Consumption, Market Price of Risk, and Wealth Accumulation under Induced Uncertainty

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Abstract

In this paper we examine implications of model uncertainty due to robustness (RB) for consumption-saving, market price of uncertainty, and aggregate wealth accumulation under limited information-processing capacity (rational inattention or RI) in an otherwise standard permanent income model. We first solve the robust permanent income models with inattentive consumers and show that RI by itself creats an additional demand for robustness that leads to higher "induced uncertainty" facing consumers. Second, we explore how the induced uncertainty composed by (i) model uncertainty due to RB and (ii) state uncertainty due to RI, affects consumption-saving decisions and the market price of uncertainty. Particurly, we find that induced uncertainty can better explain the observed market price of uncertainty. Furthermore, we explore the observational equivalence between RB and risk-sensitivity (RS) in this filtering problem. Finally, we evaluate the importance of induced uncertainty and fundamental uncertainty in determining equilibrium aggregate wealth.

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1. Introduction

Hansen and Sargent (1995) first introduced robustness (RB, a concern for model misspecification) into linear-quadratic (LQ) economic models.[1] In robust control problems, agents do not know the true data-generating process and are concerned about the possibility that their model (denoted the approximating model) is misspecified; consequently, they choose optimal decisions as if the subjective distribution over shocks was chosen by an evil nature in order to minimize their expected utility.[2] Robustness (RB) models produce precautionary savings but remain within the class of LQ models, which leads to analytical simplicity. A second class of models that produces precautionary savings but remains within the class of LQ models is the risk-sensitive (RS) model of Hansen and Sargent (1995) and Hansen, Sargent, and Tallarini (1999). In the RS model agents effectively compute expectations through a distorted lens, increasing their effective risk aversion by overweighting negative outcomes. The resulting decision rules depend explicitly on the variance of the shocks, producing precautionary savings, but the value functions are still quadratic functions of the states. As shown in Hansen and Sargent (2007), the risk-sensitivity preference can be used to interpret the desire for robustness as they lead to the same consumption-saving decisions, and similar asset pricing implications.[3]

Sims (2003) first introduce rational inattention into economics and argued that it is a plausible method for introducing sluggishness, randomness, and delay into economic models. In his formulation agents have finite Shannon channel capacity, limiting their ability to process signals about the true state of the world. As a result, an impulse to the economy induces only gradual responses by individuals, as their limited capacity requires many periods to discover just how much the state has moved. Since RI introduces additional uncertainty, the endogenous noise due to finite capacity, into economic models, RI by itself creates an additional demand for robustness. In addition, agents with finite capacity need to use a filter to update their perceived state upon receiving noisy signals, which may lead to another demand for robustness, the robust Kalman filter.[4]

In this paper we first construct a robust permanent income model with inattentive consumers who have two types of concerns about model misspecification: (i) concerns about the disturbances to the perceived permanent income (the disturbances here include both the fundamental

[1]See Hansen and Sargent (2007) for a textbook treatment on robustness and Hansen and Sargent (2010) for a recent survey.

[2]The solution to a robust decision-maker's problem is the equilibrium of a max-min game between the decision-maker and nature.

[3]It is worth noting that although both RB (or RS) and CARA preferences (i.e., Caballero 1990 and Wang 2003) increase the precautionary savings premium via the intercept terms in the consumption functions, they have distinct implications for the marginal propensity to consume out of permanent income (MPC). Specifically, CARA preferences do not alter the MPC relative to the LQ case, whereas RB or RS increases the MPC. That is, under RB, in response to a negative wealth shock, the consumer would choose to reduce consumption more than that predicted in the CARA model (i.e., save more to protect themselves against the negative shock).

[4]The key assumption in Luo and Young (2010) is that agents with finite capacity distrust their budget constraint, but still use an ordinary Kalman filter to estimate the true state; in this case, a distortion to the mean of permanent income is introduced to represent possible model misspecification. However, this case ignores the effect of the RI-induced noise on the demand for robustness.

shock and the RI-induced noise shock) and (ii) concerns about the Kalman gain. For ease of presentation, we will refer to the first type of robustness as Type I and the second as Type II. After solving the model explicitly, we first examine how different types of robustness affect optimal consumption and precautionary savings via interacting with finite capacity. Specifically, we show that given finite capacity, the two types of robustness have opposing impacts on the marginal propensity to consume out of perceived permanent income (MPC) and precautionary savings. For Type I robustness, since agents with low capacity are very concerned about the confluence of low permanent income and high consumption (meaning they believe their permanent income is high so they consume a lot and then their new signal indicates that in fact their permanent income was low), they take actions which reduce the probability of this bad event – they save more.[5] As for Type II robustness, an increase in the strength of this effect increases the Kalman gain, which leads to lower total uncertainty about the true level of permanent income and then low precautionary savings. In addition, the strength of the precautionary effect is positively related to the amount of this uncertainty that always increases as finite capacity gets smaller. Using the explicit expression for consumption dynamics, we also show that increasing Type II robustness increases the robust Kalman filter gain and thus leads to less relative volatility of consumption to income (less smooth consumption process). In contrast, Type I robustness increases the relative volatility of consumption by increasing the MPC out of changes in permanent income.[6]

Furthermore, we compare the implications of risk-sensitivity and robustness for consumption and savings when considering both control and filtering decisions of inattentive consumers. In the risk-sensitive permanent income model with information imperfections due to RI, the classical Kalman filter that extremizes the expected value of a certain quadratic objective function is still optimal. After solving the RB and RS models with filtering, we establish the observational equivalence (OE) conditions between RB and RS. We find that the simple and linear OE between RB and RS established in Hansen and Sargent (2007) and Luo and Young (2010) no longer holds, we instead have a complicated and nonlinear OE between RB and RS under RI.

We next investigate the asset pricing implications of RB and RI within the PIH setting. Following Hansen (1987) and Hansen, Sargent, and Tallarini (1999), we interpret the consumption-saving decisions in terms of a social planning problem and these decisions are equilibrium allocations for a competitive equilibrium. We can then deduce asset prices as in the consumption-based asset pricing literature by finding the shadow prices that clear security markets. Since these asset prices include information about the agent's intertemporal preferences, they measure the risk and uncertainty aversion of the agent. Given the explicit solutions for consumption and saving decisions, we can explicitly solve for the market prices of uncertainty under RB and RI.[7] We find that induced uncertainty due to RB and RI significantly increases the market price of risk. The

[5] Luo, Nie, and Young (2012) apply Type I RB in the SOE-RBC model proposed in Aguiar and Gopinath (2007) and show that this type of RB can help generate realistic relative volatility of consumption to income and the current account dynamics observed in emerging and developed small-open economies.

[6] This mechanism is similar to that examined in Luo and Young (2010).

[7] To explore how induced uncertainty due to RB and RI affects market prices of uncertainty, we follow the procedure adopted in Epstein and Wang (1994) and Hansen, Sargent, and Tallarini (1999).

mechanism is straightforward to describe. Under RB, the market price of uncertainty is related to the norm of the worst-case shock (that is, the size of the pessimistic distortion to the underlying stochastic process for income); adding rational inattention increases the size of these distortions and therefore amplifies the effect on asset prices. We find that our model, under plausible calibrations of the fear of model misspecification based on detection error probabilities (as in Hansen and Sargent 2007), produces stochastic discount factors that satisfy the Hansen-Jagannathan bounds.

We finally consider an aggregate economy with a continuum of inattentive agents who face mortality risk and have a preference for robustness. We find that in the steady state equilibrium, induced uncertainty due to RB and RI affects the level of aggregate wealth in general equilibrium. Specifically, we show that a proportionate increase in the parameters governing Type I robustness, Type II robustness, or channel capacity can significantly reduce, increase, and reduce equilibrium aggregate wealth to income ratio, respectively. Induced uncertainty generates a precautionary savings motive and also increases the propensity to consume out of income increases; in general equilibrium these two effects exactly cancel in the absence of growth. When the endowment grows over time, however, induced uncertainty can produce higher or lower aggregate wealth, depending on how these various forces add up.

2. Robust Control and Filtering under Rational Inattention

2.1. A Rational Inattention Version of the Standard Permanent Income Model

In this section we consider a rational inattention (RI) version of the standard permanent income model. In the standard permanent income model (Hall 1978, Sargent 1978, Flavin 1981), households solve the dynamic consumption-savings problem

$$v(s_0) = \max_{\{c_t\}} E_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

subject to

$$s_{t+1} = Rs_t - c_t + \zeta_{t+1}, \tag{1}$$

where $u(c_t) = -\frac{1}{2}\left(\overline{c} - c_t\right)^2$ is the period utility function, $\overline{c} > 0$ is the bliss point, c_t is consumption,

$$s_t = b_t + \frac{1}{R} \sum_{j=0}^{\infty} R^{-j} E_t \left[y_{t+j} \right] \tag{2}$$

is permanent income, i.e., the expected present value of lifetime resources, consisting of financial wealth (b_t) plus human wealth (i.e., the discounted expected present value of current and future labor income: $\frac{1}{R}\sum_{j=0}^{\infty} R^{-j} E_t\left[y_{t+j}\right]$),

$$\zeta_{t+1} \equiv \frac{1}{R} \sum_{j=t+1}^{\infty} \left(\frac{1}{R}\right)^{j-(t+1)} \left(E_{t+1} - E_t\right) \left[y_j\right], \tag{3}$$

is the time $(t+1)$ innovation to permanent income, b_t is financial wealth (or cash-on-hand), y_t is a labor income process with Gaussian white noise innovations, β is the discount factor, and $R > 1$ is the constant gross interest rate at which the consumer can borrow and lend freely.[8] In this paper, we assume that income y_t takes the following general AR(1) process with the persistence coefficient $\rho \in [0,1]$,

$$y_{t+1} = \rho y_t + x_t + \varepsilon_{t+1}, \tag{4}$$

where $x_t \equiv (g-\rho)\, g^t y_0$ is the growth component, g is the constant gross growth rate of income, y_0 is defined as the initial level of income, and ε_{t+1} is iid with mean 0 and variance ω^2.[9] Given this income specification, we have $s_t \equiv b_t + \frac{1}{R-\rho} y_t + \frac{1}{(R-g)(R-\rho)} x_t$ and $\zeta_{t+1} = \varepsilon_{t+1} / (R-\rho)$, where and $\omega_\zeta^2 \equiv \mathrm{var}(\zeta_{t+1}) = \omega^2 / (R-\rho)^2$.[10] Finally, financial wealth (b) follows the process

$$b_{t+1} = Rb_t + y_t - c_t. \tag{5}$$

This specification follows that in Hall (1978) and Flavin (1981) and implies that optimal consumption is determined by permanent income:

$$c_t = \left(R - \frac{1}{\beta R}\right) s_t - \frac{1}{R-1}\left(1 - \frac{1}{\beta R}\right) \bar{c}. \tag{6}$$

We assume for the remainder of this section that $\beta R = 1$, since this setting is the only one that implies zero drift in consumption under rational expectations. Under this assumption the model leads to the well-known random walk result of Hall (1978):

$$\Delta c_{t+1} = (R-1)\, \zeta_{t+1}; \tag{7}$$

the change in consumption depends neither on the past history of labor income nor on anticipated changes in labor income. We also point out the well-known result that the standard PIH model with quadratic utility implies the certainty equivalence property holds: uncertainty has no effect on consumption, so that there is no precautionary saving.

To motivate what follows, we now remind readers why (7) is inadequate as an empirical representation of consumption. There are many routes we could take here; we choose to follow Campbell and Deaton (1989) and run a simple bivariate VAR of (demeaned) labor income growth and the (demeaned) saving rate out of labor income, obtaining

$$\begin{bmatrix} \Delta \log (y_t) - \mu_y \\ \frac{s_t}{y_t} - \mu_s \end{bmatrix} = \begin{bmatrix} 0.0287 & 0.4686 \\ 0.9397 & 0.1429 \end{bmatrix} \begin{bmatrix} \Delta \log (y_{t-1}) - \mu_y \\ \frac{s_{t-1}}{y_{t-1}} - \mu_s \end{bmatrix} + \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix}.$$

The data we use is quarterly NIPA data on total compensation of employees and gross saving

[8] We only require that y_t and R are such that permanent income is finite.

[9] Note that when $g = 1$, this specification reduces to $y_{t+1} = \rho y_t + (1-\rho)\, y_0 + \varepsilon_{t+1}$, one of the most popular income specifications in the consumption literature.

[10] For the rest of the paper we will restrict attention to points where $c_t < \bar{c}$, so that utility is increasing and concave.

deflated using the PCE deflator, from 1947Q1-2013Q2. Campbell and Deaton (1989) show that the coefficient matrix should have the form

$$\begin{bmatrix} \delta & 0 \\ \delta & \tau^{-1} \end{bmatrix} \tag{8}$$

where $\tau \neq 0$ is the effective discount rate for future cash flows (the growth-adjusted interest rate); this matrix embodies both a test of "excess sensitivity" (that consumption responds to predictable changes in income) and a test of "excess smoothness" (consumption growth does not vary enough with income growth).[11] It is obvious that our estimated matrix is quite far from satisfying the restrictions embodied in (8).[12] One method for attacking these rejections is to assume that the information set of the agents differs systematically from that of the econometrician; rational inattention will deliver exactly this feature by assuming the econometrician has more information than the agents do.

To this end we follow Sims (2003, 2010) and incorporate rational inattention (RI) due to finite information-processing capacity into the model. Under RI, consumers have only finite Shannon channel capacity to observe the state of the world. Specifically, we use the concept of entropy from information theory to characterize the uncertainty about a random variable; the reduction in entropy is thus a natural measure of information flow.[13] With finite capacity $\kappa \in (0, \infty)$, a variable s following a continuous distribution cannot be observed without error and thus the information set at time $t+1$, $\mathcal{I}_{t+1}$, is generated by the entire history of noisy signals $\left\{s_j^*\right\}_{j=0}^{t+1}$. Following the literature, we assume the noisy signal takes the additive form $s_{t+1}^* = s_{t+1} + \xi_{t+1}$, where ξ_{t+1} is the endogenous noise caused by finite capacity. We further assume that ξ_{t+1} is an iid idiosyncratic shock and is independent of the fundamental shock. Agents with finite capacity will choose a new signal $s_{t+1}^* \in \mathcal{I}_{t+1} = \left\{s_1^*, s_2^*, \cdots, s_{t+1}^*\right\}$ that reduces their uncertainty about the state variable s_{t+1} as much as possible. Formally, this idea can be described by the information constraint

$$\mathcal{H}\left(s_{t+1} | \mathcal{I}_t\right) - \mathcal{H}\left(s_{t+1} | \mathcal{I}_{t+1}\right) \leq \kappa, \tag{9}$$

where κ is the consumer's information channel capacity, $\mathcal{H}\left(s_{t+1} | \mathcal{I}_t\right)$ denotes the entropy of the state prior to observing the new signal at $t+1$, and $\mathcal{H}\left(s_{t+1} | \mathcal{I}_{t+1}\right)$ is the entropy after observing the new signal. κ imposes an upper bound on the amount of information – that is, the change in the entropy – that can be transmitted in any given period. Finally, following the literature, we

[11] In fact, Campbell and Deaton (1989) show that the two excesses are the same – if consumption responds excessively to anticipated income changes it necessarily must respond insufficiently to unanticipated ones.

[12] There are many econometric issues related to running this VAR that we do not attempt to address here, especially the clearly nonstationarity in the savings rate series (it drifts downward over time). For this reason we do not report the test statistics for the hypotheses that the first column entries are equal.

[13] Formally, entropy is defined as the expectation of the negative of the (natural) log of the density function, $-E\left[\ln\left(f\left(s\right)\right)\right]$. For example, the entropy of a discrete distribution with equal weight on two points is simply $E\left[\ln_2\left(f\left(s\right)\right)\right] = -0.5\ln\left(0.5\right) - 0.5\ln\left(0.5\right) = 0.69$, and the unit of information contained in this distribution is 0.69 "nats". (For alternative bases for the logarithm, the unit of information differs; with log base 2 the unit of information is the 'bit' and with base 10 it is a 'dit' or a 'hartley.') In this case, an agent can remove all uncertainty about s if the capacity devoted to monitoring s is $\kappa = 0.69$ nats.

suppose that the prior distribution of s_{t+1} is Gaussian.

Under the linear-quadratic-Gaussian (LQG) setting, as has been shown in Sims (2003, 2006), the true state under RI also follows a normal distribution $s_t|\mathcal{I}_t \sim N\left(E\left[s_t|\mathcal{I}_t\right], \Sigma_t\right)$, where $\Sigma_t = E_t\left[\left(s_t - \widehat{s}_t\right)^2\right]$. In addition, given that the noisy signal takes the additive form $s^*_{t+1} = s_{t+1} + \xi_{t+1}$, the noise ξ_{t+1} should also be Gaussian. It is worth noting that the Gaussianity of the posterior variance of the true state and the noise is optimally determined by the LQG structure.[14] The information-processing constraint, (9), can then be reduced to

$$\ln\left(R^2\Sigma_t + \omega_\zeta^2\right) - \ln\left(\Sigma_{t+1}\right) \leq 2\kappa; \tag{10}$$

Since this constraint is always binding, we can compute the value of the steady state conditional variance Σ:

$$\Sigma = \frac{\omega_\zeta^2}{\exp\left(2\kappa\right) - R^2}. \tag{11}$$

Given this Σ, we can use the usual formula for updating the conditional variance of a Gaussian distribution Σ to recover the variance of the endogenous noise (Λ):

$$\Lambda = \left(\Sigma^{-1} - \Psi^{-1}\right)^{-1}, \tag{12}$$

where $\Psi = R^2\Sigma + \omega_\zeta^2$ is the posterior variance of the state. Finally, the Kalman gain (θ) can be written as

$$\theta = \Sigma\Lambda^{-1} = 1 - \frac{1}{\exp\left(2\kappa\right)}, \tag{13}$$

and $\widehat{s}_t$ is governed by the following Kalman filtering equation

$$\widehat{s}_{t+1} = \left(1-\theta\right)\left(R\widehat{s}_t - c_t\right) + \theta\left(s_{t+1} + \xi_{t+1}\right), \tag{14}$$

given $s_0 \sim N\left(\widehat{s}_0, \Sigma\right)$.

