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Abstract

This paper examines the socially optimal allocation by focusing not on the social welfare function but instead on the utility possibility frontier in exogenous growth models with a heterogeneous population. A unique balanced growth path was found on which all of the optimality conditions of all heterogeneous households are equally and indefinitely satisfied (sustainable heterogeneity). With appropriate government interventions, such a path is always achievable and is uniquely socially optimal for almost all generally usable (i.e., preferences are complete, transitive, and continuous) social welfare functions. The only exceptions are some variants in Nietzsche type social welfare functions, but those types of welfare functions will rarely be adopted in democratic societies. This result indicates that it is no longer necessary to specify the shape of the social welfare function to determine the socially optimal growth path in a heterogeneous population.

JEL Classification code: D63, D64, E20, F40, I31, I38, O41
Keywords: Sustainability; Heterogeneity; Social Optimality; Social welfare; Social welfare function; Inequality; Evolution

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1 INTRODUCTION

Problems of economic inequality, wealth disparity, and justice have long been central issues in economics and are again a hot topic in the midst of the great recession that began in 2008. The concerns of the Occupy Wall Street movement are a recent example. However, the criteria for socially optimal allocation have not been universally agreed upon because of utility's interpersonal incomparability, Arrow's general possibility theorem (Arrow, 1951), and other factors. Although the problem of utility's interpersonal incomparability was solved by Bergson (1938) and Samuelson (1947), their idea was fundamentally criticized by Arrow (1951). Arrow's criticism can be worked around if the assumptions in Arrow (1951) are modified, for example, the assumption that every individual has a single-peaked preference is added (see e.g., Black, 1958); thus, social welfare functions can be used for various analyses. Nevertheless, even if social welfare functions can be used, there is no consensus on their shape. Because of this limitation, it has been difficult to provide useful information for arguments of social optimality. Even though many people have protested that current levels of economic inequality and wealth disparity are too large, there is no theoretical basis on which to judge their arguments.

To shed light on the arguments, I take a different approach in this paper. I focus not on the nature of the social welfare function but instead on the nature of the utility possibility frontier, because if the shape of the utility possibility frontier has some special characteristics, particularly if it is very constrained by some factors, it may be able to narrow the opportunities for a socially optimal allocation, regardless of any differences in the social welfare functions.

In particular, this paper examines social optimality in dynamic models with a heterogeneous population and the condition for the state where all of the optimality conditions of all heterogeneous households are satisfied in these models. Intuitively, knowing whether the state where all of the optimality conditions of all heterogeneous households are satisfied is achieved seems to provide useful information for social optimality, but it is meaningless if we use static models because any competitive equilibrium naturally and always achieves this state even if the population is heterogeneous. It is also meaningless when dynamic models are used if the models use homogeneous populations, because such a state is naturally and always achieved and a homogeneous population generates no income inequality or wealth differential. Thus, the only remaining type of model to study is a dynamic model with a heterogeneous population. However, Becker (1980) showed that, in such models, the magnitudes of income inequality and wealth disparity eventually reach the limit; that is, the most patient household eventually will own all capital. All of the other households cannot satisfy their optimality conditions and will go bankrupt and, as it were, perish when even a very small negative shock occurs unless the authority intervenes. Consequently, examining social optimality in a heterogeneous population by using dynamic models has been regarded to be a meaningless task. As a result, little attention has been paid in the analyses of social optimality to the state where all of the optimality conditions of all heterogeneous households are satisfied.

Harashima (2010, 2012) shows that, in dynamic models with a heterogeneous population, there exists a state where all of the optimality conditions of all heterogeneous households are satisfied (i.e., "sustainable heterogeneity"), although this state is not guaranteed to be naturally and always achieved, and it is influenced by the behavior of the most advantaged household. Even though it is not naturally achievable, it can be always achieved with appropriate government intervention. The existence of this state is very important because, unlike the case with static and dynamic models with homogeneous populations, we can obtain additional meaningful and useful information about social optimality. Dynamic models with a heterogeneous population have another advantage—they describe the nature of economy far more realistically than static and dynamic models with homogeneous populations. Because little attention has been given to sustainable heterogeneity in analyses of social optimality, discoveries derived from such analyses add a new analytical tool and may help solve the previously discussed problem of the unspecifiability of social optimality. In this paper, the endogenous growth model in Harashima (2012) is modified to an exogenous growth model (a Ramsey type growth model), and social optimality is examined based on this modified model in the same manner as Harashima (2012).

A distinct feature of the models presented in this paper and Harashima (2012) is that a common nature of utility across the population is assumed to exist as a result of human evolution. Although utility functions are different across a population, some common features have been assumed, for example, a diminishing marginal rate of substitution. In this paper, an additional common nature is assumed such that extreme disutility is generated if all of the optimality conditions are not satisfied. The reason for this assumption, as described in more detail in Section 5, is that only humans who have this nature could have survived the process of natural selection. This additional common nature of utility plays an important role in the analyses of social optimality presented in this paper.

The model shows that sustainable heterogeneity is the unique socially optimal allocation for almost all generally usable (i.e., preferences are complete, transitive, and continuous) social welfare functions. This result is very important because the socially optimal allocation is uniquely determined without having to specify the shape of the social welfare function. This result therefore implies that, with the additional information provided by sustainable heterogeneity in dynamic models with a heterogeneous population, the problem of unspecifiability of social optimality can be solved.

