Munich Personal RePEc Archive

Trejos-Wright with a 2-unit bound: existence and stability of monetary steady states

Huang, Pidong and Igarashi, Yoske

4 November 2013

Online at https://mpra.ub.uni-muenchen.de/51205/
MPRA Paper No. 51205, posted 05 Nov 2013 03:08 UTC

Trejos-Wright with a 2-unit bound: existence and stability of monetary steady states

Pidong Huang*and Yoske Igarashi† ‡

November 4, 2013

Abstract

This paper investigates a Trejos-Wright random matching model of money with a consumer take-it-or-leave-it offer and with individual money holdings in the set $\{0, 1, 2\}$. It is shown that three kinds of monetary steady state exist generically: (1) pure-strategy full-support steady states, (2) mixed-strategy full-support steady states, and (3) non-full-support steady states. A full-support steady state exists if and only if a non-full-support steady state exists. Stability of these steady states is also studied. Both pure-strategy and mixed-strategy full-support steady states are locally stable. However, non-full-support steady states are unstable. (JEL classification: C62, C78, E40)

Keywords: random matching model; monetary steady state; local stability; determinacy; instability; Zhu (2003).

---

*Korea University: pidonghuang@korea.kr.ac

†University of Exeter: y.y.igarashi@exeter.ac.uk

‡The authors especially thank Neil Wallace for his guidance and encouragement. We are also grateful to Daniela Puzzello, Ed Green, Gareth Myles, Ricardo Cavalcanti, Rulin Zhou, and seminar participants at Cornell University and Penn State University for helpful comments and discussions.

1 Introduction

Trejos and Wright (1995) show the existence of a monetary steady state in a random matching model under the assumption that an agent's money holding is in $\{0, 1\}$. In the same model, for consumer take-it-or-leave-it offers and for money holdings in $\{0, 1, \cdots, B\}$, Zhu (2003) provides sufficient conditions for the existence of a full-support monetary steady state with a strictly increasing and strictly concave value function. By way of a variant of a neutrality argument, his result also implies the existence of non-full-support steady states in which all agents treat bundles of money, each bundle being $B/l \in \mathbb{N}$ units, as the smallest unit held and traded.

Among the questions that Zhu's existence result leaves open are the following. First, are his full-support steady states unique? Second, do both pure-strategy and mixed-strategy steady states exist generically?[1] Third, are full-support steady states stable? Fourth, are the above non-full-support steady states stable? The smallest set of money holdings for which these questions arise is $\{0, 1, 2\}$, which is the smallest set for which the distribution of money holdings over people depends on the trades that are made. For this set, we answer all but the first question.

Under a condition that is weaker than Zhu's sufficient conditions, a full-support steady state exists. Both pure-strategy and mixed-strategy full-support steady states exist generically. As regards stability, any full-support steady state is stable, while the nonfull-support steady state, which necessarily has support $\{0, 2\}$, is unstable. Although the two-unit bound is restrictive, it, at least, permits conjectures to be made for the general case.

One reason to study the Zhu (2003) model is that it has policy implications that differ from those of the model with money holdings in {0,1} and from models with degenerate distributions of money holdings. In particular, as shown by Molico (2006), moderate inflation improves welfare through redistributional effects in versions of the Zhu model. That cannot happen in models with money holdings in {0,1} or in models with degenerate distributions of money holdings. But prior studies of Zhu's model leave open the questions set out above, some of which are addressed here.

[1] Zhu uses a fixed-point theorem for the existence proof so the equilibrium strategy is not described.

2 The Zhu (2003) model

Time is discrete, with periods dated as $t \geq 0$. There is a unit measure of non-atomic agents who are infinitely-lived. Also, there are divisible and non-storable consumption goods at each date. Each agent maximizes expected discounted utility with discount factor $\beta \in (0, 1)$. At each date, if an agent produces an amount $q \geq 0$ of the good, the utility cost is q. If an agent consumes an amount $q \geq 0$ of the good, the period utility he gets is $u(q)$, where $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, strictly concave and continuously differentiable on $\mathbb{R}_+$. Also, $u(0) = 0$, $u'(\infty) = 0$ and $u'(0)$ is sufficiently large but finite.[2] These assumptions imply that there is a unique $\bar{x} > 0$ such that $u(\bar{x}) = \bar{x}$.

There exists a fixed stock of indivisible money that is perfectly durable. There is a bound on individual money holdings, denoted $B \in \mathbb{N}$, so the individual money-holding set is $\mathbb{B} \equiv \{0, 1, \cdots, B\}$. Let $m \in (0, 1)$ denote the per capita stock of money divided by the bound on individual money holdings so that the per capita stock is Bm.

It is assumed that agents cannot consume their own production goods so they need to trade to obtain consumption. In each period, agents are randomly matched in pairs. With probability $1/n$, where $n \geq 2$, an agent is a consumer (producer) and the partner is a producer (consumer). Such meetings are called single-coincidence meetings. With probability $1 - 2/n$, the match is a no-coincidence meeting.[3] In meetings, agents' money holdings are observable, but any other information about an agent's trading history is private.

Consider a date-t single-coincidence meeting between a consumer (potential buyer) with i units of money (pre-trade) and a producer (potential seller) with j units of money (pre-trade), an (i, j)-*meeting*. If $i > 0$ and $j < B$, the meeting is called a *trade meeting*. In trade meetings, the consumer makes a take-it-or-leave-it offer. (There are no lotteries.) The producer accepts or rejects the offer. If the producer rejects it, both sides leave the meeting and go on to the next date.

For each $k \in \mathbb{B}$, let w_k^t be the expected discounted value of holding k units of money prior to date-t matching. Using the w_k^t's, the consumer's problem

[2] The assumption $u'(0) < \infty$ is used only in the proof of proposition 5.

[3] If $n \geq 3$, one foundation is that there are n types of agents and n types of consumption goods, that type-k agents can produce type-k goods only and consume type-$(k+1)$ goods only, and that the money is symmetrically distributed across the types.

in an (i, j)-meeting is

$$\max_{p \in \Gamma(i,j), q \in \mathbb{R}_+} \{u(q) + \beta w_{i-p}^{t+1}\} \tag{1}$$

$$\text{s.t.} \ -q + \beta w_{j+p}^{t+1} \geq \beta w_j^{t+1}, \tag{2}$$

where $\Gamma(i, j) \equiv \{p \in \mathbb{B} | p \leq \min\{i, B - j\}\}$ is the set of feasible payments. As (2) holds with equality in the solution, the consumer's problem reduces to

$$f^t(i, j) \equiv \max_{p \in \Gamma(i,j)} \{u\left(\beta w_{j+p}^{t+1} - \beta w_j^{t+1}\right) + \beta w_{i-p}^{t+1}\}.$$

