by 4 voters.

In an election a vote between x and y, candidate x collects 8 + 4 = 12 votes and y collects 5 votes, so that x wins. Again a vote between y and z, candidate y collects 8 + 5 = 13 votes and z collects 4 votes, so that y wins. Again a vote between x and z, candidate x collects 8 votes and z collects 4 + 5 = 9 votes, so that z wins. We observe that there is a cycle in the voting results where x is defeated by y, and y is defeated by z, and also z is defeated by x which is a voting paradox.

Neutrality: Suppose individual preferences for w vs. z are identical to individual preferences for x vs. y, then the social preference for w vs. z must be identical to the social preference x vs. y. More specifically, for all x, y, z and w assume that for all i, xP_iy if and only if wP_iz and zP_iw if and only if yP_ix. Then wRz if and only if xRy, and zRw if and only if yRx.

Simple Diversity Preference: There exists triple of alternatives x, y, z, such that xP_iy for all i, but opinions are split on x vs. z, and on y vs. z. That is, some individuals prefer x to z and some individuals prefer z to x, and some individuals prefer y to z and some individuals prefer z to y.

Illustrative Examples in a single-profile and two Individuals Model

Let us consider there are two individuals and three alternatives, and also assume no individual indifference between any pair of alternatives. Preferences of the two individuals are indicate from top (most preferred) to bottom (least preferred) and last column indicates society's preference. Now we will illustrate 5 examples from Feldman and Serrano (2008). Observe that if one of the five examples discards, the remaining four may be mutually consistent. In all the examples 1 to 4 we use the same profile and all satisfy simple diversity. In example-5 we will modify the individual preferences.

Example 1	Individual 1	Individual 2	Social choice
	x	z	
	y	x	xPy, xIz & yIz
	z	y	

Decision: Transitivity assumption fails, since xIz and zIy should imply xIy, but society has xPy

Example 2	Individual 1	Individual 2	Social choice
	x	z	
	y	x	$xIyIz$
	z	y	

Decision: Pareto (weak or strong) fails, since xP_1y and xP_2y should imply xPy, but society has xIy.

Example 3	Individual 1	Individual 2	Social choice
	x	z	x
	y	x	z
	z	y	y

Decision: Neutrality fails, Consider preference of x vs. z, two individuals are spilt that is individual 1 prefers x to z and individual 2 prefers z to x so individual 1 gets her way. Again in y vs. z the two individuals are split and individual 2 gets her way.

Example 4

Individual 1	Individual 2	Social choice
x	z	x
y	x	y
z	y	z

Decision: Individual 1 is a dictator, so that no dictatorship assumption fails.

We have observed that although in examples 1 to 4 we use same profile but social preferences are different. Also all satisfy simple diversity assumption. Following example-5 modifies the individual preferences.

Example 5

Individual 1	Individual 2	Social choice
x	z	x
z	x	z
y	y	y

Decision: Simple diversity fails, since options are no longer split over two pairs of alternatives.

Some Elementary Arrow Paradoxes and Arrow's Theorem in a Single-profile Version when $n = 2$

Although IIA is an essential part in multi-profile Arrow's impossibility theorem but it does not play any role in single-profile case. We use neutrality assumption in single-profile instead of IIA. First we introduce Samuelson's reduction ad absurdum (Samuelson 1977, Feldman and Serrano 2008) as follows:

Example 6 (Samuelson's Chocolates) There are two individuals. There is a box of 1000 indivisible chocolates to be distributed between them. Both of the individuals are fond of chocolates, and each is hungry enough to eat all of them. The alternatives are $x_0 =$ (1000, 0); $x_1 =$ (999, 1); $x_2 =$ (998, 2),…, where the first number in each pair is the number of chocolates going to individual 1 and the second is the number going to individual 2. Here any rational individual would say that x_1 is better than x_0 i.e., $x_1 P x_0$. So that it would be legal to take a chocolate from individual 1, when he has 1000 of them, and gives it to individual 2. Let n < 1000, the individual preferences are $x_n P_1 x_{1000}$ and $x_{1000} P_2 x_n$. By the condition neutrality we get $x_{1000} P_2 x_n$! That is, society should give all the chocolates to individual 2!