The paper is organized as follows. In Section 2, a multi-economy endogenous growth model with heterogeneous population is constructed, and sustainability of heterogeneity is examined by using it. The existence of a unique balanced growth path on which all optimality conditions of all heterogeneous households are satisfied is shown. In Section 3, the endogenous model is degenerated to an exogenous growth model. The similar results as the endogenous growth model are obtained. Section 4 shows that sustainable heterogeneity is always achievable with appropriate government intervention even if the most advantaged household behaves unilaterally. In Section 5, extreme disutility to unsustainable heterogeneity is examined based on the gene theory of evolution. Section 6 introduces a utility possibility frontier and social welfare function modified to dynamic models and shows that sustainable heterogeneity represents the unique socially optimal allocation. Finally, some concluding remarks are offered in Section 7.

2 SUSTAINABLE HETEROGENEITY IN AN ENDOGENOUS GROWTH MODEL

2.1 The model

2.1.1 The base model

2.1.1.1 Production of technologies

Outputs Y_t are the sum of consumption C_t, the increase in capital, and the increase in technology such that

$$Y_t = C_t + \dot{K}_t + v\dot{A}_t \quad .$$

Thus,

$$\dot{k}_t = y_t - c_t - \frac{v\dot{A}_t}{L_t} - n_t k_t \quad ,$$

where $v(>0)$ is a constant, and a unit of K_t and v^{-1} of a unit of A_t are equivalent; that is, they are produced using the same quantities of inputs (capital, labor, and technology). This means that technologies are produced with capital, labor, and technology in the same way as consumer goods and services and capital. Unlike most idea-based growth models, no special mechanism is required for the production of technology because endogenous balanced growth (i.e., constant $\frac{A_t}{k_t}$) is not materialized by any special property of the production function of technology but by uncompensated knowledge spillovers and arbitrage between investments in capital and technology.

Because balanced growth paths are the focal point of this paper, Harrod-neutral technical progress is assumed.[1] Hence, the production function is $Y_t = K_t^{1-\alpha}(A_t L_t)^{\alpha}$; thus,

$$y_t = A_t^{\alpha} k_t^{1-\alpha} \quad .$$

It is assumed for simplicity that the population growth rate (n_t) is constant and not negative such that $n_t = n \geq 0$.

2.1.1.2 Substitution between investments in K_t and A_t

For any period,

$$m = \frac{M_t}{L_t} \quad , \tag{1}$$

where M_t is the number of firms (which are assumed to be identical) and m (> 0) is a constant. Equation (1) presents a natural assumption that the population and number of firms are proportional to each other. Equation (1) therefore indicates that any firm consists of the same number of employee regardless of L_t. Note that, unlike the arguments in Young (1998), Peretto (1998), Aghion and Howitt (1998), and Dinopoulos and Thompson (1998), M_t is not implicitly assumed to be proportional to the number of sectors or researchers in the economy (see also Jones, 1999). Equation (1) merely indicates that the average number of employees per firm in an economy is independent of the population. Hence, M_t is not essential for the amount of production of A_t. As will be shown by equations (2) and (3), production of A_t does not depend on the number of researchers but on investments in technology. In contrast, M_t plays an important role in the amount of uncompensated knowledge spillovers.

The constant m implicitly indicates that the size of a firm is, on average, unchanged even if the population increases. This assumption can be justified by Coase (1937) who argued that the size of a firm is limited by the overload of administrative information. In addition, Williamson (1967) argued that there can be efficiency losses in larger firms (see also Grossman and Hart, 1986 and Moore, 1992). Their arguments equally imply that there is an optimal firm size that is determined by factors that are basically independent of population.

Next, for any period,

$$\frac{\partial Y_t}{\partial K_t} = \frac{\varpi}{M_t^{1-\rho}} \frac{\partial Y_t}{\partial (v A_t)} \quad ; \tag{2}$$

[1] As is well known, only Harrod-neutral technological progress matches the stylized facts presented by Kaldor (1961). As Barro and Sala-i-Martin (1995) argue, technological progress must take the labor-augmenting form in the production function if the models are to display a steady state.

thus,

$$\frac{\partial y_t}{\partial k_t} = \frac{\varpi L_t^{\rho}}{m^{1-\rho} v} \frac{\partial y_t}{\partial A_t} \tag{3}$$

is always kept, where $\varpi(>1)$ and $\rho(0 \le \rho < 1)$ are constants. The parameter ρ describes the effect of uncompensated knowledge spillovers, and the parameter ϖ indicates the effect of patent protection. With patents, incomes are distributed not only to capital and labor but also to technology. For simplicity, the patent period is assumed to be indefinite, and no capital depreciation is assumed.

Equations (2) and (3) indicate that returns on investing in capital and technology for the investing firm are kept equal. The driving force behind the equations is that firms exploit all opportunities and select the most profitable investments at all times. Through arbitrage, this behavior leads to equal returns on investments in capital and technology. With substitution between investments in capital and technology, the model exhibits endogenous balanced growth. Because $\frac{\varpi}{mv} \frac{\partial y_t}{\partial A_t} = \frac{\partial y_t}{\partial k_t} \Leftrightarrow \frac{\varpi L_t^{\rho} \alpha}{m^{1-\rho} v} A_t^{\alpha-1} k_t^{1-\alpha} = (1-\alpha) A_t^{\alpha} k_t^{-\alpha}$, $A_t = \frac{\varpi L_t^{\rho} \alpha}{m^{1-\rho} v (1-\alpha)} k_t$ by equations (1) and (2), which lucidly indicates that $\frac{A_t}{k_t}$ = constant, and the model can therefore show balanced endogenous growth.

2.1.1.3 Uncompensated knowledge spillovers

Equations (2) and (3) also indicate that the investing firm cannot obtain all of the returns on its investment in technology. That is, although investment in technology increases Y_t, the investing firm's returns are only a fraction of the increase in Y_t, such that $\frac{\varpi}{M_t^{1-\rho}} \frac{\partial Y_t}{\partial (v A_t)}$, because knowledge spills over to other firms without compensation and other firms possess complementary technologies.