We define

$$P^t(i, j) \equiv \operatorname*{argmax}_{p \in \Gamma(i,j)} \{u\left(\beta w_{j+p}^{t+1} - \beta w_j^{t+1}\right) + \beta w_{i-p}^{t+1}\}. \tag{3}$$

Because the solution $P^t(i, j)$ may be multi-valued, Zhu introduces randomization. Let $\Lambda^t(i, j)$ denote the set of probability distributions on $P^t(i, j)$. A mapping λ^t is called a consumer's optimal *strategy* if it maps each $(i, j) \in \mathbb{B} \times \mathbb{B}$ to an element of $\Lambda^t(i, j)$, so that

$$\sum_{p \in P^t(i,j)} \lambda^t(p; i, j) = 1. \tag{4}$$

For each $z \in \mathbb{B}$, let π_z^t denote the fraction of agents holding z units of money at the start of period t, so that π^t is a probability distribution on $\mathbb{B}$ with mean Bm. Given a strategy, the law of motion for π^{t+1} can be expressed as

$$\pi_z^{t+1} = \frac{n-2}{n} \pi_z^t + \frac{2}{n} \sum_{i=0}^{B} \sum_{j=0}^{B} \pi_i^t \pi_j^t \frac{\lambda^t(i - z; i, j) + \lambda^t(z - j; i, j)}{2}. \tag{5}$$

The second term of (5) informs who in single-coincidence meetings will end up with z units: consumers who originally had i units and spent $i - z$ units and producers who originally had j units and acquired $z - j$ units.

The value function w^t satisfies the Bellman equation

$$w_i^t = \frac{n-1}{n} \beta w_i^{t+1} + \frac{1}{n} \sum_{j=0}^{B} \pi_j^t f^t(i, j). \tag{6}$$

The first term of the r.h.s corresponds to either entering a no-coincidence meeting or becoming a producer, who is indifferent between trading and not trading. When $i = 0$, equation (6) reduces to $w_0^t = \beta w_0^{t+1}$, so the only nonexplosive case is $w_0^t = 0, \forall t$. For this reason, we focus on equilibria in which the value from owning no money is always zero and let $w^t \equiv (w_1^t, \cdots, w_B^t)$. Finally, we allow free disposal of money and consider equilibria in which agents are not willing to throw away money. That is, the value function must be nondecreasing in every period:

$$w_B^t \geq \cdots \geq w_1^t \geq w_0^t = 0. \tag{7}$$

Definition 1 *Given π^0, an* equilibrium *is a sequence $\{(\pi^t, w^t)\}_{t=0}^{\infty}$ that satisfies the consumer's optimality condition (4), the law of motion (5), the Bellman equation (6), and non-disposal of money (7). A tuple (π, w) is a* monetary steady state *if $(\pi^t, w^t) = (\pi, w)$ for $t \geq 0$ is an equilibrium and $w \neq 0$. Full-support steady states are those for which π has a full support. Pure-strategy steady states are those for which (3) has a unique solution for all meetings. Other steady states are called mixed-strategy steady states.*

3 Monetary steady states when $B = 2$

In Trejos and Wright (1995), the case $B = 1$, a necessary and sufficient condition for existence of a monetary steady state is

$$u'(0) > \frac{n(1-\beta)}{\beta(1-m)} + 1 \quad (\equiv K)\,. \tag{8}$$

One of our propositions says that (8) is also necessary and sufficient for existence of a full-support steady state in the economy with $B = 2$. To state it, it is helpful to express π_0 and π_2 in terms of π_1 using $\sum \pi_i = 1$ and $\sum i\pi_i = Bm$. We have

$$(\pi_0, \pi_2) = \left(1 - m - \frac{\pi_1}{2}, m - \frac{\pi_1}{2}\right), \tag{9}$$

$$\pi_1 \in \Pi \equiv [0, 2\min\{m, 1-m\}]. \tag{10}$$

Throughout this paper, the dependence of π on π_1 is kept implicit to simplify the notation.

Lemma 1 *If a monetary full-support steady state exists, then* *(i) the solution set (3) is either* $\{1\}$ *or* $\{1, 2\}$ *for (2,0)-meetings and is* $\{1\}$ *for other trade meetings; and*

(ii) π_1 *satisfies* $\pi_1 \leq \pi_1^* \equiv (\sqrt{1 + 12m(1 - m)} - 1)/3$*, where the inquality is strict if and only if* $\lambda(1; 2, 0) < 1$*.*

The proof of this result and all other proofs appear in section 5. The proof of lemma 1(i) first shows optimality of one-unit payment in any trade meeting. Using that result, it then shows suboptimality of zero-unit payment in those meetings.

The next lemma gives a characterization of when full-support steady states exist and is useful for proving the main existence result.

Lemma 2 *A monetary full-support steady state exists if and only if there exists* $(\pi_1, x) \gg 0$ *such that*

$$x = \frac{\delta}{1 - \pi_2} \left[\pi_0 u(x) + \pi_1 u(\delta x)\right] \equiv h(x, \pi_1) \tag{11}$$

and

$$u[(1 + \delta)x] \leq u(x) + x, \tag{12}$$

where

$$\delta = \frac{(1 - \pi_2)\beta}{n(1 - \beta) + (1 - \pi_2)\beta} < 1 \tag{13}$$

and where (12) must hold with equality if $\pi_1 < \pi_1^*$*. (Notice that the definition of* h *takes into account the dependence of* π *and* δ *on* π_1*.)*

If $\pi_1 = \pi_1^*$ and (12) holds strictly, then the steady state is a pure-strategy steady state, and it is a mixed-strategy steady state otherwise. The necessity part of the proof uses lemma 1 to derive (11)-(13). The sufficiency part shows that $\pi_1 > 0$ and $(\beta w_1, \beta w_2) = (x, (1 + \delta)x)$ satisfy all the equilibrium conditions.

Proposition 1 $u'(0) > K$ *is necessary and sufficient for existence of a monetary full-support steady state.*

Figure 1: Parameter regions for full-support steady states

This figure shows the regions of (m, β) for existence of pure-strategy full-support steady states (PF) and mixed-strategy full-support steady states (MF).

The proof uses the following main ideas. For a given $\pi_1 \in [0, \pi_1^*]$, the function $h(\cdot, \pi_1)$ is strictly concave and differentiable on $\mathbf{R}_+$, $h(0, \pi_1) = 0$, and $h_1(\infty, \pi_1) = 0$. Therefore, (11) has a positive solution if and only if $h_1(0, \pi_1) > 1$.

Proposition 2 *Generically, both pure-strategy and mixed-strategy full-support steady states exist. (That is, there exists open regions in the parameter space in which pure-strategy full-support steady states exist and open regions in which mixed-strategy full-support steady states exist.)*

It is helpful to have an example of cases where pure-strategy and mixed-strategy full-support steady states exist. Let $n = 2$ and $u(y) = y^{1/2}$. For such a utility function, (11) can be explicitly solved, and the inequality (12) for (m, β), although complicated, can be explicitly derived. Figure 1 displays open regions of (m, β) for which each of the two types of full-support steady states exist.[4] The figure shows that when money is sufficiently scarce buyers always pay just one unit, whereas when the money supply is high, buyers sometimes spend two units of money in one trade. The threshold depends on how much the future matters, namely β.[5]

We end this section with a lemma about a monetary steady state with a non-full-support distribution.