Samuelson's chocolates example is a serious problem on neutrality. Neutrality would have implied that all the x's are socially indifferent. So that it is illogical if $x_1 P x_0$. In fact, any social decision procedure that simply counts instances of $xP_i y$, $yP_i x$ and $xI_i y$, but does not weigh strength of feelings, satisfies neutrality. On the basis of Samuelson's chocolates example we will introduce a proposition from Feldman and Serrano (2008) which is a strong result and we hope our proof is clearer than theirs.

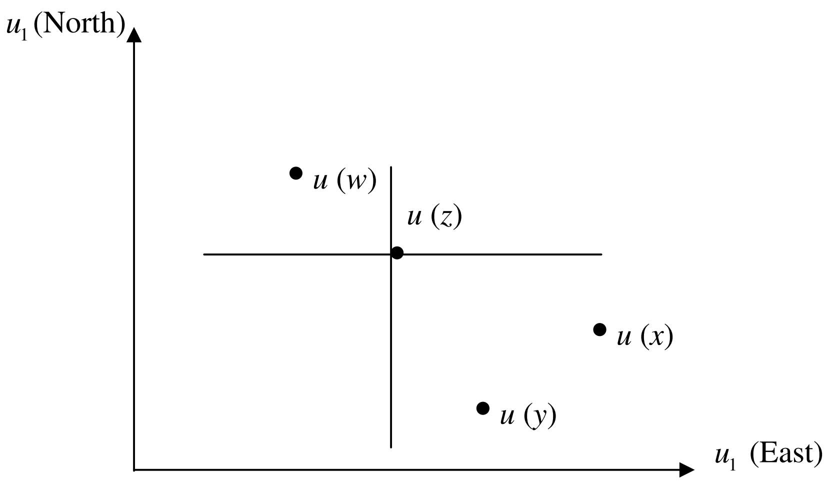
Proposition 1: *Assume $n = 2$, and the strong Pareto principle and neutrality hold. Suppose there are two alternatives x and y, and for the two individuals i and j, xP_iy and yP_jx. Suppose that social preference is xPy. Then individual i is a dictator.*

Proof: Let p and q be two alternatives in the society and let individual i's preference is pP_iq. We will show that social preference is pPq. Let contrary, so that qP_jp. Strong Pareto implies qR_jp for all j and qP_jp for some j, then social preference is qPp. So that person j is a dictator but we have person i is a dictator, a contradiction. Again neutrality implies that for four alternatives p, q, w and z, wP_iz implies pP_iq. Hence wPz implies pPq, so that individual i is a dictator. Q. E. D.

Fleurbaey and Mongin Graphical Arrow Impossibility Argument

Samuelson (1977) offered a graphical argument against Arrow's theorem with neutrality, an argument that was simplified and improved by Fleubaey and Mongin (2005). Assume

Figure 2 Fleurbaey and Morgin's Arrow impossibility argument.



that there are two individuals, and a set of alternatives x, y, z,…. Let u_1 and u_2 be the utilities of individuals 1 and 2 respectively. Consider the graph in figure-2. Each alternative shows up in the graph as a utility pair, for example $u(x) = (u_1(x), u_2(x))$ represents utility of alternative x. Now draw horizontal and vertical straight lines through u (z) creating four quadrants. North-east directions are indicated in figure-2. We assume that complete and transitivity social preferences, strong Pareto and neutrality are satisfied.

