An analysis of portfolio selection with multiplicative background risk

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Abstract: This paper investigates the impact of multiplicative background risk on an investor's portfolio choice in a mean-variance framework. We also study the efficient boundary frontiers with and without risk-free security.

Keywords: Background risk; Portfolio selection; VaR; CVaR

1 Introduction

Gollier and Pratt (1996) provide the necessary and sufficient conditions under which adding an unfair background risk to wealth makes risk-averse individuals behave in a more risk-averse way with respect to any other independent risk. They call this property as “risk vulnerability”. Eeckhoudt, Gollier, and Schlesinger (1996) establish conditions on preferences under which some changes in the distribution of the background wealth entail more risk-averse behavior towards endogenous risk. Hara, Huang and Kuzmics (2011) examine necessary and sufficient conditions on an individual’s expected utility function under which any zero-mean idiosyncratic risk increases cautiousness. Tsetlin and Winkler (2005) analyze optimal investment decisions in the correlated background risk setting, and show that it may be optimal for a risk-averse agent to undertake a project with zero or even negative expected returns in the presence of an additive negatively correlated background risk.

Lajeri-Chaherli (2002) proves that proper risk aversion is equivalent to both quasi-concavity of a MV utility function and DARA, while Lajeri-Chaherli (2005) shows that standard risk aversion defined in the MV framework holds if and only if MV functions display DARA and DAP. Eichner and Wagener (2003) define the concept of variance vulnerability to characterize the property that an agent with MV deviation preferences reduces his/her risky activities when facing an increase in the variance of an independent background risk. Eichner (2008) transfers the concept of risk vulnerability into MV preferences, and shows that risk vulnerability is equivalent to the slope of the MV indifference curve being decreasing in mean and increasing in variance. Eichner and Wagener (2009) document the comparative statics with both an endogenous risk and a background risk for an agent with MV preferences in a generic decision model, and confirm that the agent becomes less risk-averse in response to an increase in the expected value of the background risk or a decrease in its variability if the preferences exhibit DARA or variance vulnerability. Wagener (2002) analyze the comparative statics of optimal decisions under uncertainty when preferences are represented by two-moment, MV utility functions.

Franke, Schlesinger and Stapleton (2006) are the first authors to analyze the effect of the presence of a multiplicative background risk but they restrict themselves to the conditions under which this background risk makes an agent behave in a more risky manner after the introduction of the background risk. Sévi (2010) studies the single-period newsvendor problem when the newsvendor faces a multiplicative neutral independent background risk in an expected utility framework. It is shown that multiplicative risk vulnerability is a sufficient condition to guarantee a decrease in the optimal order. A weaker sufficient condition which has more interpretability is also provided and discussed. Li (2011) examines the demand for a risky asset in the presence of two risks: a financial risk and a background risk which need not be financial.

Baptista (2008) explores optimal delegated portfolio management with background risk and provides conditions under which investors delegate their wealth to portfolio managers with mean and tracking error variance functions. Baptista (2012) considers an investor with multiple accounts. He assumes that for each account the investor seeks to maximize the account's expected return subject to a constraint that reflects the account's motive. Jiang et al. (2010) investigate the impact of background risk on an investor's portfolio choice in a MV framework, and analyzes the properties of efficient portfolios as well as the investor's hedging behaviour in the presence of background risk. Heaton and Lucas (2000) focus on how the presence of background risks - from sources such as labour and entrepreneurial income-influences portfolio allocations.

This paper extend the work in this area by studying the impact of multiplicative background risk on the efficient portfolio selection in the MV framework. We also study the efficient boundary frontiers with and without risk-free security.

2 Mean-Variance with Multiplicative Background Risk

We first assume no risk-free security. Supposing to have n assets with return $r = (r_1, r_2, \cdots, r_n)^\tau$, the return of the portfolio is $r_\omega = \omega^\tau r$ in which $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^\tau$ with $\sum_{i=1}^n \omega_i = 1$. Denote r_b the return of the multiplicative background asset and assume r_b and r are independent, the mean and variance of total return is

$$r_p = \omega^\tau r \cdot r_b \tag{2.1}$$

with $E(r_p) = \omega^\tau E(r) \cdot E(r_b)$ and $\sigma^2(r_p) = \omega^\tau M \omega$ where $M = VE(r_b^2) + E(r)E^\tau(r)\mathrm{Var}(r_b)$ and V is non-singular.