Lemma 3 *A monetary steady state with support $\{0, 2\}$ exists if and only if (8) holds, and it is unique. It has $w_1 = 0$ and w_2 that is a unique positive solution to*

$$w = \frac{n-1+m}{n}\beta w + \frac{1-m}{n}u(\beta w). \tag{14}$$

Moreover, $P(1,0) = \{0,1\}$, $P(1,1) = P(2,1) = \{1\}$, and $P(2,0) = \{2\}$.

[4] Although the full-support steady states computed in figure 1 seem to be unique, we have been unable to establish such uniqueness in general. Nor do we have an example of co-existence.

[5] On the boundary, one-unit payment and two-unit payment are indifferent for buyers but the former is assigned probability one.

4 Stability

In this section we study stability of our three kinds of steady states. Our stability criterion is as follows.

Definition 2 *A steady state* (π, w) *is locally stable if there is a neighborhood of* π *such that for any initial distribution in the neighborhood, there is an equilibrium path such that* $(\pi^t, w^t) \to (\pi, w)$. *A locally stable steady state is determinate, if for each initial distribution in this neighborhood, there is only one equilibrium that converges to it.*

This definition of stability only requires convergence of some equilibria, not all equilibria. This is because there are always equilibria that do not converge to a given monetary steady state. In particular, a non-monetary equilibrium always exists from any initial condition.

Notice that the above definition of local stability implies that the valued-money steady state in the Trejos-Wright $\{0, 1\}$ model is stable, because there is no 'neighborhood' of the steady state. Also, for that model, the only non-explosive path converging to that steady state is the one in which the value of money remains constant, which implies determinacy of that steady state.[6] The following are our stability results for the $\{0, 1, 2\}$ economy.

Proposition 3 *Mixed-strategy full-support steady states are generically locally stable.*

Proposition 4 *Pure-strategy full-support steady states are generically locally stable and determinate.*

Proposition 5 *Non-full-support steady states are unstable.*

The stability of the mixed-strategy steady state is proved by showing that if the initial distribution is sufficiently close to the steady state distribution, then the mixed-strategy steady state can be attained in one step, except in a nongeneric situation in which $P(2, 0) = \{1, 2\}$ but $\lambda(1; 2, 0) = 1$ so the distribution is equal to that of the pure-strategy full-support steady state.

The standard approach to stability analysis for difference equation systems (see, for example, [6]) is to compare the number of eigenvalues of the

[6]For the Trejos-Wright $\{0, 1\}$ model, Lomeli and Temzelides (2002) show that the non-monetary steady state is indeterminate.

dynamical system that are strictly smaller than one in absolute value, say a, and the number of initial conditions, say b. If $a = b$ ($a > b$), then there is a unique (an infinity of) convergent path(s). If $a < b$, then there is no convergent solution. This standard approach is applied to establish local stability and determinacy of the pure-strategy full-support steady state except for a nongeneric situation in which the Bellman equation is not time-invertible at the steady state.

The statement about non-full-support steady state shows that if the economy starts with a positive measure of people holding one unit of money, then the economy does not converge to the steady state in which a bundle of two units of money is treated as one in $\{0, 1\}$ model. The proof is by way of contradiction and relies on two features. First, the dynamical system necessarily involves *unit-root convergence* because the outflow from holdings of 1 unit, which comes from $(1, 1)$-meetings, approaches zero rapidly as the frequency of such meetings goes to zero. Second, the non-full-support steady state is on the *boundary* of the state space in two senses: the distribution does not have full support and the value of money is not strictly increasing. Hence, a convergent sequence must at all dates satisfy $\pi_1^t \geq 0$ and (7).

5 Proofs

Before turning to the proofs, we set out some steady state consequences that we use in the proofs. The steady-state law of motion reduces to

$$(\pi_1)^2 \lambda(1; 1, 1) = \left(1 - m - \frac{\pi_1}{2}\right)\left(m - \frac{\pi_1}{2}\right)\lambda(1; 2, 0), \tag{15}$$

which equates outflows from holdings of 1 (the lefthand side) to inflows into holdings of 1 (the righthand side). The Bellman equations are

$$\begin{aligned} w_1 = {} & \frac{n - 1 + \pi_2}{n}\beta w_1 + \frac{\pi_0}{n}\max[u(\beta w_1), \beta w_1] \\ & + \frac{\pi_1}{n}\max[u(\beta w_2 - \beta w_1),\ \beta w_1], \quad \text{and} \end{aligned} \tag{16}$$

$$\begin{aligned} w_2 = {} & \frac{n - 1 + \pi_2}{n}\beta w_2 + \frac{\pi_1}{n}\max[u(\beta w_2 - \beta w_1) + \beta w_1,\ \beta w_2] \\ & + \frac{\pi_0}{n}\max[u(\beta w_2),\ u(\beta w_1) + \beta w_1,\ \beta w_2]. \end{aligned} \tag{17}$$

As for full-support steady states, lemma 1 establishes that zero-unit payment is suboptimal and one-unit payment is optimal in all trade meetings in any full-support steady state, two-unit payment in $(2,0)$-meetings being also optimal for a mixed-strategy full-support steady state. The corresponding inequalities are

$$\begin{aligned}
(1,1)\text{-meeting}\ u(\beta w_2-\beta w_1) &> \beta w_1 && (18)\\
(1,0)\text{-meeting}\qquad\quad u(\beta w_1) &> \beta w_1 && (19)\\
(2,1)\text{-meeting}\ u(\beta w_2-\beta w_1) &> \beta w_2-\beta w_1 && (20)\\
\&\ (2,0)\text{-meeting}\ u(\beta w_1)+\beta w_1 &> \beta w_2 && (21)\\
u(\beta w_1)+\beta w_1 &\geq u(\beta w_2). && (22)
\end{aligned}$$

The proof of lemma 1 is composed of two steps. Step 1 shows that being a full-support monetary steady state (i.e., $\pi_1 > 0$ and $w_2 > 0$) implies that both $\lambda(1;1,1)$ and $\lambda(1;2,0)$ are strictly positive and that (18)-(22) hold at least weakly. Step 2 shows that when (18)-(22) hold weakly, the solution to (16)-(17) satisfies (18)-(21) strictly.

Proof of lemma 1. (i) Being a monetary steady state implies $w_2 > 0$ and having a full-support distribution implies $\pi_1 > 0$. Then (16) implies $w_1 > 0$.

Step 1: Any full-support monetary steady state satisfies (18)-(22) at least weakly.

