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Dynamics and equilibria under incremental horizontal differentiation on the Salop circle

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Abstract

We study product differentiation on a Salop circle when firms relocate incrementally due to bounded rationality. We prove that, under common assumptions on demand, firms relocate only when two or more firms target the same niche. In any other case, there is no incentive for any firm to relocate incrementally. We prove that all distributions in which firms are sufficiently far apart in product space are unstable Nash equilibria. We prove, in particular, that the classical equidistant distribution is an unstable Nash equilibrium that cannot emerge from another distribution. However, we show that if each firm is engaged in head-on rivalry with one other competitor, the industry converges to an 'equidistantesque' equilibrium of clusters of rivals.

Keywords: product differentiation, bounded rationality, Salop circle, equidistant equilibrium, maximum differentiation
JEL: L13, C73, L22

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1. Introduction

Firms differentiate their products to exploit differences in consumers' preferences. By producing a product in a niche that differs from the niches that competitors target, firms enjoy local monopolies. Product differentiation models are extensively applied to industry and service sectors of various sorts, used in policy studies, and extended upon in fundamental economic and economic geography research.
In a prominent body in product differentiation research literature, products are represented as locations in a low-dimensional space of product characteristics like the Hotelling line or the Salop circle. These models capture real-world situations like shops located on the main street or different sweetness of cider (Hotelling, 1929), shops located along the

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ring road (Gupta et al., 2004) or departure times of flights at airports (Borenstein and Netz, 1999). However, the findings on locations of firms in product space in equilibrium are also taken as indications of rational product-market divisions in more complex (real-life) situations.

In the classical two-stage game (pick location, then price or quantity), firms maximize their profit by maximum differentiation. Firms are then evenly spread across the Salop circle in a so-called equidistant equilibrium. However, in this classical game, firms act perfectly rational and are perfectly informed on location decisions of competitors. In this paper, we follow up on the call of Anderson et al. (1992) to study product differentiation under *bounded* rationality. We assume that firms have imperfect information on locations and strategies of competitors and display *boundedly* rather than *perfectly* rational (re)location behavior. We assume that, given the inherent incapabilities of poor information and uncertainties about competitors' moves, firms reposition *incrementally* over the product landscape following a myopic strategy. We study the dynamics and emerging equilibria in firm locations on the Salop landscape. In contrast to the classical findings, we find that the equidistant equilibria are in fact *unstable* and do not emerge dynamically. Moreover, we find that when firms are randomly scattered over the product space, even relocation is not very likely.

2. Literature

Product differentiation is a competitive positioning strategy that exploits consumer heterogeneity. If consumers differ in that they prefer different products and have different willingness to pay depending on the extent to which their preferences are met, firms can soften competition by providing a product to niches of consumers that are yet underserviced. Literature provides several models to study product differentiation equilibria (See Anderson et al. (1992) for a detailed discussion of the various streams). One stream of product differentiation models studies so-called location models that require a specification of both product features and consumer preferences in terms of a *location* in a product characteristics landscape. Particularly popular location models are the Hotelling line and Salop circle in which the product space is a line piece and a circle respectively. Consumers experience disutility from a mismatch of 'product specifications' with 'product preferences', so the models feature an 'attractiveness', 'utility', or 'travel cost' measure to reflect the fit of characteristics with preferences. The greater the distance from product (or firm) to preference (or consumer), the greater the consumption disutility or 'traveling costs'. Utility is generally assumed to decrease linearly (Hotelling, 1929) or quadratically (D'Aspremont et al., 1979; Perloff and Salop, 1985; Tabuchi and Thisse, 1995; Tyagi, 1999; Tirole, 1988). Common assumptions are that consumers maximize their utility (i.e. pick one of the products with the best fit) and that firms select product specifications that maximize profit given the product specifications selected by competitors. To assure that product specifications chosen by firms are not due to asymmetries in the consumer preference distribution, but purely due to strategic positioning vis-a-vis competitors, it is common to assume that all possible consumer preferences occur equally often. In both of these location models, researchers have studied whether and when the equilibrium is that of minimum or maximum differentiation or neither one (see Lerner and Singer, 1937; Graitson, 1982; Anderson et al., 1992; Economides, 1986; D'Aspremont et al., 1979; Böckem, 1994).

In the Salop model of product differentiation, consumers are distributed uniformly across the perimeter of a circle. Products (or firms offering the products) are positioned on the circle as well. Each period, each consumer purchases one of the products based on the attractiveness of the various products on offer. The attractiveness of a product to a consumer is a decreasing function of the arc distance from consumer to product. A common assumption is that demand is inelastic, i.e. all consumers buy a product every period (See e.g. Hotelling, 1929; D'Aspremont et al., 1979; Perloff and Salop, 1985). In the classical Salop circle model, firms pick a location on the circle (given firms already present and future entrants' location picking strategy) and a price to optimize their expected profit. The typical results are that the equidistant distribution of firms across the circle is a location-price equilibrium (Economides, 1989) (under inelastic demand with quadratic disutility), a location-quantity equilibrium (Pal, 1998; Yu, 2007) (under firm-borne transportation costs), and maximally entry deterring (Salop, 1979). Recently, Gupta et al. (2004) discovered a further wide range of non-equipriced non-equidistant Nash equilibria. In the models discussed, firms are perfectly rational and make an optimal location choice. Given that competitors are perfectly rational, a firm can anticipate the competitors' strategies in its own price and location decisions (see e.g. Häckner, 1995; Capozza and Order, 1980).