As argued in Sims (2010), instead of using fixed finite channel capacity to model limited information-processing ability, one could assume that the marginal cost of information processing (i.e., the shadow price of information-processing ability) is constant. That is, the Lagrange multiplier on (10) is constant. In the univariate case, the objective of the agent with finite capacity in the filtering problem is to minimize $E\left[\sum_{t=0}^{\infty}\beta^t\left(s_t - s_t^*\right)^2\right]$, subject to the information-processing constraint, or

$$\min_{\{\Sigma_t\}}\left\{\sum_{t=0}^{\infty}\beta^t\left[\Sigma_t + \lambda\ln\left(\frac{R^2\Sigma_{t-1}+\omega_\zeta^2}{\Sigma_t}\right)\right]\right\},$$

where Σ_t is the conditional variance at t, λ is the Lagrange multiplier corresponding to (10), and we impose the restriction that $\beta R = 1$ for simplicity. Solving this problem yields the optimal

[14]This result is often assumed as a matter of convenience in signal extraction models with exogenous noises, and RI can rationalize this assumption.

steady state conditional variance:

$$\Sigma = \frac{-\left(1-R\widetilde{\lambda}\right)+\sqrt{\left(1-\widetilde{\lambda}R\right)^2+4\widetilde{\lambda}R^2}}{2R^2}\omega_\zeta^2, \tag{15}$$

where $\widetilde{\lambda}=\lambda/\omega_\zeta^2$ is the normalized shadow price of information-processing capacity. It is straightforward to show that as λ goes to 0, $\Sigma = 0$; and as λ goes to ∞, $\Sigma = \infty$. Note that $\frac{\partial \ln \Sigma}{\partial \ln \omega_\zeta^2} < 1$ if we adopt the assumption that λ is fixed, while $\frac{\partial \ln \Sigma}{\partial \ln \omega_\zeta^2} = 1$ in the fixed κ case. Comparing (15) with (11), it is clear that the two RI modeling strategies are observationally equivalent in the sense that they lead to the same conditional variance if the following equality holds:

$$\kappa = \ln R + \frac{1}{2}\ln\left(1+\frac{2}{-\left(1-R\widetilde{\lambda}\right)+\sqrt{\left(1-\widetilde{\lambda}R\right)^2+4R^2\widetilde{\lambda}}}\right). \tag{16}$$

In this case, the Kalman gain is

$$\theta = 1-\frac{1}{R}\left\{1+\frac{2}{-\left(1-R\widetilde{\lambda}\right)+\sqrt{\left(1-\widetilde{\lambda}R\right)^2+4R^2\widetilde{\lambda}}}\right\}^{-1}. \tag{17}$$

It is obvious that κ converges to its lower limit $\underline{\kappa} = \ln(R) \approx R-1$ as λ goes to ∞; and it converges to ∞ as λ goes to 0. In other words, using this RI modeling strategy, the consumer is allowed to adjust the optimal level of capacity in such a way that the marginal cost of information-processing for the problem at hand remains constant. Note that this result is consistent with the concept of 'elastic' capacity proposed in Kahneman (1973). From 1, it is also clear that if λ is fixed, an increase in ω_ζ^2 will lead to more capacity being devoted to monitoring the evolution of the state. Figure (1) illustrates how κ and θ change with λ when $R = 1.03$ and $\omega_\zeta^2 = 1$.

Note that after substituting (1) into (14), we have an alternative expression of the regular Kalman filter:

$$\widehat{s}_{t+1} = R\widehat{s}_t - c_t + \eta_{t+1}, \tag{18}$$

where

$$\eta_{t+1} = \theta R\left(s_t - \widehat{s}_t\right) + \theta\left(\zeta_{t+1}+\xi_{t+1}\right) \tag{19}$$

is the innovation to the mean of the distribution of perceived permanent income,

$$s_t - \widehat{s}_t = \frac{(1-\theta)\,\zeta_t}{1-(1-\theta)R\cdot L} - \frac{\theta\xi_t}{1-(1-\theta)R\cdot L}, \tag{20}$$

and $E_t\left[\eta_{t+1}\right] = 0$ because the expectation is conditional on the perceived signals and inattentive agents cannot perceive the lagged shocks perfectly.[15] The variance of the innovation to the perceived state can be written as

$$\omega_\eta^2 = \operatorname{var}(\eta_{t+1}) = \frac{\theta}{1-(1-\theta)R^2}\omega_\zeta^2,$$

which means that ω_η^2 reflects two sources of uncertainty facing the consumer: (i) fundamental uncertainty, ω_ζ^2 and (ii) induced uncertainty, i.e., state uncertainty due to RI, $\left[\frac{\theta}{1-(1-\theta)R^2}-1\right]\omega_\zeta^2$. Therefore, as κ decreases, the relative importance of induced uncertainty to fundamental uncertainty increases.

In the next section, we will discuss alternative ways to robustify this RI-PIH model and their different implications for consumption, precautionary savings, and the welfare costs of uncertainty. The RB-RI model proposed here encompasses the hidden state (HS) model discussed in Hansen, Sargent, and Wang (2002) and Hansen and Sargent (2005); the main difference is that agents in the RB-RI model cannot observe the entire state vector perfectly, whereas agents in the RB-hidden state model can observe some part of the state vector (in particular, the part they control).

2.2. Concerns about the Fundamental Shock and the Noise Shock

As shown in Hansen and Sargent (2007), we can robustify the permanent income model by assuming agents with finite capacity distrust their model of the data-generating process (i.e., their income process), but still use an ordinary Kalman filter to estimate the true state. Note that without the concern for model misspecification, the consumer has no doubts about the probability model used to form the conditional expectation of permanent income (s). It is clear that the Kalman filter under RI, (18), is not only affected by the fundamental shock (ζ_{t+1}), but also affected by the endogenous noise (ξ_{t+1}) induced by finite capacity; these noise shocks could be another source of the demand for robustness. We therefore need to consider this demand for robustness in the RB-RI model. By adding the additional concern for robustness developed here, we are able to strengthen the effects of robustness on decisions.[16] Specifically, we assume that the agent thinks that (18) is the approximating model.

A simple version of robust optimal control considers the question of how to make decisions when the agent does not know the probability model that generates the data. Specifically, an agent with a preference for robustness considers a range of models surrounding the given approximating model, (18):

$$\widehat{s}_{t+1} = R\widehat{s}_t - c_t + \omega_\eta w_t + \eta_{t+1}. \tag{21}$$

[15]In order that the variance of η be finite we need $\kappa > \ln(R) \approx R-1$. For short time periods this requirement is obviously not very restrictive. Since $R>1$, some minimum level of capacity is needed to control the conditional mean of permanent income.

[16]Luo, Nie, and Young (2012) use this approach to study the joint dynamics of consumption, income, and the current account in emerging and developed countries.

where w_t distorts the mean of the innovation, and makes decisions that maximize lifetime expected utility given this worst possible model (i.e., the distorted model).[17] To make that model (18) is a good approximation when (21) generates the data, we constrain the approximation errors by an upper bound ψ_0:

$$E_0\left[\sum_{t=0}^{\infty}\beta^{t+1}w_t^2\right]\leq\psi_0, \tag{22}$$

where $E_0[\cdot]$ denotes conditional expectations evaluated with model, and the left side of this inequality is a statistical measure of the discrepancy between the distorted and approximating models. Note that the standard full-information RE case corresponds to $\psi_0 = 0$. In the general case in which $\psi_0 > 0$, the evil agent is given an intertemporal entropy budget $\eta_0 > 0$ which defines the set of models that the agent is considering. Following Hansen and Sargent (2007), we compute robust decision rules by solving the following two-player zero-sum game: a minimizing decision maker chooses the optimal consumption process $\{c_t\}$ and a maximizing evil agent chooses the model distortion process $\{w_t\}$.

Following Hansen and Sargent (2007a), a simple robustness version of the PIH model proposed above can be written as

$$v(s_t)=\max_{c_t}\min_{\nu_t}\left\{-\frac{1}{2}(\bar{c}-c_t)^2+\beta\left[\frac{1}{2}\vartheta_0 w_t^2+E_t\left[v(s_{t+1})\right]\right]\right\} \tag{23}$$

subject to the distorted transition equation (i.e., the worst-case model), (21), where $\vartheta_0 > 0$ is the Lagrange multiplier on the constraint specified in (22) and controls how bad the error can be. (23) is a standard dynamic programming problem and can be easily solved using the standard procedure.[18] The following proposition summarizes the solution to the RB-RI model.

Proposition 1. *Given ϑ_0 and κ, the consumption function under RB and RI is*

$$c_t=\frac{R-1}{1-\Pi}\widehat{s}_t-\frac{\Pi\bar{c}}{1-\Pi} \tag{24}$$

with $\Pi < 1$, the mean of the worst-case shock is

$$\omega_\eta w_t=\frac{(R-1)\Pi}{1-\Pi}\widehat{s}_t-\frac{\Pi\bar{c}}{1-\Pi}, \tag{25}$$

and $\widehat{s}_t$ is governed by

$$\widehat{s}_{t+1}=\rho_s\widehat{s}_t+\frac{\Pi\bar{c}}{1-\Pi}+\eta_{t+1}, \tag{26}$$

where $\rho_s=\frac{1-R\Pi}{1-\Pi}\in(0,1)$,

$$\Pi=\frac{R\omega_\eta^2}{\vartheta_0}\in(0,1), \tag{27}$$

[17] Formally, this setup is a game between the decision-maker and a malevolent nature that chooses the distortion process w_t.

[18] There is a one-to-one correspondence between ψ_0 in (22) and ϑ_0 in (23).

η_{t+1} *is defined in (19),*

$$\omega_\eta^2 = \frac{\theta}{1-(1-\theta)R^2}\omega_\zeta^2,$$

and $\theta = 1 - 1/\exp(2\kappa)$.

Proof. See Online Appendix 8.1. $\Pi < 1$ can be obtained because the second-order condition for the optimization problem is

$$\frac{R(R-1)}{2\left(1 - R\omega_\eta^2/\vartheta_0\right)} > 0, \text{ i.e., } \Pi < 1.$$

■

Following Hansen and Sargent (2007), we can also use multiplier preferences to represent a fear of model misspecification:

$$\widehat{v}(\widehat{s}_t) = \max_{c_t}\left\{-\frac{1}{2}(c_t - \overline{c})^2 + \beta \min_{m_{t+1}} E_t\left[m_{t+1}\widehat{v}(\widehat{s}_{t+1}) + \vartheta_0 m_{t+1}\ln(m_{t+1})\right]\right\}, \tag{28}$$

where m_{t+1} is the likelihood ratio, $E_t[m_{t+1}\ln(m_{t+1})]$ is defined as the relative entropy of the distribution of the distorted model with respect to that of the approximating model, and $\vartheta_0 > 0$ is the shadow price of capacity that can reduce the distance between the two distributions, i.e., the Lagrange multiplier on the constraint:

$$E_t[m_{t+1}\ln(m_{t+1})] \leq \eta,$$

where $\eta \geq 0$ defines an entropy ball of the distribution of the distorted model with respect to that of the approximating model. Minimizing with respect to m_{t+1} yields

$$m_{t+1} = \frac{\exp(-v(\widehat{s}_{t+1})/\vartheta_0)}{E_t[\exp(-v(\widehat{s}_{t+1})/\vartheta_0)]}, \tag{29}$$

and it is straightforward to show that substituting m_{t+1} into (28) yields the following Bellman equation:

$$\widehat{v}(\widehat{s}_t) = \max_{c_t}\left\{-\frac{1}{2}(c_t - \overline{c})^2 + \beta\mathcal{R}_t[\widehat{v}(\widehat{s}_{t+1})]\right\}, \tag{30}$$

where

$$\mathcal{R}_t[\widehat{v}(\widehat{s}_{t+1})] = -\vartheta_0 \log E_t[\exp(-\widehat{v}(\widehat{s}_{t+1})/\vartheta_0)],$$

subject to (18). The following proposition summarizes the solution to the RB-RI model when $\beta R = 1$.

Proposition 2. *Given ϑ_0 and θ, the consumption function under RB and RI is*

$$c_t = \frac{R-1}{1-\Pi}\widehat{s}_t - \frac{\Pi\bar{c}}{1-\Pi}, \tag{31}$$

the value function is

$$\widehat{v}\left(\widehat{s}_t\right) = \Omega\left(\widehat{s}_t - \frac{\bar{c}}{R-1}\right)^2 + \rho, \tag{32}$$

where $\Omega = -\frac{R(R-1)}{2(1-\Pi)}$, $\rho = \frac{\vartheta_0}{2(R-1)}\ln\left(1 - \frac{(R-1)\Pi}{1-\Pi}\right)$, and $\widehat{s}_t$ is governed by (26).

Proof. See Online Appendix 8.2. ∎

Comparing (24) with (31), it is clear that the two specifications of RB, the two-player minmax game and multiplier preferences, lead to the same consumption-saving decisions.[19] Equations (31) and (27) determine the effects of model uncertainty due to RB and state uncertainty due to RI on the marginal propensity to consume out of perceived permanent income (MPC_η) and the constant precautionary saving premium. It is clear from these two expressions that Π governs how RB and RI interact and then affect the consumption function and precautionary savings. Since Π is increasing with the degrees of both RB (smaller ϑ_0) and RI (smaller κ and θ), it is straightforward to show that either RB or RI leads to more constant precautionary savings and higher marginal propensity to consume, holding other factors constant and given that $\Pi < 1$.

However, RB and RI affect consumption and precautionary savings through distinct channels. RI affects Π by increasing the variance of the innovation to the perceived state, ω_η^2, whereas RB affects Π via changing the structure of the response of consumption to income shocks. Furthermore, if we consider the marginal propensity to consume out of true permanent income,

$$MPC_\zeta \equiv \frac{R-1}{1 - R\theta/\left[\vartheta_0\left(1-(1-\theta)R^2\right)\right]\omega_\zeta^2}\theta, \tag{33}$$

we can immediately see that

$$\frac{\partial\left(MPC_\zeta\right)}{\partial\vartheta_0} < 0, \frac{\partial\left(MPC_\zeta\right)}{\partial\theta} > 0.$$

That is, both an increase in the demand for robustness and an increase in inattention reduces the marginal propensity to consume out of true (but unobserved) permanent income.

To examine the relative importance of the two informational frictions in determining the consumption function and precautionary savings, we compare the effects from proportionate shifts in ϑ_0 governing RB and κ governing RI. Specifically, the marginal effects on Π from an increase in

[19] In Section (5), we show that the two specifications have slightly different asset pricing implications.

ϑ_0 and κ are given by

$$\frac{\partial \Pi}{\partial \kappa} = \frac{R\left(1-R^2\right)\exp\left(-2\kappa\right)}{\vartheta_0\left[1-\exp\left(-2\kappa\right)R^2\right]^2}\omega_\zeta^2,$$
$$\frac{\partial \Pi}{\partial \vartheta_0} = -\frac{R\omega_\eta^2}{\vartheta_0^2},$$

respectively. Therefore, the marginal rate of transformation between proportionate changes in ϑ and changes in κ can be written as

$$\text{MRT} = -\frac{\partial \Pi/\partial \kappa}{\left(\partial \Pi/\partial \vartheta_0\right)\vartheta_0} = \frac{2\left(R^2-1\right)\exp\left(-2\kappa\right)}{\left(1-\exp\left(-2\kappa\right)\right)\left(1-\exp\left(-2\kappa\right)R^2\right)} > 0. \tag{34}$$

This expression gives the proportionate reduction in ϑ_0 (i.e., a stronger preference for RB) that compensates, at the margin, for a decrease in κ (i.e., more inattentive) — in the sense of preserving the same effect on the consumption function for a given $\widehat{s}_t$. Equation (34) shows that this compensating change depends on the interest rate (R) and the degree of inattention (κ). Figure 2 clearly shows that MRT is decreasing with κ for any given R. Since $\partial\left(\text{MRT}\right)/\partial\kappa < 0$, consumers with lower capacity will ask for higher compensation in an proportionate increase in model uncertainty facing them for an increase in capacity. For example, when $R = 1.03$, MRT $= 0.256$ when $\kappa = 0.5$ bits, while MRT $= 0.054$ when $\kappa = 1$ bit. In other words, to maintain the same effect on the consumption function, a decrease in κ by 50 percent (from 1 bit to 0.5 bits) matches up approximately with a proportional decline in ϑ_0 of 2.7 percent. We will show later that there is a model-independent procedure for estimating ϑ; the tradeoff here could in principle be used to discipline the choice for κ.[20]

It is also instructive to examine exactly what agents "fear" – that is, what are the dynamics of permanent income under the worst-case model? Substituting (25) and (19) into the law of motion for $\widehat{s}_t$ under the worst-case model yields

$$\widehat{s}_{t+1} = \left(1-\theta R\right)\widehat{s}_t + \theta R s_t + \theta\left(\zeta_{t+1}+\xi_{t+1}\right), \tag{35}$$

as compared to the actual process

$$\widehat{s}_{t+1} = \left(\frac{1-R^2\omega_\eta^2/\vartheta_0}{1-R\omega_\eta^2/\vartheta_0} - \theta R\right)\widehat{s}_t + \frac{\left(R\omega_\eta^2/\vartheta_0\right)\overline{c}}{1-R\omega_\eta^2/\vartheta_0} + \theta R s_t + \theta\left(\zeta_{t+1}+\xi_{t+1}\right). \tag{36}$$

The key difference between the two processes is the autocorrelation parameter; since

$$\frac{1-R^2\omega_\eta^2/\vartheta_0}{1-R\omega_\eta^2/\vartheta_0} < 1,$$

[20] κ (or θ) are difficult to estimate outside the model; the literature on processing information provides estimates of the total ability of humans, but little guidance on how much of that ability would be dedicated to monitoring economic data. Obviously it would not be feasible to model all the competing demands for attention.

the worst case model is more persistent than the true process. As noted in Kasa (2006), the most destructive distortions are low-frequency ones, so naturally the agents in the model design their decision rules to be robust against precisely those kinds of processes. ϑ_0 does not appear in (35), as it only determines the size of the distortion process $\{w_t\}$ needed to achieve the worst-case model.[21]

2.3. Robust Kalman Filter Gain

Another source of robustness could arise from the Kalman filter gain. In Section 2.2, we assumed that the agent distrusts the innovation to the perceived state but trusts the regular Kalman filter gain. Following Hansen and Sargent (Chapter 17, 2007), in this section we consider a situation in which the agent pursues a robust Kalman gain. Specifically, assume that at t the agent observes the noisy signal

$$s_t^* = s_t + \xi_t, \tag{37}$$

where s_t is the true state and ξ_t is the iid endogenous noise. The variance of the noise term, $\lambda^2 = \text{var}(\xi_t)$, is determined by

$$\lambda^2 = \text{var}(\xi_t) = \frac{\left(\omega_\zeta^2 + R^2\sigma^2\right)\sigma^2}{\omega_\zeta^2 + (R^2 - 1)\sigma^2},$$

and $\sigma^2 = \frac{\omega_\zeta^2}{\exp(2\kappa) - R^2}$ is the steady state conditional variance. Given the budget constraint,

$$s_{t+1} = Rs_t - c_t + \zeta_{t+1}, \tag{38}$$

we consider the following time-invariant robust Kalman filter equation,

$$\widehat{s}_{t+1} = (1-\theta)\left(R\widehat{s}_t - c_t\right) + \theta\left(s_{t+1} + \xi_{t+1}\right), \tag{39}$$

where $\widehat{s}_{t+1}$ is the estimate of the state using the history of the noisy signals, $\left\{s_j^*\right\}_{j=0}^{t+1}$. We want θ to be robust to unstructured misspecifications of Equations (37) and (38). To obtain a robust Kalman filter gain, the agent considers the following distorted model:

$$s_{t+1} = Rs_t - c_t + \zeta_{t+1} + \omega\nu_{1,t+1}, \tag{40}$$

$$s_{t+1}^* = s_{t+1} + \xi_{t+1} + \varrho\nu_{2,t+1}, \tag{41}$$

where $\nu_{1,t+1}$ and $\nu_{2,t+1}$ are distortions to the conditional means of the two shocks, ζ_{t+1} and ξ_{t+1}, respectively.