Proof of Step 1
First we show $\lambda(1;1,1) > 0$ and that (18), (21) and (22) hold at least weakly. Suppose by way of contradiciton that $\lambda(1;1,1) = 0$ so $u(\beta w_2 - \beta w_1) \leq \beta w_1$. Then (19) must hold, because substituting (19) with a reversed weak inequality and the supposition into (16) gives $w_1 = 0$, a contradiction to $w_1 > 0$. Then the supposition and (19) gives

$$\beta w_2 - \beta w_1 < \beta w_1. \tag{23}$$

Note that (19) implies $0 < \beta w_1 < \bar{x}$, with $\bar{x} = u(\bar{x})$. Thus we have $0 \leq \beta w_2 - \beta w_1 < \bar{x}$, which in turn implies (20) with weak inequality. This weak inequality and (23) gives (21). Because u is strictly concave, that (18) does not hold implies $u(\beta w_2) - u(\beta w_1) < \beta w_1$. This together with (21) implies $\lambda(1;2,0) = 1$. For π_1 to be strictly positive in (15), we must have

$\lambda(1;1,1) > 0$, a contradiction. Therefore we have $\lambda(1;1,1) > 0$ and hence the weak (18). But if $\lambda(1;1,1) > 0$, then (15) and $\pi_1 > 0$ imply $\lambda(1;2,0) > 0$. This implies (21) at least weakly and (22).

Next we show that (19) and (20) hold at least weakly. Suppose by way of contradiction that $u(\beta w_1) < \beta w_1$. Then the weak (18) implies $\beta w_2 - \beta w_1 > \beta w_1$. Combining this with the weak (21) gives $u(\beta w_1) > \beta w_1$, which is a contradiction. Suppose now by way of contradiction that (20) does not hold even weakly: $u(\beta w_2 - \beta w_1) < \beta w_2 - \beta w_1$. Then the weak (21) implies $\beta w_2 - \beta w_1 < \beta w_1$. But the weak (18) and supposition imply $\beta w_2 - \beta w_1 > \beta w_1$, which is a contradiction. (End of proof of Step 1)

Step 2: If (18)-(22) hold weakly, then (18)-(21) hold strictly.

Proof of Step 2

When (18)-(22) hold at least weakly, we can eliminate 'max' operators from (16)-(17). Then subtracting (16) from (17) gives

$$w_2 - w_1 = \frac{(1-\pi_2)\beta}{n(1-\beta)+(1-\pi_2)\beta} w_1, \tag{24}$$

and βw_1 is a solution to

$$\beta w_1 = \frac{\beta}{n(1-\beta)+(1-\pi_2)\beta}\left[\pi_0 u(\beta w_1) + \pi_1 u\left(\frac{(1-\pi_2)\beta}{n(1-\beta)+(1-\pi_2)\beta}\beta w_1\right)\right], \tag{25}$$

Suppose by way of contradiction that (18) does not hold:

$$u(\beta w_2 - \beta w_1) = u\left(\frac{(1-\pi_2)\beta}{n(1-\beta)+(1-\pi_2)\beta}\beta w_1\right) \leq \beta w_1.$$

Then, we have

$$\begin{aligned}
\beta w_1 &\leq \frac{\pi_0\beta}{n(1-\beta)+\pi_0\beta} u(\beta w_1) \\
&< u\left(\frac{\pi_0\beta}{n(1-\beta)+\pi_0\beta}\beta w_1\right) \\
&< u\left(\frac{(1-\pi_2)\beta}{n(1-\beta)+(1-\pi_2)\beta}\beta w_1\right) = u(\beta w_2 - \beta w_1),
\end{aligned}$$

where the first inequality is by substituting the supposition into (25) and the second is by $u(0) = 0$ and strict concavity of u. This is contradiction and thus (18) should hold. Inequality (21) follows from $u(\beta w_1) > u(\beta w_2 - \beta w_1) > \beta w_1 > \beta w_2 - \beta w_1$, where the first and the third inequalities are by (24) and the second is (18).

Suppose by way of contradiction that (19) does not hold: $u(\beta w_1) \leq \beta w_1$. Then (18) implies $\beta w_2 - \beta w_1 > \beta w_1$. Combining this with (21) gives $u(\beta w_1) > \beta w_1$, which is a contradiction. Suppose now by way of contradiction that (20) does not hold: $u(\beta w_2 - \beta w_1) \leq \beta w_2 - \beta w_1$. Then (21) implies $\beta w_2 - \beta w_1 \leq \beta w_1$. But (18) and supposition imply $\beta w_2 - \beta w_1 > \beta w_1$, which is a contradiction. In summary, (18)-(21) hold strictly. (End of proof of Step 2)

(ii) Letting $\lambda(1;1,1) = 1$ and $\lambda(1;2,0) = \gamma$ in (15) and solving it for π_1 yields

$$\pi_1 = \sqrt{\left(\frac{\gamma}{4-\gamma}\right)^2 + 4m(1-m)\frac{\gamma}{4-\gamma}} - \frac{\gamma}{4-\gamma}. \tag{26}$$

Here $\pi_1 \in [0, \pi_1^*]$ is strictly increasing in $\gamma \in [0,1]$ and is equal to π_1^* iff $\gamma = 1$. ■

Proof of lemma 2. (Necessity) By lemma 1, inequalities (18)-(22) hold for any full-support steady state. Under these inqualities, the Bellman equations (16)-(17) become (11) and (13) with $x = \beta w_1$ and $(1+\delta)x = \beta w_2$. Also, (22) implies (12).

(Sufficiency) The proof resembles a guess and verify argument. Suppose we have such (π_1, x). Let $\beta w_1 = x$ and $\beta w_2 = (1+\delta)x$. Then we have (24)-(25). The same arguments as step 2 of the proof of lemma 1 show that (18)-(21) hold. Also, (22) is given by (12), Therefore we have (18)-(22) with equality in (22) if and only if (12) holds with equality. That is, such $(\beta w_1, \beta w_2)$ satisfies the optimality of lemma-1 trade. Under the lemma-1 trade, the Bellman equation (16)-(17) is equivalent to (11) and (13) with $x = \beta w_1$ and $(1+\delta)x = \beta w_2$, so the Bellman equation trivially holds. If (12) holds strictly and hence $\lambda(1;2,0) = 1$, then $\pi_1 = \pi_1^*$ satisfies the law of motion (15). If (12) holds with equality and hence $\pi_1 < \pi_1^*$, then (15) holds with some unique $\lambda(1;2,0)$ due to lemma 1(ii) That is, all the equilibrium conditions are satisfied. ■

The proof of proposition 1 uses the intermediate value theorem to construct $(\pi_1, x) \gg 0$ in lemma 2.