However, this perfect rationality is merely a 'normative model of an idealized decision maker, not a description of the behavior of real people' (Tversky and Kahneman, 1986, p.S251). People suffer *bounded* rationality and this hampers people in deciding optimally (Simon, 1955). Bounded rationality also affects managerial cognition and thereby (strategic) decisions (see e.g. Johnson and Hoopes, 2003). As such, management resorts to heuristics and routines (see e.g. Nelson and Winter, 1982).
We follow the call by Anderson et al. (1992) to study product differentiation under such *bounded* rationality. In this case, firms have imperfect information on locations and strategies of competitors and display *boundedly* rational (re)location behavior. Given that bounded rationality and competitive pressures have firms focus on immediate competitors (Johnson and Hoopes, 2003), relocation is expected to occur merely locally and -given the uncertainty about competitors' moves- incrementally. With such deviations from perfect rationality, we expect structurally different equilibria than the classical results (see e.g. Akerlof and Yellen, 1985).

3. Product differentiation model

In this paper, we study the Salop circular product differentiation model in which both (product preferences of) customers and (specifications of products made by) firms are associated with locations on the circle perimeter. We study the industry dynamics and equilibria when firms make boundedly rational (re)location decisions. We study a repeated two-stage game consisting of a sales round in which consumers buy a unit of product and a relocation round in which firm may pick a new location on the circle perimeter. Prior to the first game round, we place M firms uniform randomly on the circle. During the repeated game, firms do not enter or exit. We operationalize imperfect information by having firms not anticipate competitors relocation strategy. Firms relocate on sales prospects, which is affected by the locations of the other firms. To reflect imperfect information on the relocation decision of competitors, we assume firms (re)locate simultaneously. We operationalize bounded rationality by having firms not (re)locate somewhere on the circle freely, but by having them do so incrementally. Firms move in the clockwise or counterclockwise direction step by step through a myopic, ceteris paribus strategy. We assume that the steps taken by firms are of size $2\pi/N$, thereby dividing the Salop circular landscape in N discrete 'niches' (cf. Krugman, 1992; Huang and Levinson, 2007; Camacho-Cuena et al., 2005, for other discretized landscapes). Variable $\theta_i(t)$ is the location of firm i on the circle during period t. To study the industry dynamics and equilibrium outcome caused by the incremental nature of relocation and not mediated by other factors, we assume that firms charge the same price ('equal price at the mill') and have the same relocation strategy. We furthermore assume that consumers -rather than firms- suffer disutility from the distance to the firms ('consumers pay the travel costs to the mill'). To study firm relocation dynamics that is not affected by asymmetries in consumer distributions, we assume that consumers are uniformly distributed over the circle.

Each period starts with a sales round in which each consumer purchases a single unit of product, hereby sensitive to local attractiveness of products (related to arc distances to the firms and their prices). As is common in spatial competition models, we assume that consumers are uniformly distributed over the landscape, buy a single unit of product each period and are anchored to their location (i.e. their preferences do not change over time). We study the case that consumers maximize their utility. In period t, firm k thus sells:

$$s_k(t) = \sum_{1 \leq n \leq N} d_{kn}(t) \tag{1}$$

where d_{kn} is the demand realization of consumers at location n purchasing product k. We assume that sales and production equal demand.

In the relocation round, all of the M firms make a move simultaneously. The relocation step is dictated by a myopic strategy taking into account sales prospects without accounting for competitors' moves. The firm relocates into the direction that increases the expected sales most, ceteris paribus. As consumers are utility maximizers, each firm gets demand exclusively from consumers in niches up and until halfway to neighboring competitors. As consumers maximize their utility, they pick one of the nearest products. So, a consumer located at niche n simply picks the product k^* that is at minimum distance:

$$k^* = \arg\min_k \Delta(\theta_k, n) \tag{2}$$

The distance function $\Delta(\theta, n)$ on a *circular* landscape is the *minimum* number of steps, clockwise and counterclockwise, between niche n and niche θ. In analysis, we also use the distance function $\vec{\Delta}(\theta, n)$, which is the number of steps in clockwise direction (see Figure 1).

Since a consumer maximizes its utility (minimizes the travel distance), it picks the (product of) the firm nearest to it. As such, firms only compete directly with head-on rivals at the same niche and the nearest neighbor(s) in clockwise and the nearest neighbor(s) in counterclockwise direction. In case products[2] reside at the same distance to certain

[2]As firms have only one product, we use k and θ_k as index and location respectively for both the product and the firm.

consumers (or even at the same location), consumers have no further preference for one or the other and each of the products gets an equal fraction of the consumers. The thus deterministic demand d_{kn} from consumers at niche n for the product (of firm) k is:

$$d_{kn} := \begin{cases} |M^*(n)|^{-1} & \text{if } k \in M^*(n) \\ 0 & \text{else} \end{cases}$$

Without loss of generality, we assume that each niche contains only one consumer. The set $M(n, \delta) := \{1 \leq j \leq M \mid \Delta(\theta_j, n) = \delta\}$ is the set of firms at distance δ from n, and the set $M^*(n) := M(n, \Delta^*(n))$ with $\Delta^*(n) = \min_j \Delta(\theta_j, n)$ is the set of products (firms) at the minimum distance from n. Figure 2 gives an illustration of the demand curves. The height of the demand curve to the circle reflects the demand for that product in that particular niche.

4. Analysis of incremental differentiation behavior

We prove several basic lemmas on dynamics and emerging equilibria under boundedly rational relocation that show how most initial distributions are in fact already equilibria and prove that the classical equidistant equilibrium cannot emerge dynamically but has to be initialized as such. We furthermore show the existence of an 'equidistantesque' equilibrium of clusters of firms.