Combining (38), (39), (40) with (41) gives the following dynamic equation for the estimation

[21] If $\theta = 1$ (so that $\widehat{s}_t = s_t$) then the worst-case model is a random walk.

error:

$$e_{t+1} = (1-\theta) R e_t + (1-\theta)\zeta_{t+1} - \theta\xi_{t+1} + (1-\theta)\omega\nu_{1,t+1} - \theta\varrho\nu_{2,t+1}, \tag{42}$$

where $e_t = s_t - \widehat{s}_t$.[22] We can then solve for the robust Kalman filter gain corresponding to this problem by solving the following deterministic optimal linear regulator problem:

$$e_0^T P e_0 = \max_{\{\nu_{t+1}\}} \sum_{t=0}^{\infty} \left(e_t^T e_t - \vartheta \nu_{t+1}^T \nu_{t+1} \right), \tag{43}$$

subject to

$$e_{t+1} = (1-\theta) R e_t + D\nu_{t+1}, \tag{44}$$

where $D = \begin{bmatrix} (1-\theta)\omega & -\theta\varrho \end{bmatrix}$ and $\nu_{t+1} = \begin{bmatrix} \nu_{1,t+1} & \nu_{2,t+1} \end{bmatrix}^T$. We can compute the worst-case shock by solving the corresponding Bellman equation

$$\nu_{t+1}^* = Q e_t, \tag{45}$$

where

$$Q = \left(\vartheta I - D^T P D\right)^{-1} D^T P (1-\theta) R. \tag{46}$$

Here I is the identity matrix, and Q depends on robustness (ϑ) and channel capacity (κ).

For arbitrary Kalman filter gain θ and (45), the error in reconstructing the state s can be written as

$$e_{t+1} = \{(1-\theta) R + [(1-\theta)\omega - \theta\varrho] Q\} e_t + (1-\theta)\zeta_{t+1} - \theta\xi_{t+1}. \tag{47}$$

Taking unconditional mean on both sides of (47) gives

$$\Sigma_{t+1} = \{(1-\theta) R + [(1-\theta)\omega - \theta\varrho] Q\} \Sigma_t + (1-\theta)^2 \omega_\zeta^2 + \theta^2 \omega_\xi^2, \tag{48}$$

where $\Sigma_{t+1} = E\left[e_{t+1}^2\right]$. From (48), it follows directly that in the steady state

$$\Sigma(\theta; Q) = \frac{(1-\theta)^2 \omega_\zeta^2 + \theta^2 \omega_\xi^2}{1-\chi^2},$$

where $\chi = (1-\theta) R + [(1-\theta)\omega - \theta\varrho] Q$, and the robust Kalman filter gain $\theta(\vartheta, \kappa)$ minimizes the variance of e_t, $\Sigma(\theta; Q)$:

$$\theta(\vartheta, \kappa) = \arg\min\left(\Sigma(\theta; Q(\vartheta, \kappa))\right). \tag{49}$$

Figure 3 illustrates how robustness (measured by ϑ) and inattention (measured by κ) affect the robust Kalman gain when $R = 1.02$ and $\omega_\zeta = 1$.[23] It clearly shows that holding the degree of attention (i.e., channel capacity κ) fixed, increasing robustness (reducing ϑ) increases the Kalman gain (θ). In addition, for given robustness (ϑ), the Kalman gain is increasing with capacity. For

[22] Note that control variable, c, does not affect the estimation error equation.

[23] We use the program rfilter.m provided in Hansen and Sargent (2007) to compute the robust Kalman filter gain $\theta(\vartheta, \kappa)$.

example, when $\log(\vartheta) = 3$, the robust Kalman gain will increase from 60.17 percent to 77.35 percent when capacity κ increases from 0.6 bits to 1 bit; when $\kappa = 0.6$ bits, the robust Kalman gain will increase from 58.31 percent to 60.17 percent if ϑ falls from $\log(\vartheta) = 4$ to 3.[24]

After obtaining the robust Kalman gain $\theta(\vartheta, \kappa)$, we can solve the Bellman equation proposed in Section 2.2 using the Kalman filtering equation with robust θ. The following proposition summarizes the solution to this problem:

Proposition 3. *Given ϑ_0, ϑ, and κ, the consumption function is*

$$c_t = \frac{R-1}{1-\Pi}\widehat{s}_t - \frac{\Pi\bar{c}}{1-\Pi}, \tag{50}$$

where

$$\Pi = \frac{R\omega_\eta^2}{\vartheta_0} \in (0,1), \tag{51}$$
$$\omega_\eta^2 = \operatorname{var}[\eta_{t+1}] = \frac{\theta(\vartheta,\kappa)}{1-(1-\theta(\vartheta,\kappa))R^2}\omega_\zeta^2,$$

and $\widehat{s}_t$ is governed by

$$\widehat{s}_{t+1} = \rho_s \widehat{s}_t + \eta_{t+1}, \tag{52}$$

where $\rho_s = \frac{1-R\Pi}{1-\Pi} \in (0,1)$.

Proof. The proof is the similar to that provided in Online Appendix 8.2. We just need to replace $\theta(\kappa) = 1 - \exp(-2\kappa)$ with $\theta(\vartheta, \kappa)$. ■

Note that here θ is a function of both ϑ (concerns about Kalman gain) and κ (channel capacity), rather than simply $1 - 1/\exp(2\kappa)$ as obtained in Section 2.2. In this case the agent has two types of concerns about model misspecification: (i) concerns about the disturbances to the perceived permanent income (ϑ_0) and (ii) concerns about the Kalman gain (ϑ). It is clear from (50) and (51) that the two types of robustness have opposing effects on both the marginal propensity to consume out of permanent income, i.e., the responsiveness of c_t to $\widehat{s}_t$ $\left(MPC_\eta = \frac{R-1}{1-\Pi}\right)$ and precautionary savings, i.e., the intercept of the consumption profile $\left(PS = \frac{\Pi\bar{c}}{1-\Pi} = -\bar{c} + \frac{\bar{c}}{1-\Pi}\right)$.[25] Specifically, the less the value of ϑ_0 (Type I robustness) the larger the MPC and the larger the precautionary saving increment, since

$$\frac{\partial(MPC_\eta)}{\partial\vartheta_0} < 0 \text{ and } \frac{\partial(PS)}{\partial\vartheta_0} < 0.$$

For the effects of Type II robustness (ϑ), the less the value of ϑ the less the MPC and the less the

[24] This result is consistent with that obtained in a continuous-time setting discussed in Kasa (2006).

[25] Note that given the consumption function Π has the same effect on the marginal propensity to consume and precautionary savings.

precautionary saving increment

$$\frac{\partial \left(MPC_{\eta}\right)}{\partial \vartheta} > 0 \text{ and } \frac{\partial \left(PS\right)}{\partial \vartheta} > 0$$

because $\frac{\partial \omega_{\eta}^{2}}{\partial \theta} < 0$ and $\frac{\partial \theta}{\partial \vartheta} < 0$.

From (50) and (51), it is clear that the precautionary savings increment in the RB-RI model is determined by the interaction of three factors: labor income uncertainty, preferences for robustness (RB), and finite information-processing capacity (RI). We first present some intuition about the effects of Type I robustness (ϑ_0) on precautionary savings. Since agents with low capacity are very concerned about the confluence of low permanent income and high consumption (meaning they believe their permanent income is high so they consume a lot and then their new signal indicates that in fact their permanent income was low), they take actions which reduce the probability of this bad event – they save more. The strength of the precautionary effect is positively related to the amount of uncertainty regarding the true level of permanent income, and this uncertainty increases as θ gets smaller.

We now provide some intuition about the effects of Type II robustness (ϑ) on precautionary savings. An increase in Type II robustness (a reduction in ϑ) will increase the Kalman gain θ, which leads to lower ω_{η}^{2} and then low precautionary savings. Figure 4 illustrates how Type II robustness (ϑ) and channel capacity (κ) affect ω_{η}^{2}. In addition, since

$$\frac{\partial \omega_{\eta}^{2}}{\partial \vartheta} > 0, \frac{\partial \omega_{\eta}^{2}}{\partial \kappa} < 0, \frac{\partial \omega_{\eta}^{2}}{\partial \vartheta_0} = 0.$$

it is clear that under certain conditions a greater reaction to the shock can either be interpreted as an increased concern for robustness in the presence of model misspecification, or an increase in information-processing ability when agents only have finite channel capacity. Figure 5 illustrates Π as functions of ϑ_0 and ϑ. It clearly shows that how increasing the robustness preference for the shock to the perceived state (decreasing ϑ_0) and reducing the preference for a robust gain (increasing ϑ) increases Π and then increase the effects of the two types of robustness on consumption and precautionary savings.

3. Consumption and Saving Dynamics

3.1. Sensitivity and Smoothness of Consumption Process

We will now discuss the effect of RI-RB on the dynamics of consumption, in particular the excess smoothness and sensitivity puzzles noted earlier. Since the deterministic growth component does not affect the stochastic properties of the model, for simplicity, here we assume that there is no growth component ($g = 1$). Combining (50) with (52) yields the change in individual consumption in the RI-RB economy:

$$c_t = \rho_s c_{t-1} + \frac{(R-1)\,\Pi}{1-\Pi}\bar{c} + \frac{R-1}{1-\Pi}\left[\frac{\theta\zeta_t}{1-(1-\theta)R\cdot L} + \theta\left(\xi_t - \frac{\theta R\xi_{t-1}}{1-(1-\theta)R\cdot L}\right)\right], \tag{53}$$

where L is the lag operator and we assume that $(1-\theta)R < 1$. This expression shows that consumption growth is a weighted average of all past permanent income and noise shocks. In addition, it is also clear from (53) that the propagation mechanism of the model is determined by the robust Kalman filter gain, $\theta(\vartheta,\kappa)$. Figure 6 illustrates that consumption in the RB-RI model reacts gradually to income shocks, with monotone adjustments to the corresponding RB asymptote. Note that when $\vartheta_0 = 2$ and $\log(\vartheta) = 3$, the robust Kalman gain is $\theta = 42.56$ percent. This case is illustrated by the dash-dotted line in Figure 4. Similarly, the dotted line corresponds to the case in which $\vartheta_0 = 4$ and $\log(\vartheta) = 5$ ($\theta = 0.3541$). With a stronger preference for robustness, the precautionary savings increment is larger and thus an income shock that is initially undetected would have larger impacts on consumption during the adjustment process.

Using (53), we can obtain the expression for the relative volatility of consumption growth relative to income growth. The following proposition provides the expression of this relative volatility.

Proposition 4. *The relative volatility of consumption growth relative to income growth is*

$$\mu \equiv \frac{\mathrm{sd}(\Delta c_t)}{\mathrm{sd}(\Delta y_t)} = \frac{\theta}{1-\Pi}\sqrt{\sum_{j=0}^{\infty}\Upsilon_j^2 + \frac{1-\theta}{\theta\left(1-(1-\theta)R^2\right)}\sum_{j=0}^{\infty}\left(\Upsilon_j - R\Upsilon_{j-1}\right)^2}, \tag{54}$$

where we use the fact that $\omega_\xi^2 = \mathrm{var}(\xi_t) = \frac{1-\theta}{\theta(1-(1-\theta)R^2)}\omega_\zeta^2$, $\rho_s = \frac{1-R\Pi}{1-\Pi} \in (0,1)$, $\rho_\theta = (1-\theta)R \in (0,1)$, *and*

$$\Upsilon_j = \sum_{k=0}^{j}\left(\rho_s^{j-k}\rho_\theta^k\right) - \sum_{k=0}^{j-1}\left(\rho_s^{j-1-k}\rho_\theta^k\right), \text{ for } j \geq 1,$$

and $\Upsilon_0 = 1$.

Proof. See Online Appendix 8.3. ■

Figure (7) illustrates how the combination of the two types of robustness, ϑ_0 and ϑ, affects the relative volatility of consumption growth to income growth when $\kappa = 0.3$ bits. It clearly shows that given ϑ_0, the relative volatility μ is increasing with ϑ. The intuition is that reducing ϑ (i.e., increasing Type II robustness) increases the robust Kalman gain θ and reduces ω_η^2 and Π, which leads to a smoother consumption process. Note that θ is independent of ϑ_0. Hence, given ϑ, μ is decreasing with ϑ_0 because Π is increasing with ϑ_0. To explore the intuition behind this result, we consider the perfect-state-observation case in which $\kappa = \infty$. In this case, the relative volatility of consumption growth to income growth reduces to

$$\mu = \frac{1}{1-\Pi}\sqrt{\frac{2}{1+\rho_s}}, \tag{55}$$

which clearly shows that ϑ_0 increases the relative volatility via two channels. First, a higher ϑ_0 increases the marginal propensity to consume out of permanent income $\left(\frac{R-1}{1-\Pi}\right)$, and second, it increases consumption volatility by reducing the persistence of permanent income measured by ρ_s: $\frac{\partial \rho_s}{\partial \Pi} < 0$.

In the presence of robustness, rational inattention measured by κ affects consumption volatility via two channels: (i) the gradual and smooth responses to income shocks (i.e., the $1-\rho_\theta \cdot L$ term in (53) and (ii) the RI-induced noises (ξ_t). Specifically, a reduction in capacity κ decreases the Kalman gain θ, which strengthens the smooth responses to income shock and increases the volatility of the RI-induced noise. Luo (2008) shows that the noise effect dominates the smooth response effect, and the volatility of consumption growth decreases with κ. Figure 8 illustrates how the combination of ϑ_0 and ϑ affects the relative volatility of consumption growth to income growth when $\kappa = 0.3$ bits and there is no noise term. In this case, $\Delta c_t = \frac{(1-R)\Pi}{1-\Pi}\left(c_{t-1}-\overline{c}\right) + \frac{R-1}{1-\Pi}\frac{\theta\zeta_t}{1-\rho_\theta \cdot L}$. Figure 8 shows that given ϑ_0, the relative volatility μ is decreasing with ϑ. The intuition is that reducing ϑ (i.e., increasing Type I robustness) increases the robust Kalman gain θ, which leads to a more volatile consumption process because the smooth response effect completely dominates the noise effect.

3.2. Saving Process

Combining the original budget constraint, $b_{t+1} = Rb_t + y_t - c_t$, with the consumption function (50), we can obtain the following expression for individual saving d_t:

$$\begin{aligned} d_t &\equiv b_{t+1} - b_t \\ &= \frac{-\Pi(R-1)}{1-\Pi}\left(b_t - \overline{b}\right) + \left(1 - \frac{R-1}{(1-\Pi)(R-\rho)}\right)(y_t - \overline{y}) + \varsigma_{t+1}, \end{aligned} \tag{56}$$

where the evolution of individual financial wealth (b_t) follows

$$b_{t+1} = \rho_s b_t + \frac{\Pi}{1-\Pi}\left(\overline{c} - \overline{y}\right) + \left(1 - \frac{R-1}{(1-\Pi)(R-\rho)}\right)(y_t - \overline{y}) + \varsigma_{t+1}, \tag{57}$$

where $\varsigma_{t+1} = \frac{R-1}{1-\Pi}\left(s_t - \widehat{s}_t\right)$ is determined by the estimation error, $s_t - \widehat{s}_t = \frac{(1-\theta)\zeta_t}{1-(1-\theta)R\cdot L} - \frac{\theta\xi_t}{1-(1-\theta)R\cdot L}$ and $\overline{b} = \frac{\overline{c}-\overline{y}}{R-1}$ is the steady state value of b_t. From (56), it is straightforward to show that the both the unconditional and conditional means of individual saving are zero:

$$E\left[d_t\right] = 0 \text{ or } E_t\left[d_t\right] = 0.$$

That is, induced uncertainty due to the interaction of RB and RI does not affect the amount of individual saving on average. Using (56), we can compute the cross-sectional variance of individual savings. The following proposition provides the expression of this variance.

Proposition 5. *The variance of individual savings is*

$$\operatorname{var}(d_t) = \left[\begin{array}{c} \frac{1-\rho}{1+\rho} + \frac{\Gamma^2}{1-\rho_s^2} + \left(\frac{R-1}{1-\Pi}\right)^2 \frac{1-\theta}{1-R^2(1-\theta)} \\ +\frac{2(1-\rho)\Gamma}{1-\rho\rho_s} + \left(\frac{R-1}{1-\Pi}\right) \frac{2(1-\rho)(1-\theta)}{1-\rho\rho_\theta} + \left(\frac{R-1}{1-\Pi}\right) \frac{2\Gamma(1-\theta)}{1-\rho_s\rho_\theta} \end{array} \right] \omega_\zeta^2, \tag{58}$$

where $\Gamma = -\frac{(R-1)\Pi}{1-\Pi} < 0$, $\rho_s = \frac{1-R\Pi}{1-\Pi}$, *and* $\Pi = \frac{\theta(\vartheta,\kappa)}{1-(1-\theta(\vartheta,\kappa))R^2} R\omega_\zeta^2/\vartheta_0$.

Proof. See Online Appendix 8.5. ■

The complexity of this expression prevents us from obtaining clear results about how RI and RB affect the variance of individual savings. Figure 9 illustrates the effects of RI (κ) and RB (ϑ_0, ϑ) on the relative volatility of d_t to ω_ζ^2, $\mu_d \equiv \sqrt{\operatorname{var}(d_t)/\omega_\zeta^2}$ when $\omega_\zeta^2 = 1$ and $\rho = 0.9$. It is clear from the figure that for a given ϑ_0 (Type I robustness), this relative volatility is decreasing with the degree of attention (θ) that is determined by ϑ and κ. In other words, μ_d is decreasing with channel capacity (κ) and is increasing with Type II robustness (ϑ) because $\frac{\partial\theta(\vartheta,\kappa)}{\partial\kappa} > 0$ and $\frac{\partial\theta(\vartheta,\kappa)}{\partial\vartheta} < 0$.

If we now consider an aggregate economy with a continuum of ex ante identical inattentive consumers with the same preference for robustness and each of them has the consumption function (50), then the total saving demand in the economy is equal to zero. The intuition is simple. The saving function can be expressed as a combination of different types of income and noise shocks: ς, ε or ζ, and ξ, and all of these shocks are idiosyncratic. These idiosyncratic shocks cancel out after aggregating across consumers and therefore have no effect on aggregate savings. In Section 6 we study an economy with mortality that generates positive aggregate savings.