First consider for two utility vectors $u(x)$ and $u(y)$ which are on south-east quadrant and choose them so that $u(x)$ is north-east of $u(y)$. In society zIx is impossible, since if zIx then zIy must satisfy. Again if yIz and zIx then by transitivity we can write xIy. Finally $u(x)$ is north-east of $u(y)$ so that by Pareto optimality xPy. Hence social choice will be zPx or xPz. Let social choice be xPz and let w be another alternative such that $u(w)$ is in the north-west quadrant then wPz, also if $u(w)$ is in the south-west quadrant then zPw. All the cases above show that social preferences are always depend on individual 1's choice that is individual 1 is a dictator. Similarly if social preference be zPx then individual 2 would have been the dictator. So that Fleurbaey and Mongin graph produces Arrow impossibility. But there are two drawbacks to Fleurbaey and Mongin graphical impossibility argument as follows:

- It has the disadvantage that it requires the use of the utility functions u_1 and u_2, and it is cleaner to dispense with utility functions and simply use preference relations for individuals.
- It incorporates a crucial diversity assumption without being explicit about it. Assume the existence of the triple of utility vectors $u(x)$, $u(y)$ and $u(z)$, with their respective locations in the utility diagram, is in fact exactly the assumption of simple diversity; both 1 and 2 prefer x to y, but opinions are split on x vs. z and y vs. z.

Single-profile Arrow Impossibility Theorem 1

Let us consider there are two individuals say i =1, 2 and four alternatives x, y, z and w in the society. Here we only discuss single-profile version of Arrow's impossibility theorem (Feldman and Serrano 2008). We will show how five conditions above are mutually inconsistent. By strong and weak Pareto principle we can say for i =1, 2; xP_iy implies xPy and both individuals are dictators. Again by simple diversity xP_iy for i =1, 2 but opinions are split on x vs. z and y vs. z. Since opinions are split on x vs. z, one individual prefer x to z, while the other prefers z to x. If xPz, then by proposition-1 the individual who prefers x to z is a dictator. Similarly, if zPx then by proposition-1 the individual who prefers z to x is a dictator. Again if zIx, then with xPy we yield transitivity result zPy. But opinions are split on y vs. z. Then as before one person prefers y to z and another person prefers z to y. By proposition-1 the individual who prefers y to z is a dictator and similarly who prefers z to y is another dictator. Neutrality implies for any four alternatives x, y, z and w for i =1, 2, xP_iy if and only if wP_iz and zP_iw if and only if yP_ix. Then wRz if and only if xRy, and zRw if and only if yRx. In both cases as before we find a dictator. Now we are in a position to introduce Arrow's Impossibility theorem for a single-profile and two-individuals case as follows:

Theorem 1 (Feldman and Serrano 2008): *Assume n = 2. The assumptions of complete and transitive social preferences, strong Pareto, neutrality, simple diversity and no dictator are mutually inconsistent.*

Proof: Assume there are two individuals in the society and they are i = 1 and 2. For three individuals $x, y, z \in X$ we can write xP_iy. By simple diversity preference we can split xP_iy as follows: Some individuals prefer x to z, some individuals prefer z to x. By proposition-1 if xPz then individual 1 is a dictator and if zPx then individual 2 is a dictator. Also some individuals prefer y to z and some individuals prefer z to y, so that by proposition-1 if yPz then individual 1 is a dictator and if zPy then individual 2 is a dictator.

Again completeness implies if xIy then yIx. The transitivity implies xIy and yPz implies xPz. Now we split this relations between x and y then if individual 1 prefers x to y then individual 2 prefers y to x. By proposition-1 if xPy then individual 1 is a dictator and if yPx then individual 2 is a dictator.

Again if both individual i's preference relation is $xP_i y$ then by weak or strong Pareto optimality we have xPy. Now we can split this preference relation as follows: If we consider the split between x vs. z we can write if individual 1 prefers x to z then individual 2 prefers z to x. So that by proposition-1 if xPz then individual 1 is a dictator and if zPx then individual 2 is a dictator.

Now it is clear that in every case we have a dictator. So that if all the assumptions except no dictatorship condition are satisfied, then we have a dictator always. Hence the conditions are mutually inconsistent. Q. E. D.