To explain firm behavior, we need two variables: the number $m(n)$ of firms residing at niche n (where these firms are then head-on rivals) and the distance Δ of our focal firm θ_k to the neighbor(s) in clockwise and counterclockwise direction. We call firms (and their products) 'head-on rivals' if they reside at the same niche. We say that one firm 'imitates' another firm if it moves into the niche of that other firm.

4.1. Conditions for relocation to occur

We prove that for any relocation to occur in an industry, some firms need to be already engaged in head-on rivalry or need to imitate another firm to then engage in head-on rivalry.

Suppose firm k is located at θ_k, while the nearest neighbors on either side are located at $\theta_L < \theta_k - 1$ and $\theta_R > \theta_k + 1$ (see the illustration in Figure 1). Note that since $\vec{\Delta}(\theta_L, \theta_k) \geq 2 \leq \vec{\Delta}(\theta_k, \theta_R)$, firm k cannot end up in niche θ_L or θ_R in one step.

From the range of niches $[\theta_L, \ldots \theta_k]$, firm k gets the consumers from the rightmost $(\vec{\Delta}(\theta_L, \theta_k) + 1)/2$ niches if $\vec{\Delta}(\theta_L, \theta_k)$ is odd, and the rightmost $\vec{\Delta}(\theta_L, \theta_k)/2$ niches plus a fraction $1/(1 + m(\theta_L))$ of the consumers in the middle niche if $\vec{\Delta}(\theta_L, \theta_k)$ is even. If the firms at θ_L and θ_R now remain at their location -as firm k assumes in its relocation consideration-, a step by firm k does change the number of consumers served in at most two niches. The actual change in number of consumers depends on the numbers $m(\theta_L)$ and $m(\theta_R)$ of firms in the niches of neighboring firms. While a single step by firm k may create more sales, any subsequent step (in either direction) ceteris paribus would lower the sales to the initial level. The following lemma formalizes that in a ceteris paribus situation, firm k would indeed move at most only once.

Lemma 1 (Relocate at most only once) Ceteris paribus, a firm located at θ_k at distance of at least two of other firms relocates at most only once. A necessary condition

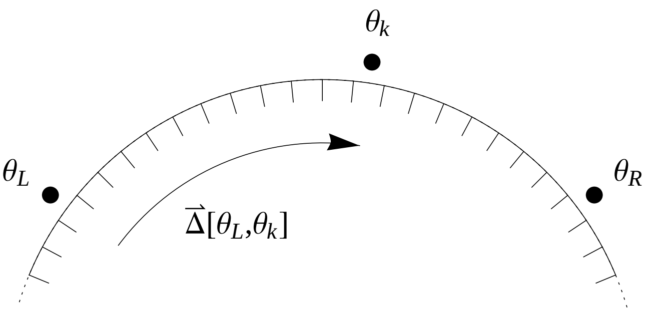


Figure 1: Illustration of firm locations θ_L, θ_k, θ_R and distance $\vec{\Delta}(\theta_L, \theta_k)$.

for a relocation is that $m(\theta_L) > 1$ or $m(\theta_R) > 1$, i.e. there is head-on rivalry in at least one of the niches θ_L or θ_R .

Proof: We study when firm k moves toward θ_R, which is the case when $s_k^+ > s_k$. Following the sales division insights described above, we find:

$$s_k = \frac{\vec{\Delta}(\theta_L, \theta_R)}{2} - 1 + \begin{cases} \frac{1}{m(\theta_L)+1} & \text{if } \vec{\Delta}(\theta_L, \theta_k) \text{ even} \\ \frac{1}{2} & \text{if } \vec{\Delta}(\theta_L, \theta_k) \text{ odd} \end{cases} + \begin{cases} \frac{1}{m(\theta_R)+1} & \text{if } \vec{\Delta}(\theta_k, \theta_R) \text{ even} \\ \frac{1}{2} & \text{if } \vec{\Delta}(\theta_k, \theta_R) \text{ odd} \end{cases}$$

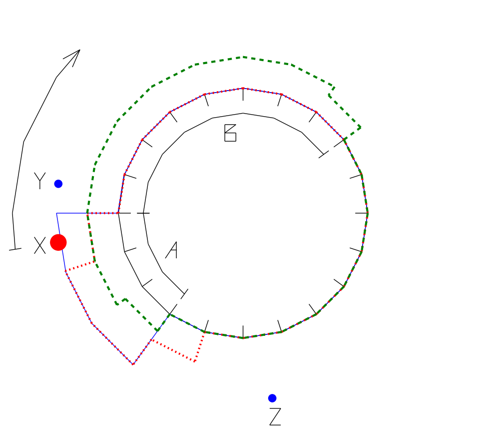
while

$$s_k^+ = \frac{\vec{\Delta}(\theta_L, \theta_R)}{2} - 1 + \begin{cases} \frac{1}{2} & \text{if } \vec{\Delta}(\theta_L, \theta_k) \text{ even} \\ \frac{1}{m(\theta_L)+1} & \text{if } \vec{\Delta}(\theta_L, \theta_k) \text{ odd} \end{cases} + \begin{cases} \frac{1}{2} & \text{if } \vec{\Delta}(\theta_k, \theta_R) \text{ even} \\ \frac{1}{m(\theta_R)+1} & \text{if } \vec{\Delta}(\theta_k, \theta_R) \text{ odd} \end{cases}$$

