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Emura, Takeshi and Lin, Yi-Shuan

National Central University, Graduate Institute of Statistics,
National Central University, Graduate Institute of Accounting

5 August 2013

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Takeshi Emura[1] and Yi-Shuan Lin[2]

Control charts, known for more than 80 years, have been important tools for business and industrial manufactures. Among many different types of control charts, the attribute control chart (*np*-chart or *p*-chart) is one of the most popular methods to monitor the number of observed defects in products, such as semiconductor chips, automobile engines, and loan applications. The attribute control chart requires that the sample size n is sufficiently large and the defect rate p is not too small so that the normal approximation to the binomial works well. Some rules for the required values for n and p are available in the textbooks of quality control and mathematical statistics. However, these rules are considerably different and hence it is less clear which rule is most appropriate in practical applications. In this paper, we perform a comparison of five frequently used rules for n and p required for the normal approximation to the binomial. Based on this result, we also refine the existing rules to develop a new rule that has a reliable performance. Datasets are analyzed for illustration.

KEY WORDS: attribute control chart; binomial distribution; np-chart; p-chart; statistical process control

[1] Correspondence to: Takeshi Emura, Graduate Institute of Statistics, National Central University, TAIWAN, Email: emura@stat.ncu.edu.tw

[2] Graduate Institute of Accounting, National Central University, TAIWAN

1. Introduction

With the advance development of mass production technologies, quality control has been essential and convenient tools for manufacturers. As factories produce items in mass production, they necessarily encounter some defective (nonconforming) items. The basic idea of the quality control is to keep the number of defective items within an in-control range. Consequently, the manufacturers can control the loss of their business. In 1924, Walter Shewhart provided the most important tool to detect whether the manufacturing process is under control or not, called control chart[1]. Since then, control charts have been applied extensively for more than 80 years.

Among many different types of control charts, the attribute control chart (*np*-chart or *p*-chart) is one of the most widely used control charts. The attribute chart is originally developed for industrial manufactures that monitor the number of observed defects in products, such as semiconductor chips and automobile engines. Here, a defect (or nonconforming) item refers to a product that does not meet the specifications set by manufactures. Nowadays, the attribute control chart can also be applied to service industry to monitor the number of incomplete operations such as unexpected shipping errors[2] and filed complaints from taxpayers[3]. In addition, the applications of the attribute control chart extend to health care in which researchers monitor the number of adverse events[4].

The attribute control chart utilizes the binomial distribution to model the distribution of the observed number of nonconforming items. Specifically, let X denote the observed number of nonconforming items in n inspected items. If the items are produced and inspected independently, the distribution of X follows a binomial distribution $Bin(n, p)$, where p ($0 < p < 1$) is the fraction nonconforming. For this reason, the attribute chart is more often called *np*-chart (or *p*-chart). The false alarm rate is the probability that the nonconforming occurs under the in-control state, which is defined as

$$\alpha^* = P(X < \text{LCL} \quad \text{or} \quad X > \text{UCL}),$$

where LCL and UCL are the lower and upper control limits chosen so that α^* is close to the nominal level α. For the quality control work, one typically sets $\alpha = 0.0027$ so that

99.73% of all data fall between the control limits. This leads to the traditional ± 3-sigma charts with control limits

$$\mathrm{LCL} = np - 3\sqrt{np(1-p)} \ ,$$

$$\mathrm{UCL} = np + 3\sqrt{np(1-p)} \ .$$

The control chart that monitors the observed values of X together with the LCL and LCL is called *np*-chart. For more details, readers are referred to the standard textbooks for quality control such as Wetherill and Brown[5], Montgomery[2] and Ryan[6].

Note that the ± 3-sigma limits are obtained by using the normal approximation to the binomial distribution whose reliability is due to the central limit theorem. Hence, in theory, the closeness of α^* to α is guaranteed when n is sufficiently large. In practice, the normal approximation is numerically satisfactory if np is large enough. This is because the value of p very close to 0 produces a highly skewed binomial distribution, leading to the poor approximation even for a very large n.

Although the study of the normal approximation to the binomial has a quite long history[7], the problem has not been sufficiently discussed in the quality control literature. A recent study of Duran and Albin[3] gives the cases where a careful application of the normal approximation still produces a large error in the false alarm rate and the average run length (ARL). Specifically, even for a quite large n (n = 1000) that fulfills the rules of thumb, the ARL of the resultant *np*-chart is far from the desired value.

Several different criteria for n and p have been proposed for the normal approximation. Most criteria suggest the normal approximation when p, np, $n(1-p)$ or $np(1-p)$ is greater than some thresholds, as shown below in Section 2. Accordingly, the suggested thresholds considerably differ among the literature and hence it is often unclear which criterion is most reliable for a given dataset.