Proof of proposition 1. In this proof, we denote δ as δ_{π_1} to make the dependence on π_1 explicit. First we show necessity of (8). Suppose that a full-support steady state exists. By lemma 1, full-support steady states satisfy (18)-(22). If all these optimality conditions are substituted into (16) and (17) and $(\beta w_2, \beta w_1)$ is replaced by $(x, (1+\delta_{\pi_1})x)$, then one obtains (11). That is, it is necessary for (11) to have a positive solution x, which as remarked in the text implies $h_1(0,\pi_1) > 1$, where

$$h_1(0,\pi_1) = \delta_{\pi_1}\left[\frac{\pi_0}{1-\pi_2} + \frac{\pi_1\delta_{\pi_1}}{1-\pi_2}\right]u'(0) \equiv J_{\pi_1}u'(0). \tag{27}$$

Therefore, $h_1(0,\pi_1) > 1$ is equivalent to $u'(0) > 1/J_{\pi_1}$. By some algebra, one can show

$$1/J_{\pi_1} - K = \frac{n(1-\beta)}{\beta} \cdot \frac{\pi_1 n(1-\beta) + \beta\pi_1\pi_0}{[\pi_0 n(1-\beta) + \beta(1-\pi_2)^2](2-2m)} \geq 0,$$

with equality if and only if $\pi_1 = 0$. Therefore (8) is necessary for (11) to have a positive $x > 0$.

Next we show sufficiency of (8). Let x_0 be the unique positive solution for x to $x = h(x,0)$. A consequence is $h_1(x_0,0) < 1$. Because $h(x,0) = \delta_0 u(x)$, where δ_0 denotes δ when $\pi_1 = 0$, we have $\delta_0 u'(x_0) < 1$. Applying the concavity theorem, we have

$$u[(1+\delta_0)x_0] - u(x_0) < u'(x_0)\delta_0 x_0 < x_0. \tag{28}$$

Now, consider the function $h_1(0,\pi_1)$ on the domain $[0,\pi_1^*]$. By (13) and (27), $h_1(0,\pi_1)$ is continuous. We also know that $h_1(0,0) = \delta_0 u'(0) = u'(0)/K > 1$. Now, there are two cases.

Case 1: There exists $\bar{\pi}_1 \in (0,\pi_1^*)$ such that $h_1(0,\bar{\pi}_1) = 1$ and $h_1(0,\pi_1) > 1$ for all $\pi_1 \in (0,\bar{\pi}_1)$ (That is, (11) has a potive solution $x > 0$ for all $\pi_1 \in (0,\bar{\pi}_1)$.) Let x_{π_1} be the unique positive solution to $x = h(x,\pi_1)$ for each $\pi_1 \in (0,\bar{\pi}_1)$. Note first that because δ_{π_1} is decreasing in n and is equal to one when $n = 0$, the expression in the square brackets in (27) is smaller than one. Therefore, (27) for $\pi_1 = \bar{\pi}_1$ implies $1 = h_1(0,\bar{\pi}_1) < \delta_{\bar{\pi}_1}u'(0)$. Thus

$$\begin{aligned} &\frac{u(x_{\pi_1}) + x_{\pi_1} - u((1+\delta_{\pi_1})x_{\pi_1})}{x_{\pi_1}} < \frac{x_{\pi_1} - u'((1+\delta_{\pi_1})x_{\pi_1})\delta_{\pi_1}x_{\pi_1}}{x_{\pi_1}} \\ \to\ & 1 - u'(0)\delta_{\bar{\pi}_1} < 0, \text{ as } \pi_1 \to \bar{\pi}_1, \end{aligned} \tag{29}$$

where the first inequality follows from concavity of u and the limit operation uses the fact that $x_{\pi_1} \to 0$ as $\pi_1 \to \bar{\pi}_1$. Therefore we have $u(x_{\pi_1}) + x_{\pi_1} < u((1+\delta_{\pi_1})x_{\pi_1})$ for $\pi_1 < \bar{\pi}_1$ that is sufficiently close to $\bar{\pi}_1$. Recalling (28), we can apply the intermediate value theorem, so there exists $\hat{\pi}_1$ such that

$$u(x_{\hat{\pi}_1}) + x_{\hat{\pi}_1} = u((1+\delta_{\hat{\pi}_1})x_{\hat{\pi}_1}). \tag{30}$$

By lemma 2, such a pair $(\hat{\pi}_1, x_{\hat{\pi}_1})$ forms a mixed-strategy full-support steady state.

Case 2: $h_1(0, \pi_1) > 1$ for all $\pi_1 \in [0, \pi_1^*]$. In this case, (11) has a positive solution for all $\pi_1 \in [0, \pi_1^*]$. If $u(x_{\pi_1^*}) + x_{\pi_1^*} \leq u((1+\delta_{\pi_1^*})x_{\pi_1^*})$, then with (28) the intermediate value theorem is applied and we have (30) for some $\hat{\pi}_1$. By lemma 2, the pair $(\hat{\pi}_1, x_{\hat{\pi}_1})$ forms a mixed-strategy full-support steady state. If $u(x_{\pi_1^*}) + x_{\pi_1^*} > u((1+\delta_{\pi_1^*})x_{\pi_1^*})$, then lemma 2 implies that there is a (pure-strategy) full-support steady state. Lemma 1 and 2 imply that it is a unique pure-strategy full-support steady state.

Overall, a mixed-strategy steady state exists when a pure-strategy steady state does not. ■

Proof of Proposition 2. When β is sufficiently close to one, the pure-strategy full-support steady state exists. To see this, fix all parameters except β and let $\pi_1 = \pi_1^*$. As $\beta \to 1$, equation (11) approaches $x_{\pi_1^*} = u(x_{\pi_1^*})$, (13) approaches $\delta_{\pi_1^*} = 1$, and (12) approaches $u(2x_{\pi_1^*}) < u(x_{\pi_1^*}) + x_{\pi_1^*}$. By strict concavity of u, this last inequality holds. As remarked after lemma 2, x and $(1+\delta_{\pi_1})x$ represent βw_1 and βw_2, respectively, so the pure-strategy full-support steady state exists.

For the generic existence of mixed-strategy full-support steady states, note that in the proof of proposition 1 we saw that a mixed-strategy full-support steady state exists if a pure-strategy one does not exist. By (**??**) we saw that $u'(0) > 1/J_{\pi_1^*}$ is necessary for the existence of a pure-strategy full-support steady state. Therefore, if $u'(0) \in (K, 1/J_{\pi_1^*})$, then a mixed-strategy full-support steady state exists. ■

Proof of Lemma 3. (Necessity) Suppose there is a monetary steady state with support $\{0, 2\}$. We have $\pi_1 = 0$ and hence $\lambda(1; 2, 0) = 0$ by (15). Then equations (16)-(17) imply that both w_1 and w_2 must satisfy

$$w = \frac{n-1+m}{n}\beta w + \frac{1-m}{n}\max[u(\beta w), \beta w]. \tag{31}$$

This equation has at most two solutions: $\beta w = 0$ and $\beta w \in (0, \bar{x})$. For a monetary steady state, (31) (or (14)) must have a positive solution. This requires $u'(0) > K$.