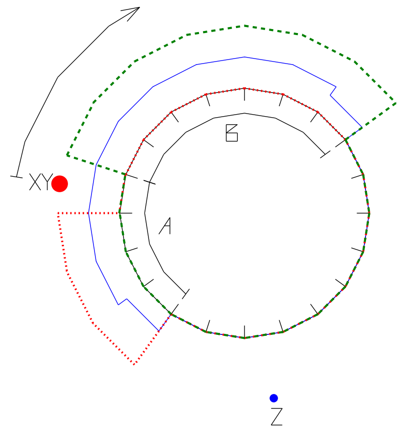
We find the following conditions under which firm k will move toward θ_R, i.e. $s_k^+ > s_k$:

Case	$s_k - \frac{\vec{\Delta}(\theta_L,\theta_R)}{2} + 1$	$s_k^+ - \frac{\vec{\Delta}(\theta_L,\theta_R)}{2} + 1$	When is $s_k^+ > s_k$
$\vec{\Delta}(\theta_L, \theta_k)$ even, $\vec{\Delta}(\theta_k, \theta_R)$ odd	$\frac{1}{m(\theta_L)+1} + \frac{1}{2}$	$\frac{1}{2} + \frac{1}{m(\theta_R)+1}$	$m(\theta_R) < m(\theta_L)$
$\vec{\Delta}(\theta_L, \theta_k)$ even, $\vec{\Delta}(\theta_k, \theta_R)$ even	$\frac{1}{m(\theta_L)+1} + \frac{1}{m(\theta_R)+1}$	$\frac{1}{2} + \frac{1}{2}$	$m(\theta_R) + m(\theta_L) > 2$
$\vec{\Delta}(\theta_L, \theta_k)$ odd, $\vec{\Delta}(\theta_k, \theta_R)$ even	$\frac{1}{2} + \frac{1}{m(\theta_R)+1}$	$\frac{1}{m(\theta_L)+1} + \frac{1}{2}$	$m(\theta_L) < m(\theta_R)$
$\vec{\Delta}(\theta_L, \theta_k)$ odd, $\vec{\Delta}(\theta_k, \theta_R)$ odd	$\frac{1}{2} + \frac{1}{2}$	$\frac{1}{m(\theta_L)+1} + \frac{1}{m(\theta_R)+1}$	Never

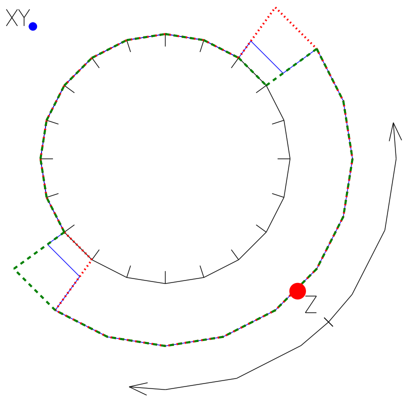
If a condition in the last column of the table is met (which requires head-on rivalry in either θ_L, θ_R or both), a move is made, thereby increasing θ_k with one. This flips the odds and evens, and with that the condition to be met to make a move in the same direction. Ceteris paribus, this last condition can never be met as it is the logical counterpart of the first condition. Ceteris paribus, after such a move, moving from $\theta_k + 1$ back to θ_k does not occur as $s^-(\theta_k + 1) = s(\theta_k) < s^+(\theta_k) = s(\theta_k + 1)$.
The analysis for a move toward θ_L is analogous and yields the same conclusions. □



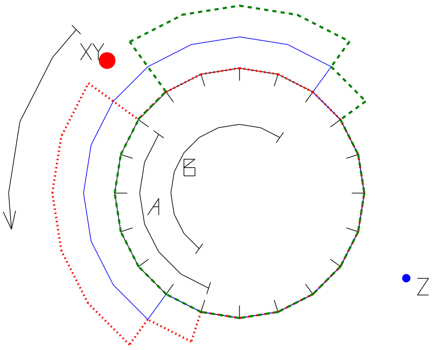
(a) Firm X will move to Y to share region A and B rather than to just monopolize region A



(b) Firms X and Y are co-located and move clockwise



(c) Any move will improve sales of firm Z



(d) Firms X and Y can improve after move of Z

Figure 2: Plots of simple demand curves d_k (continuous), $\mathbb{E}d_k^+$ (dashed) and $\mathbb{E}d_k^-$ (dotted) in case consumers are utility maximizers, here $M = 3$ and $N = 20$. The height of the curves to the circle reflects that demand.

Lemma 1 implies that to have more than one period in which firms relocate, it is a necessary condition that firms in different niches relocate; otherwise there would be relocation at most in the first period. We also see that for relocation to happen, some head-on rivalry is required. This head-on rivalry need not be initialized as such, as the relocation in the first period may also cause one firm to imitate another and thereby establish head-on rivalry. The following lemma concerns the formal conditions under which imitation occurs. We assume that $m(\theta_L), m(\theta_R) \leq 2$ to simplify further analysis.

Lemma 2 (Imitation) Let firms k and $k+1$ be located at niches $\theta_k = \theta_{k+1} - 1$ with $m(\theta_k) = m(\theta_{k+1}) = 1$. Hereby θ_L is the first niche counterclockwise of firm k and θ_R is the first niche clockwise of firm $k+1$ containing one or multiple firms.
Firm k $(k+1)$ located at θ_k (θ_{k+1}) will imitate the immediate neighbor located at θ_{k+1} (θ_k) if the distance from θ_{k+1} to θ_R (from θ_L to θ_k) is larger than the distance from θ_{k+1} to θ_L (from θ_k to θ_R). Firm k will not imitate $k+1$ if the distance is smaller.