Broadly speaking, there are two types of uncompensated knowledge spillovers: intra-sectoral knowledge spillovers (MAR externalities: Marshall, 1890; Arrow, 1962; Romer, 1986) and inter-sectoral knowledge spillovers (Jacobs externalities: Jacobs, 1969). MAR theory assumes that knowledge spillovers between homogenous firms are the most effective and that spillovers will primarily emerge within sectors. As a result, uncompensated knowledge spillovers will be more active if the number of firms within a sector is larger. On the other hand, Jacobs (1969) argues that knowledge spillovers are most effective among firms that practice different activities and that diversification (i.e., a variety of sectors) is more important in influencing spillovers. As a result, uncompensated knowledge spillovers will be more active if the number of sectors in the economy is larger. If all sectors have the same number of firms, an increase in the number of firms in the economy results in more knowledge spillovers in any case, as a result of either MAR or Jacobs externalities.

As uncompensated knowledge spillovers increase, the investing firm's returns on investment in technology decrease. $\frac{\partial Y_t}{\partial A_t}$ indicates the total increase in Y_t in the economy by an increase in A_t, which consists of increases in both outputs of the firm that invested in the new technologies and outputs of other firms that utilize the newly invented technologies, regardless of whether the firms obtained the technologies by compensating the originating firm or through uncompensated knowledge spillovers. If the number of firms increases and uncompensated knowledge spillovers increase, the compensated fraction in $\frac{\partial Y_t}{\partial A_t}$ that the investing firm can obtain becomes smaller, as do its returns on the investment in technology. The parameter ρ describes the magnitude of this effect. If $\rho = 0$, the investing firm's returns are reduced at the same rate as the increase of the number of firms. $0 < \rho < 1$ indicates that the investing firm's returns diminish as the number of firms increase but not to the same extent as when $\rho = 0$.

Both types of externalities predict that uncompensated knowledge spillovers will increase as the number of firms increases, and scale effects have not actually been observed (Jones, 1995a), which implies that scale effects are almost canceled out by the effects of MAR and Jacobs externalities. Thus, the value of ρ is quite likely to be very small. From the point of view of a firm's behavior, a very small ρ appears to be quite natural. Because firms intrinsically seek profit opportunities, newly established firms work as hard as existing firms to profit from knowledge spillovers. An increase in the number of firms therefore indicates that more firms are trying to obtain the investing firm's technologies.

Because of the non-rivalness of technology, all firms can equally benefit from uncompensated knowledge spillovers, regardless of the number of firms. Because the size of firms is independent of population and thus constant as argued in Section 2.1.1.2, each firm's ability to utilize the knowledge that has spilled over from each of the other firms will not be reduced by an increase in population. In addition, competition over technologies will increase as the number of firms increases, and any firm will completely exploit all opportunities to utilize uncompensated knowledge spillovers as competition increases.[2] Hence, it is quite likely that the probability that a firm can utilize a unit of new technologies developed by each of the other firms without compensation will be kept constant even if the population and the number of firms increase. As a result, uncompensated knowledge spillovers will increase eventually to the point that they increase at the same rate as the increase in the number of firms.

The investing firm's fraction of $\frac{\partial Y_t}{\partial A_t}$ that it can obtain will thereby be reduced at the same rate as the increase in the number of firms, which means that ρ will naturally decrease to zero as a result of firms' profit-seeking behavior. Based on $\rho = 0$,

$$\frac{\partial Y_t}{\partial K_t} = \frac{\varpi}{M_t} \frac{\partial Y_t}{\partial (v A_t)} \tag{4}$$

by equations (2) and (3); thus,

$$\frac{\partial y_t}{\partial k_t} = \frac{\varpi}{m v} \frac{\partial y_t}{\partial A_t} \tag{5}$$

is always maintained.

Complementary technologies also reduce the fraction of $\frac{\partial Y_t}{\partial A_t}$ that the investing firm can obtain. If a new technology is effective only if it is combined with other technologies, the returns on investment in the new technology will belong not only to the investing firm but also to the firms that possess the other technologies. For example, an innovation in computer software technology generated by a software company increases the sales and profits of

[2] Moreover, a larger number of firms indicates that firms are more specialized. More specialized and formerly neglected technologies may become valuable to the larger number of specialized firms. Hence, knowledge spillovers will increase.

computer hardware companies. The economy's productivity increases because of the innovation but the increased incomes are attributed not only to the firm that generated the innovation but also to the firms that possess complementary technologies. A part of $\frac{\partial Y_t}{\partial A_t}$ leaks to these firms, and the leaked income is a kind of rent revenue that unexpectedly became obtainable because of the original firm's innovation. Most new technologies will have complementary technologies. Because of both complementary technologies and uncompensated knowledge spillovers, the fraction of $\frac{\partial Y_t}{\partial A_t}$ that an investing firm can obtain on average will be very small; that is, ϖ will be far smaller than M_t except when M_t is very small.[3]

2.1.1.4 The optimization problem

Because $A_t = \frac{\varpi\alpha}{mv(1-\alpha)}k_t$, then $y_t = \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}k_t$ and $\dot{A}_t = \frac{\varpi}{mv}\dot{k}_t\left(\frac{\alpha}{1-\alpha}\right)$. Hence, $\dot{k}_t = y_t - c_t - \frac{v\dot{A}_t}{L_t} - n_t k_t = \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}k_t - c_t - \frac{\varpi}{mL_t}\dot{k}_t\left(\frac{\alpha}{1-\alpha}\right) - n_t k_t$ and

$$\dot{k}_t = \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}k_t - c_t - n_t k_t\right] . \tag{6}$$

As a whole, the optimization problem of the representative household is to maximize the expected utility

$$E\int_0^\infty u(c_t)\exp(-\theta t)dt$$

subject to equation (6) where $u(\bullet)$ is a constant relative risk aversion (CRRA) utility function and E is the expectation operator.