Diversity in Single-Profile World

Let X be any non-empty set of individuals in the society, X' be the compliment of X which may be empty. X is partitioned into two non-empty subsets X_1 and X_2. Now we introduce standard Arrow array as follows:

Individual 1	Individual 2	Individual 3
x	z	y
y	x	z
z	y	x.

We have defined above simple-diversity and now we will define stronger diversity. The stronger diversity assumption was in fact used by Parks (1976) and Pollak (1979). Pollak diversity is as follows:

"Imagine *any logically possible sub-profile* of individual preferences over three *hypothetical* alternatives x, y and z. Then there exists three actual alternatives a, b and c for which the sub-profile of preferences exactly matches that *logically possible sub-profile* over x, y and z".

To clarify the Pollak diversity we consider two individuals and three alternatives (Feldman and Serrano 2008). Pollak diversity would require that every one of the following arrays be represented somewhere in the actual preference profile of the two people over the actual alternatives:

Figure 3: Pollack diversity arrays for $n = 2$.

1	2	1	2	1	2	1	2	1	2	1	2
x	x	x	x	x	y	x	y	x	z	x	z
y	y	y	z	y	z	y	x	y	x	y	y
z	z	z	y	z	x	z	z	z	y	z	x

Here the number of arrays in the table above is 3!= 6. If n = 3 we would have triples of columns instead of pairs, and there would have to be $(3!)^2 = 36$ such triples. So that with n individuals, the number of required n triples would be $(3!)^{n-1}$. Obviously, the number of arrays required for Pollak diversity raises exponentially with n. Parks' (1976) diversity in society is very similar to Pollak's, although not so clear, and he indicates that "it requires at least 3^n alternatives. . . ".

Now we define decisive set and minimally sized decisive set as follows:

Decisive set: We say that a set of individuals *X* is decisive if it is non-empty and if for all alternatives x and *y*, if xP_iy for all *i* in *X*, then *xPy*. *X* is a minimally sized decisive set if there is no decisive set of smaller cardinality.

According to the definition of decisive set the followings are appropriate:

- If an individual *i* is a dictator, then *i* by himself is a minimally sized decisive set.
- Although without two or more members, and any set strictly containing *i* is also decisive, but not minimally sized.
- By the Pareto principle (weak or strong) implies that the set of all people is decisive.

In the light of above discussion we can modify diversity assumption as follows:

Complex diversity: For any decisive set *X* with two or more members, there exists a triple of alternatives x, *y* and *z* such that xP_iy for all *i* in *X*; such that yP_iz and zP_ix for everyone outside of *X*; and such that *X* can be partitioned into non-empty subsets X_1 and X_2, where the members of X_1 all put *z* last in their rankings over the triple, and the members of X_2 all put z first in their rankings over the triple. Comparisons between simple diversity and complex diversity are given below:

- If $n = 2$ and weak Pareto holds then both of them are equivalent.
- If $n > 2$, neither are implied the other, but they are both implied by Pollak diversity.

The following example shows that for decisive set $X = \{2, 3\}$ i.e., $X_1 = \{2\}$, $X_2 = \{3\}$, $X' = \{1\}$ then the preference profile is consistent with complex diversity and an Arrow impossibility yields.

Example 7	Individual 1	Individual 2	Individual 3	Social choice
	x	*z*	*y*	
	y	*x*	*z*	*xPy, yPz, zPx*
	z	*y*	*w*	*xPw, yPw, zPw*
	w	*w*	*x*	

Decision: Transitivity for social preferences fails, with a *P* cycle among x, *y*, and *z*.

A Single-profile Arrow/Pollak Impossibility Theorem 2 when $n \geq 2$

For $n \geq 2$ we need to strengthen neutrality assumption, which we call neutrality/monotonicity and is defined as follows:

Neutrality/monotonicity: For all *x*, *y*, *z* and *w*, assume that for all *i*, xP_iy implies wP_iz and for all *i*, zP_iw implies yP_ix. Then xPy implies wPz.