Numerical investigations of the normal approximation to the binomial distribution has been considered in many papers (see Schader and Schmid[8] and references therein) and books (e.g., Hald[9, 10]; Duncan[11]). However, most numerical experiments are designed for a general purpose from mathematical point of view, which is not oriented for applications to the quality control work. For instance, Raff [12] and Schader and Schmid [8] study the maximum error of the normal approximation to the binomial data in the entire range of the observed number of defects X. In the quality control work, however, it is more relevant to study the accuracy of the normal approximation only at the ± 3-sigma limits to attain a good approximation to the false alarm rate $\alpha = 0.0027$ [13] or the desired ARL of 370[3].

The major objective of this paper is to perform a comparison of five frequently used rules for the normal approximation to the binomial. In section 2, we review the five different rules for the normal approximation to the binomial. Section 3 describes our methods for evaluating the performance of the normal approximation to the binomial. In Section 4, we compare the performance of the five different rules. Based on the comparative results, we also consider suggesting a new rule that has a reliable performance. Section 5 analyzes datasets for illustration. Section 6 concludes the paper.

2. Five different rules for the normal approximation

Most existing rules for the normal approximation to the binomial are defined in terms f n and p , where n is the number of inspected items and $0 < p < 1$ is the fraction nonconforming. Table 1 summarizes five well-known rules for the normal approximation to the binomial. More specifically,

- **Rule A** requires that both $np > 10$ and $p \geq 0.1$ holds (Wetherill and Brown[5]; Montgomery[2]). Rule A is stricter than other rules due to the strong condition $p \geq 0.1$. The rationale behind the condition is that the Poisson approximation to the binomial works better than the normal approximation for $p < 0.1$ (Wetherill and Brown[5]; Montgomery[2]).
- **Rule B** requires $np > 15$ (Johnson[14]). Compared to Rule A, Rule B drops the condition $p \geq 0.1$ but instead imposes a more stringent condition on np.
- **Rule C** requires both $np > 5$ and $n(1-p) > 5$. This rule is very popular and is often called "rule of thumb" (Schader and Schmid[8] Ryan[6]). Many standard textbooks for mathematical statistics (e.g., Casella and Berger[15]) and quality control (e.g., Ryan[6]) follow this rule. In quality control, the defective rate is small and so $p < 0.5$ usually

holds. Then, Rule C is equivalent to $np > 5$ only. Hence, the rule is much weaker than Rule B.

- **Rule D** requires both $np > 10$ and $n(1-p) > 10$, which is a stringent version of Rule C. The rule is suggested by Hahn and Meeker[16], the handbook on statistical intervals for engineering applications. More precisely, they suggest applying the rule when $n\overline{p} > 10$ and $n(1-\overline{p}) > 10$, where $\overline{p}$ is a data-driven estimate of p. Perhaps due to estimation bias, the rule becomes more stringent than Rule C.
- **Rule E** requires $np(1-p) > 9$, which is suggested by Hald[9, 10] in his statistics textbooks on engineering applications. The rule is called "rule of thumb" (Schader and Schmid[8]) and often appears in the textbooks on quality control (e.g., p.33 of Wetherill and Brown[5]).

With the considerable differences among these rules, it is often unclear whether the normal approximation to the binomial should be used for a given dataset. We explain this problem using Example 7.1 of Montgomery[2], in which p = 0.23 and n = 50. Since np = 11.5, $n(1\text{-}p)$ = 38.5, and $np(1\text{-}p)$ = 8.86, Rules A, C and D hold while Rules B and E do not.

3. Performance assessment for normal approximation rules

In this section, we describe our methods for evaluating the performance of the normal approximation to the binomial. We define the performance of the normal approximation to the binomial in terms of the absolute error defined as

$$\text{Error}(\,n,\,p\,) \;=\; |\,\alpha^{*} - \alpha\,| = \left| \sum_{x<\text{LCL}} \binom{n}{x} p^{x}(1-p)^{n-x} + \sum_{x>\text{UCL}} \binom{n}{x} p^{x}(1-p)^{n-x} - \alpha \right|,$$

where $\alpha = 0.0027$, $\text{LCL} = np - 3\sqrt{np(1-p)}$ and $\text{UCL} = np + 3\sqrt{np(1-p)}$. Although a relative error might be used, the absolute error $\text{Error}(\,n,\,p\,)$ may be of more practical interest[8]. Larger values of $\text{Error}(\,n,\,p\,)$ correspond to poorer performance. We evaluate the value of $\text{Error}(\,n,\,p\,)$ under $24 \times 20 = 480$ pairs of

$$n = (\,\underbrace{10, 20, ..., 90, 100, 150, ..., 750, 800}_{\times 24}\,),$$

$$p = (\underbrace{0.01, 0.02, ..., 0.19, 0.20}_{\times 20}).$$

Here, we focus on small values of p as it is common in real applications.