2.1.2 A model with heterogeneous households

Heterogeneous time preference is examined in an endogenous growth model, which is a modified version of the model shown in Section 2.1.1 (See Harashima (2012) for other heterogeneities—risk aversion and productivity). First, suppose that there are two economies—economy 1 and economy 2—that are identical except for time preference. The population growth rate is zero (i.e., $n_t = 0$). The economies are fully open to each other, and goods, services, and capital are freely transacted between them, but labor is immobilized in each economy.

Each economy can be interpreted as representing either a country (the international interpretation) or a group of identical households in a country (the national interpretation). Because the economies are fully open, they are integrated through trade and form a combined

[3] If M_t is very small, the value of ϖ will be far smaller than that for sufficiently large M_t because the number of firms that can benefit from an innovation is constrained owing to the very small M_t. The very small number of firms indicates that the economy is not sufficiently sophisticated, and thereby the benefit of an innovation cannot be fully realized. This constraint can be modeled as $\varpi = \tilde{\varpi}\left[1-\left(1-\tilde{\varpi}^{-1}\right)^{M_t}\right]$, where $\tilde{\varpi}(\geq 1)$ is a constant. Nevertheless, for sufficiently large M_t (i.e., in sufficiently sophisticated economies), the constraint is removed such that $\lim_{M_t\to\infty}\tilde{\varpi}\left[1-\left(1-\tilde{\varpi}^{-1}\right)^{M_t}\right] = \tilde{\varpi} = \varpi$.

economy. The combined economy is the world economy in the international interpretation and the national economy in the national interpretation. In the following discussion, a model based on the international interpretation is called an international model and that based on the national interpretation is called a national model. Usually, the concept of the balance of payments is used only for the international transactions. However, because both national and international interpretations are possible, this concept and terminology are also used for the national models in this paper.

In this section, a model in which the two economies are identical except for time preference is constructed.[4] The rate of time preference of the representative household in economy 1 is θ_1 and that in economy 2 is θ_2, and $\theta_1 < \theta_2$. The production function in economy 1 is $y_{1,t} = A_t^{\alpha} f(k_{1,t})$ and that in economy 2 is $y_{2,t} = A_t^{\alpha} f(k_{2,t})$, where $y_{i,t}$ and $k_{i,t}$ are, respectively, output and capital per capita in economy i in period t for i = 1, 2. The population of each economy is $\frac{L_t}{2}$; thus, the total for both is L_t, which is sufficiently large. Firms operate in both economies, and the number of firms is M_t. The current account balance in economy 1 is τ_t and that in economy 2 is - τ_t. Because a balanced growth path requires Harrod neutral technological progress, the production functions are further specified as

$$y_{i,t} = A_t^{\alpha} k_{i,t}^{1-\alpha} \quad ;$$

thus, $Y_{i,t} = K_{i,t}^{1-\alpha}(A_t L_t)^{\alpha} \; (i = 1,2)$.

Because both economies are fully open, returns on investments in each economy are kept equal through arbitration such that

$$\frac{\partial y_{1,t}}{\partial k_{1,t}} = \frac{\varpi}{2mv}\frac{\partial(y_{1,t} + y_{2,t})}{\partial A_t} = \frac{\partial y_{2,t}}{\partial k_{2,t}} \quad . \tag{7}$$

Equation (7) indicates that an increase in A_t enhances outputs in both economies such that $\frac{\partial Y_{i,t}}{\partial K_{i,t}} = \frac{\varpi}{M_t}\frac{\partial(Y_{1,t} + Y_{2,t})}{\partial(vA_t)}$, and because the population is equal ($\frac{L_t}{2}$), $\frac{\partial Y_{i,t}}{\partial K_{i,t}} = \frac{\partial y_{i,t}}{\partial k_{i,t}} =$ $\frac{\varpi}{M_t}\frac{\partial(Y_{1,t} + Y_{2,t})}{\partial(vA_t)} = \frac{\varpi}{mL_t}\frac{\partial(y_{1,t} + y_{2,t})}{\partial(vA_t)}\frac{L_t}{2} = \frac{\varpi}{2mv}\frac{\partial(y_{1,t} + y_{2,t})}{\partial A_t}$. Therefore,

$$A_t = \frac{\varpi\alpha[f(k_{1,t}) + f(k_{2,t})]}{2mvf'(k_{1,t})} = \frac{\varpi\alpha[f(k_{1,t}) + f(k_{2,t})]}{2mvf'(k_{2,t})} \quad .$$

Because equation (7) is always held through arbitration, equations $k_{1,t} = k_{2,t}$, $\dot{k}_{1,t} = \dot{k}_{2,t}$, $y_{1,t} = y_{2,t}$ and $\dot{y}_{1,t} = \dot{y}_{2,t}$ are also held. Hence,

$$A_t = \frac{\varpi\alpha f(k_{1,t})}{mvf'(k_{1,t})} = \frac{\varpi\alpha f(k_{2,t})}{mvf'(k_{2,t})} \quad .$$

[4] This type of endogenous growth model of heterogeneous time preference was originally shown by Harashima (2009).

In addition, because $\frac{\partial(y_{1,t}+y_{2,t})}{\partial A_{1,t}}=\frac{\partial(y_{1,t}+y_{2,t})}{\partial A_{2,t}}$ through arbitration, then $\dot{A}_{1,t}=\dot{A}_{2,t}$ is held.