Suppose W be the set of portfolios and, for any $\bar{E} \in \mathbb{R}$, $W(\bar{E}) = \{\omega \in W : E[r_p] = \bar{E}\}$ be the set of portfolios with expected return equal to $\bar{E}$. Then

Definition 2.1 *A portfolio $\bar{\omega} \in W(\bar{E})$ belongs to the MV boundary with multiplicative background risk if and only if for some $\bar{E} \in \mathbb{R}$, $\bar{\omega}$ is the solution of solving $\min_{\omega \in W(\bar{E})} \sigma_p^2$ where σ_p^2 is defined in (2.1).*

We first establish the following proposition.

Proposition 2.1 *Portfolio ω belongs to the MV boundary with multiplicative background risk if and only if*

$$\frac{\sigma_p^2}{1/C} - \frac{(E(r_p)/E(r_b) - A/C)^2}{D/C^2} = 1 \ , \tag{2.2}$$

where $a = I^\tau V^{-1}E(r), b = E(r)^\tau V^{-1}E(r), c = I^\tau V^{-1}I, d = bc - a^2,\ A = I^\tau M^{-1}E(r) = a/E(r_b^2) - Kab, B = E(r)^\tau M^{-1}E(r) = b/E(r_b^2) - Kb^2,\ C = I^\tau M^{-1}I = c/E(r_b^2) - Ka^2, D = BC - A^2 = d/E(r_b^2)(1/E(r_b^2) - Kb),$ *and* $K = \mathrm{Var}(r_b)/\Big[E(r_b^2)\big(E(r_b^2) + b\mathrm{Var}(r_b)\big)\Big].$

We then develop the following proportion:

Proposition 2.2 *If there exists a multiplicative background risk exists, then the minimum variance portfolio has the following property:*

$$\omega_{mvp} = -\frac{a^2K}{C} \cdot q_1 + (1 + \frac{a^2K}{C}) \cdot q_2.$$

where $q_1 = V^{-1}E(r)/I^\tau V^{-1}E(r)$ *and* $q_2 = V^{-1}I/I^\tau V^{-1}I.$

From Proposition 2.2, we obtain the following corollary:

Corollary 2.1 *For any portfolio ω (not necessary on the MV boundary),* $\mathrm{Cov}(\omega^\tau r, \omega_{mvp}^\tau r) = 1/[CE(r_b^2)] - aK\mu/C$ *is a linear function of the mean.*

We turn to develop the impact of background risk on the two-fund separation theorem:

Proposition 2.3

1. *For any portfolio on the MV boundary, we have*

$$\omega \ =: \ \Big[\frac{(c\mu - a)a}{d} + \frac{a^2K(b - a\mu)E(r_b^2)}{d}\Big] \cdot \omega_d + \frac{C(b - a\mu)E(r_b^2)}{d} \cdot \omega_{mvp}.$$

   where $\omega_d = q_1$.

2. *Let ω_u and ω_v be two different portfolios on the MV boundary. Then, for any portfolio on the MV boundary ω, it can be represented as a convex combination of ω_u and ω_v.*

3 Adding a Risk-free Security

3.1 Adding a Risk-free Lending but No Borrowing

Assuming there is risk-free security with rate of return $r_f \geq 0$ at which agents can lend but cannot borrow, we let $W_f = \{(\omega, \omega_f) \in \mathbb{R}^n \times \mathbb{R} : \sum_{j=1}^n \omega_j + \omega_f = 1\}$ and the mean and variance of total return

$$r_p = (\omega_f r_f + \omega^\tau r) \cdot r_b \tag{3.1}$$

become $E(r_p) = (\omega_f r_f + \omega^\tau E(r)) \cdot E(r_b)$ and $\sigma^2(r_p) = \omega^\tau H \omega + 2\omega^\tau (E(r) - r_f I) r_f \mathrm{Var}(r_b) + r_f^2 \mathrm{Var}(r_b)$ where $H = VE(r_b^2) + (E(r) - r_f I)(E(r) - r_f I)^\tau \mathrm{Var}(r_b)$.