Proof: We prove θ_k imitates θ_{k+1} if $\vec{\Delta}(\theta_{k+1}, \theta_R) > \vec{\Delta}(\theta_L, \theta_{k+1})$ by showing that $s_k < s_k^+$, such that firm k indeed relocates to niche θ_{k+1}. The proof for the counterpart (θ_{k+1} imitating θ_k) is analogous.
Define $\delta := \vec{\Delta}(\theta_L, \theta_{k+1}) - \vec{\Delta}(\theta_{k+1}, \theta_R)$, which is the difference in the number of niches counterclockwise and clockwise of θ_{k+1}. As $\vec{\Delta}(\theta_L, \theta_k) = \vec{\Delta}(\theta_{k+1}, \theta_R) - 1 + \delta$:

$$\begin{aligned} s_k &= \frac{\vec{\Delta}(\theta_L, \theta_k)}{2} + \begin{cases} \frac{1}{m(\theta_L)+1} & \text{if } \vec{\Delta}(\theta_L, \theta_k) \text{ even} \\ \frac{1}{2} & \text{if } \vec{\Delta}(\theta_L, \theta_k) \text{ odd} \end{cases} \\ &= \frac{1}{2}\vec{\Delta}(\theta_{k+1}, \theta_R) - \frac{1}{2} + \frac{\delta}{2} + \begin{cases} \frac{1}{m(\theta_L)+1} & \text{if } \vec{\Delta}(\theta_L, \theta_k) \text{ even} \\ \frac{1}{2} & \text{if } \vec{\Delta}(\theta_L, \theta_k) \text{ odd} \end{cases} \end{aligned}$$

As furthermore $\vec{\Delta}(\theta_L, \theta_R) = 2\vec{\Delta}(\theta_{k+1}, \theta_R) + \delta$, we know that if firm k at niche θ_k moves to niche $\theta_{k+1} = \theta_k + 1$, this would change sales for firm k to:

$$\begin{aligned} s_k^+ &= \frac{\vec{\Delta}(\theta_{k+1}, \theta_R)}{2} + \frac{\delta}{4} - \frac{1}{2} \\ &+ \begin{cases} \frac{1}{4} & \text{if } \vec{\Delta}(\theta_L, \theta_{k+1}) \text{ odd} \\ \frac{1}{m(\theta_L)+2} & \text{if } \vec{\Delta}(\theta_L, \theta_{k+1}) \text{ even} \end{cases} + \begin{cases} \frac{1}{4} & \text{if } \vec{\Delta}(\theta_{k+1}, \theta_R) \text{ odd} \\ \frac{1}{m(\theta_R)+2} & \text{if } \vec{\Delta}(\theta_{k+1}, \theta_R) \text{ even} \end{cases} \end{aligned}$$

Under the assumption that $m(\theta_L), m(\theta_R) \leq 2$:

$$\begin{aligned} s_k^+ - s_k = -\frac{\delta}{4} &+ \begin{cases} \frac{1}{4} - \frac{1}{m(\theta_L)+1} & \text{if } \vec{\Delta}(\theta_L, \theta_{k+1}) \text{ odd} \\ \frac{1}{m(\theta_L)+2} - \frac{1}{2} & \text{if } \vec{\Delta}(\theta_L, \theta_{k+1}) \text{ even} \end{cases} \\ &+ \begin{cases} \frac{1}{4} & \text{if } \vec{\Delta}(\theta_{k+1}, \theta_R) \text{ odd} \\ \frac{1}{m(\theta_R)+2} & \text{if } \vec{\Delta}(\theta_{k+1}, \theta_R) \text{ even} \end{cases} = \begin{cases} > 0 & \text{if } \delta < 0 \\ \geq 0 & \text{if } \delta = 0 \\ \leq 0 & \text{if } \delta > 0 \end{cases} \end{aligned}$$

So, if the number of niches clockwise of firm $k+1$ is higher than the the number of niches counterclockwise of him, firm k will imitate firm $k+1$. If the number is lower, it will not. $\square$

Remark 1 Lemma 2 concerns only cases when the number of niches counterclockwise of firm $k+1$ is different from the number of niches clockwise of firm $k+1$. Given that $s_k^+ - s_k \geq 0$ when $\delta = 0$, there are cases in which firm k imitates firm $k+1$ even if the number of niches counterclockwise is equal to the number of niches clockwise. After all, this depends on the number of firms at both θ_R and θ_L. As $\delta = 0$, the number counterclockwise and clockwise might be both odd, or both even. In the odd-odd case, $s_k^+ > s_k$ only if $-\delta/4 + (1/4 - 1/(m(\theta_L)+1)) + 1/4 > 0$, i.e. only if $m(\theta_L) > 1$. In the even-even case, $-\delta/4 + (1/(m(\theta_L)+2) - 1/2) + 1/(m(\theta_R)+2) > 0$, we see that $s_k^+ > s_k$ only if $1/(2+m(\theta_L)) + 1/(2+m(\theta_R)) > 1/2$.

We now know that one firm decides to imitate another if the number of niches on the other side of the immediate neighbor is higher or equal but with fewer remote competitors. A simultaneous imitation by two neighboring firms would have them trade places. However, close inspection of the conditions required for this reveals this is not possible. This is formalized in the following lemma.

Lemma 3 Imitation never has two firms trade places.