The accumulated current account balance $\int_0^t \tau_s \, ds$ mirrors capital flows between the two economies. The economy with current account surpluses invests them in the other economy. Since $\frac{\partial y_{1,t}}{\partial k_{1,t}}\left(=\frac{\partial y_{2,t}}{\partial k_{2,t}}\right)$ are returns on investments, $\frac{\partial y_{1,t}}{\partial k_{1,t}}\int_0^t \tau_s ds$ and $\frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds$ represent income receipts or payments on the assets that an economy owns in the other economy. Hence,

$$\tau_t-\frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds$$

is the balance on goods and services of economy 1, and

$$\frac{\partial y_{1,t}}{\partial k_{1,t}}\int_0^t \tau_s ds-\tau_t$$

is that of economy 2. Because the current account balance mirrors capital flows between the economies, the balance is a function of capital in both economies such that

$$\tau_t=\kappa(k_{1,t},k_{2,t}) \quad .$$

The government (or an international supranational organization) intervenes in activities of economies 1 and 2 by transferring money from economy 1 to economy 2. The amount of transfer in period t is g_t and it is assumed that g_t depends on capitals such that

$$g_t=\bar{g}k_{1,t}$$

where $\bar{g}$ is a constant. Because $k_{1,t}=k_{2,t}$ and $\dot{k}_{1,t}=\dot{k}_{2,t}$,

$$g_t=\bar{g}k_{1,t}=\bar{g}k_{2,t} \quad .$$

The representative household in economy 1 maximizes its expected utility

$$E\int_0^\infty u_1(c_{1,t})\exp(-\theta_1 t)dt \quad ,$$

subject to

$$\begin{aligned}\dot{k}_{1,t}&=y_{1,t}+\frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s \, ds-\tau_t-c_{1,t}-g_t-v\dot{A}_{1,t}\left(\frac{L_t}{2}\right)^{-1}\\&=y_{1,t}+\frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s \, ds-\tau_t-c_{1,t}-\bar{g}k_{1,t}-v\dot{A}_{1,t}\left(\frac{L_t}{2}\right)^{-1} \quad ,\end{aligned} \tag{8}$$

and the representative household in economy 2 maximizes its expected utility

$$E\int_0^\infty u_2(c_{2,t})\exp(-\theta_2 t)dt \quad ,$$

subject to

$$\begin{aligned}\dot{k}_{2,t} &= y_{2,t} - \frac{\partial y_{1,t}}{\partial k_{1,t}}\int_0^t \tau_s ds + \tau_t - c_{2,t} + g_t - v\dot{A}_{2,t}\left(\frac{L_t}{2}\right)^{-1} \\ &= y_{2,t} - \frac{\partial y_{1,t}}{\partial k_{1,t}}\int_0^t \tau_s ds + \tau_t - c_{2,t} + \bar{g}k_{2,t} - v\dot{A}_{2,t}\left(\frac{L_t}{2}\right)^{-1} \quad , \end{aligned} \tag{9}$$

where $u_{i,t}$, $c_{i,t}$, and $\dot{A}_{i,t}$, respectively, are the utility function, per capita consumption, and the increase in A_t by R&D activities in economy i in period t for i = 1, 2; E is the expectation operator; and $\dot{A}_t = \dot{A}_{1,t} + \dot{A}_{2,t}$. Equations (8) and (9) implicitly assume that each economy does not have foreign assets or debt in period t = 0.

Because the production function is Harrod neutral and because $A_t = \frac{\varpi\alpha f(k_{1,t})}{mvf'(k_{1,t})}$ $= \frac{\varpi\alpha f(k_{2,t})}{mvf'(k_{2,t})}$ and $f = k_{i,t}^{1-\alpha}$, then

$$A_t = \frac{\varpi\alpha}{mv(1-\alpha)}k_{i,t}$$

and

$$\frac{\partial y_{i,t}}{\partial k_{i,t}} = \left(\frac{\varpi\alpha}{mv}\right)^\alpha (1-\alpha)^{1-\alpha} \quad .$$

Since $\dot{A}_{1,t} = \dot{A}_{2,t}$ and $\frac{\partial y_{1,t}}{\partial k_{1,t}} = \frac{\partial y_{2,t}}{\partial k_{2,t}}$, then

$$\begin{aligned}\dot{k}_{1,t} &= y_{1,t} + \frac{\partial y_{1,t}}{\partial k_{1,t}}\int_0^t \tau_s ds - \tau_t - c_{1,t} - \bar{g}k_{1,t} - \frac{v\dot{A}_t}{2}\left(\frac{L_t}{2}\right)^{-1} \\ &= \left(\frac{\varpi\alpha}{mv}\right)^\alpha (1-\alpha)^{-\alpha}k_{1,t} + \left(\frac{\varpi\alpha}{mv}\right)^\alpha (1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t} - \bar{g}k_{1,t} - \frac{\varpi\alpha}{mL_t(1-\alpha)}\dot{k}_{1,t}\end{aligned}$$

and

$$\dot{k}_{1,t} = \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}\left[\left(\frac{\varpi\alpha}{mv}\right)^\alpha (1-\alpha)^{-\alpha}k_{1,t} + \left(\frac{\varpi\alpha}{mv}\right)^\alpha (1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t} - \bar{g}k_{1,t}\right] \quad .$$

Because L_t is sufficiently large and ϖ is far smaller than M_t, the problem of scale effects vanishes and thereby $\frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}=1$.