Assuming the tangency portfolio associated with the risk-free lending rate, denoted by w_1, lies above the minimum variance portfolio in the absence of the risk-free security, we obtain the following proposition:

Proposition 3.1 *Portfolio ω belongs to the MV boundary with multiplicative background risk if and only if*

$$\begin{cases} \frac{\sigma_p^2}{1/C} - \frac{(E(r_p)/E(r_b) - A/C)^2}{D/C^2} = 1 & \textit{if} \quad E(r_\omega) > E(r_{\omega_1}) \ , \\ \frac{\sigma_p^2}{a^*} - \frac{(E(r_p)/E(r_b) - (r_f - r_f \mathrm{Var}(r_b) J))^2}{Ja^*} = 1 & \textit{otherwise} \ , \end{cases} \tag{3.2}$$

where $a = I^\tau V^{-1} E(r), b = E(r)^\tau V^{-1} E(r),\ c = I^\tau V^{-1} I, j = b - 2r_f a + r_f^2 c,\ A = I^\tau H^{-1} E(r) = a/E(r_b^2) - K(a - r_f c)(b - r_f a)$, $B = E(r)^\tau H^{-1} E(r) = b/E(r_b^2) - K(b - r_f a)^2$, $C = I^\tau H^{-1} I = c/E(r_b^2) - K(a - r_f c)^2$, $J = B - 2r_f A + r_f^2 C = j/E(r_b^2) - KL$, $K = \mathrm{Var}(r_b) / \Big[E(r_b^2)\big(E(r_b^2) + j\mathrm{Var}(r_b)\big)\Big]$, $L = b^2 - 4abr_f + 2bcr_f^2 + 4a^2 r_f^2 - 4acr_f^3 - c^2 r_f^4$, *and* $a^* = r_f^2 \mathrm{Var}(r_b)(1 - \mathrm{Var}(r_b) J) > 0$.

3.2 Allowing for Both Risk-Free Lending and Borrowing

Assuming the borrowing rate r_{fb} is higher than the risk-free lending rate r_{fl}, supposing the set of portfolios with expected rates of returns is $W_f = \{(\omega, \omega_f) \in R^n \times R_+ \times R_- : \sum_{j=1}^n \omega_j + \omega_{fl} + \omega_{fb} = 1\}$, where ω_{fl} and ω_{fb} are the proportions of wealth lend and borrowed at r_{fl} and r_{fb}, we establish the following proposition:

Proposition 3.2 *Portfolio ω belongs to the MV boundary with both background risk and* *risk-free security if and only if*

$$\begin{cases} \frac{\sigma_p^2}{a_2^*} - \frac{(E(r_p)/E(r_b)-(r_{fb}-r_{fb}\mathrm{Var}(r_b)J_2))^2}{J_2 a_2^*} = 1 & \textit{if} \;\; E(r_\omega) > E(r_{\omega_2}) \;, \\ \frac{\sigma_p^2}{1/C} - \frac{(E(r_p)/E(r_b)-A/C)^2}{D/C^2} = 1 & \textit{if} \;\; E(r_{\omega_1}) < E(r_\omega) < E(r_{\omega_2}) \;, \\ \frac{\sigma_p^2}{a_1^*} - \frac{(E(r_p)/E(r_b)-(r_{fl}-r_{fl}\mathrm{Var}(r_b)J_1))^2}{J_1 a_1^*} = 1 & \textit{if} \;\; E(r_\omega) < E(r_{\omega_1}) \;, \end{cases}$$

in which a_i^* and J_i $(i = 1, 2)$ can be obtained from replacing with r_{fl} and r_{fb}, respectively, in the corresponding equations in Proposition 3.1.

4 Conclusion

In this paper, we study the impact of multiplicative background risk on the efficient portfolio selection in the MV framework. We also study the efficient boundary frontiers with and without risk-free security.

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