4. Comparison with Risk-sensitive Control and Filtering

Risk-sensitivity (RS) was first introduced into the LQ-Gaussian framework by Jacobson (1973) and extended by Whittle (1981, 1990). Exploiting the recursive utility framework of Epstein and Zin (1989), Hansen and Sargent (1995) introduce discounting into the RS specification and show that the resulting decision rules are time-invariant. In the RS model agents effectively compute expectations through a distorted lens, increasing their effective risk aversion by overweighting negative outcomes. The resulting decision rules depend explicitly on the variance of the shocks, producing precautionary savings, but the value functions are still quadratic functions of the states.[26] Hansen, Sargent, and Tallarini (1999) and Hansen and Sargent (2007) interpret RS preferences in terms of a concern about model uncertainty (robustness or RB) and argue that RS introduces precautionary

[26] Formally, one can view risk-sensitive agents as ones who have non-state-separable preferences, as in Epstein and Zin (1989), but with a value for the intertemporal elasticity of substitution equal to one.

savings because RS consumers want to protect themselves against model specification errors. In the corresponding risk-sensitive filtering LQ problem, the problem is that when the state cannot be observed perfectly, is the classical Kalman filter that minimizes the expected loss function still optimal? In our LQ-PIH model setting, we can easily see that the regular Kalman filter is still optimal given the quadratic forms of the utility function and the value function.[27]

In this section we will explore how the RS filtering affects consumption dynamics and precautionary savings and show that the OE between RB and RS is no longer linear, but takes a more complicated non-linear form. The RI version of risk-sensitive control based on recursive preferences with an exponential certainty equivalence function can be formulated as follows:

$$\widehat{v}\left(\widehat{s}_t\right) = \max_{c_t}\left\{-\frac{1}{2}\left(c_t - \overline{c}\right)^2 + \beta\mathcal{R}_t\left[\widehat{v}\left(\widehat{s}_{t+1}\right)\right]\right\} \tag{59}$$

subject to the Kalman filter equation (18).[28] The distorted expectation operator is now given by

$$\mathcal{R}_t\left[\widehat{v}\left(\widehat{s}_{t+1}\right)\right] = -\frac{1}{\alpha}\log E_t\left[\exp\left(-\alpha\widehat{v}\left(\widehat{s}_{t+1}\right)\right)\right],$$

where $s_0|\,\mathcal{I}_0 \sim N\left(\widehat{s}_0, \overline{\sigma}^2\right)$, $\widehat{s}_t = E_t\left[s_t\right]$ is the perceived state variable, θ is the optimal weight on the new observation of the state, and ξ_{t+1} is the endogenous noise. The optimal choice of the weight θ is given by $\theta\left(\kappa\right) = 1 - 1/\exp(2\kappa) \in [0,1]$. It is worth noting that given that the value function in the RS model is quadratic, the regular Kalman filter is still optimal because the objective function in the filtering problem is the square of the estimation error.

Following the same procedure used in Hansen and Sargent (1995) and Luo and Young (2010), we can solve this risk-sensitive control problem explicitly. The following proposition summarizes the solution to the RI-RS model when $\beta R = 1$:

Proposition 6. *Given finite channel capacity κ and the degree of risk-sensitivity α, the consumption function of a risk-sensitive consumer under RI is*

$$c_t = \frac{R-1}{1-\Pi}\widehat{s}_t - \frac{\Pi\overline{c}}{1-\Pi}, \tag{60}$$

[27] As has been shown in Moore, Elliott, and Dey (1997), even if the agent has the risk-sensitive preference when filtering,

$$\min \ln E_t\left\{\exp\left[-\vartheta\left(s_t - \widehat{s}_t^{RS}\right)^2\right]\right\},$$

the risk-senstive estimate $\widehat{s}_t^{RS}$ is identical to the minimum variance estimate $\widehat{s}$ obtained from solving

$$\min E_t\left[\left(s_t - \widehat{s}_t\right)^2\right].$$

[28] Given the quadratic form of the value function, introducing risk-sensitivity does not change the optimality of the *ex post* Gaussianity of the true state and the induced noise; see Luo and Young (2010) for more discussion.

where

$$\Pi = R\alpha\omega_{\eta}^{2} \in (0,1), \tag{61}$$

$$\omega_{\eta}^{2} = \mathrm{var}(\eta_{t+1}) = \frac{\theta}{1-(1-\theta)R^{2}}\omega_{\zeta}^{2}, \tag{62}$$

η_{t+1} *is defined in (19), and* $\theta(\kappa) = 1 - 1/\exp(2\kappa)$.

Proof. See Online Appendix 8.3. ■

Comparing (31) obtained from the model with only concerns about the innovation to the perceived state (i.e., without robust Kalman filtering) in Section 2.2 and (60), it is straightforward to show that RB and RS under RI are indistinguishable using only consumption-savings decisions if

$$\alpha = \frac{1}{\vartheta_0}. \tag{63}$$

Note that (63) is exactly the same as the observational equivalence condition obtained in the full-information RE model (see Backus, Routledge, and Zin 2004). That is, under the assumption that the agent trusts the Kalman filter equation, the OE result obtained under full-information RE still holds under RI.

Hansen, Sargent, and Tallarini (1999) show that as far as the quantity observations on consumption and savings are concerned, the robustness version ($\vartheta_0 > 0$ or $\alpha > 0, \widetilde{\beta}$) of the PIH model is observationally equivalent to the standard version ($\vartheta_0 = \infty$ or $\alpha = 0, \beta = 1/R$) of the PIH model for a unique pair of discount factors.[29] The intuition is that introducing a preference for risk-sensitivity (RS) or a concern about robustness (RB) increases savings in the same way as increasing the discount factor, so that the discount factor can be changed to offset the effect of a change in RS or RB on consumption and investment.[30] Alternatively, holding all parameters constant except the pair (α, β), the RI version of the PIH model with RB consumers ($\vartheta_0 > 0$ and $\beta R = 1$) is observationally equivalent to the standard RI version of the model ($\vartheta_0 = \infty$ and $\widetilde{\beta} > 1/R$). To do so, we compare the consumption function obtained from the RI model ($\vartheta_0 = \infty$ and $\widetilde{\beta} > 1/R$),

$$c_t = \left(R - \frac{1}{\beta R}\right)\widehat{s}_t - \frac{1}{R-1}\left(1 - \frac{1}{\beta R}\right)\overline{c}.$$

with (31)and (60), and obtain the following OE expression for the discount factor:

Proposition 7. *Let*

$$\widetilde{\beta} = \frac{1}{R}\frac{1 - R\omega_{\eta}^{2}/\vartheta_0}{1 - R^{2}\omega_{\eta}^{2}/\vartheta_0} = \frac{1}{R}\frac{1 - R\alpha\omega_{\eta}^{2}}{1 - R^{2}\alpha\omega_{\eta}^{2}} > \frac{1}{R}.$$

[29] Hansen, Sargent, and Tallarini (1999) derive the observational equivalence result by fixing all parameters, including R, except for the pair (α, β).

[30] As shown in Hansen, Sargent, and Tallarini (1999), the two models have different implications for asset prices because continuation valuations change as one varies (α, β) within the observationally-equivalent set of parameters.

Then consumption and savings are identical in the RI, RB-RI, and RS-RI models.

However, if we compare (50) obtained from the model with both concerns about the innovation to the perceived state and concerns about Kalman gain with (60), it is obvious that the observational equivalence between RB and RS under RI, (63), no longer holds. Given the same value of κ, the Kalman gain only depends on κ in the RS model, whereas it depends on both κ and ϑ (the preference for robust Kalman gain) in the RB model. The two Kalman gains are therefore different for any finite value of ϑ. If we allow for different values of κ, the models are observationally equivalent when $\alpha = \vartheta_0^{-1}$ and

$$\theta\left(\vartheta, \kappa_{RB}\right) = 1 - \frac{1}{\exp(2\kappa_{RS})}.$$

Figure (10) illustrates how κ_{RS} varies with ϑ and κ_{RB} when the OE between RB and RS holds under RI. It clearly shows that given the level of ϑ, κ_{RS} is increasing with κ_{RB}.

5. Market Price of Induced Uncertainty

The PIH model presented in Section 2.2 is usually regarded as a partial equilibrium model. However, as noted in Hansen (1987) and Hansen, Sargent, and Tallarini (1999), it can be interpreted as a general equilibrium model with a linear production technology and an exogenous income process. Given the expression of optimal consumption in terms of the state variables derived from the robust version of the PIH model with inattentive agents, we can price assets by treating the process of aggregate consumption that solves the model as though it were an endowment process. In this setup, equilibrium prices are shadow prices that leave the agent content with that endowment process. Hansen, Sargent, and Tallarini (1999) study how robustness and risk-sensitivity affect the predicted market price of risk within a PIH model with shocks to both labor income and preferences, and find that RB or RS significantly alter the model's predictions on the market price of risk and thus provides an alternative explanation for the equity premium puzzle.

In this section, using the optimal consumption and saving decisions derived in the previous sections, we will explore how induced uncertainty due to the interactions of RB and RI with income shocks affect the market price of uncertainty. We first consider the single-period asset pricing case. In this case, we assume that the agent purchases a security at period t at a price q_t, holds it for one period, and then sells it at $t+1$ for a total payoff ϕ_{t+1} in terms of the consumption good after collecting the dividend. Under this assumption, the following Euler equation holds:

$$q_t = \widetilde{E}_t\left[\left(\beta \frac{u'\left(\widehat{s}_{t+1}\right)}{u'\left(\widehat{s}_t\right)}\right)\phi_{t+1}\right], \tag{64}$$

where $\beta \frac{u'(\widehat{s}_{t+1})}{u'(\widehat{s}_t)}$ is the stochastic discount factor (SDF) and $\widetilde{E}_t\left[\cdot\right]$ is the distorted conditional expectations operator. Note that here SDF depends on the perceived states because optimal consumption is a linear function of perceived permanent income; because $\widehat{s}_{t+1}$ is a function of the true state s_{t+1},

s_t will also affect the SDF. The corresponding formula for q_t in terms of the original conditional expectations operator can be written as

$$q_t = E_t \left[m_{t,t+1} \phi_{t+1} \right], \tag{65}$$

where $m_{t,t+1}$ depends not only on the usual SDF but also on robustness. As has been shown in Hansen, Sargent, and Tallarini (1999), RB or RS are reflected in the usual measure of the SDF being scaled by a random variable with conditional mean 1. They also show that this multiplicative adjustment to the SDF increases the volatility of the SDF and thus drives up the risk premium. To explore the effects of induced uncertainty on the market price of uncertainty, we write (65) as

$$q_t = E_t \left[\phi_{t+1} \right] E_t \left[m_{t,t+1} \right] + \operatorname{cov}_t \left(m_{t,t+1}, \phi_{t+1} \right),$$

which leads to the following price bound:

$$q_t \geq E_t \left[\phi_{t+1} \right] E_t \left[m_{t,t+1} \right] - \operatorname{sd}_t \left(m_{t,t+1} \right) \operatorname{sd}_t \left(\phi_{t+1} \right),$$

where $\operatorname{sd}_t(\cdot)$ denotes the conditional standard deviation. If we define the market price of uncertainty (MPU) as $\mathrm{MPU} \equiv \frac{\operatorname{sd}_t(m_{t,t+1})}{E_t[m_{t,t+1}]}$, the pricing bound can be rewritten as

$$\mathrm{MPU} \geq \frac{E_t \left[\phi_{t+1}/q_t \right]}{\operatorname{sd}_t \left(\phi_{t+1}/q_t \right)},$$

where the RHS is the Sharpe ratio and is above 0.2 for most industrial countries.[31] In the standard full-information state- and time-separable utility model, the value of MPU is an order of magnitude lower than what is required for this inequality to be satisfied, which is just another manifestation of the equity premium puzzle: consumption growth is smooth, uncorrelated with returns, and positively autocorrelated, leading to a small cost of bearing uncertainty.[32]

5.1. MPU under RB and RI

The SDF, $m_{t,t+1}$, can be decomposed into

$$m_{t,t+1} = m_{t,t+1}^{f} m_{t,t+1}^{rb},$$

where $m_{t,t+1}^{f} \equiv \beta \frac{u'(\widehat{s}_{t+1})}{u'(\widehat{s}_t)}$ is the "familiar" stochastic discount factor ($\vartheta_0 = \infty$) and $m_{t,t+1}^{rb}$ is the Radon-Nikodym derivative, or the likelihood ratio of the distorted conditional probability of $\widehat{s}_{t+1}$ with respect to the approximating conditional probability. Under the two-player game specification of RB, (23), asset prices are computed using the pessimistic view of the next period's shock: $\widetilde{\eta}_{t+1} = \omega_\eta \widetilde{\epsilon}_{t+1} = \omega_\eta \left(\epsilon_{t+1} + w_t \right)$, where ϵ_{t+1} is a normally distributed variable with mean $\omega_\eta w_t$

[31]In the U.S. data documented in Campbell (2002), the Sharpe ratio is about 0.52 (annualized) during $1947 - 1998$. Using a longer annual U.S. time series put together by Shiller yields a similar value of the Sharpe ratio.

[32]The standard deviation of aggregate consumption growth is 0.84 percent, the autocorrelation is 0.08, and the correlation with real returns on the S&P500 Index is 0.22.

and variance ω_η^2. In this case, the Radon-Nikodym derivative can be written as

$$m_{t,t+1}^{rb} \equiv \frac{\exp\left(-\left(\widetilde{\epsilon}_{t+1}-w_t\right)^2/2\right)}{\exp\left(-\widetilde{\epsilon}_{t+1}^2/2\right)} = \exp\left(\widetilde{\epsilon}_{t+1}w_t - w_t^2/2\right).$$

By construction, we obtain $E_t\left[m_{t,t+1}^{rb}\right] = 1$. By straightforward calculations, we obtain the following conditional second moment of $m_{t,t+1}^{rb}$ as a means for computing its conditional variance:

$$E_t\left[\left(m_{t,t+1}^{rb}\right)^2\right] = \exp\left(w_t^2\right). \tag{66}$$

The following proposition summarizes the result on how induced uncertainty affects the market price of uncertainty.

Proposition 8. *The expression for the market price of induced uncertainty is*

$$\mathrm{sd}_t\left(m_{t,t+1}^{rb}\right) = \sqrt{\exp\left(w_t^2\right)-1} \cong |w_t| \tag{67}$$

for small distortions, where w_t is the mean of the worse-case shock:

$$w_t = \Theta\left[(R-1)\,\widehat{s}_t - \bar{c}\right] \tag{68}$$

and $\Theta = \frac{\Pi/\omega_\eta}{1-\Pi}$.

Proof. Given $E_t\left[m_{t,t+1}^{rb}\right] = 1$, we can obtain (67) using (66). It is also straightfoward to derive (68) using (25). ∎

Expression (67) clearly shows that the amount of market price of uncertainty contributed by the Radon-Nikodym derivative is approximately equal to the norm of the mean of the worse-case shock (w).[33] Note that Θ can be used to measure the importance of RB and RI in determining the market price of uncertainty for given $\widehat{s}_t$ and $\partial\Theta/\partial\Pi > 0$. Specifically, both ϑ_0, ϑ, and κ influence $m_{t,t+1}^{rb}$ through their effects on Π.[34] Lowering ϑ_0 strengthens the preference for robustness and then drive $m_{t,t+1}^{rb}$ away from 1 by increasing Π. Lowering κ reduces the Kalman gain, and then increases ω_η and Π. However, since the evolution of $\widehat{s}_t$ is also affected by ω_η and Π, we have to take both the Θ term and the $(R-1)\,\widehat{s}_t - \bar{c}$ term in (68) into account when evaluating how the interaction of RB and RI affects the market price of uncertainty.

To fully explore how induced uncertainty due to RB and RI affects the market price of uncertainty, we adopt the calibration procedure outlined in Hansen, Sargent, and Wang (2002), Anderson, Hansen, and Sargent (2003), and Hansen and Sargent (Chapter 9, 2007) to calibrate the

[33] In other words, $|w_t|$ is an upper bound on the approximate enhancement to the market price of uncertainty caused by the interaction of RB and RI.

[34] When $\kappa = \infty$, i.e., no RI, (68) reduces to $w_t = \frac{\Pi/\omega_\eta}{1-\Pi}\left[(R-1)\,s_t - \bar{c}\right]$. Without RB, $w_t = 0$.

value of Π that summarizes the interaction between RB and RI. Specifically, we calibrate Π by using the notion of a model detection error probability that is based on a statistical theory of model selection. We can then infer what values of the RB parameter imply reasonable fears of model misspecification for empirically-plausible approximating models. The model detection error probability denoted by p is a measure of how far the distorted model can deviate from the approximating model without being discarded; low values for this probability mean that agents are unwilling to discard very many models, implying that the cloud of models surrounding the approximating model is large (since agents want errors to be rare, they push the two models very far apart). The value of p is determined by the following procedure. Let model A denote the approximating model, (18), and model B be the distorted model, (21). Define p_A as

$$p_A = \text{Prob}\left(\ln\left(\frac{L_A}{L_B}\right) < 0 \middle| A \right), \tag{69}$$

where $\ln\left(\frac{L_A}{L_B}\right)$ is the log-likelihood ratio. When model A generates the data, p_A measures the probability that a likelihood ratio test selects model B. In this case, we call p_A the probability of the model detection error. Similarly, when model B generates the data, we can define p_B as

$$p_B = \text{Prob}\left(\log\left(\frac{L_A}{L_B}\right) > 0 \middle| B \right). \tag{70}$$

The detection error probability, p, is defined as the average of p_A and p_B:

$$p\left(\vartheta_0; \Pi\right) = \frac{1}{2}\left(p_A + p_B\right), \tag{71}$$

where ϑ_0 is the robustness parameter used to generate model B. Given this definition, we can see that $1 - p$ measures the probability that econometricians can distinguish the approximating model from the distorted model. Now we show how to compute the model detection error probability in the RB model. The general idea of the calibration exercise is to find a value of ϑ_0 (or Π) such that $p\left(\vartheta_0; \Pi\right)$ equals a given value (for example, 5 percent or 10 percent) after simulating model A, (18), and model B, (21).[35]

Following the consumption and saving literature, we set $R = 1.02$, $\omega/y_0 = 0.15$, $\rho = 0.8$, and $\bar{c} = 3.5y_0$. Using these parameter values, Figures 11 and 12 show how p affects the mean and median of MPU under RB and RI, respectively. It is clear from the figures that the effects of RB and RI on the mean and median of MPU are quite similar. We therefore focus on the mean of MPU in our subsequent analysis. Using either the data set documented in Campbell (2002) or that provided by Shiller, the estimated Sharpe ratio for the postwar U.S. time series is greater than 50 percent a year. Figure 13 plots the Hansen-Jagannathan (HJ) bound under RB and RI when $\theta = 0.1$ using the Shiller data set. It is clear from this figure that the model's predicted MPU can enter the HJ bound when both the preference for RB and the degree of RI are strong, i.e., when

[35]The number of periods used in the simulation, T, is set to be the actual length of the data we study. For example, if we consider the post-war U.S. annual time series data provided by Shiller from $1946 - 2010$, $T = 65$.

$p = 0.05$ or $p = 0.1$ and $\theta = 0.1$. If we consider the international developed-country data set in Campbell (2002), the Sharpe ratio is between 15 percent and 20 percent for Australia and Italy, between 20 percent and 30 percent for Canada and Japan, and above 30 percent for all the other countries. In the long-run annual data sets the lower bound on the standard deviation exceeds 30% for all three countries. From Figure 13, we can see that the theoretical MPU satisfies the HJ bound for higher values of p. Figure (14) clearly shows that our results about how the interaction of RB and RI with income uncertainty affects the market price of uncertainty are robust to the changes in the values of R, ω/y_0, ρ, and $\bar{c}$.