Putting the above elements together, the optimization problem of economy 1 can be rewritten as

$$Max\ E\int_0^\infty u_1(c_{1,t})\exp(-\theta_1 t)dt \quad ,$$

subject to

$$\dot{k}_{1,t}=\left(\frac{\varpi\alpha}{mv}\right)^\alpha (1-\alpha)^{-\alpha} k_{1,t}+\left(\frac{\varpi\alpha}{mv}\right)^\alpha (1-\alpha)^{1-\alpha}\int_0^t \tau_s ds-\tau_t-c_{1,t}-\bar{g}k_{1,t} \quad .$$

Similarly, that of economy 2 can be rewritten as

$$Max\ E\int_0^\infty u_2(c_{2,t})\exp(-\theta_2 t)dt \quad ,$$

subject to

$$\dot{k}_{2,t}=\left(\frac{\varpi\alpha}{mv}\right)^\alpha (1-\alpha)^{-\alpha} k_{2,t}-\left(\frac{\varpi\alpha}{mv}\right)^\alpha (1-\alpha)^{1-\alpha}\int_0^t \tau_s ds+\tau_t-c_{2,t}+\bar{g}k_{2,t} \quad .$$

2.2 *The multilateral path*

Heterogeneity is defined as being sustainable if all the optimality conditions of all heterogeneous households are satisfied indefinitely. Although the previously discussed state of Becker (1980) is Pareto efficient, by this definition, the heterogeneity is not sustainable because only the most patient household can achieve optimality. Sustainability is therefore the stricter criterion for welfare than Pareto efficiency.

In this section, the growth path that makes heterogeneity sustainable is examined. First, the basic natures of the models presented in Section 2.1 when the government does not intervene, i.e., $\bar{g}=0$ are examined.

2.2.3 Sustainability

Because balanced growth is the focal point for the growth path analysis, the following analyses focus on the steady state such that $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}$, $\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$, $\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}$, $\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}$, and $\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}$ are constants. The balanced growth path in the heterogeneous time preference model has the following properties.

Lemma 1: In the model of heterogeneous time preference, if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant, then

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} = \lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} \quad .$$

Proof: See Harashima (2010)

Proposition 1: In the model of heterogeneous time preference, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}$ $= \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant, all the optimality conditions of both economies are satisfied at steady state.

The path on which $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant has the following properties.

Proof: See Harashima (2010)

Corollary 1: In the model of heterogeneous time preference, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} =$ $\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant, then

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} = \lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}} = \lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}} = \lim_{t\to\infty}\frac{\dot{A}_t}{A_t} = \text{constant.}$$

Proof: See Harashima (2010)

Note that the limit of the growth rate on this path is

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\theta_1+\theta_2}{2}\right] . \tag{10}$$ [5]

Corollary 2: In the model of heterogeneous time preference, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} =$ $\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant,

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\int_0^t \tau_s ds}{dt}}{\int_0^t \tau_s ds} = \lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}$$

[5] See Harashima (2010)

$$= \lim_{t\to\infty} \frac{\dot{y}_{1,t}}{y_{1,t}} = \lim_{t\to\infty} \frac{\dot{y}_{2,t}}{y_{2,t}} = \lim_{t\to\infty} \frac{\dot{A}_t}{A_t} = \text{constant.}$$

Proof: See Harashima (2010)

Because current account imbalances eventually grow at the same rate as output, consumption, and capital on the multilateral path, the ratios of the current account balance to output, consumption, and capital do not explode, but they stabilize as shown in the proof of Proposition 1; that is, $\lim_{t\to\infty} \frac{\tau_t}{k_{1,t}} = \lim_{t\to\infty} \frac{\tau_t}{k_{2,t}} = \varXi$.

On the balanced growth path satisfying Proposition 1 and Corollaries 1-1 and 2-1, heterogeneity in time preference is sustainable by definition because all the optimality conditions of the two economies are indefinitely satisfied. The balanced growth path satisfying Proposition 1 and Corollaries 1-1 and 2-1 is called the "multilateral balanced growth path" or (more briefly) the "multilateral path" in the following discussion. The term "multilateral" is used even though there are only two economies, because the two-economy models shown can easily be extended to the multi-economy models shown in Section 2.2.6.

Because technology will not decrease persistently (i.e., $\lim_{t\to\infty} \frac{\dot{A}_t}{A_t} > 0$), only the case such that $\lim_{t\to\infty} \frac{\dot{A}_t}{A_t} > 0$ (i.e., $\lim_{t\to\infty} \frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty} \frac{\dot{c}_{2,t}}{c_{2,t}} > 0$ on the multilateral path by Corollary 1) is examined in the following discussion.

2.2.4 The balance of payments

As shown in the proof of Proposition 1, $\lim_{t\to\infty} \frac{\tau_t}{k_{1,t}} = \lim_{t\to\infty} \frac{\tau_t}{k_{2,t}} = \varXi$ and $\lim_{t\to\infty} \frac{\int_0^t \tau_s ds}{k_{1,t}} = \lim_{t\to\infty} \frac{\int_0^t \tau_s ds}{k_{2,t}} = \varXi \left(\lim_{t\to\infty} \frac{\dot{c}_{1,t}}{c_{1,t}} \right)^{-1}$ on the multilateral path. Because $k_{i,t}$ is positive, if the sign of $\varXi$ is negative, the current account of economy 1 will eventually show permanent deficits and vice versa.

Lemma 2: In the model of heterogeneous time preference,

$$\varXi = \frac{\theta_1 - \theta_2}{2} \left\{ \varepsilon \left(\frac{\varpi \alpha}{mv} \right)^{\alpha} (1-\alpha)^{1-\alpha} \left[\left(\frac{\varpi \alpha}{mv} \right)^{\alpha} (1-\alpha)^{-\alpha} - \frac{\theta_1 + \theta_2}{2} \right]^{-1} - 1 \right\}^{-1} .$$

Proof: See Harashima (2010)

Lemma 2 indicates that the value of $\varXi$ is uniquely determined on the multilateral path, and the sign of $\varXi$ is also therefore uniquely determined.