3.1 Numerical performance assessment

We describe a method to compare the aforementioned five different rules (Rules A - E). One can define the range of (n, p) that satisfies each rule:

$$R^{\mathrm{A}} = \{ (n, p) : p \geq 0.1, \text{ and } np > 10 \},$$

$$R^{\mathrm{B}} = \{ (n, p) : np > 15 \},$$

$$R^{\mathrm{C}} = \{ (n, p) : np > 5, \text{ and } n(1-p) > 5 \},$$

$$R^{\mathrm{D}} = \{ (n, p) : np > 10, \text{ and } n(1-p) > 10 \},$$

$$R^{\mathrm{E}} = \{ (n, p) : np(1-p) > 9 \}.$$

Our evaluation criteria for a rule R are

$$\mathrm{MeanError}(\mathrm{R}) = \frac{\sum_{(n,p)\in R} \mathrm{Error}(n, p)}{\sum_{(n,p)\in R} 1},$$

$$\mathrm{MaxError}(\mathrm{R}) = \max_{(n,p)\in R} \mathrm{Error}(n, p),$$

where $(n, p) \in R$ implies that (n, p) satisfy the condition defined by the rule R.

3.2 Graphical performance assessment

More insight can be gained using a heat map, a very popular display for visualizing the error[17]. Specifically, we assign the color (reddish brown, red, orange, yellow and blue) to the equally spaced ranges of errors as:

(reddish brown) for $0.0020 \leq \mathrm{Error}(n, p)$

(red) for $0.0012 \leq \mathrm{Error}(n, p) < 0.0020$

(orange) for $0.0008 \leq \mathrm{Error}(n, p) < 0.0012$

(yellow) for $0.0004 < \mathrm{Error}(n, p) < 0.0008$

(blue) for $0 \leq \mathrm{Error}(n, p) < 0.0004$.

For instance, one can easily identify the pairs of (n, p) that produce largest error (reddish brown color). The heat map is useful not only for comparing the five different rules, but also for developing a new decision rule that effectively eliminates the pairs of (n, p) that have large error.

4. Numerical results

4.1 Comparison of the five different rules

We compare the performance of the five different rules and summarize the results in Table II and Figures 1-5.

Table II shows that Rule A has the best performance in terms of both $\mathrm{MeanError}(\mathrm{R})$ and $\mathrm{MaxError}(\mathrm{R})$. Indeed, Figure 1 shows that more than half of (n, p) produces minor error (error < 0.0004; blue color). However, this rule is the most stringent one since it permits the narrowest ranges of (n, p) among the five rules (Figure 1). Hence, Rule A the most conservative one.

The second best is achieved by Rule B (Table II). The value of $\mathrm{MaxError}(\mathrm{R})$ is the same as Rule A but it has only slightly inferior value of $\mathrm{MeanError}(\mathrm{R})$. However, Rule B is more liberal than Rule A as Rule B permits wider ranges of (n, p) (Figure 2).

The third and fourth are Rule D and Rule E, respectively. Their performances are very similar to each other (Table II). The reason is that their allowable ranges of (n, p) are only slightly different (compare Figures 4 and 5). Both Rule D and Rule E are inferior to Rule B, leading to slight increase in $\mathrm{MeanError}(\mathrm{R})$ and $\mathrm{MaxError}(\mathrm{R})$.

Rule C produces the worst performance, which results in substantially larger values of $\mathrm{MeanError}(\mathrm{R})$ and $\mathrm{MaxError}(\mathrm{R})$ compared with the other rules. While this rule allows the widest allowable ranges of (n, p), it inflates the error on the boundary $np \approx 5$ that appears in reddish brown color (Figure 3). Hence, even though Rule C is often called the rule of thumb, we do not recommend Rule C for the quality control work. This conclusion will also be supported by real data analyses below.

4.2 Newly proposed rule

The previous error comparison shows that Rule A performs the best among the five rules. However, Rule A is too stringent for practical applications as it permit smallest allowable ranges of (n, p) (Figure 1) compared to other rules (Figures 2 - 6). In light of this result, we propose a new rule that refines Rule A.

Our approach is to combine Rule A with Rule B that is the second best rule among the five rules. We propose a rule that allows the normal approximation when either Rule A or or Rule B is satisfied. Hence, the new rule is defined to be

$$(np > 10 \text{ and } p \geq 0.1) \text{ or } (np > 15).$$

Table II shows the performance of the new rule in terms of both $\mathrm{MeanError}(R)$ and $\mathrm{MaxError}(R)$. $\mathrm{MeanError}(R)$ of the new rule is only slightly larger than that of Rules A and B. Also, $\mathrm{MaxError}(R)$ remains the same as Rules A and B. Hence, although the new rule relaxes both Rules A and B, the performance is still competitive. In addition, the performance of the new rule is better than that of Rules C, D and E.