(Sufficiency) If $u'(0) > K$, equation (14) has a (unique) positive solution. The rest of the proof proceeds by guess and verify. Let (π, w, λ) satisfy the following: $(\pi_0, \pi_1, \pi_2) = (1 - m, 0, m)$, $\lambda(1; 1, 0) = \lambda(1; 1, 1) = \lambda(1; 2, 1) = \lambda(2; 2, 0) = 1$, $w_1 = 0$, and w_2 is the unique positive solution to (14). We show these form a monetary steady state with support $\{0, 2\}$. Under such λ, the above π satisfies (15) and w satisfies (16)-(17). By (14), $u(\beta w_2) > \beta w_2$ holds. This and $w_1 = 0$ ensure strict optimality of the above λ in all trade meetings but $(1, 0)$-meetings. That is, we have $P(2, 0) = \{2\}$, $P(1, 0) = \{0, 1\}$, and $P(1, 1) = P(2, 1) = \{1\}$. It is not hard to check that there is no other monetary steady state with a non-full-support distribution. ■

Proof of Proposition 3. Denote the mixed-strategy steady state by (π, w, λ). We show that if the initial distribution π_1^0 is sufficienty close to π_1 and if $\pi_1 < \pi_1^*$, then the economy can jump to the steady state in one period. By (5), the date-1 distribution is given by

$$\pi_1^1 \;=\; \pi_1^0 - \frac{2(\pi_1^0)^2}{n} + \frac{2}{n}\left(1 - m - \frac{\pi_1^0}{2}\right)\left(m - \frac{\pi_1^0}{2}\right)\lambda^0(1; 2, 0) \qquad (32)$$

where $\lambda^0(1; 2, 0) \in [0, 1]$ is the date-0 randomization. That is, the distribution can jump to π_1 in one period if

$$\pi_1 \in \left[\pi_1^0 - \frac{2(\pi_1^0)^2}{n},\;\; \pi_1^0 - \frac{2(\pi_1^0)^2}{n} + \frac{2}{n}\left(1 - m - \frac{\pi_1^0}{2}\right)\left(m - \frac{\pi_1^0}{2}\right)\right]. \qquad (33)$$

The lower bound is smaller than π_1 if π_1^0 is sufficiently close to π_1. The upper bound can be rewritten as $\pi_1^0 + \xi(\pi_1^0)$ where $\xi(\pi_1^0)$ is positive, zero, or negative iff $\pi_1^0 < \pi_1^*$, $\pi_1^0 = \pi_1^*$ or $\pi_1^0 > \pi_1^*$, respectively. Therefore, the upper bound of (33) is greater than π_1 if we have $\pi_1 < \pi_1^*$ and π_1^0 is sufficiently close to π_1. That is, generically, there exists an open neighborhood of π_1 from which the economy can choose $\lambda^0(1; 2, 0)$ to jump to π_1 in one period. Afterwards, the economy can have $(\pi^t, w^t, \lambda^t) = (\pi, w, \lambda)$ for all $t \geq 1$. Such randomization is the optimal choice by the agents, because $w^1 = w$ satisfies the indifference condition for date-0 trades. Then the initial value w^0 can be determined from the initial distribution π_1^0 and w^1 via the Bellman equation. (Note that w^0 does not affect agents' decisions.) Thus the mixed-strategy steady state is locally stable generically (i.e., if $\pi_1 \neq \pi_1^*$). ■

The proof of stability of the pure-strategy full-support steady state and the proof of instability of the non-full-support steady state share some common procedures. The following lemma derives a dynamical system which governs equilibrium paths, if any, that converge to these steady states.

Lemma 4 *Let* (π_1, w_1, w_2) *be a steady state and let*

$$\Psi_\pi^\gamma = 1 - \frac{\sqrt{1+12m(1-m)}}{n}\gamma, \tag{34}$$

$$\phi_\pi^\gamma = \begin{bmatrix} \frac{-\beta w_1 - u(\beta w_1) + 2u(\beta \Delta w)}{2n} \\ \frac{-\beta w_2 - [\gamma\{u(\beta w_1)+\beta w_1\} + (1-\gamma)u(\beta w_2)] + 2[u(\beta\Delta w)+\beta w_1]}{2n} \end{bmatrix}, \tag{35}$$

and

$$\phi_w^\gamma = \begin{pmatrix} \frac{(n-1+\pi_2)\beta}{n} + \frac{\pi_0 \beta u'(\beta w_1)}{n} - \frac{\pi_1\beta u'(\beta\Delta w)}{n} & \frac{\pi_1 \beta u'(\beta \Delta w)}{n} \\ \frac{\gamma \pi_0 (u'(\beta w_1)+1)\beta}{n} + \frac{\pi_1(1-u'(\beta\Delta w))\beta}{n} & \frac{(n-1+\pi_2)\beta}{n} + \frac{\pi_0(1-\gamma)\beta u'(\beta w_2)}{n} + \frac{\pi_1 \beta u'(\beta\Delta w)}{n} \end{pmatrix}, \tag{36}$$

where $\Delta w \equiv w_2 - w_1$ *and* $\gamma \in \{0, 1\}$. *Suppose that* ϕ_w^γ *has an inverse. If a sequence converges to either the pure-strategy full-support or non-full-support steady state, then it satisfies* $x^{t+1} = F(x^t)$, *where* $x^t \equiv (\pi_1^t, w_1^t, w_2^t)$, *and the Jacobian of* F *evaluated at the steady state is given by*

$$A^\gamma \equiv \begin{bmatrix} \Psi_\pi^\gamma & O \\ -(\phi_w^\gamma)^{-1}\phi_\pi^\gamma & (\phi_w^\gamma)^{-1} \end{bmatrix}, \tag{37}$$

where $\gamma = 1$ *for the pure-strategy full-support steady state and* $\gamma = 0$ *for the non-full-support steady state.*

Note that (36) is always invertible for the non-full-support steady state $(0 = \gamma = \pi_1 = w_1)$. For the pure-strategy full-support steady state $(\gamma = 1)$, the determinant of (36) is zero iff

$$(n-1+\pi_2)(n-1+\pi_2+\pi_0 u'(\beta w_1)) = (1-\pi_2)\pi_1 u'(\beta\Delta w), \tag{38}$$

where $\pi_1 = \pi_1^*$, and $(w_1, \Delta w) = (x, \delta x)$ from Lemma 2 is an implicit but well-defined function of parameters (n, m, β, u). Equation (38) is not implied by (11) and hence the set of parameters for which (36) is singular has measure zero.

Proof of Lemma 4. For the pure-strategy full-support steady state, trading one unit in all trade meetings is a strictly preferred strategy at the steady state (see Definition 1 and Lemma 1), so it is also optimal in its neighborhood. That is, $\lambda^t(1;1,0) = \lambda^t(1;1,1) = \lambda^t(1;2,1) = \lambda^t(1;2,0) = 1$ for all $t \geq 0$.