Proposition 2: In the model of heterogeneous time preference, $\varXi < 0$ if

$$\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}\left[1-(1-\alpha)\varepsilon\right]<\frac{\theta_1+\theta_2}{2}.$$

Proof: See Harashima (2010)

Proposition 2 indicates that the current account deficit of economy 1 and the current account surplus of economy 2 continue indefinitely on the multilateral path. The condition $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}\left[1-(1-\alpha)\varepsilon\right]<\frac{\theta_1+\theta_2}{2}$ is generally satisfied for reasonable parameter values.

Conversely, the opposite is true for the trade balance.

Corollary 3: In the model of heterogeneous time preference, $\lim_{t\to\infty}\left(\tau_t-\frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds\right)>0$ if $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}\left[1-(1-\alpha)\varepsilon\right]<\frac{\theta_1+\theta_2}{2}$.

Proof: See Harashima (2010)

Corollary 3 indicates that, on the multilateral path, the trade surpluses of economy 1 continue indefinitely and vice versa. That is, goods and services are transferred from economy 1 to economy 2 in each period indefinitely in exchange for the returns on the accumulated current account deficits (i.e., debts) of economy 1.

Nevertheless, the trade balance of economy 1 is not a surplus from the beginning. Before Corollary 3 is satisfied, negative $\int_0^t \tau_s ds$ should be accumulated. In the early periods, when $\int_0^t \tau_s ds$ is small, the balance on goods and services of economy 1 ($\tau_t-\frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds$) continues to be a deficit. After a sufficient negative amount of $\int_0^t \tau_s ds$ is accumulated, the trade balances of economy 1 shift to surpluses.

Current account deficit of economy 1 means for example that a firm that is owned by economy 1 borrows money from a bank in which economy 2 deposits money. Economy 1 indirectly borrows money from economy 2. This situation can be easily understood if you see the current account deficit of the United States.

2.2.5 A model with heterogeneities in multiple elements

Three heterogeneities—heterogeneous time preference, risk aversion, and productivity—are not exclusive. It is particularly likely that heterogeneities in time preference and productivity coexist. Many empirical studies conclude that the rate of time preference is negatively correlated with income (e.g., Lawrance, 1991; Samwick, 1998; Ventura, 2003); this indicates that the economy with the higher productivity has a lower rate of time preference and vice versa. In this section, the models are extended to include heterogeneity in multiple elements. Suppose that there are H economies that are identical except for time preference. Let the degree of relative risk aversion of economy i be $\varepsilon_i=-\frac{c_{i,t}\,u_i''}{u_i'}$, the production function of economy i be $y_{i,t}=\omega_i^{\alpha}A_t^{\alpha}f(k_{i,t})$, and $\tau_{i,j,t}$ be the current account balance of economy i with economy j, where i = 1, 2, … , H, j = 1, 2, … , H, and $i \neq j$.

Proposition 3: If and only if

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\left(\frac{\sum_{q=1}^{H}\varepsilon_q\omega_q}{\sum_{q=1}^{H}\omega_q}\right)^{-1}\left\{\left[\frac{\varpi\alpha\sum_{q=1}^{H}\omega_q}{Hmv(1-\alpha)}\right]^{\alpha}-\frac{\sum_{q=1}^{H}\theta_q\omega_q}{\sum_{q=1}^{H}\omega_q}\right\} \tag{11}$$

for any i (= 1, 2, … , H), all the optimality conditions of all heterogeneous economies are satisfied at steady state such that $\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}$, $\lim_{t\to\infty}\frac{\dot{k}_{i,t}}{k_{i,t}}$, and $\lim_{t\to\infty}\frac{\dot{\tau}_{i,j,t}}{\tau_{i,j,t}}$ are constants, and

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\lim_{t\to\infty}\frac{\dot{k}_{i,t}}{k_{i,t}}=\lim_{t\to\infty}\frac{\dot{y}_{i,t}}{y_{i,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\lim_{t\to\infty}\frac{\dot{\tau}_{i,j,t}}{\tau_{i,j,t}}=\lim_{t\to\infty}\frac{\frac{d\int_0^t\tau_{i,j,s}ds}{dt}}{\int_0^t\tau_{i,j,s}ds}$$

for any i and j ($i\neq j$).

Proof: See Harashima (2012)

Proposition 3 implies that the concept of the representative household in a heterogeneous population implicitly assumes that all households are on the multilateral path.

2.3 The unilateral path

The multilateral path satisfies all the optimality conditions, but that does not mean that the two economies naturally select the multilateral path. Ghiglino (2002) predicts that it is likely that, under appropriate assumptions, the results of Becker (1980) still hold in endogenous growth models. Farmer and Lahiri (2005) show that balanced growth equilibria do not exist in a multi-agent economy in general, except in the special case that all agents have the same constant rate of time preference. How the economies behave in the environments described in Sections 2.1 and 2.3 when the government does not intervene, i.e., $\bar{g}=0$. is examined in this section.

The multilateral path is not the only path on which all the optimality conditions of economy 1 are satisfied. Even if economy 1 behaves unilaterally, it can achieve optimality, but economy 2 cannot.