We use the heat map to graphically explain how the new rule attains the good performance (Figures 1 – 6). The new rule (Figure 6) permits significantly wider ranges of (n, p) compared to Rule A (Figure 1) and Rule B (Figure 2). In addition, the new rule (Figure 6) successfully eliminates the (n, p)'s having high $\mathrm{Error}(n, p)$ compared to Rule D (Figure 4) and Rule E (Figure 5) (compare places with red color in Figures 4, 5 and 6).

5. Data analysis

We compare the five rules for the normal approximation to the binomial using real datasets. We make an exhaustive list of datasets for the *np*-chart (or *p*-chart) from Chapter 7 of Montgomery[2], which results in five datasets (denoted by Data I -V). We examine whether the five rules of the normal approximation holds for each dataset.

The results are summarized in Table III. For Data I, three rules (Rules A, C and D) hold while the other rules (Rules B and E) do not. To resolve the disagreement, we consult the newly proposed criterion that is satisfied if either Rule A or Rule B holds. Then, we conclude that the normal approximation to the binomial is acceptable for Data I. For other data (Data II - V), all the rules except Rule C provide the same results. For instance, Rule C suggests the normal approximation to Data II while the others (Rules A, B, D and E) do not. In fact, Rule C suggests the normal approximation for all the data. Clearly, Rule C is too liberal to be applied to these data.

We also perform the Shapiro-Wilk normality test to see the concordance between the results of the normal approximation rules and the normality tests (Table III). If the results between the normal approximation rule and the normality test are the same, the rule has a good performance. For instance, Rule A admits the normal approximation and the Shapiro-Wilk test accepts the normality test at 5% significance (Table III). Hence, the Rule A is concordant with the normality test for Data I. The concordance percentages of the five rules across all the data are given in the bottom notes of Table III. Rule A, Rule D, and the new rule give 80% concordance while Rule B, Rule C and Rule E gives 60% concordance. Therefore, the new rule provides reliable results in accordance with the normality test.

6. Conclusion

In this paper, we perform a comparison of the five common rules for n and p required for the normal approximation to the binomial. Here, the study design is oriented for applications to the np-chart (or p-chart) unlike most existing studies for a general mathematical point of view. Our comparison shows that the two rules (Rules A and B) perform better than the other three rules (Rules C, D and E). Somewhat surprisingly, the rule of thumb (Rule C) performs worst and is too liberal to be used for the np-chart. Therefore, Rule A and Rule B are generally recommended. Our comparative results may give some application guidelines for the np-chart in routine industrial practice and health care monitoring.

Furthermore, we propose a new rule that combines Rule A with Rule B. The new rule allows wider ranges for n and p compared to Rules A and B while it still preserves their competitive performance and hence outperforms Rules C, D and E. Remarkably, we find a data example where the two recommended rules (Rules A and B) lead to different conclusions on whether the normal approximation is acceptable. Our proposed rule can be reliably applied for making a decision for the normal approximation under this type of difficult cases.

Acknowledgements

This research is supported by the grant NSC 101-2118-M-008-002-MY2 from the National Science Council of Taiwan.

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Authors' biographies

Takeshi Emura is Assistant Professor of the Graduate Institute of Statistics, National Central University, Taiwan. He teaches quality control in the master's program and mathematical statistics in the Ph.D program in Statistics. His research interest is in developing statistical methodologies that are useful in analysis of survival data, reliability data and process control data.

Yi-Shuan Lin is a student currently working toward her master degree in the Graduate Institute of Accounting, National Central University, Taiwan. Her research interest includes quality management and statistical process control.

Table I. Five well-known rules for the normal approximation to the binomial

Rule	Reference
Rule A. $np > 10$ and $p \geq 0.1$	Wetherill and Brown (1991), Montgomery (2009)
Rule B. $np > 15$	Johnson (2009)
Rule C. $np > 5$ and $n(1-p) > 5$	Schader and Schmid (1989), Casella and Berger (2002), Ryan (2011)
Rule D. $np > 10$ and $n(1-p) > 10$	Hahn and Meeker (1991)
Rule E. $np(1-p) > 9$	Hald (1952, 1978), Schader and Schmid (1989)

Table II. Numerical comparison of the five different rules for n and p required for the normal approximation. Larger values of $\mathrm{MeanError}(R)$ and $\mathrm{MaxError}(R)$ correspond to poorer performance.