Proof: From remark 1 and lemma 2, we know that for firm k to imitate firm $k+1$, at least $\vec{\Delta}(\theta_{k+1}, \theta_R) \geq \vec{\Delta}(\theta_L, \theta_{k+1})$ (ignore further restrictions in the equality case). Similarly, for firm $k+1$ to imitate k, $\vec{\Delta}(\theta_L, \theta_k) \geq \vec{\Delta}(\theta_k, \theta_R)$. Suppose θ_k and θ_{k+1} are such that firm k and firm $k+1$ trade places, then:

$$\vec{\Delta}(\theta_k, \theta_R) - 1 = \vec{\Delta}(\theta_{k+1}, \theta_R) \geq \vec{\Delta}(\theta_L, \theta_{k+1}) = \vec{\Delta}(\theta_L, \theta_k) + 1 \geq \vec{\Delta}(\theta_k, \theta_R) + 1$$

This is a contradiction. There are not θ_k and θ_{k+1} such that firms k and $k+1$ trade places. □

An illustration of imitation is given in figure 2a. Firms Y and Z cannot improve sales, but firm X can do so by moving toward the niche of Y. Firm X gives up its monopoly in region A with three full niches, to then share region A and region B with firm Y and increase sales to 9 half niches and 2 one-third niches.

4.2. Head-on rivals and equidistantesque equilibria

If firms are engaged in head-on rivalry or get engaged in head-on rivalry through imitation, multiple steps of relocation may follow. If firms are head-on rivals, they both face the same industry conditions and take the same decisions. As they do not take into account the strategic interaction with the head-on rival, they end up relocating in the same way toward or away from the nearest neighboring firms. In fact, in moving away form his head-on rival, each of the rivals moves away from the nearest neighbor(s) to divide up the niches between him and move toward the immediate neighbor(s) furthest away. The following lemma formalizes that the two rivals thus move toward the middle of the nearest niches with neighbors on either side.

Lemma 4 (Head-on rivalry drives differentiation from nearest neighbors) Two firms k and $k+1$ rivaling at the same niche θ move toward the niche(s) in the middle of the niches θ_L ($\theta_L + 1 < \theta$) and θ_R ($\theta_R - 1 > \theta$) with the nearest competitors in the counterclockwise and clockwise direction.

If the distance from θ_L to θ_R is odd, there are two middle niches. If the two rival firms are in one of these two niches, the two rival firms jump from one to the other middle niche if $m(\theta_L) = m(\theta_R)$, i.e. the number of competitors at θ_L and θ_R are equal. If $m(\theta_L) \neq m(\theta_R)$, the two rivals end up in the right or left one of the two middle niches, closer to θ_L if $m(\theta_L) < m(\theta_R)$ and closer to θ_R if $m(\theta_L) > m(\theta_R)$.

Proof: Firms k and $k+1$ are head-on rivals, so both have sales s. Suppose that θ is closer to θ_L than to θ_R, we then prove that the sales upon moving clockwise s^+ is larger than the current sales s and larger than s^- to establish that both firms k and $k+1$ indeed move in direction of the middle niche(s). The analysis for the counterpart in which θ is closer to θ_R is analogous.
Note that:

$$s = \frac{\vec{\Delta}(\theta_L, \theta_R)}{4} - \frac{1}{2} + \begin{cases} \frac{1}{m(\theta_L)+2} & \text{if } \vec{\Delta}(\theta_L, \theta) \text{ even} \\ \frac{1}{4} & \text{if } \vec{\Delta}(\theta_L, \theta) \text{ odd} \end{cases} + \begin{cases} \frac{1}{m(\theta_R)+2} & \text{if } \vec{\Delta}(\theta, \theta_R) \text{ even} \\ \frac{1}{4} & \text{if } \vec{\Delta}(\theta, \theta_R) \text{ odd} \end{cases}$$

The sales s^+ each of the firms naively expects to get upon a move to niche $\theta + 1$ is:

$$s^+ = \frac{\vec{\Delta}(\theta, \theta_R)}{2} + \begin{cases} -\frac{1}{2} + \frac{1}{m(\theta_R)+1} & \text{if } \vec{\Delta}(\theta+1, \theta_R) \text{ even} \\ 0 & \text{if } \vec{\Delta}(\theta+1, \theta_R) \text{ odd} \end{cases}$$

Firstly, we show that $s^- < s^+$, hence that firms k and $k+1$ both prefer moving in the clockwise direction over moving in the counterclockwise direction. The sales s^- upon a move to niche $\theta - 1$ is:

$$s^- = \frac{\vec{\Delta}(\theta_L, \theta)}{2} + \begin{cases} -\frac{1}{2} + \frac{1}{m(\theta_L)+1} & \text{if } \vec{\Delta}(\theta_L, \theta) \text{ odd} \\ 0 & \text{if } \vec{\Delta}(\theta_L, \theta) \text{ even} \end{cases}$$

By assumption $\vec{\Delta}(\theta_L, \theta) < \vec{\Delta}(\theta, \theta_R)$. As $m(\theta_L) \geq 1$, we find:

$$s^+ \geq \frac{\vec{\Delta}(\theta, \theta_R)}{2} - \frac{1}{2} \geq \frac{\vec{\Delta}(\theta_L, \theta)}{2} + \frac{1}{2} > \frac{\vec{\Delta}(\theta_L, \theta)}{2} \geq s^-$$

Secondly, we show that $s < s^+$, hence that firms k and $k+1$. Using $\vec{\Delta}(\theta_L, \theta) < \vec{\Delta}(\theta, \theta_R)$, there is some $\delta \in \mathbb{N}$ for which:

$$\vec{\Delta}(\theta, \theta_R) = \vec{\Delta}(\theta_L, \theta_R)/2 + \begin{cases} \frac{1}{2} + \delta & \text{if } \vec{\Delta}(\theta_L, \theta_R) \text{ odd} \\ 1 + \delta & \text{if } \vec{\Delta}(\theta_L, \theta_R) \text{ even} \end{cases}$$