5.2. MPU under RS and RI

Under the risk-sensitivity (RS) specification of RB, (28), we can also compute the corresponding market price of uncertainty. The intertemporal marginal rate of substitution between t and $t+1$ can be written as

$$m_{t,t+1} = m^f_{t,t+1} m^{rs}_{t,t+1},$$

where $m^f_{t,t+1} \equiv \beta \frac{u'(\widehat{s}_{t+1})}{u'(\widehat{s}_t)}$ and

$$m^{rs}_{t,t+1} \equiv \frac{\exp\left(-v_{t+1}/\vartheta_0\right)}{E_t\left[\exp\left(-v_{t+1}/\vartheta_0\right)\right]} = \frac{\exp\left(-\left(\Omega \widehat{s}^2_{t+1} + \rho\right)/\vartheta_0\right)}{\exp\left(-\left(\widehat{\Omega}\widehat{s}^2_t + \widehat{\rho}\right)/\vartheta_0\right)}.$$

Using a formula found in Jacobson (1973) and used in Hansen, Sargent, and Tallarini (1999), we have

$$E_t\left[\left(m^{rs}_{t,t+1}\right)^2\right] = \exp\left(-2\left[\left(\widetilde{\Omega} - \widehat{\Omega}\right)\widehat{s}^2_t + \left(\widetilde{\rho} - \widehat{\rho}\right)\right]/\vartheta_0\right),$$

because $E_t\left[\exp\left(-2\left(\Omega\widehat{s}^2_{t+1} + \rho\right)/\vartheta_0\right)\right] = \exp\left(-2\left(\widetilde{\Omega}\widehat{s}^2_t + \widetilde{\rho}\right)/\vartheta_0\right)$. The following proposition summarizes the result on how induced uncertainty affects the amount of market price of uncertainty under the RS specification of RB.

Proposition 9. *Under this specification, the Radon-Nikodym derivative is*

$$E_t\left[\left(m^{rs}_{t,t+1}\right)^2\right] = \exp\left(\Xi^{rs}\left[(R-1)\,\widehat{s}_t - \bar{c}\right]^2\right)\Upsilon, \tag{72}$$

where $\Xi^{rs} \equiv \frac{R\Pi(1-R\Pi)}{\vartheta_0(1-\Pi)^2[1-(2R-1)\Pi]}$ *and* $\Upsilon \equiv \frac{1-(R-1)\Pi/(1-\Pi)}{\sqrt{1-2(R-1)\Pi/(1-\Pi)}}$. *The market price of induced uncertainty is*

$$\mathrm{sd}_t\left(m^{rs}_{t,t+1}\right) = \sqrt{\exp\left(\Xi^{rs}\left((R-1)\,\widehat{s}_t - \bar{c}\right)^2\right)\Upsilon - 1} \cong \left|\sqrt{\Xi^{rs}}\left((R-1)\,\widehat{s}_t - \bar{c}\right)\right| \tag{73}$$

Proof. See Online Appendix 8.4. ∎

Denote $\Xi^{rb} = \Theta^2$, we have

$$\Delta \equiv \sqrt{\frac{\Xi^{rs}}{\Xi^{rb}}} = \sqrt{\frac{1-R\Pi}{1-(2R-1)\,\Pi}}, \tag{74}$$

which is close 1 when R is close 1 and Π is not sufficiently high. Using (74), it is straightforward to show that

$$\frac{\partial \Delta}{\partial \Pi} > 0 \text{ and } \frac{\partial \Delta}{\partial \theta} < 0,$$

i.e., the stronger the degree of RI, the larger the difference of the market price of uncertainty under the RB and RS specifications. Figure 15 illustrates how Δ varies with Π for given values of R. It is clear that theoretically the difference of the market price of uncertainty between RB and RS under RI can be very significant. For example, when $R = 1.02$, $\Delta = 1.25$ when $\Pi = 0.93$. In other words, the MPU under the RS-RI specification is 25 percent higher than that under the RB-RI specification when the two models are observationally equivalent. However, after calibrating empirically-plausible ϑ_0 (and Π) using the DEP, $\mathrm{sd}_t\left(m^{rb}_{t,t+1}\right)$ and $\mathrm{sd}_t\left(m^{rs}_{t,t+1}\right)$ are very close because R is close to 1 and the calibrated values of Π are between 0.1 and 0.2 for $p = 0.05$.[36] Figure (16) clearly shows that the two specifications have similar effects on the mean of for MPU different values of p and θ. In addition, Figure 17 plots the HJ bound under RS and RI when $\theta = 0.1$ using the same data set as in the above RB-RI specification, and it is clear that this HJ bound is similar to that shown in Figure 13.

6. Implications for Aggregate Wealth Accumulation

In this section, following Caballero (1991), Irvine and Wang (1994), and Wang (2000), we consider an aggregate economy with a continuum of infinitely-lived inattentive agents having the preference for robustness. The aggregate economy we consider is an overlapping generations model with a constant population size (normalized to 1) and mortality risk. The individual consumers in the aggregate economy are *ex ante* identical in the sense that they face the same exogenously given income process, the same initial level of financial wealth, and have the same degrees of inattention and concerns about model misspecification. Within this setting, we can examine how two types of induced uncertainty, model uncertainty due to RB and state uncertainty due to RI, affect aggregate wealth accumulation.

Specifically, we assume that δ is the probability of surviving through period $t+1$, given that one is alive at t. The unconditional probability of reaching age t is δ^{t-1}, and the effective discount factor can be written as $\widetilde{\beta} = \delta\beta$. Incorporating the survival rate into the above RI-RB model, the objective function becomes

$$v(s_0) = \max_{\{c_t\}} \left\{ E_0 \left[\sum_{t=0}^{\infty} \widetilde{\beta}^t u(c_t) \right] \right\}.$$

For simplicity, we also impose the restriction that $\widetilde{\beta} R = 1$. Note that given the survival rate (δ) and the population size (1) for the aggregate economy in any period, the population size of individuals who die in any period is $1-\delta$. Since the population size of the newborn in any period is also $1-\delta$, the population size of the aggregate economy remains constant over time.

Using the income process with the growth component ($g > 1$), (4), the budget constraint, (5),

[36] The calibrated values of Π are lower for higher values of p.

and the consumption function, (50), the dynamic equation of financial wealth (b_t) can be written as

$$b_{t+1} = \rho_s b_t + \left(1 - \frac{R-1}{(1-\Pi)(R-\rho)}\right)(y_t - \overline{y}_t) + \left[1 - \frac{(R-1)(R-g+1-\rho)}{(1-\Pi)(R-\rho)(R-g)}\right]\overline{y}_t + \frac{\Pi\overline{c}}{1-\Pi} + \varsigma_{t+1}, \tag{75}$$

where $\varsigma_{t+1} = \frac{R-1}{1-\Pi}(s_t - \widehat{s}_t)$ and $\overline{y}_t = E[y_t] = \frac{g-\rho}{1-\rho}g^t y_0$. (See Online Appendix 8.5 for the derivation.) Writing (75) recursively to yield

$$\begin{aligned} b_{t+1} = {} & \rho_s^{t+1} b_0 + \left(1 - \frac{R-1}{(1-\Pi)(R-\rho)}\right)\sum_{j=0}^{t}\rho_s^{t-j}\left(y_j - \overline{y}_j\right) \\ & + \left\{\left[1 - \frac{(R-1)(R-g+1-\rho)}{(1-\Pi)(R-\rho)(R-g)}\right]\overline{y}_t + \frac{\Pi\overline{c}}{1-\Pi}\right\}\frac{1-\rho_s^{t+1}}{1-\rho_s} + \sum_{j=0}^{t+1}\rho_s^{t+1-j}\varsigma_j \end{aligned} \tag{76}$$

The aggregate economy is populated by a continuum of ex ante identical, but ex post heterogeneous consumers because consumers face both idiosyncratic income and noise shocks. Sun (2006) presents an exact law of large numbers for this type of economic models and then characterizes the cancellation of individual risk via aggregation. In this model, we adopt this law of large numbers (LLN) and assume that the initial cross-sectional distributions of the income and noise shocks are their stationary distributions. Provided that we construct the space of agents and the probability spaces appropriately, all idiosyncratic shocks are canceled out, and aggregate income is constant and aggregate noise is zero. That is, the cross-sectional mean of ς_j for any j is constant.[37] As the population size of the age t group is $(1-\delta)\delta^{t-1}$, aggregate wealth in the economy, A, can be written as

$$A = (1-\delta)\sum_{t=1}^{\infty}\left(\delta^{t-1}\widetilde{b}_t\right), \tag{77}$$

where $\widetilde{b}_t = b_t - \sum_{j=0}^{t}\rho_s^{t-j}\varsigma_j$. At any period, there are $1-\delta$ newborns entering the economy and demanding a total endowment, $(1-\delta)b_0$. At the same time, $1-\delta$ individuals die, leaving a total accidental bequest $(1-\delta)A$. Following Caballero (1991), the equilibrium condition is that the total supply of wealth equals the total demand for wealth:

$$b_0 = A. \tag{78}$$

To determine a steady state equilibrium, we need the following assumption.

Assumption 1 $\delta g < 1$ and $R > g$.

We assume $R > g$ to guarantee that the x component is positively related to permanent income: $s \equiv b + \frac{1}{R-\rho}y + \frac{1}{(R-g)(R-\rho)}x$. Here we also impose the restriction that $\delta g < 1$ so that the turnover of the population is sufficiently higher than the growth rate of income. Substituting (76) and (78) into (77), we can solve for A and have the following proposition:

[37]Wang (2003) applied the same LLN in the Caballero-type model with idiosyncratic income shocks.

Proposition 10. *Given that $\widetilde{\beta}R = 1$, in the steady state equilibrium, aggregate wealth is*

$$A^* = F(\Pi) \frac{g(g-\rho)(1-\delta)(1-\delta\rho_s)}{(1-\rho)(1-\delta g)(1-\delta g\rho_s)} y_0 + \frac{1}{R-1}\bar{c}, \tag{79}$$

where $F(\Pi) = \frac{1}{\Pi}\left[\frac{1-\Pi}{R-1} - \frac{R-g+1-\rho}{(R-\rho)(R-g)}\right]$, $\rho_s = \frac{1-R\Pi}{1-\Pi}$, *and* $\Pi = \frac{\theta(\vartheta,\kappa)}{1-(1-\theta(\vartheta,\kappa))R^2} R\omega_\zeta^2/(2\vartheta_0)$. *Therefore, equilibrium aggregate wealth in this model depends on the interactions of RB and RI with fundamental income uncertainty. Given that aggregate income* (Y) *is* $Y = (1-\delta)\sum_{t=1}^{\infty}\left(\delta^{t-1}y_t\right) = \frac{g(1-\delta)}{1-pg}y_0$, *the wealth-income ratio is*

$$\frac{A^*}{Y} = F(\Pi)\frac{(g-\rho)(1-\delta\rho_s)}{(1-\rho)(1-\delta g\rho_s)} + \frac{1-\delta g}{g(1-\delta)(R-1)}\frac{\bar{c}}{y_0}. \tag{80}$$

Proof. See Online Appendix 8.5. ∎

As we found above, the expressions (79) and (80) do not lend themselves to clear results regarding how RI and RB affect A^* or A^*/Y. Figure 18 illustrates the effect of Π on A^*/Y, when $\delta = 0.96$, $R = 1.03$, $\omega_\zeta^2 = 1$, and $\rho = 0.9$.[38] It is clear from the figure that for a given $g > 1$, the equilibrium aggregate wealth-income ratio is increasing with Π:

$$\frac{\partial(A^*/Y)}{\partial\Pi} > 0.$$

Similarly, we also have $\frac{\partial A^*}{\partial\Pi} > 0$. Since Π is a function of ϑ_0, ϑ, and κ, equilibrium aggregate wealth depends on both RB and RI. We now consider the following numerical example to see how induced uncertainty due to RB and RI affects A^*/Y. When we set $g = 1.02$, $\vartheta_0 = 1$, and $R\omega_\zeta^2/(2\vartheta_0) = 0.62$, A^*/Y (or A^*) will fall by 85 percent, increase by 3 percent, and fall by 7 percent as Type I robustness (ϑ_0), Type II robustness (ϑ), and capacity κ are increased by 10 percent, respectively.[39] These results clearly show that induced uncertainty due to the interactions of RI, RB, and fundamental uncertainty does have significant effects on aggregate wealth accumulation in equilibrium. Furthermore, it is not surprising that Type I robustness has the greatest effect on aggregate wealth because it enters the Π function directly while Type II robustness and capacity only affect Π through the robust Kalman gain.

It is worth noting that when there is no income growth component ($g = 1$), (79) reduces to $A^* = \frac{\bar{c}-y_0}{R-1}$. In other words, in this special case without the growth component the level of equilibrium aggregate wealth is independent of induced uncertainty. The intuition behind this result is as follows. As we have discussed in the last section, the consumption function, (50), implies that induced uncertainty measured by Π affects individual saving behavior via two opposing channels: $-MPC_\eta\widehat{s}_t$ ($MPC_\eta = \frac{R-1}{1-\Pi}$) and $PS = \frac{\Pi\bar{c}}{1-\Pi}$ because $\frac{\partial(-MPC_\eta)}{\partial\Pi} < 0$ and $\frac{\partial(PS)}{\partial\Pi} > 0$. When there is no income growth, these two effects cancel out in general equilibrium and thus induced uncertainty has no effect on aggregate wealth accumulation.

[38] These parameter values are chosen based on stylized facts in US annual data on aggregate income.

[39] This pattern is robust for different parameter values.

7. Conclusion

This paper has provided a characterization of the consumption-savings behavior of agents who have a preference for robustness (worries about model misspecification) and limited information-processing ability. After obtaining the optimal individual decisions, we explore how two types of induced uncertainty, state uncertainty due to RI and model uncertainty due to RB, affect consumption and saving decisions as well as the market prices of uncertainty. Specifically, we show that different types of about model misspecification – (i) concerns about the disturbances to the perceived permanent income and (ii) concerns about the Kalman gain – can have opposite effects on consumption, savings, and asset prices via interacting with finite capacity in the control and filtering problems. In addition, we show that once allowing RB consumers to use the robust Kalman filter to update the perceived state, the simple observational equivalence (OE) between RB and RS obtained in Hansen, Sargent, and Tallarini (1999) and Luo and Young (2010) no longer holds; instead, we find a more complicated OE between RB and RS. Finally, we explore how the two types of informational frictions affect the market price of risk and aggregate wealth accumulation.

8. Online Appendix (Not for Publication)

8.1. Solving the Two-Player Game Version of the Robust Model

To solve the Bellman equation (23) subject to (21), $\widehat{s}_{t+1} = R\widehat{s}_t - c_t + \omega_\eta w_t + \eta_{t+1}$, we conjecture that

$$v\left(\widehat{s}_t\right) = -C - B\widehat{s}_t - A\widehat{s}_t^2, \tag{81}$$

where A, B, and C are constants to be determined. Substituting this guessed value function into the Bellman equation (23) gives

$$-C - B\widehat{s}_t - A\widehat{s}_t^2 = \max_{c_t}\min_{w_t}\left\{-\frac{1}{2}\left(c_t - \overline{c}\right)^2 + \beta E_t\left[\widetilde{\vartheta}_0 w_t^2 - C - B\widehat{s}_{t+1} - A\widehat{s}_{t+1}^2\right]\right\}, \tag{82}$$

where $\widetilde{\vartheta}_0 = \vartheta_0/2$. We can do the min and max operations in any order, so we choose to do the minimization first. The first-order condition for w_t is

$$2\vartheta\nu_t - 2AE_t\left(\omega_\eta w_t + R\widehat{s}_t - c_t\right)\omega_\eta - B\omega_\eta = 0,$$

which means that

$$w_t = \frac{B + 2A\left(R\widehat{s}_t - c_t\right)}{2\left(\widetilde{\vartheta}_0 - A\omega_\eta^2\right)}\omega_\eta. \tag{83}$$

Substituting (83) back into (82) gives

$$-A\widehat{s}_t^2 - B\widehat{s}_t - C = \max_{c_t}\left\{-\frac{1}{2}\left(\bar{c}-c_t\right)^2 + \beta E_t\left[\widetilde{\vartheta}_0\left[\frac{B+2A\left(Rs_t-c_t\right)}{2\left(\vartheta - A\omega_\eta^2\right)}\omega_\eta\right]^2 - As_{t+1}^2 - Bs_{t+1} - C\right]\right\},$$

where $\widehat{s}_{t+1} = R\widehat{s}_t - c_t + \omega_\eta w_t + \eta_{t+1}$. The first-order condition for c_t is

$$\left(\bar{c}-c_t\right) - 2\beta\widetilde{\vartheta}_0\frac{A\omega_\eta}{\vartheta - A\omega_\eta^2}w_t + 2\beta A\left(1+\frac{A\omega_\eta^2}{\widetilde{\vartheta}_0 - A\omega_\eta^2}\right)\left(R\widehat{s}_t - c_t + \omega_\eta w_t\right) + \beta B\left(1+\frac{A\omega_\eta^2}{\widetilde{\vartheta}_0 - A\omega_\eta^2}\right) = 0.$$

Using the solution for ν_t the solution for consumption is

$$c_t = \frac{2A\beta R}{1-A\omega_\eta^2/\widetilde{\vartheta}_0 + 2\beta A}\widehat{s}_t + \frac{\bar{c}\left(1-A\omega_\eta^2/\widetilde{\vartheta}_0\right)+\beta B}{1-A\omega_\eta^2/\widetilde{\vartheta}_0 + 2\beta A}.$$

Substituting the above expressions into the Bellman equation gives

$$\begin{aligned}
&-A\widehat{s}_t^2 - B\widehat{s}_t - C \\
&= -\frac{1}{2}\left(\frac{2A\beta R}{1-A\omega_\eta^2/\widetilde{\vartheta}_0 + 2\beta A}\widehat{s}_t + \frac{-2\beta A\bar{c}+\beta B}{1-A\omega_\eta^2/\widetilde{\vartheta}_0 + 2\beta A}\right)^2 \\
&+ \frac{\beta\widetilde{\vartheta}_0\omega_\eta^2}{\left(2\left(\widetilde{\vartheta}_0 - A\omega_\eta^2\right)\right)^2}\left[\frac{2AR\left(1-A\omega_\eta^2/\widetilde{\vartheta}_0\right)}{1-A\omega_\eta^2/\widetilde{\vartheta}_0 + 2\beta A}\widehat{s}_t + B - \frac{2\bar{c}\left(1-A\omega_\eta^2/\widetilde{\vartheta}_0\right)A + 2\beta AB}{1-A\omega_\eta^2/\widetilde{\vartheta}_0 + 2\beta A}\right]^2 \\
&- \beta A\left\{\left[\frac{R}{1-A\omega_\eta^2/\widetilde{\vartheta}_0 + 2\beta A}\widehat{s}_t - \frac{-B\omega_\eta^2/\widetilde{\vartheta}_0 + 2c + 2B\beta}{2\left(1-A\omega_\eta^2/\widetilde{\vartheta}_0 + 2\beta A\right)}\right]^2 + \omega_\eta^2\right\} \\
&- \beta B\left[\frac{R}{1-A\omega_\eta^2/\widetilde{\vartheta}_0 + 2\beta A}\widehat{s}_t - \frac{-B\omega_\eta^2/\widetilde{\vartheta}_0 + 2c + 2B\beta}{2\left(1-A\omega_\eta^2/\widetilde{\vartheta}_0 + 2\beta A\right)}\right] - \beta C.
\end{aligned}$$

Collecting and matching terms, the constant coefficients turn out to be

$$A = \frac{\beta R^2 - 1}{2\beta - \omega_\eta^2/\widetilde{\vartheta}_0}, B = \frac{\left(\beta R^2 - 1\right)\bar{c}}{(R-1)\left(\omega_\eta^2/\left(\widetilde{\vartheta}_0\right) - \beta\right)},$$
$$C = \frac{R\left(\beta R^2 - 1\right)}{2\left(\beta R - R\omega_\eta^2/\widetilde{\vartheta}_0\right)(R-1)^2}\left((R-1)\,\omega_\eta^2 + \bar{c}^2\right),$$

where $\widetilde{\vartheta}_0 = \vartheta_0/2$. When $\beta R = 1$, we obtain the consumption function (24) in the text.