A single-profile Arrow/Pollak impossibility theorem when $n \geq 2$ is not restricted to a two individuals society as like theorem-1. First we proof the following proposition which is parallel to proposition-1 (Feldman and Serrano 2008).

Proposition 2 *Assume $n \geq 2$ neutrality/monotonicity. Assume there is a non-empty group of individuals and a pair of alternatives x and y, such that xP_iy for all i in X and yP_ix for all i not in X. Suppose that xPy then X is decisive.*

Proof: Let we have a set of four alternatives $X = \{x, y, z, w\}$. By the definition of neutrality/monotonicity for all $i \in X$, xP_iy implies wP_iz. Then *xPy* implies *wPz*. Therefore *X* is decisive set. Again by neutrality/monotonicity for all $i \in X$, yP_ix implies zP_iw. Then *yPx* implies *zPw*. In this case also *X* is decisive. Q. E. D.

Now we will discuss some assumptions to state a single-profile Arrow/Pollak impossibility theorem as follows: Pareto principle states that the set of all individuals is decisive. Hence decisive sets exist. By the definition minimal decisive set we can write, it is the smallest decisive set. Let *X* be a minimal decisive set. If *X* is not minimal decisive then *X* has two or more members. Let $X = \{x, y, z\}$, a partition of *X* are two non-empty subsets X_1 and X_2. Since *X* is decisive which implies *xPy*. Now we consider the social preference for *x* vs. *z*. Let *zRx*, we have *xPy*, then by transitivity *zPy*. Then X_2 becomes decisive by proposition-2, which is impossible because we consider *X* as a minimal decisive set. Again if *xPz* by completeness, in this case as like above X_1 is decisive which is impossible. Hence in *X* there is only one individual which makes that individual a dictator. Now we are in a position to state a single-profile Arrow/Pollak impossibility theorem as follows:

Theorem 2 (Feldman and Serrano 2008): *Assume n = 2. The assumptions of complete and transitive social preferences, weak Pareto, neutrality/monotonicity, complex diversity, and no dictator are mutually inconsistent.*

For the proof of theorem-2 see Feldman and Serrano (2008).

Innocuous Dictators in Single-profile Arrow Theorem

We have seen that in multi-profile case a dictator may be dangerous. In single-profile case, a dictator may be innocuous sometimes but not always. Now we introduce examples of such dictators as follows:

- If an individual *i* is indifferent between all pairs of alternatives by definition he is a dictator.
- If weak Pareto is satisfied by the society then everyone is a dictator.
- If in a society of 100 individuals 80 have identical preferences, then these 80 individuals are dictators.

Concluding Remarks

In this paper we have analyze Arrow's impossibility theorem of multi-profile preference version and two single-profile preference versions. We have tried our best to explain the paper in easier ways by introducing examples, definitions and easier mathematical calculations. In the single-profile case we used simple-decisive set, diversity, neutrality (for theorem-1) and a stronger assumption neutrality/monotonicity, and complex-diversity (for theorem-2). In every case we have obtained a dictator. We have observed that in multi-

Mohajan, H.K. (2012n), Social Welfare and Social Choice in Different Individuals' Preferences, *International Journal of Human Development and Sustainability*, 5(1): 11–22.

profile case a dictator may be dangerous but in single-profile case, a dictator may be innocuous sometimes but not always.

References

Arrow K.J. (1963), *Social Choice and Individual Values*, (*2nd edition*), New York: John Wiley & Sons.

Barbera, S. (1980), Pivotal Voters: A New Proof of Arrow's Theorem. *Economics letter*, 6(1): 13-16.

Barbera, S. (2010), Strategy-proof Social Choice: In Arrow K. J., Sen A. K.and Suzumura K. *A Handbook of Social Choice and Welfare,* Elsevier, Amsterdam (forthcoming).

Bossert, W. and Weymark, J. (2003), Utility in Social Choice, in *Handbook of Utility Theory*, Vol-2 : *Extensions*, Barbera S. Hammond P. Seidl, C. Eds., Kluwer: Dordrecht.