Similarly, we can also pin down the optimal trading strategy that is constantly played along a path that converges to the non-full-support steady state from $\pi_1^0 \neq 0$, if there is any such path. By lemma 3, trading one unit is strictly preferred in $(1,1)$- and $(2,1)$-meetings, and paying two units is strictly preferred in $(2,0)$-meetings at (π, w). Therefore, they are also optimal in the neighborhood of (π, w), so $\lambda^t(1;1,1) = \lambda^t(1;2,1) = \lambda^t(2;2,0) = 1$ for all $t \geq 0$. Moreover, the following argument shows $\lambda^t(1;1,0) = 1$ should be the case for all $t \geq 0$. When the economy is close to but not equal to (π, w), we have $\pi_1^t > 0$ for all $t \geq 0$ so (6) implies $w_1^t > 0$ for all $t > 0$, because there is always a positive probability that a consumer with one unit meets a producer with one unit and the consumer can get a positive amount of utility from such a meeting.[7] Moreover, (14) implies $u(x) > x$ for all $x < \beta w_2$ and therefore $u(\beta w_1^t) > \beta w_1^t$ holds all along the path. So, in $(1,0)$-meetings, paying one unit is strictly preferred to paying nothing along the path.

Therefore in both cases, a unique strategy is constantly played along any potential convergent path. Under that strategy, the law of motion and Bellman equation reduces to

$$\pi_1^{t+1} = \pi_1^t - \frac{2(\pi_1^t)^2}{n} + \frac{2}{n}\left(1 - m - \frac{\pi_1^t}{2}\right)\left(m - \frac{\pi_1^t}{2}\right)\gamma \tag{39}$$

$$w_1^t = \frac{n-1+\pi_2^t}{n}\beta w_1^{t+1} + \frac{\pi_0^t}{n}u(\beta w_1^{t+1}) + \frac{\pi_1^t}{n}u(\beta w_2^{t+1} - \beta w_1^{t+1}) \tag{40}$$

$$\begin{aligned} w_2^t = {} & \frac{n-1+\pi_2^t}{n}\beta w_2^{t+1} + \frac{\pi_0^t}{n}[\gamma(u(\beta w_1^{t+1}) + \beta w_1^{t+1}) + (1-\gamma)u(\beta w_2^{t+1})] \\ & + \frac{\pi_1^t}{n}[u(\beta w_2^{t+1} - \beta w_1^{t+1}) + \beta w_1^{t+1}], \end{aligned} \tag{41}$$

where $\gamma = 1$ for the pure-strategy full-support steady state and $\gamma = 0$ for the non-full-support steady state. Denote (39) by $\pi_1^{t+1} = \Psi^\gamma(\pi_1^t) : \Pi \to \Pi$ and (40)-(41) by $w^t = \phi^\gamma(\pi_1^t, w^{t+1}) : \Pi \times W \to W$, where $w^t \equiv (w_1^t, w_2^t)$ and

[7] One can consider the dynamic version of (16) with $\pi_1^t > 0$ and $w_2^{t+1} > 0$.

$W \equiv \{(w_1, w_2) | 0 \le w_1 \le w_2\}$. Then, our dynamical system is

$$\begin{pmatrix} \pi_1^{t+1} \\ w^{t+1} \end{pmatrix} = \begin{pmatrix} \Psi^\gamma(\pi_1^t) \\ \Phi^\gamma(\pi_1^t, w^t) \end{pmatrix}, \tag{42}$$

where Φ^γ is the inverse of ϕ^γ (inverse in terms of w) and is obtained by applying the implicit function theorem in the vicinity of the steady state.[8] This is F in the statement of this lemma.

Finally, straightforward differentiation of (39)-(41) and the implicit function theorem imply that the Jacobian of F at the steady state is given by (34)-(37). ■

The proofs of propositions 4 and 5 look into the properties of (37).

Proof of Propositions 4. For the pure-strategy full-support steady state, we set $\gamma = 1$ in (37) and denote the steady state by (π^*, w^*). Because the top-right submatrix of (37) is a zero matrix, one eigenvalue of (37) is given by (34), which is smaller than one, and the other two eigenvalues are those of $(\phi_w^\gamma)^{-1}$, which are the reciprocals of eigenvalues of ϕ_w^γ. In what follows, we show that eigenvalues of ϕ_w^γ are smaller than one in absolute value.

Because $h_1(\beta w_1^*, \pi_1^*) < 1$, we have

$$\frac{n(1-\beta) + (1-\pi_2^*)\beta}{\beta} > \pi_0^* u'(\beta w_1^*) + \pi_1^* \frac{(1-\pi_2^*)\beta}{n(1-\beta) + (1-\pi_2^*)\beta} u'(\beta \Delta w^*). \tag{43}$$

The eigenvalues of a general 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are given by

$$\eta_+, \eta_- = \frac{a + d \pm \sqrt{(a-d)^2 + 4bc}}{2}.$$

[8]Because the non-full-support steady state lies on the boundary of $\Pi \times W$, the domain of Ψ^γ, ϕ^γ and hence the domain of u are extended to allow for an open neighborhood around the steady state before applying the implicit function theorem. This is the only place where the assumption $u'(0) < \infty$ is used.

Because

$$\begin{aligned}
& (a-d)^2+4bc \\
= & \left[\frac{\pi_0^*}{n}\beta u'(\beta w_1^*)-2\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right]^2 \\
& +4\left[\frac{1-\pi_2^*}{n}\beta+\frac{\pi_0^*}{n}\beta u'(\beta w_1^*)-\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right]\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*) \\
= & \left[\frac{\pi_0^*}{n}\beta u'(\beta w_1^*)\right]^2+4\frac{1-\pi_2^*}{n}\beta\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)>0,
\end{aligned}$$

both eigenvalues are real. They are smaller than one in absolute value if and only if $a+d<2$ and $(1-a)(1-d)-bc>0$. Checking these conditions for (36) gives

$$\begin{aligned}
& 1-a+1-d \\
= & 2\left(1-\frac{n-1+\pi_2^*}{n}\beta\right)-\frac{\pi_0^*}{n}\beta u'(\beta w_1^*) \\
> & 2\frac{n(1-\beta)+(1-\pi_2^*)\beta}{n}-\frac{\pi_0^*}{n}\beta u'(\beta w_1^*)-\frac{\pi_1^*}{n}\frac{(1-\pi_2^*)\beta}{n(1-\beta)+(1-\pi_2^*)\beta}\beta u'\left(\beta\Delta w^*\right) \\
> & \frac{n(1-\beta)+(1-\pi_2^*)\beta}{n}>0;\ \text{and}
\end{aligned}$$