Rule	$\mathrm{MeanError}(R)$	$\mathrm{MaxError}(R)$	Rank*
Rule A: $np > 10$ and $p \geq 0.1$	0.0003368	0.001558	1
Rule B: $np > 15$	0.0003425	0.001558	2
Rule C: $np > 5$ and $n(1-p) > 5$	0.0004845	0.002981	5
Rule D: $np > 10$ and $n(1-p) > 10$	0.0003812	0.001887	4
Rule E: $np(1-p) > 9$	0.0003759	0.001887	3
New rule**: ($np > 10$ and $p \geq 0.1$) or ($np > 15$)	0.0003551	0.001558	/

* Smaller rank corresponds to better performance in terms of the error.

** The new rule is derived from Rules A and B.

Table III. The analysis of five datasets in Chapter 7 of Montgomery (2009).

We examine whether the five rules of the normal approximation holds for each dataset.

The Shapiro-Wilk normality test with 5% significance is also applied to each dataset.

	Data I Example 7.1 $\bar{p} = 0.231$ $n = 50$	Data II Exercise 7.2 $\bar{p} = 0.023$ $n = 150$	Data III Exercise 7.3 $\bar{p} = 0.059$ $n = 100$	Data IV Exercise 7.5 $\bar{p} = 0.008$ $n = 500$	Data V Exercise 7.13 $\bar{p} = 0.164$ $n = 100$
Rule A: $np > 10$ and $p \geq 0.1$	V	×	×	×	V
Rule B: $np > 15$	×	×	×	×	V
Rule C: $np > 5$ and $n(1-p) > 5$	V	V	V	V	V
Rule D: $np > 10$ and $n(1-p) > 10$	V	×	×	×	V
Rule E: $np(1-p) > 9$	×	×	×	×	V
New rule: ($np > 10$ and $p \geq 0.1$) or ($np > 15$)	V	×	×	×	V
Shapiro-Wilk normality test (P-value)	Accept (0.2037)	Reject (0.0020)	Accept (0.6175)	Reject (0.0104)	Accept (0.4005)

- $\bar{p}$ is a data-driven estimate of p.
- Concordance probabilities between the results of the rule and the normality test are Rule A = 80%, Rule B = 60%, Rule C = 60%, Rule D = 80%, Rule E = 60%, New rule = 80%.

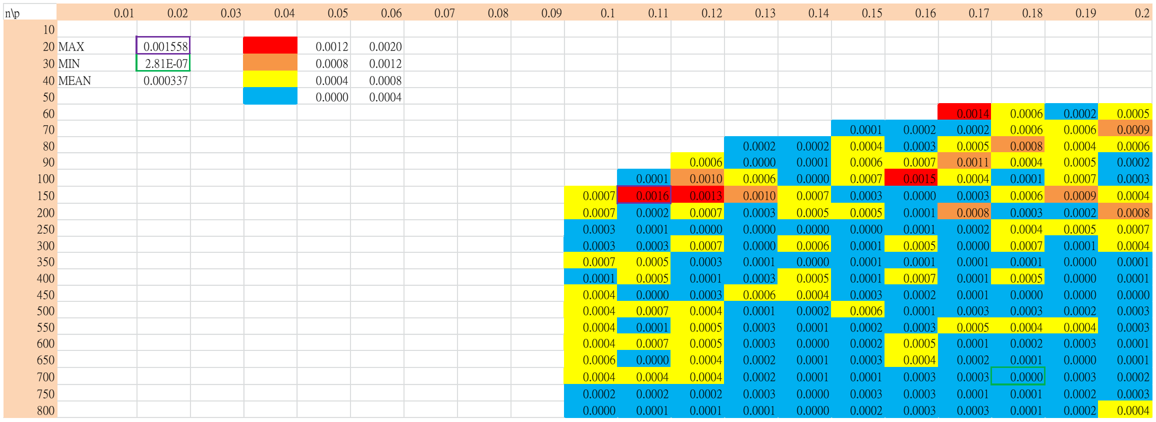

Figure 1. The absolute errors of the normal approximation with $np > 10$ and $p \geq 0.1$ (Rule A).