8.2. Solving the Multiplier Preference Version of the RB Model

To solve the Bellman equation (30) subject to (18), we conjecture that

$$v\left(\widehat{s}_t\right) = -C - B\widehat{s}_t - A\widehat{s}_t^2, \tag{84}$$

where A, B, and C are constants to be determined. We can then evaluate $E_t\left[\exp\left(-\alpha v\left(\widehat{s}_{t+1}\right)\right)\right]$ to obtain

$$\begin{aligned}
&E_t\left[\exp\left(-\alpha v\left(\widehat{s}_{t+1}\right)\right)\right] \\
&= E_t\left[\exp\left(\alpha A\widehat{s}_{t+1}^2 + \alpha B\widehat{s}_{t+1} + \alpha C\right)\right] \\
&= E_t\left[\exp\left(\alpha A\left(R\widehat{s}_t - c_t\right)^2 + \alpha B\left(R\widehat{s}_t - c_t\right) + \left[2\alpha A\left(R\widehat{s}_t - c_t\right) + \alpha B\right]\eta_{t+1} + \alpha A\eta_{t+1}^2 + \alpha C\right)\right] \\
&= \left(1-2c\right)^{-1/2}\exp\left(a + \frac{b^2}{2\left(1-2c\right)}\right),
\end{aligned}$$

where $\alpha = 1/\vartheta_0$,

$$\begin{aligned}
a &= \alpha A\left(R\widehat{s}_t - c_t\right)^2 + \alpha B\left(R\widehat{s}_t - c_t\right) + \alpha C, \\
b &= \left[2\alpha A\left(R\widehat{s}_t - c_t\right) + \alpha B\right]\omega_\eta, \\
c &= \alpha A\omega_\eta^2.
\end{aligned}$$

Thus, the distorted expectations operator can be written as

$$\begin{aligned}
\mathcal{R}_t\left[v\left(\widehat{s}_{t+1}\right)\right] &= -\frac{1}{\alpha}\left\{-\frac{1}{2}\log\left(1-2c\right) + a + \frac{b^2}{2\left(1-2c\right)}\right\} \\
&= \frac{1}{2\alpha}\log\left(1-2\alpha A\omega_\eta^2\right) - \frac{A}{1-2\alpha A\omega_\eta^2}\left(R\widehat{s}_t - c_t\right)^2 - \frac{B}{1-2\alpha A\omega_\eta^2}\left(R\widehat{s}_t - c_t\right) - \left[C + \frac{\alpha B^2\omega_\eta^2}{2\left(1-2\alpha A\omega_\zeta^2\right)}\right]
\end{aligned} \tag{85}$$

Maximizing the RHS of (85) with respect to c_t yields the first-order condition

$$-\left(c_t - \overline{c}\right) + \frac{2\beta A}{1-2\alpha A\omega_\eta^2}\left(R\widehat{s}_t - c_t\right) + \frac{B\beta}{1-2\alpha A\omega_\eta^2} = 0,$$

which means that

$$c_t = \frac{2A\beta R}{1-2\alpha A\omega_\eta^2 + 2A\beta}\widehat{s}_t + \frac{\overline{c}\left(1-2\alpha A\omega_\eta^2\right) + B\beta}{1-2\alpha A\omega_\eta^2 + 2A\beta}. \tag{86}$$

Substituting (86) and (84) into (59), and collecting and matching terms, the constant coefficients turn out to be

$$A = \frac{\beta R^2 - 1}{2\beta - 2\alpha\omega_\eta^2}, \tag{87}$$

$$B = \frac{\left(\beta R^2 - 1\right)\bar{c}}{(R-1)\left(\alpha\omega_\eta^2 - \beta\right)}, \tag{88}$$

$$C = \frac{1}{2}\frac{\beta R^2 - 1}{(R-1)^2\left(\beta - \alpha\omega_\eta^2\right)}\bar{c}^2 - \frac{\beta}{1-\beta}\frac{1}{\alpha}\ln\left(1 - \frac{\left(\beta R^2 - 1\right)\alpha R\omega_\eta^2}{\beta R - \alpha R\omega_\eta^2}\right),$$

where $\alpha = 1/\vartheta_0$. Setting $\beta R = 1$, substituting (87) and (88) into (86) yields the consumption function (31) in the text.

8.3. Deriving the Stochastic Properties of Individual Consumption and Saving

8.3.1. Deriving the Volatility of Consumption

Given that

$$c_t = \rho_1 c_{t-1} + \frac{R-1}{1-\Pi}\left[\frac{\theta\zeta_t}{1-\rho_2\cdot L} + \theta\left(\xi_t - \frac{\theta R\xi_{t-1}}{1-\rho_2\cdot L}\right)\right],$$

c_t can be rewritten as

$$c_t = \theta\frac{R-1}{1-\Pi}\frac{\zeta_t + (\xi_t - R\xi_{t-1})}{(1-\rho_1\cdot L)(1-\rho_2\cdot L)}.$$

Taking the unconditional variance on both sides yields

$$\begin{aligned}\operatorname{var}(c_t) &= \operatorname{var}\left(\theta\frac{R-1}{1-\Sigma}\frac{\zeta_t + (\xi_t - R\xi_{t-1})}{(1-\rho_1\cdot L)(1-\rho_2\cdot L)}\right)\\ &= \left(\theta\frac{R-1}{1-\Sigma}\right)^2\sum_{k=0}^{\infty}\left\{\left[\sum_{j=0,j\le k}^{k}\left(\rho_1^{k-j}\rho_2^j\right)^2\right]\omega_\zeta^2 + \left[\sum_{j=0,j\le k}^{k}\left(\rho_1^{k-j}\rho_2^j\right) - R\sum_{j=0,j\le k-1}^{k-1}\left(\rho_1^{k-1-j}\rho_2^j\right)\right]^2\omega_\xi^2\right\}.\end{aligned}$$

8.3.2. Deriving the Variance of Individual Saving under RB-RI

Given (56) in the text, we can derive the variance of individual saving (d) as follows:

$$
\begin{aligned}
\operatorname{var}(d_t) &= \operatorname{var}\left(\frac{-\Pi(R-1)}{1-\Pi}\left(b_t-\overline{b}\right)+\left(1-\frac{R-1}{(1-\Pi)(R-\rho)}\right)(y_t-\overline{y})+\varsigma_{t+1}\right)\\
&= \operatorname{var}\left(\frac{(1-\rho)\zeta_t}{1-\rho\cdot L}+\frac{\Gamma\zeta_t}{1-\rho_s\cdot L}+\frac{R-1}{1-\Pi}\left(\frac{(1-\theta)\zeta_t}{1-\rho_\theta\cdot L}-\frac{\theta\xi_t}{1-\rho_\theta\cdot L}\right)\right)\\
&= (1-\rho)^2\frac{\omega_\zeta^2}{1-\rho^2}+\Gamma^2\frac{\omega_\zeta^2}{1-\rho_s^2}+\left(\frac{R-1}{1-\Pi}\right)^2\operatorname{var}\left(\frac{(1-\theta)\zeta_t}{1-\rho_\theta\cdot L}-\frac{\theta\xi_t}{1-\rho_\theta\cdot L}\right)\\
&\quad+2\operatorname{cov}\left(\frac{(1-\rho)\zeta_t}{1-\rho\cdot L},\frac{\Gamma\zeta_t}{1-\rho_s\cdot L}\right)+2\operatorname{cov}\left(\frac{(1-\rho)\zeta_t}{1-\rho\cdot L},\frac{R-1}{1-\Pi}\left(\frac{(1-\theta)\zeta_t}{1-\rho_\theta\cdot L}-\frac{\theta\xi_t}{1-\rho_\theta\cdot L}\right)\right)\\
&\quad+2\operatorname{cov}\left(\frac{\Gamma\zeta_t}{1-\rho_s\cdot L},\frac{R-1}{1-\Pi}\left(\frac{(1-\theta)\zeta_t}{1-\rho_\theta\cdot L}-\frac{\theta\xi_t}{1-\rho_\theta\cdot L}\right)\right)\\
&= \left\{\begin{array}{l}\frac{1-\rho}{1+\rho}+\frac{\Gamma^2}{1-\rho_s^2}+\left(\frac{R-1}{1-\Pi}\right)^2\left[\frac{(1-\theta)^2}{1-\rho_\theta^2}+\frac{\theta\lambda^2}{1-\rho_\theta^2}\frac{1}{(1/(1-\theta)-R^2)}\right]\\ +\frac{2(1-\rho)\Gamma}{1-\rho\rho_s}+\left(\frac{R-1}{1-\Pi}\right)\frac{2(1-\rho)(1-\theta)}{1-\rho\rho_\theta}+\left(\frac{R-1}{1-\Pi}\right)\frac{2\Gamma(1-\theta)}{1-\rho_s\rho_\theta}\end{array}\right\}\omega_\zeta^2,
\end{aligned}
$$

which is just (58) in the text.

8.4. Computing the Market Price of Uncertainty

Given the value function we obtained in Section (2.2),

$$
v(\widehat{s}_t)=\Omega\left(\widehat{s}_t-\frac{\overline{c}}{R-1}\right)^2+\rho, \tag{89}
$$

where $\Omega=-\frac{R(R-1)}{2(1-\Pi)}$ and $\rho=\frac{\vartheta_0}{2(R-1)}\ln\left(1-\frac{(R-1)\Pi}{1-\Pi}\right)$, it follows from Jacobson (1973) and HST (1999) that the risk-sensitivity operator can be written as

$$
\begin{aligned}
\mathcal{R}_t\left[\widehat{v}(\widehat{s}_{t+1})\right] &= -\vartheta_0\log E_t\left[\exp\left(-\widehat{v}(\widehat{s}_{t+1})/\vartheta_0\right)\right]\\
&= \widehat{\Omega}\left(\widehat{s}_t-\frac{\overline{c}}{R-1}\right)^2+\widehat{\rho},
\end{aligned}
$$

where

$$
\begin{aligned}
\widehat{\Omega} &= \rho_s^2\Omega\left(1-\frac{2}{\vartheta_0}\Omega\omega_\eta^2\left(1+\frac{2}{\vartheta_0}\Omega\omega_\eta^2\right)^{-1}\right)\\
&= -\frac{R(R-1)(1-R\Pi)}{2(1-\Pi)^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{\rho} &= \rho + \frac{\vartheta_0}{2} \ln\left(1 + \frac{2}{\vartheta_0}\Omega\omega_\eta^2\right) \\
&= \frac{\vartheta_0}{2(R-1)} \ln\left(1 - \frac{2(R-1)\Pi}{1-\Pi}\right) + \frac{\vartheta_0}{2}\ln\left(1 - \frac{(R-1)\Pi}{(1-\Pi)}\right),
\end{aligned}
$$

where we assume that $\frac{2(R-1)\Pi}{1-\Pi} < 1$.

Given that

$$
m_{t,t+1}^{rs} \equiv \frac{\exp(-v_{t+1}/\vartheta_0)}{E_t[\exp(-v_{t+1}/\vartheta_0)]} = \frac{\exp\left(-\left(\Omega\widehat{s}_{t+1}^2 + \rho\right)/\vartheta_0\right)}{\exp\left(-\left(\widehat{\Omega}\widehat{s}_t^2 + \widehat{\rho}\right)/\vartheta_0\right)},
$$

we have $\left(m_{t,t+1}^{rs}\right)^2 = \frac{\exp\left(-2\left(\Omega\widehat{s}_{t+1}^2+\rho\right)/\vartheta_0\right)}{\exp\left(-2\left(\widehat{\Omega}\widehat{s}_t^2+\widehat{\rho}\right)/\vartheta_0\right)}$. Multiplying the numerator and denominator by the time t conditional mean of the exponential term in the numerator, $E_t\left[\exp\left(-2\left(\Omega\widehat{s}_{t+1}^2 + \rho\right)/\vartheta_0\right)\right]$, gives

$$
\left(m_{t,t+1}^{rs}\right)^2 = \frac{E_t\left[\exp\left(-2\left(\Omega\widehat{s}_{t+1}^2 + \rho\right)/\vartheta_0\right)\right]}{\exp\left(-2\left(\widehat{\Omega}\widehat{s}_t^2 + \widehat{\rho}\right)/\vartheta_0\right)} \frac{\exp\left(-2\left(\Omega\widehat{s}_{t+1}^2 + \rho\right)/\vartheta_0\right)}{E_t\left[\exp\left(-2\left(\Omega\widehat{s}_{t+1}^2 + \rho\right)/\vartheta_0\right)\right]},
$$

where the exponential term, $E_t\left[\exp\left(\sigma\left(\Omega\widehat{s}_{t+1}^2 + \rho\right)\right)\right]$, can be computed using a formula found in Jacobson (1973):

$$
E_t\left[\exp\left(-2\left(\Omega\widehat{s}_{t+1}^2 + \rho\right)/\vartheta_0\right)\right] = \exp\left(-2\left(\widetilde{\Omega}\widehat{s}_t^2 + \widetilde{\rho}\right)/\vartheta_0\right),
$$

where

$$
\begin{aligned}
\widetilde{\Omega} &= \rho_s^2\Omega\left(1 - \frac{4}{\vartheta_0}\Omega\omega_\eta^2\left(1 + \frac{4}{\vartheta_0}\Omega\omega_\eta^2\right)^{-1}\right) = \frac{R\rho_s^2\Omega}{R + 4\Omega\Pi}, \\
\widetilde{\rho} &= \rho + \frac{\vartheta_0}{4}\ln\left(1 + \frac{1}{\vartheta_0}\Omega\omega_\eta^2\right) = \rho + \frac{\vartheta_0}{4}\ln\left(1 - \frac{2(R-1)\Pi}{(1-\Pi)}\right).
\end{aligned}
$$

Therefore, we obtain

$$
E_t\left[\left(m_{t,t+1}^{rs}\right)^2\right] = \exp\left(-2\left(\left(\widetilde{\Omega} - \widehat{\Omega}\right)\widehat{s}_t^2 + \left(\widetilde{\rho} - \widehat{\rho}\right)\right)/\vartheta_0\right),
$$

which yields (72) in the main text.

8.5. Deriving Aggregate Wealth Accumulation

Substituting (50) into (5), we have

$$
\begin{aligned}
& b_{t+1} - b_t \\
&= (R-1)\, b_t + y_t - \left(\frac{R-1}{1-\Pi} \widehat{s}_t - \frac{\Pi \overline{c}}{1-\Pi} \right) \\
&= (R-1)\, b_t + y_t - \left[\frac{R-1}{1-\Pi} \left(b_t + \frac{1}{R-\rho} y_t + \frac{1}{(R-g)\,(R-\rho)} x_t \right) - \frac{\Pi \overline{c}}{1-\Pi} \right] + \frac{R-1}{1-\Pi} \left(s_t - \widehat{s}_t \right) \\
&= \frac{-\Pi\,(R-1)}{1-\Pi} b_t + \left(1 - \frac{R-1}{(1-\Pi)\,(R-\rho)} \right) (y_t - \overline{y}_t) + \left[1 - \frac{(R-1)\,(R-g+1-\rho)}{(1-\Pi)\,(R-\rho)\,(R-g)} \right] \overline{y}_t \\
&+ \frac{\Pi \overline{c}}{1-\Pi} + \frac{R-1}{1-\Pi} \left(s_t - \widehat{s}_t \right),
\end{aligned}
$$

which is just (75) in the text. Note that here we use the following facts:

$$
\begin{aligned}
s_t &\equiv b_t + \frac{1}{R-\rho} y_t + \frac{1}{(R-g)\,(R-\rho)} x_t, \\
x_t &= (g-\rho)\, g^t y_0, y_t = g^t y_0, \overline{y}_t = \frac{g-\rho}{1-\rho} g^t y_0.
\end{aligned}
$$

In the steady state in which $y_t = \overline{y}_t$ and b and y grow at the same rate, g, (75) leads to

$$
\overline{b}_{t+1} = \rho_s \overline{b}_t + \left[1 - \frac{(R-1)\,(R-g+1-\rho)}{(1-\Pi)\,(R-\rho)\,(R-g)} \right] \overline{y}_t + \frac{\Pi \overline{c}}{1-\Pi},
$$

which reduces to $\overline{b}_t = \frac{1}{g-\rho_s} \left\{ \left[1 - \frac{(R-1)(R-g+1-\rho)}{(1-\Pi)(R-\rho)(R-g)} \right] \overline{y}_t + \frac{\Pi}{1-\Pi} \overline{c} \right\}$. Substituting (76) and (78) into (77) yields

$$
\begin{aligned}
\left[1 - (1-\delta) \sum_{t=1}^{\infty} \left(\delta^{t-1} \rho_s^t \right) \right] A &= \left(\frac{1-\delta}{1-\rho_s} \right) \sum_{t=1}^{\infty} \left(\delta^{t-1} \left(1-\rho_s^t\right) \left\{ \left[1 - \frac{(R-1)\,(R-g+1-\rho)}{(1-\Pi)\,(R-\rho)\,(R-g)} \right] \overline{y}_t + \frac{\Pi \overline{c}}{1-\Pi} \right\} \right) \\
A &= \frac{1}{\Pi} \left[\frac{1-\Pi}{R-1} - \frac{R-g+1-\rho}{(R-\rho)\,(R-g)} \right] \frac{g\,(g-\rho)\,(1-\delta)\,(1-\delta\rho_s)}{(1-\rho)\,(1-\delta g)\,(1-\delta g \rho_s)} y_0 + \frac{1}{R-1} \overline{c},
\end{aligned}
$$

which is just (79) in the text. Here we use the facts that

$$
\begin{aligned}
\sum_{t=1}^{\infty} \left(\delta^{t-1} \rho_s^t \right) &= \frac{\rho_s}{1-\delta\rho_s}, \sum_{t=1}^{\infty} \left[\delta^{t-1} \left(1-\rho_s^t\right) \right] = \frac{1-\rho_s}{(1-\delta)\,(1-\delta\rho_s)}, \\
\sum_{s=0}^{\infty} \sum_{j=0}^{s} \left(\delta^s \rho_s^{s-j} \right) &= \sum_{s=0}^{\infty} \delta^s \left(\sum_{j=0}^{s} \rho_s^{s-j} \right) = \frac{1}{\delta\,(1-\delta\rho_s)}, \\
\sum_{t=1}^{\infty} \left[\delta^{t-1} \left(\overline{y}_t \frac{1-\rho_s^t}{1-\rho_s} \right) \right] &= \frac{g}{1-\rho_s} \frac{g-\rho}{1-\rho} \frac{1-\rho_s}{(1-\delta g)\,(1-\delta g \rho_s)} y_0.
\end{aligned}
$$