$$\begin{aligned}
& (1-a)(1-d)-bc \\
= & \left(1-\frac{n-1+\pi_2^*}{n}\beta-\frac{\pi_0^*}{n}\beta u'(\beta w_1^*)+\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right)\left(1-\frac{n-1+\pi_2^*}{n}\beta-\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right) \\
& -\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\left[\frac{1-\pi_2^*}{n}\beta+\frac{\pi_0^*}{n}\beta u'(\beta w_1^*)-\frac{\pi_1^*}{n}\beta u'(\beta\Delta w^*)\right] \\
= & \frac{(n(1-\beta)+(1-\pi_2^*)\beta)\beta}{n^2}\times \\
& \quad\left(\frac{n(1-\beta)+(1-\pi_2^*)\beta}{\beta}-\pi_0^*u'(\beta w_1^*)-\pi_1^*\frac{(1-\pi_2^*)\beta}{n(1-\beta)+(1-\pi_2^*)\beta}u'(\beta\Delta w^*)\right) \\
> & 0,
\end{aligned}$$

where the last inequalities of the above two conditions follow from (43). Therefore, the eigenvalues of $(\phi_w^\gamma)^{-1}$ are greater than one in absolute value. The pure-strategy full-support steady state has a one-dimensional stable manifold. Because we have one initial condition, this full-support steady state is locally stable and determinate. ■

Proof of Propositions 5. To establish the instability of the non-full-support steady state, suppose by way of contradiction that there is an equilibrium path that converges to that steady state. By assumption, the economy starts with $\pi_1^0 > 0$, which in turn means $w_1^0 > 0$ as was shown in the proof of lemma 4. By lemma 4, such a path must have Jacobian (37) with $\gamma = 0$. We analize the local trajectory implied by the eigenvectors.

Equation (34) gives a unit eigenvalue for the law of motion. The unit

Figure 2: Unit-root convergence of the law of motion

root does not immediately imply instability: As figure 2 illustrates, the law of motion (39) has slope at the fixed point that is unity, but it still displays convergence. Note also that this steady state is on the boundary of the state space $\Pi \times W$, which makes it necessary to explictly study the limiting behavior by seeing the eigenspace of the linearized system (37) to check $(\pi_1^t, w^t) \in \Pi \times W$ all along the path.[9]

As $0 = \gamma = \pi_1 = w_1$, the Jacobian (37) reduces to

$$A^\gamma = \begin{bmatrix} 1 & 0 & 0 \\ -r/a' & 1/a' & 0 \\ -s/d' & 0 & 1/d' \end{bmatrix}, \tag{44}$$

where $r \equiv \frac{1}{n}u(\beta w_2)$, $s \equiv \frac{1}{2n}[u(\beta w_2) - \beta w_2] > 0$, $a' \equiv \frac{(n-1+m)\beta}{n} + \frac{1-m}{n}\beta u'(0)$, and $d' \equiv \frac{(n-1+m)\beta}{n} + \frac{1-m}{n}u'(\beta w_2)\beta$. Note that because w_2 is a positive solution to (14), $a' > 1$ and $d' \in (0,1)$ hold. Eigenvalues of (44) are simply its diagonal elements.

Since $\pi_1^0 > 0$, $w_1^0 > 0$, and the law of motion has unit-root convergence, the convergent trajectory will eventually be parallel to the eigenspace of (44) associated with the unit eigenvalue[10]. The associated eigenvector, which

[9] Note that this analysis is not needed for the pure-strategy full-support steady state because that steady state is in the interior of $\Pi \times W$.

[10] See Subsection "Dominant Eigenvector" on page 165 of [4].

constitutes a base of the space, has the form

$$\begin{bmatrix} 1 \\ \frac{-r}{a'-1} \\ \frac{s}{1-d'} \end{bmatrix}.$$

The fact that convergent trajectory of $(\pi_1^t, w_1^t, w_2^t - w_2)$ will be parallel to the above eigenvector implies that π_1^t and w_1^t will eventually have different signs, in contradiction to $\pi_1^t, w_1^t > 0$ for all t. ■

6 Concluding remarks

We show that the necessary and sufficient condition for existence of the monetary steady state of the Trejos-Wright $\{0, 1\}$ economy, namely (8), is also necessary and sufficient for existence of a full-support steady state of the $\{0, 1, 2\}$ economy. Hence, Zhu's (2003) sufficient condition is not necessary for the bound of two. Moreover, both the pure-strategy and mixed-strategy full-support steady states are generic. Given our result, a reasonable conjecture would be that even for a higher bound, the condition (8) is necessary and sufficient for the existence of full-support steady states. For values of parameters that lead to lower values of money (i.e., high n, low β and high m), randomization may occur.

The sharp contrast we find in our stability analysis, namely the local stability of full-support steady states and instability of non-full-support steady states, also permits conjectures for the case of general bound.[11] Generalizing Proposition 4-5 to a higher bound case, however, is not simple. When the bound is two, we can identify candidate strategies that support steady states and get explicit expressions for the relevant difference-equation system. For a general bound, we do not know the supporting strategies. Therefore, if analogous proofs are to be provided, they must be constructed differently.[12]

[11] In this respect, our result is consistent with the equilibrium refinement studied by Wallace and Zhu (2004), who show that a preference perturbation by means of commodity money rules out the non-full-support steady states while the full-support steady state survives such a refinement. The multiplicity of steady states bears some resemblance to that in Green-Zhou (2002). However, the models are very different, as are the stability results.

[12] In our companion paper, we generalize the result concerning instability of non-full-support steady states (Proposition 5) to a general bound case. See Huang and Igarashi (2013).

References

[1] **Green, E. J., Zhou, R.** (2002) :"Dynamic monetary equilibrium in a random matching economy," *Econometrica*, Vol. 70, No. 3, 929-969

[2] **Huang, P., Igarashi, Y.** (2013) :"Why ten \$1's are not treated as a \$10," unpublished manuscript

[3] **Lomeli, H., Temzelides, T.** (2002): "Discrete time dynamics in a random matching monetary model," *Economic Theory*, 20, 259-269

[4] **Luenberger, David G.** (1979): *Introduction to Dynamic Systems: Theory, Models, and Applications*, Wiley

[5] **Molico, M.** (2006): "The distribution of money and prices in search equilibrium," *International Economic Review*, Vol. 47, No. 3, 701-722

[6] **Stokey, N., Lucas, R., with Prescott, E.** (1989): *Recursive Methods in Economic Dynamics*, Harvard University Press

[7] **Trejos, A., and Wright, R.** (1995): "Search, bargaining, money and prices," *Journal of Political Economy*, Vol. 103, No. 1, 118-141

[8] **Wallace, N., and Zhu, T.** (2004): "A commodity-money refinement in matching models," *Journal of Economic Theory*, 117, 246-258

[9] **Zhu, T.** (2003): "Existence of a monetary steady state in a matching model: indivisible money," *Journal of Economic Theory*, 112, 307-324