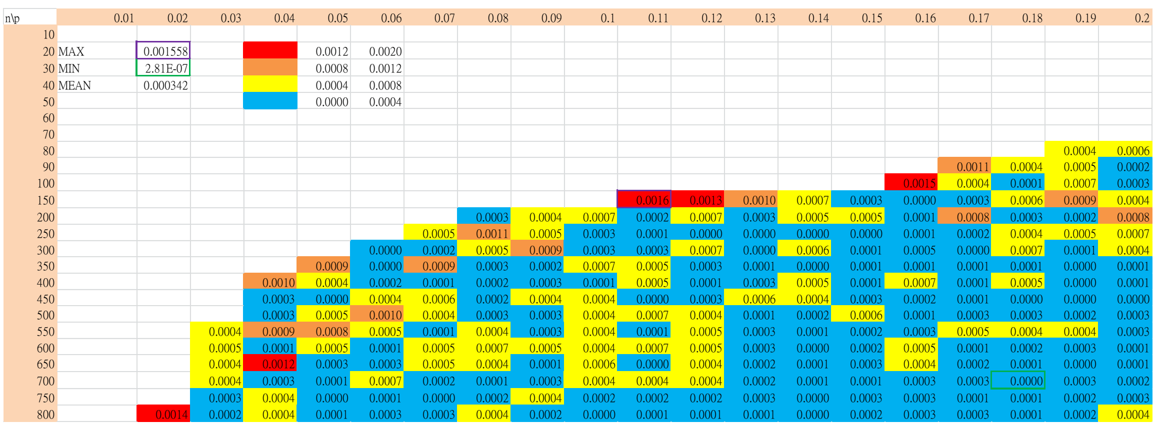

Figure 2. The absolute errors of the normal approximation with $np > 15$ (Rule B).

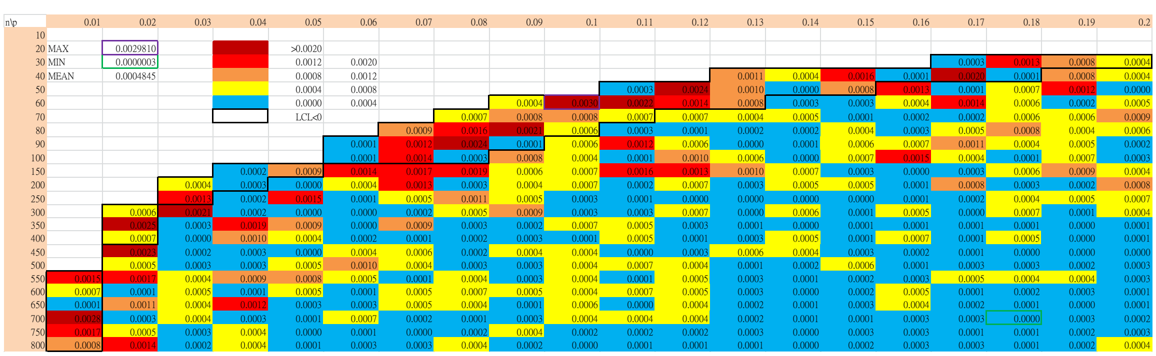

Figure 3. The absolute errors of the normal approximation with $np > 5$ and $n(1-p) > 5$ (Rule C).

n\p	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
10																				
20	MAX	0.001887			0.0012	0.0020														
30	MIN	2.81E-07			0.0008	0.0012														
40	MEAN	0.000381			0.0004	0.0008														
50					0.0000	0.0004														
60																	0.0014	0.0006	0.0002	0.0005
70															0.0001	0.0002	0.0002	0.0006	0.0006	0.0009
80													0.0002	0.0002	0.0004	0.0003	0.0005	0.0008	0.0004	0.0006
90												0.0006	0.0000	0.0001	0.0006	0.0007	0.0011	0.0004	0.0005	0.0002
100											0.0001	0.0010	0.0006	0.0000	0.0007	0.0015	0.0004	0.0001	0.0007	0.0003
150							0.0017	0.0019	0.0006	0.0007	0.0016	0.0013	0.0010	0.0007	0.0003	0.0000	0.0003	0.0006	0.0009	0.0004
200						0.0004	0.0013	0.0003	0.0004	0.0007	0.0002	0.0007	0.0003	0.0005	0.0005	0.0001	0.0008	0.0003	0.0002	0.0008
250					0.0015	0.0001	0.0005	0.0011	0.0005	0.0003	0.0001	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0004	0.0005	0.0007
300				0.0002	0.0000	0.0000	0.0002	0.0005	0.0009	0.0003	0.0003	0.0007	0.0000	0.0006	0.0001	0.0005	0.0000	0.0007	0.0001	0.0004
350			0.0003	0.0019	0.0009	0.0000	0.0009	0.0003	0.0002	0.0007	0.0005	0.0003	0.0001	0.0000	0.0001	0.0001	0.0001	0.0001	0.0000	0.0001
400			0.0000	0.0010	0.0004	0.0002	0.0001	0.0002	0.0003	0.0001	0.0005	0.0001	0.0003	0.0005	0.0001	0.0007	0.0001	0.0005	0.0000	0.0001
450			0.0002	0.0003	0.0000	0.0004	0.0006	0.0002	0.0004	0.0004	0.0000	0.0003	0.0006	0.0004	0.0003	0.0002	0.0001	0.0000	0.0000	0.0000
500			0.0003	0.0003	0.0005	0.0010	0.0004	0.0003	0.0003	0.0004	0.0007	0.0004	0.0001	0.0002	0.0006	0.0001	0.0003	0.0003	0.0002	0.0003
550		0.0017	0.0004	0.0009	0.0008	0.0005	0.0001	0.0004	0.0003	0.0004	0.0001	0.0005	0.0003	0.0001	0.0002	0.0003	0.0005	0.0004	0.0004	0.0003
600		0.0001	0.0005	0.0001	0.0005	0.0001	0.0005	0.0007	0.0005	0.0004	0.0007	0.0005	0.0003	0.0000	0.0002	0.0005	0.0001	0.0002	0.0003	0.0001
650		0.0011	0.0004	0.0012	0.0003	0.0003	0.0005	0.0004	0.0001	0.0006	0.0000	0.0004	0.0002	0.0001	0.0003	0.0004	0.0002	0.0001	0.0000	0.0001
700		0.0003	0.0004	0.0003	0.0001	0.0007	0.0002	0.0001	0.0003	0.0004	0.0004	0.0004	0.0002	0.0001	0.0001	0.0003	0.0003	0.0000	0.0003	0.0002
750		0.0005	0.0003	0.0004	0.0000	0.0001	0.0000	0.0002	0.0004	0.0002	0.0002	0.0002	0.0003	0.0000	0.0003	0.0003	0.0001	0.0001	0.0002	0.0003
800		0.0014	0.0002	0.0004	0.0001	0.0003	0.0003	0.0004	0.0002	0.0000	0.0001	0.0001	0.0001	0.0000	0.0002	0.0003	0.0003	0.0001	0.0002	0.0004