Breton, M. Le and Weymark, J. (2006), Arrovian Social Choice Theory on Economics Domains: In Arrow K. J., Sen A. K.and Suzumura K. *A Handbook of social Choice and Welfare,* 2, North Holland: Amsterdam.

Condorcet, M. (1785), *Essai sur l'Application de l Analyse a la Probabilite des De'cisions Rendues $\dot{a}$ la Pluralite' des vopix*, Paris.

Feldman, M. A. (1974), A very Unsubtle Version of Arrow's Impossibility Theorem, *Economic Inquiry*, XII(4): 534-546.

Feldman, M. A. and Serrano, R. (2006), *Welfare Economics and Social Choice Theory*, (2nd Edition), New York: Springer.

Feldman, M. A. and Serrano, R. (2008), Arrow's Impossibility Theorem: Two Simple Single-Profile Version, *Harvard College Mathematics Review*, 2 (2): 46-57.

Fleurbaey M. and Mongin, P. (2005), The News of the Death of Welfare Economics is Greatly Exaggerated, *Social Choice and Welfare*, 25, 381–418.

Geonokoplos, J. (2005), The Three Brief Proof of Arrow's Impossibility Theorem, *Economic Theory,* 26(1): 211-215.

Hammond, P. (1976), Why Ethical Measures of Inequality Need Interpersonal Comparisons, *Theory and Decision*, 7: 263-274.

Islam, J. N. (1997), Aspects of Mathematical Economics and Social Choice Theory. *Proceedings of the Second Chittagong Conference on Mathematical Economics and its Relevance for Development (*Ed.) J.N. Islam. University of Chittagong, Chittagong: Bangladesh.

Islam, J. N. (2008), An *Introduction to Mathematical Economics and Social Choice Theory*, Book to appear.

Islam, J. N.; Mohajan, H. K. and Moolio, P. (2009a), Preference of Social Choice in Mathematical Economics, *Indus Journal of management & Social Sciences*, 3(1): 17-38.

Islam, J. N.; Mohajan, H. K. and Moolio, P. (2009b), Political Economy and Social Welfare with Voting Procedure: *KASBIT Business Journal,* 2(1&2): 42-66.

Mohajan, H.K. (2012n), Social Welfare and Social Choice in Different Individuals' Preferences, *International Journal of Human Development and Sustainability*, 5(1): 11–22.

Islam, J. N.; Mohajan, H. K. and Moolio, P. (2011),, Method of Voting System and Manipulation of Voting: *International Journal of Management and Transformation*, Brown Walker Press 23331 Water Circle, Boca Raton, FL 33486-8540, USA. 5(1): 10-34. Web: www.brownwalker.com/ASMT-journals.php

Kemp, M. and Ng Yew-Kwang, (1976), On the Existence of Social Welfare Functions, Social Orderings and Social Decision Functions," *Economica*, 43: 59-66.

Miller, M. K. (2009), Social Choice Theory without Pareto: The Pivotal Voter Approach. *Mathematical Social Sciences* (Accepted 20 Feb, 2009).

Parks, R. (1976), An Impossibility Theorem for Fixed Preferences: A Dictatorial Bergson-Samuelson Welfare Function: *Review of Economic Studies*, 43: 447–450.

Pollak, R. A. (1979), Bergson-Samuelson Social Welfare Functions and the Theory of Social Choice, *Quarterly Journal of Economics*, 93: 73–90.

Roberts, K. (1980), Social Choice Theory: The Single-profile and Multi-profile Approaches, *Review of Economic Studies*, 47: 441-450.

Rubinstein, A. (1984), The Single Profile Analogues to Multi-Profile Theorems: Mathematical Logic's Approach: *International Economic Review*, 25: 719-730.

Samuelson, P. (1977), Reaffirming the Existence of 'Reasonable' Bergson-Samuelson Social Welfare Functions, *Economica*, 44: 81–88.

Sen, A. K. (1970), *Collective Choice and Social Welfare*, Holden Day: Oliver and Boyd.