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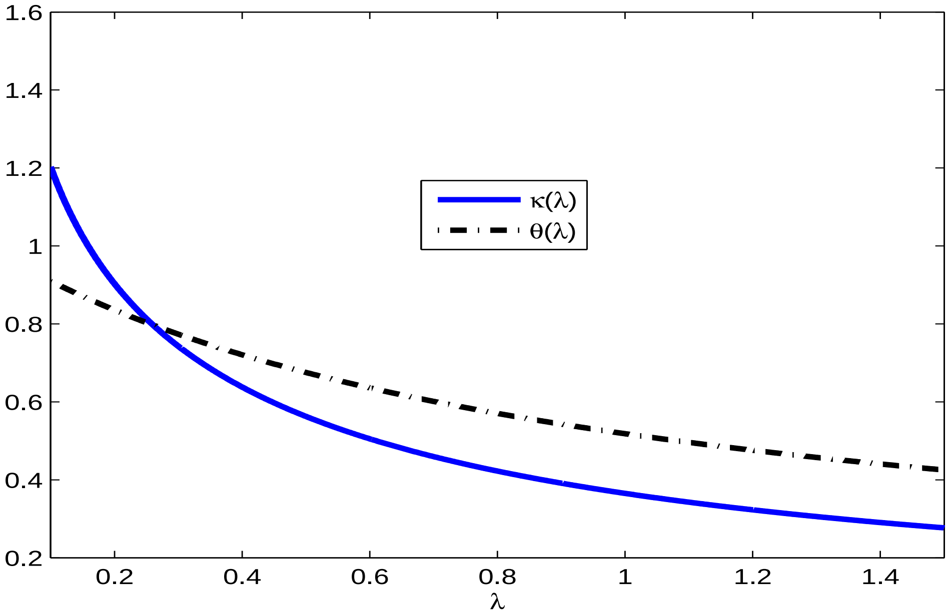

Figure 1. *Elastic Capacity* (κ) *and Kalman Gain* (θ)

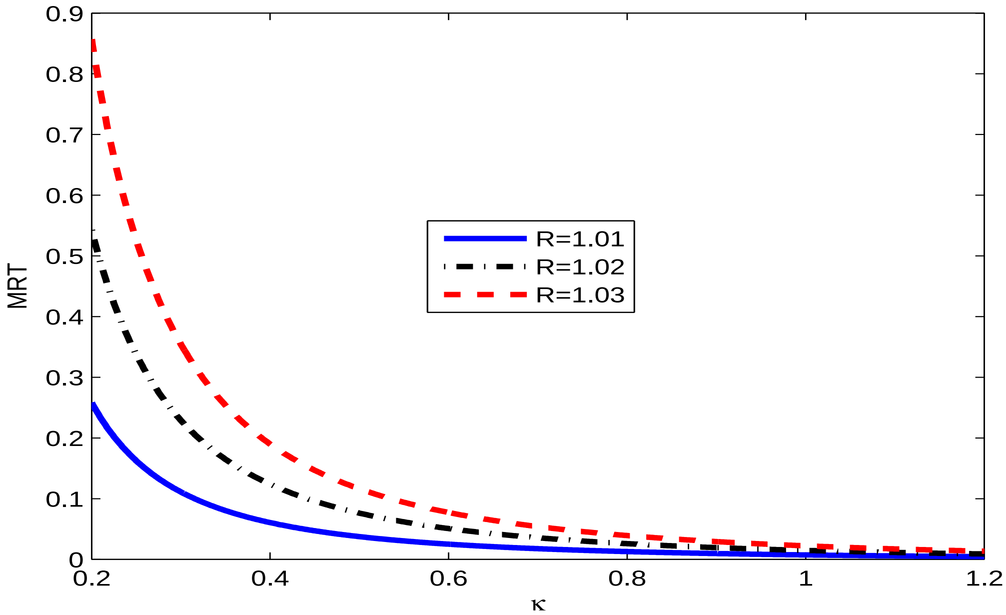

Figure 2. *MRT between RB and RI*

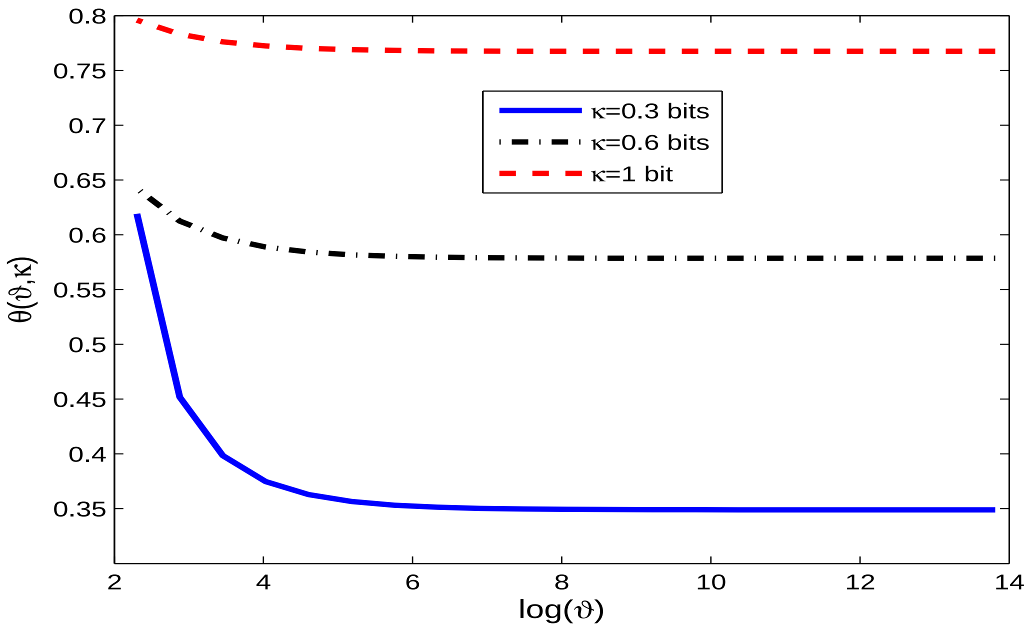

Figure 3. *Effects of RB and RI on Robust Kalman Gain* θ

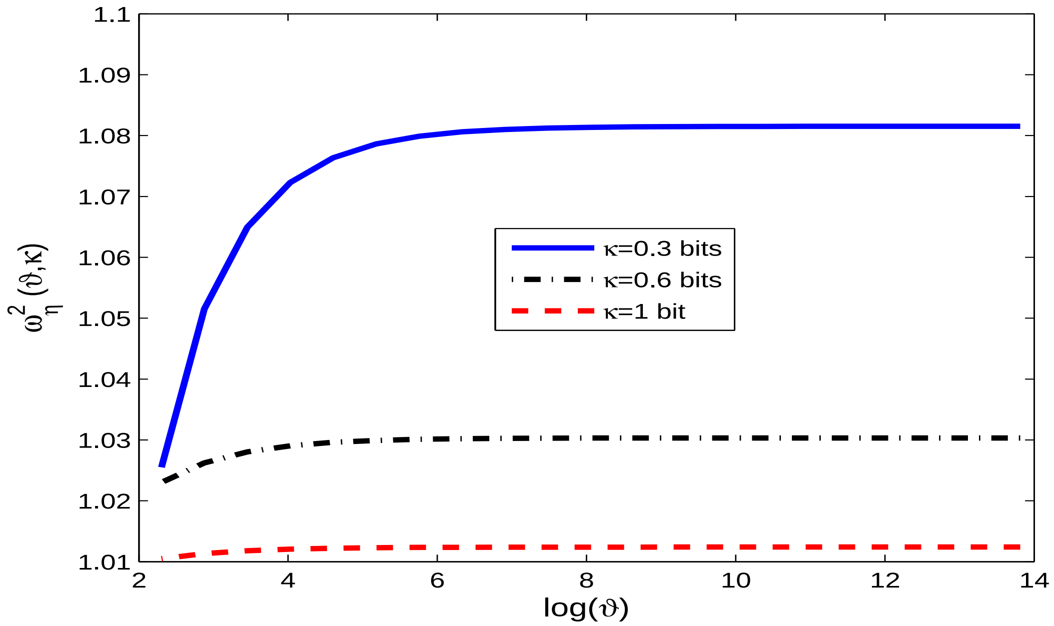

Figure 4. *Effects of RB* (ϑ) *and Capacity* (κ) *on* ω_η^2

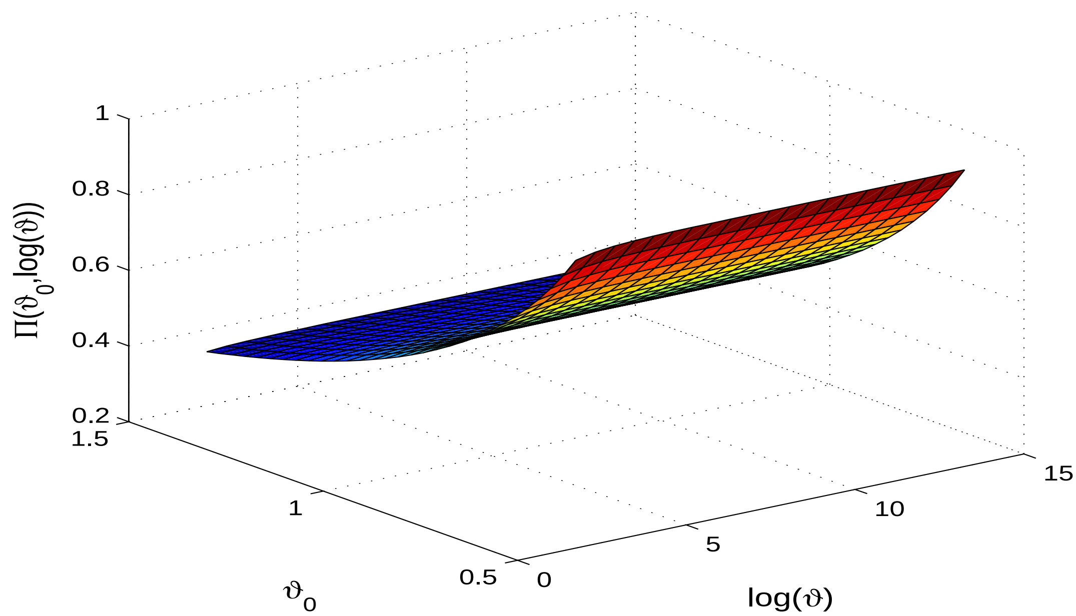

Figure 5. *Effects of Two Types of Robustness,* ϑ_0 *and* ϑ*, on* Π

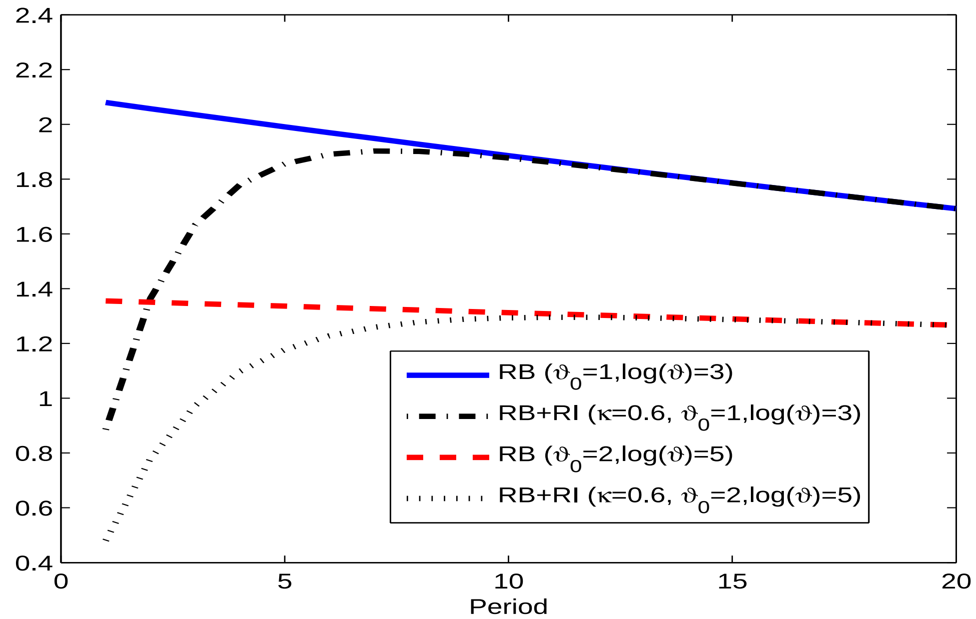

Figure 6. *Impulse Responses of Consumption to Income Shock*

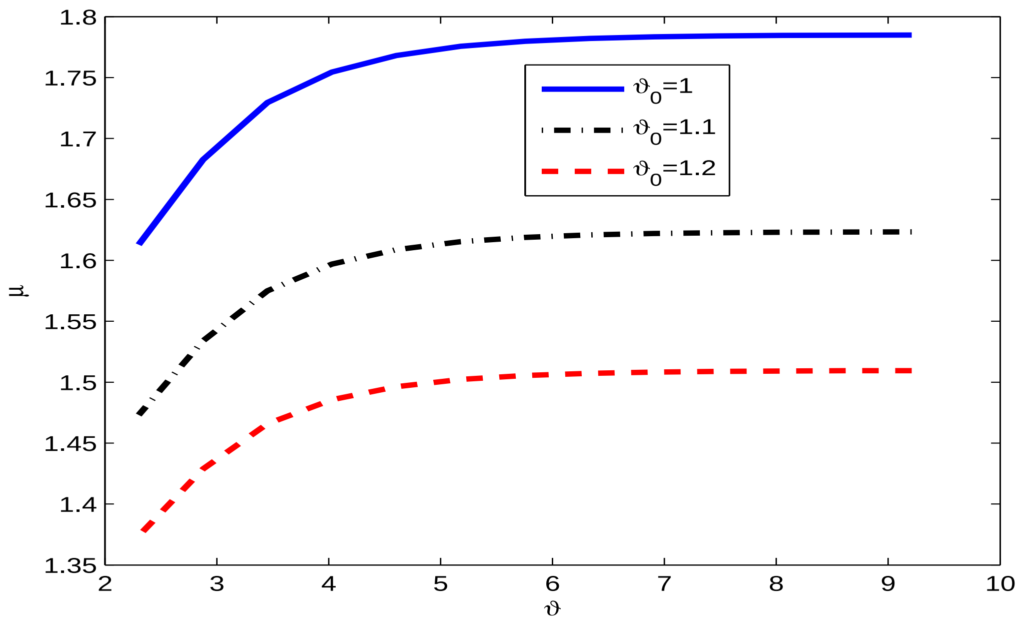

Figure 7. *Relative Volatility of Consumption to Income under RB-RI*

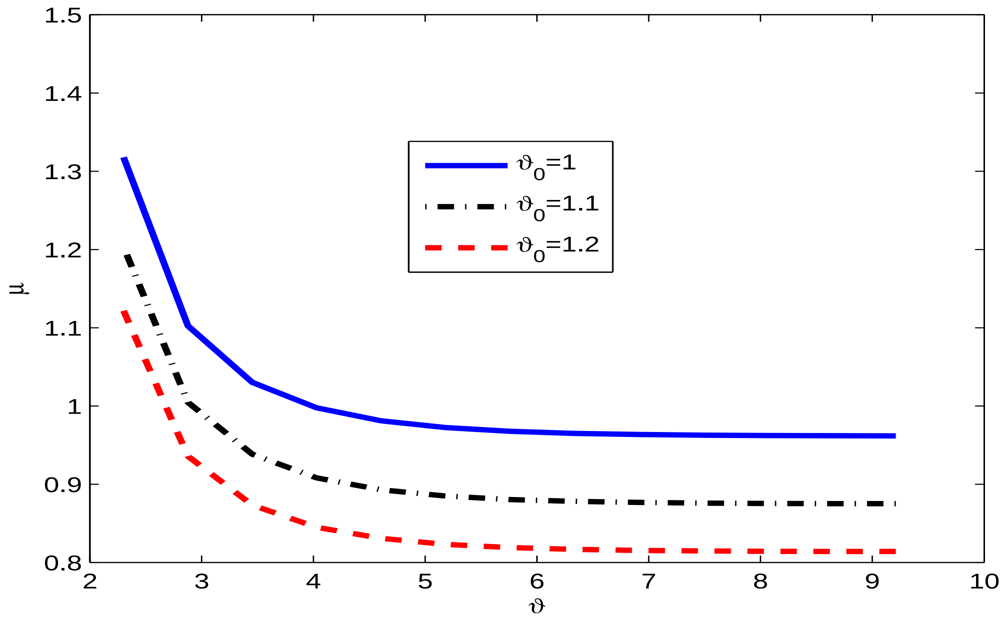

Figure 8. *Relative Volatility of Consumption to Income under RB-RI without Noise*