Figure 4. The absolute errors of the normal approximation with $np > 10$ and $n(1-p) > 10$ (Rule D).

n\p	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
10																				
20	MAX	0.001887			0.0012	0.0020														
30	MIN	2.81E-07			0.0008	0.0012														
40	MEAN	0.000376			0.0004	0.0008														
50					0.0000	0.0004														
60																			0.0002	0.0005
70																0.0002	0.0002	0.0006	0.0006	0.0009
80													0.0002	0.0002	0.0004	0.0003	0.0005	0.0008	0.0004	0.0006
90												0.0006	0.0000	0.0001	0.0006	0.0007	0.0011	0.0004	0.0005	0.0002
100											0.0001	0.0010	0.0006	0.0000	0.0007	0.0015	0.0004	0.0001	0.0007	0.0003
150							0.0017	0.0019	0.0006	0.0007	0.0016	0.0013	0.0010	0.0007	0.0003	0.0000	0.0003	0.0006	0.0009	0.0004
200					0.0000	0.0004	0.0013	0.0003	0.0004	0.0007	0.0002	0.0007	0.0003	0.0005	0.0005	0.0001	0.0008	0.0003	0.0002	0.0008
250				0.0002	0.0015	0.0001	0.0005	0.0011	0.0005	0.0003	0.0001	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0004	0.0005	0.0007
300				0.0002	0.0000	0.0000	0.0002	0.0005	0.0009	0.0003	0.0003	0.0007	0.0000	0.0006	0.0001	0.0005	0.0000	0.0007	0.0001	0.0004
350			0.0003	0.0019	0.0009	0.0000	0.0009	0.0003	0.0002	0.0007	0.0005	0.0003	0.0001	0.0000	0.0001	0.0001	0.0001	0.0001	0.0000	0.0001
400			0.0000	0.0010	0.0004	0.0002	0.0001	0.0002	0.0003	0.0001	0.0005	0.0001	0.0003	0.0005	0.0001	0.0007	0.0001	0.0005	0.0000	0.0001
450			0.0002	0.0003	0.0000	0.0004	0.0006	0.0002	0.0004	0.0004	0.0000	0.0003	0.0006	0.0004	0.0003	0.0002	0.0001	0.0000	0.0000	0.0000
500		0.0005	0.0003	0.0003	0.0005	0.0010	0.0004	0.0003	0.0003	0.0004	0.0007	0.0004	0.0001	0.0002	0.0006	0.0001	0.0003	0.0003	0.0002	0.0003
550		0.0017	0.0004	0.0009	0.0008	0.0005	0.0001	0.0004	0.0003	0.0004	0.0001	0.0005	0.0003	0.0001	0.0002	0.0003	0.0005	0.0004	0.0004	0.0003
600		0.0001	0.0005	0.0001	0.0005	0.0001	0.0005	0.0007	0.0005	0.0004	0.0007	0.0005	0.0003	0.0000	0.0002	0.0005	0.0001	0.0002	0.0003	0.0001
650		0.0011	0.0004	0.0012	0.0003	0.0003	0.0005	0.0004	0.0001	0.0006	0.0000	0.0004	0.0002	0.0001	0.0003	0.0004	0.0002	0.0001	0.0000	0.0001
700		0.0003	0.0004	0.0003	0.0001	0.0007	0.0002	0.0001	0.0003	0.0004	0.0004	0.0004	0.0002	0.0001	0.0001	0.0003	0.0003	0.0000	0.0003	0.0002
750		0.0005	0.0003	0.0004	0.0000	0.0001	0.0000	0.0002	0.0004	0.0002	0.0002	0.0002	0.0003	0.0000	0.0003	0.0003	0.0001	0.0001	0.0002	0.0003
800		0.0014	0.0002	0.0004	0.0001	0.0003	0.0003	0.0004	0.0002	0.0000	0.0001	0.0001	0.0001	0.0000	0.0002	0.0003	0.0003	0.0001	0.0002	0.0004