We use this to rewrite s^+ and find:

Case	s	s^+
$\vec{\Delta}(\theta_L, \theta)$ odd, $\vec{\Delta}(\theta, \theta_R)$ odd	$\vec{\Delta}(\theta_L, \theta_R)/4$	$\vec{\Delta}(\theta, \theta_R)/2 - \frac{1}{2} + \frac{1}{m(\theta_R)+1} = \vec{\Delta}(\theta_L, \theta_R)/4 + \frac{\delta}{2} + \frac{1}{m(\theta_R)+1}$
$\vec{\Delta}(\theta_L, \theta)$ odd, $\vec{\Delta}(\theta, \theta_R)$ even	$\vec{\Delta}(\theta_L, \theta_R)/4 - \frac{1}{4} + \frac{1}{m(\theta_R)+2}$	$\vec{\Delta}(\theta, \theta_R)/2 = \vec{\Delta}(\theta_L, \theta_R)/4 + \frac{1}{4} + \frac{\delta}{2}$
$\vec{\Delta}(\theta_L, \theta)$ even, $\vec{\Delta}(\theta, \theta_R)$ even	$\vec{\Delta}(\theta_L, \theta_R)/2 - \frac{1}{2} + \frac{1}{m(\theta_L)+2} + \frac{1}{m(\theta_R)+2}$	$\vec{\Delta}(\theta, \theta_R)/2 = \vec{\Delta}(\theta_L, \theta_R)/4 + \frac{1}{2} + \frac{\delta}{2}$
$\vec{\Delta}(\theta_L, \theta)$ even, $\vec{\Delta}(\theta, \theta_R)$ odd	$\vec{\Delta}(\theta_L, \theta_R)/4 - \frac{1}{4} + \frac{1}{m(\theta_L)+2}$	$\vec{\Delta}(\theta, \theta_R)/2 - \frac{1}{2} + \frac{1}{m(\theta_R)+1} = \vec{\Delta}(\theta_L, \theta_R)/4 - \frac{1}{4} + \frac{\delta}{2} + \frac{1}{m(\theta_R)+1}$

For the first three cases, the inequality $s < s^+$ clearly holds. In the last case, the inequality only holds if $\delta > 0$, i.e. if firms k and $k+1$ are far enough from the middle two niches. This proves the first part of the lemma.
For the second part of the lemma, about when firms are already in one of the two middle niches, we use the derived results in the table. If $\delta = 0$, firms k and $k+1$ are in the left one of the two middle niches. If $m(\theta_L) \leq m(\theta_R)$, firm k (and $k+1$) moves toward θ_L (but stay otherwise). If firm k and $k+1$ move toward the right one of the two middle niches, we get into the counterpart case where θ is closer to θ_R. In that case, whether or not $s^- > s$ determines whether or not the two firms move back to the left one of the two middle niches. This is the case if $m(\theta_R) \leq m(\theta_L)$. Consequently, if $m(\theta_L) = m(\theta_R)$, the two firms jump to the other of the two middle niches. In any other case, ceteris paribus, they stay in either the left or the right one of the two middle niches. Given that the jump will be made toward the niche θ_R or θ_L with the lowest number of firms, the head-on rivals end up, ceteris paribus, on the left or right one of the two middle niches closer to the niche with the fewest competitors. $\square$

Note that by assuming $\theta_L < \theta - 1$ and $\theta_R > \theta + 1$, we need not evaluate the case that the rivaling firms consider imitating the firms at θ_L or θ_R. In case of imitation, we also need information on the firms and number of niches clockwise of θ_R or counterclockwise of θ_L.
In real cases, the neighboring competitors in niches θ_L or θ_R may also decide to relocate. This would change the situation for the two rivals, but not the actual rule that they move toward the niches in the middle of θ_L and θ_R.

An illustration for $M = 3$ of the dynamics is given in Figure 2. In Figure 2a, we see that firm X imitates firm Y as it prefers sharing its currently monopolized region A to also share region B, in line with lemma 2. As soon as firm X and Y share the location, they have the same incentives vis-a-vis consumers and (other) competitors, so, in line with lemma 4, differentiate from firm Z. Due to the number of niches being even, there is a dynamic equilibrium. Firm Z can improve sales by moving either clockwise or counterclockwise (see Figure 2c), after which firm X and Y can again improve their sales by moving in the same direction (see Figure 2d).

According to Lemma 4, head-on rivals are engaged in relocating to the middle of their immediate neighbors. If *each* firm is engaged in head-on rivalry, each firm displays 'middle seeking' behavior. If all firms are engaged in head-on rivalry with only one other firm, and these 'clusters of rivals' are at distance more than one of each other, these clusters all seek the middle of the immediately neighboring clusters. So, under these conditions the industry converges to equidistantly distributed clusters. However, in general, this does not yield a *static* equilibrium with all the inter-cluster distances equal to $N/(2M)$ even if $N/(2M) \in \mathbb{N}$. Often, this yields a *dynamic* equilibrium. As relocation by multiple firms at once reset industry conditions for the other in the next round, the clusters might relocate in opposite directions, thereby resetting the condition for one another to revert the step just taken. This then gives rise to a dynamic equilibrium with a periodicity of two. With $N = 20$ and $M = 8$ (4 clusters of 2 rivals), an example of this is the alternation between inter-cluster distances $(6, 4, 6, 4)$ and $(4, 6, 4, 6)$. Another example is the alternation between inter-cluster distances $(6, 3, 6, 5)$ and $(4, 5, 4, 7)$. We refer to such emerging (dynamic) equilibria as 'equidistantesque'; the clusters of rivals seek maximum differentiation (equidistance).