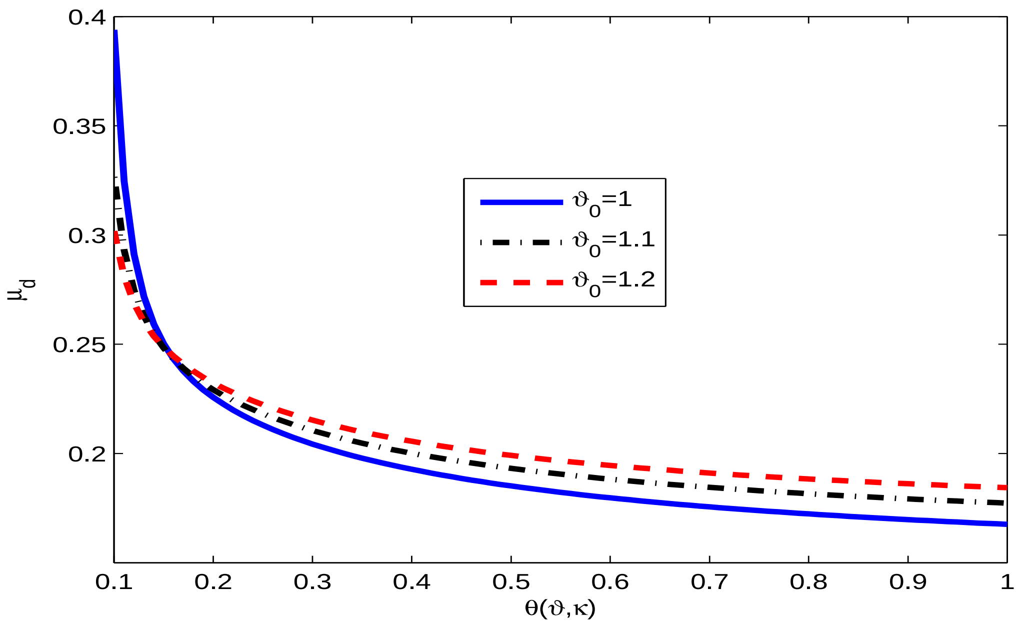

Figure 9. *Relative Volatility of Individual Saving*

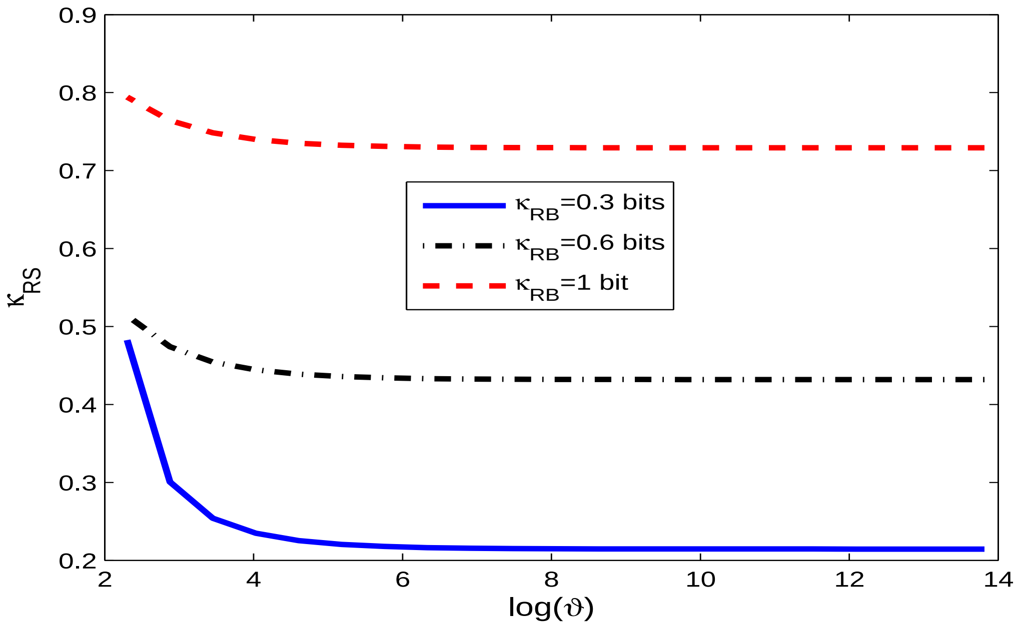

Figure 10. *The OE between RB and RS*

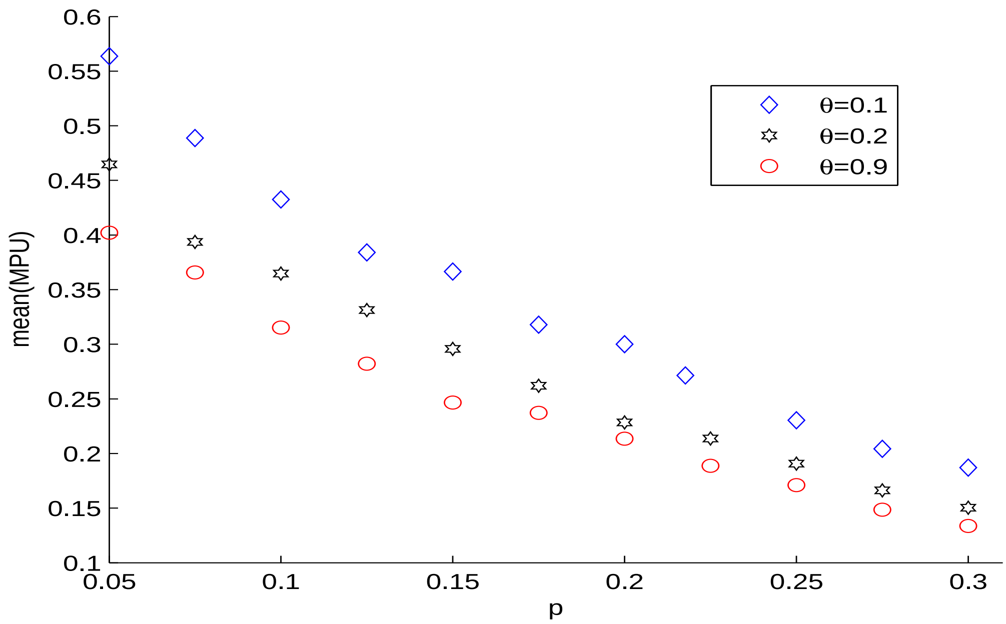

Figure 11. *Effects of p on the mean of* MPU *for Different θ*

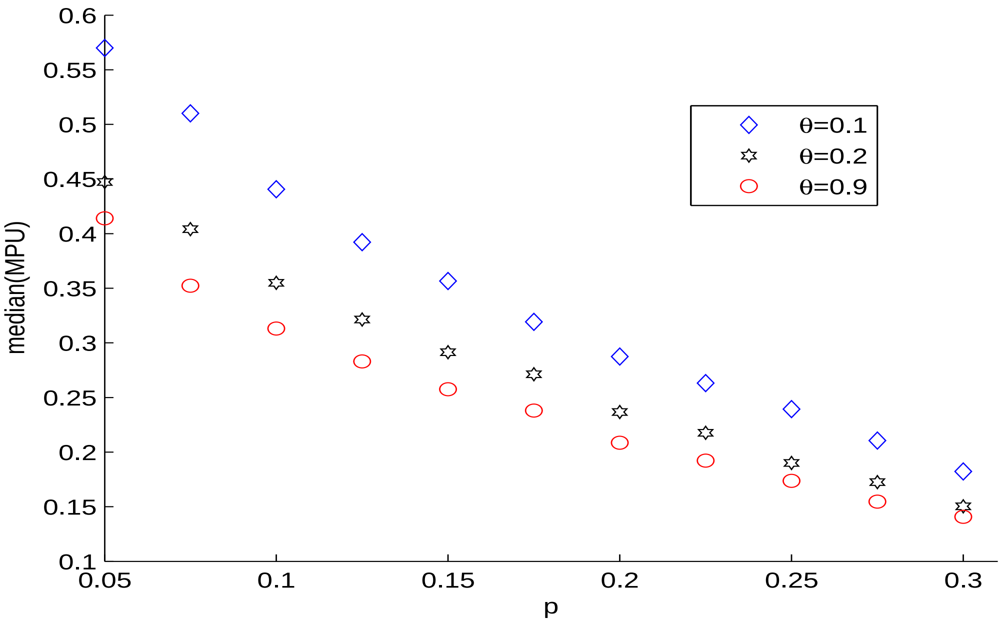

Figure 12. *Effects of p on the median of* MPU *for Different θ*

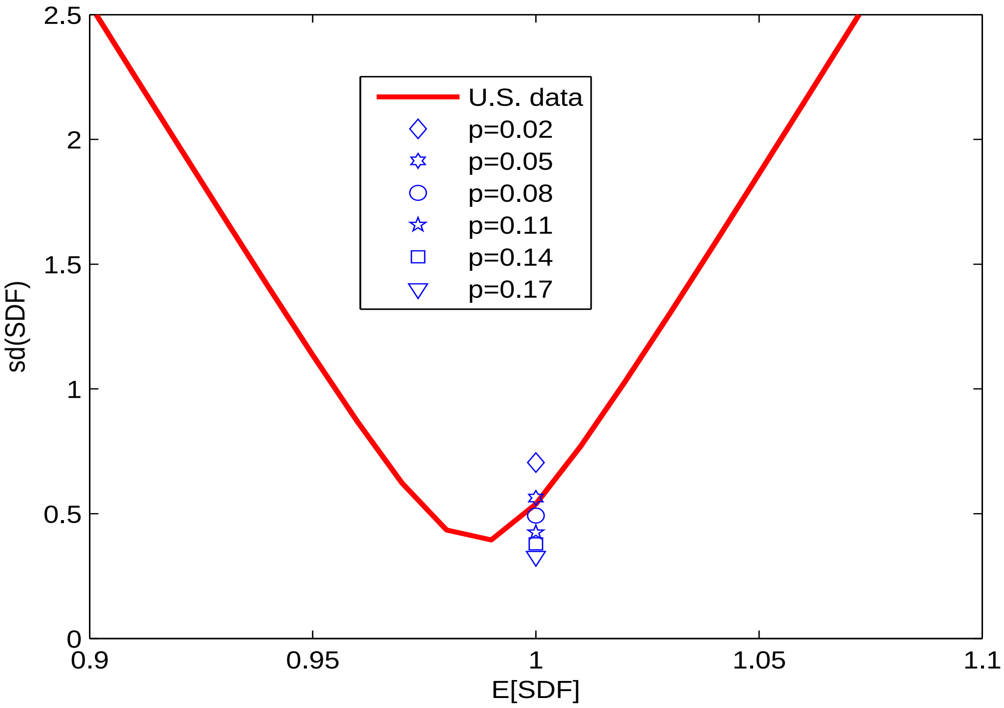

Figure 13. *HJ Bound under RB and RI*

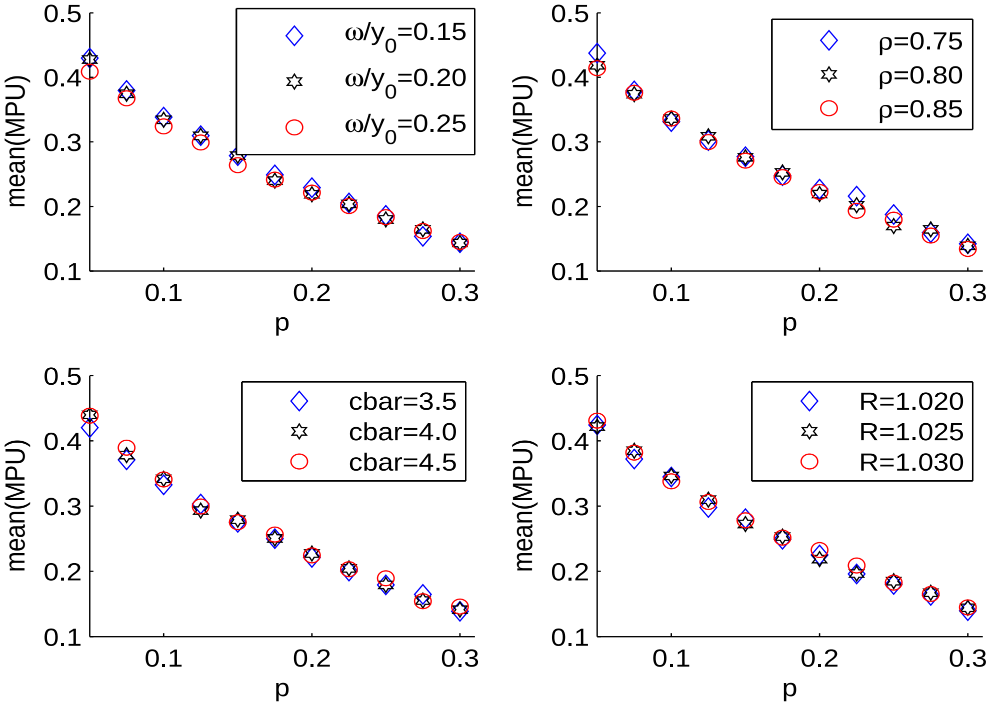

Figure 14. *Sensitivity Analysis of the Effects of RB on* MPU *($\theta = 0.3$)*

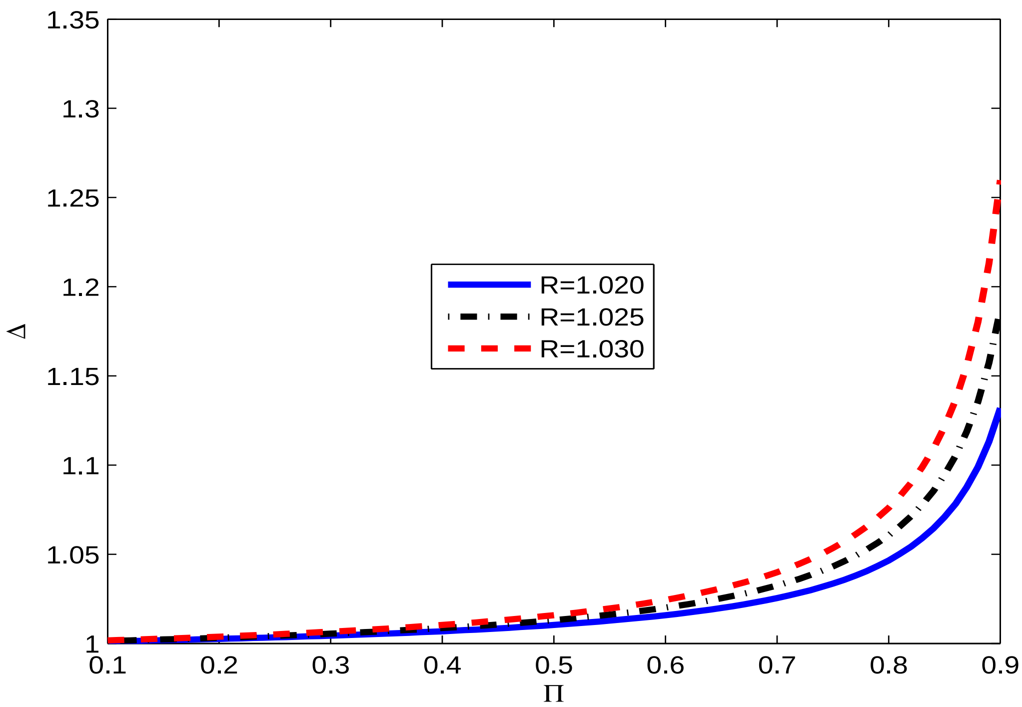

Figure 15. *Difference of* MPU *under RB+RI and RS+RI*

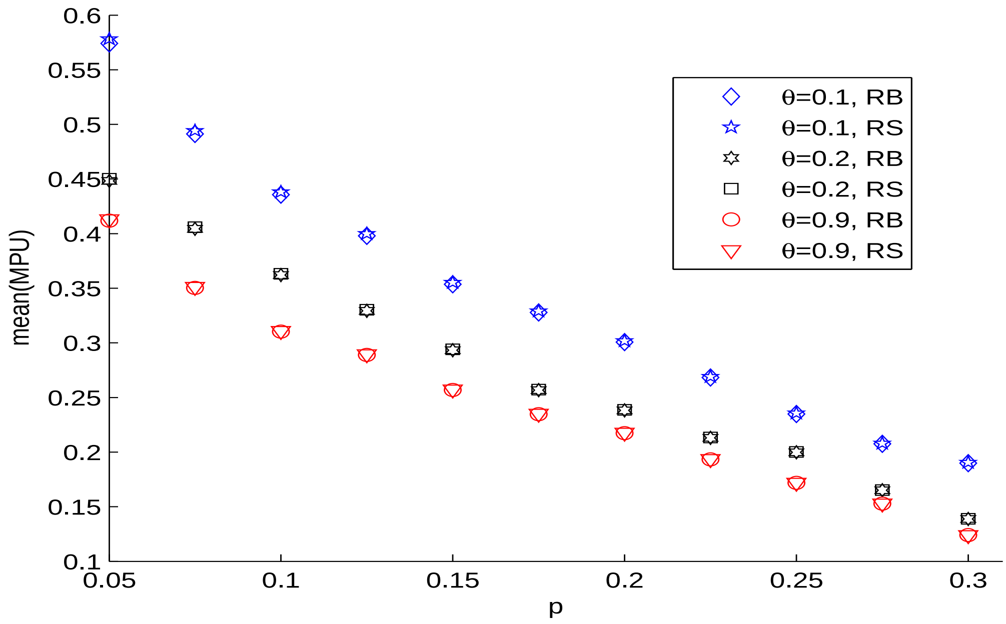

Figure 16. *Comparison of the Effects of p on* MPU *under RB+RI and RS+RI*

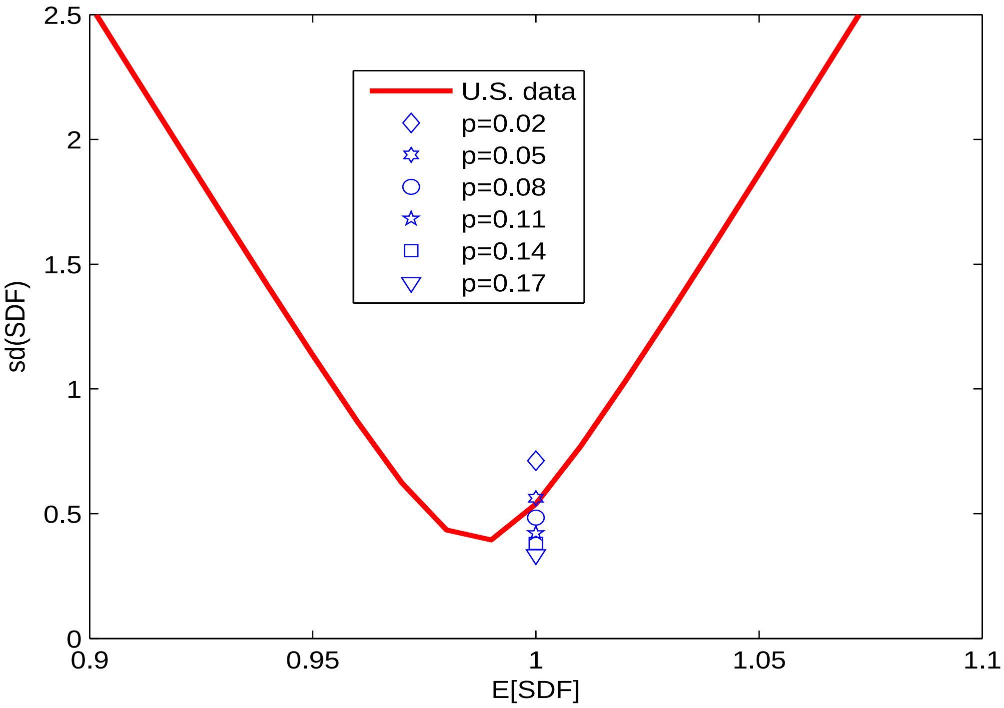

Figure 17. *HJ Bound under RS and RI*

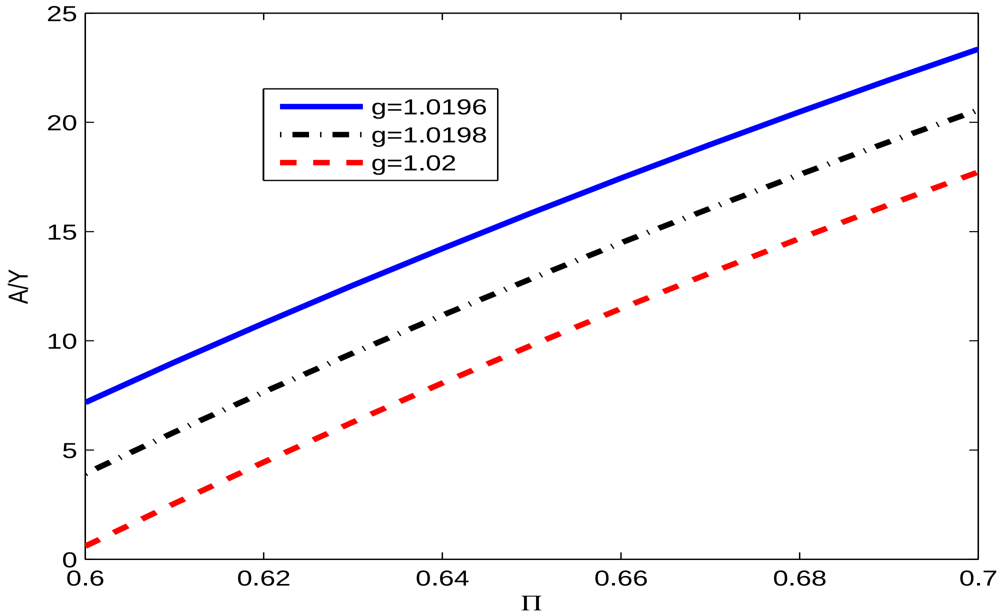

Figure 18. *Effects of* Π *on Aggregate Wealth-Income Ratio*