Figure 5. The absolute errors of the normal approximation with $np(1-p) > 9$ (Rule E)

n\p	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
10																				
20	MAX	0.001558			0.0012	0.0020														
30	MIN	2.81E-07			0.0008	0.0012														
40	MEAN	0.000355			0.0004	0.0008														
50					0.0000	0.0004														
60																	0.0014	0.0006	0.0002	0.0005
70															0.0001	0.0002	0.0002	0.0006	0.0006	0.0009
80													0.0002	0.0002	0.0004	0.0003	0.0005	0.0008	0.0004	0.0006
90												0.0006	0.0000	0.0001	0.0006	0.0007	0.0011	0.0004	0.0005	0.0002
100											0.0001	0.0010	0.0006	0.0000	0.0007	0.0015	0.0004	0.0001	0.0007	0.0003
150										0.0007	0.0016	0.0013	0.0010	0.0007	0.0003	0.0000	0.0003	0.0006	0.0009	0.0004
200								0.0003	0.0004	0.0007	0.0002	0.0007	0.0003	0.0005	0.0005	0.0001	0.0008	0.0003	0.0002	0.0008
250							0.0005	0.0011	0.0005	0.0003	0.0001	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0004	0.0005	0.0007
300						0.0000	0.0002	0.0005	0.0009	0.0003	0.0003	0.0007	0.0000	0.0006	0.0001	0.0005	0.0000	0.0007	0.0001	0.0004
350					0.0009	0.0000	0.0009	0.0003	0.0002	0.0007	0.0005	0.0003	0.0001	0.0000	0.0001	0.0001	0.0001	0.0001	0.0000	0.0001
400				0.0010	0.0004	0.0002	0.0001	0.0002	0.0003	0.0001	0.0005	0.0001	0.0003	0.0005	0.0001	0.0007	0.0001	0.0005	0.0000	0.0001
450				0.0003	0.0000	0.0004	0.0006	0.0002	0.0004	0.0004	0.0000	0.0003	0.0006	0.0004	0.0003	0.0002	0.0001	0.0000	0.0000	0.0000
500				0.0003	0.0005	0.0010	0.0004	0.0003	0.0003	0.0004	0.0007	0.0004	0.0001	0.0002	0.0006	0.0001	0.0003	0.0003	0.0002	0.0003
550			0.0004	0.0009	0.0008	0.0005	0.0001	0.0004	0.0003	0.0004	0.0001	0.0005	0.0003	0.0001	0.0002	0.0003	0.0005	0.0004	0.0004	0.0003
600			0.0005	0.0001	0.0005	0.0001	0.0005	0.0007	0.0005	0.0004	0.0007	0.0005	0.0003	0.0000	0.0002	0.0005	0.0001	0.0002	0.0003	0.0001
650			0.0004	0.0012	0.0003	0.0003	0.0005	0.0004	0.0001	0.0006	0.0000	0.0004	0.0002	0.0001	0.0003	0.0004	0.0002	0.0001	0.0000	0.0001
700			0.0004	0.0003	0.0001	0.0007	0.0002	0.0001	0.0003	0.0004	0.0004	0.0004	0.0002	0.0001	0.0001	0.0003	0.0003	0.0000	0.0003	0.0002
750			0.0003	0.0004	0.0000	0.0001	0.0000	0.0002	0.0004	0.0002	0.0002	0.0002	0.0003	0.0000	0.0003	0.0003	0.0001	0.0001	0.0002	0.0003
800		0.0014	0.0002	0.0004	0.0001	0.0003	0.0003	0.0004	0.0002	0.0000	0.0001	0.0001	0.0001	0.0000	0.0002	0.0003	0.0003	0.0001	0.0002	0.0004

Figure 6. The absolute errors of the normal approximation with

[($np \geq 10$ and $p \geq 0.1$) or ($np > 15$)] (New rule).