4.3. Equidistant equilibrium

While an 'equidistantesque' equilibrium emerges under the specific conditions that each firm engages in head-on rivalry, in the absence of head-on rivalry or imitation, there is no relocation, as we show in this subsection. For further analysis, we study cases with $m(\theta_j) = 1$ for all j and in which firms are still at a distance of at least two to other firms. In that case, the table of conditions for a singular move presented in the proof of Lemma 1 can be used immediately to establish that this situation is a Nash equilibrium. After all, it is obvious that moving one step to either side, moves this firm *toward* one of his single neighbors and *away* from the other immediate single neighbor. This move thereby *decreases* the number of niches served by these two firms by half a niche on one side, and *increases* the number of niches served on the other side with an equal number, so the net increase in sales is zero and there is no incentive to move. The following lemma formalizes this result.

Lemma 5 Each distribution of firms over the circle with distance between two firms no less than two niches is a Nash equilibrium.

Proof: The necessary condition in lemma 1 is not met. Using the table in lemma 1 with $m(\theta_L) = m(\theta_R) = 1$, we see that none of the conditions for $s_k^+ > s_k$ is ever met. The same holds for s_k^-. As the conditions are similar for each of the firms, no firm has a unilateral incentive to move. □

So, in our model, there is no incentive for a firm to relocate if there are no head-on rival(s) and if no firm will imitate a competitor. As firms relocate only under specific conditions, the classical equidistant equilibrium found in literature is very rare. The exact condition under which such an equilibrium occurs is formalized in the following theorem.

Theorem 1 For $\frac{N}{M} \in \mathbb{N}^+$, an equidistant equilibrium emerges only if the initial distribution is equidistant.

Proof: If condition $\frac{N}{M} \in \mathbb{N}^+$ is not met, there is no equal distance between firms, so there is no *equidistant* equilibrium. According to Lemma 5, an equidistant distribution is a (Nash) equilibrium, if it occurs. Clearly, once $m(\theta_k) > 1$, the firms residing at θ_k have similar interests and will also act similarly. So, equidistance does not emerge once there is head-on rivalry. Consequently, there may not be head-on rivalry and conditions should not be so that imitation occurs. Using lemma 2 with $m(\theta_L) = m(\theta_R) = 1$, we know that there either is imitation if two firms are in consecutive niches or they do not move at all (which further stifles convergence to equidistance). So, firms should not be in consecutive niches or in the same niche. Furthermore, one of the firms should not have an incentive to relocate immediately next to another firm. From lemma 1, we know that a necessary but not sufficient condition is that $m(\theta_L) > 1$ or $m(\theta_R) > 1$ for relocation to occur. In case $m(\theta_L) = m(\theta_R) = 1$, none of the conditions for a relocation in the table of the proof of lemma 1 is ever met.
So, as the initial industry conditions must be such that there is no head-on rivalry or imitation, there are no incentives to move at all and an equidistant equilibrium does never *dynamically* emerge, but must be initialized as such. □

The probability that an equidistant equilibrium is installed when firms' initial locations are drawn uniform randomly is $N^{-M+1}(M-1)!$. For large N and small M, this probability is very small and rapidly decreasing in N and M. Note that the industry does not converge back to an equidistant distribution upon any incremental perturbation of the firm away from that equidistant distribution (but without violating the conditions of Lemma 5). The equidistant equilibrium hence even is an unstable Nash equilibrium. Note that even a single step of relocation is not very likely to occur: it may happen only if at least some firms are head-on rivals or imitate another. When firms are randomly placed on the circle one by one, the probability that at least one firm is placed in the same niche or in a niche immediately next to another firm is approximately $N^{-M}\Pi_{i=1}^{M-1}(N-3i)$. This probability becomes ever more accurate with increasing N. This expression reveals that the probability that at least one firm relocates after initialization is small whenever N is large and M is small.

5. Conclusions and further research

In product differentiation literature, authors commonly (implicitly) assume firms are perfectly rational, endowed with prior information on competitors' strategies and as such capable of positioning and pricing their product to maximize profit. In game-theoretic approaches to product differentiation, the equidistant distribution of firms on the one-dimensional circular product characteristic landscape often emerges as the ultimate outcome in which firms cannot improve profit unilaterally by relocating or changing price.
In our research, we answered a call by Anderson et al. (1992) to study product differentiation under *bounded* rationality. We operationalized this by having firms relocate *incrementally* purely on ceteris paribus sales prospects, without anticipating moves by competitors. We show that when consumers are utility maximizers, their demand is inelastic and firms charge uniform prices, the distribution of firms changes only if two or more firms (are) engage(d) in head-on rivalry. In any other case, there is no incentive for any firm to relocate incrementally. An equidistant distribution thus is an unstable Nash equilibrium as it can and does not develop from another industry state. In fact, *all* distributions in which firms have a distance > 1 to other firms are (unstable) Nash equilibria. Consequently, the random scattering of firms at the onset of the mature phase is also often the emerging static equilibrium.
Arguably, the completely free relocation strategy in the perfectly rational case and the incremental relocation strategy in the boundedly rational case are both extreme cases. Further research should shed light on what type of information on competitors' relocation strategy is known and what realistic entry location and relocation strategies are followed. Another interesting avenue for further research is the type of dynamics and equilibria when the utility maximization of consumers is relaxed.

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