Lemma 3: In the heterogeneous time preference model, if each economy sets τ_t without regarding the other economy's optimality conditions, then it is not possible to satisfy all the optimality conditions of both economies.
Proof: See Harashima (2010)

Since $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\varepsilon^{-1}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}+\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}\left(\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}\right)^{-1}-\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}-\theta_1\right\}$

at steady state, all the optimality conditions of economy 1 can be satisfied only if either

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}=\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} \tag{12}$$

or

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}=\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha} \quad . \tag{13}$$

That is, $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}$ can be constant only when either equation (12) or (13) is satisfied. Conversely, economy 1 has two paths on which all its optimality conditions are satisfied. Equation (12) indicates that $\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}=$ constant, and equation (13) indicates that $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\left(\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}\right)^{-1}-1=0$ for any $\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}$. Equation (12) corresponds to the multilateral path. On the path satisfying equation (13), $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\neq\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}$, and $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}>\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$. Here, by equations (8) and (9),

$$c_{1,t}-c_{2,t}=2\left(\frac{\partial y_{1,t}}{\partial k_{1,t}}\int_0^t \tau_s ds-\tau_t\right)=2\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds-\tau_t\right] \quad ,$$

and

$$\lim_{t\to\infty}\left(c_{1,t}-c_{2,t}\right)=0$$

is required because $\lim_{t\to\infty}\frac{\tau_t}{\int_0^t \tau_s ds}=\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}$. However, because $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}>\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$, economy 2 must initially set consumption such that $c_{2,0}=\infty$, which violates the optimality condition of economy 2. Therefore, unlike with the multilateral path, all the optimality conditions of economy 2 cannot be satisfied on the path satisfying equation (13) even though those of economy 1 can. Hence, economy 2 has only one path on which all its optimality conditions can be satisfied—the multilateral path. The path satisfying equation (13) is called the “unilateral balanced growth path” or the “unilateral path” in the following discussion. Clearly, heterogeneity in time preference is not sustainable on the unilateral path.

How should economy 2 respond to the unilateral behavior of economy 1? Possibly, both economies negotiate for the trade between them, and some agreements may be reached. If no agreement is reached, however, and economy 1 never regards economy 2's optimality conditions, economy 2 generally will fall into the following unfavorable situation.

Remark 1: In the model of heterogeneous time preference, if economy 1 does not regard the optimality conditions of economy 2, the ratio of economy 2's debts (owed to economy 1) to its consumption explodes to infinity while all the optimality conditions of economy 1 are satisfied.

The reasoning behind Remark 1 is as follows. When economy 1 selects the unilateral path and sets $c_{1,0}$ so as to achieve this path, there are two options for economy 2. The first option is for economy 2 to also pursue its own optimality without regarding economy 1: that is, to select its own unilateral path. The second option is to adapt to the behavior of economy 1 as a follower. If economy 2 takes the first option, it sets $c_{2,0}$ without regarding $c_{1,0}$. As the proof of Lemma 3 indicates, unilaterally optimal growth rates are different between the two economies and $\frac{\dot{c}_{1,t}}{c_{1,t}} > \frac{\dot{c}_{2,t}}{c_{2,t}}$; thus, the initial consumption should be set as $c_{1,0} < c_{2,0}$. Because $\frac{\partial y_{1,t}}{\partial k_{1,t}} = \frac{\varpi}{2mv} \frac{\partial (y_{1,t} + y_{2,t})}{\partial A_t} = \frac{\partial y_{2,t}}{\partial k_{2,t}}$ and $k_{1,t} = k_{2,t}$ must be kept, capital and technology are equal and grow at the same rate in both economies. Hence, because $c_{1,0} < c_{2,0}$, more capital is initially produced in economy 1 than in economy 2 and some of it will need to be exported to economy 2. As a result, $\frac{\dot{c}_{1,t}}{c_{1,t}} > \frac{\dot{k}_{1,t}}{k_{1,t}} = \frac{\dot{k}_{2,t}}{k_{2,t}} > \frac{\dot{c}_{2,t}}{c_{2,t}}$, which means that all the optimality conditions of both economies cannot be satisfied. Since $\lim_{t\to\infty} \frac{\dot{c}_{1,t}}{c_{1,t}} > \lim_{t\to\infty} \frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty} \frac{\dot{k}_{2,t}}{k_{2,t}} > \lim_{t\to\infty} \frac{\dot{c}_{2,t}}{c_{2,t}}$, capital soon becomes abundant in economy 2, and excess goods and services are produced in that economy. These excess products are exported to and utilized in economy 1. This process escalates as time passes because $\lim_{t\to\infty} \frac{\dot{c}_{1,t}}{c_{1,t}} > \lim_{t\to\infty} \frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty} \frac{\dot{k}_{2,t}}{k_{2,t}} > \lim_{t\to\infty} \frac{\dot{c}_{2,t}}{c_{2,t}}$, and eventually almost all consumer goods and services produced in economy 2 are consumed by households in economy 1. These consequences will be unfavorable for economy 2.

If economy 2 takes the second option, it should set $c_{2,0} = \infty$ to satisfy all its optimality conditions, as the proof of Lemma 3 indicates. Setting $c_{2,0} = \infty$ is impossible, but economy 2 as the follower will initially set $c_{2,t}$ as large as possible. This action gives economy 2 a higher expected utility than that of the first option, because consumption in economy 2 in the second case is always higher. As a result, economy 2 imports as many goods and services as possible from economy 1, and the trade deficit of economy 2 continues until $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha} (1-\alpha)^{1-\alpha} \int_0^t \tau_s ds = \tau_t$ is achieved; this is, $\frac{\dot{\tau}_t}{\tau_t} = \frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}$ is achieved. The current account deficits and the accumulated debts of economy 2 will continue to increase indefinitely. Furthermore, they will increase more rapidly than the growth rate of outputs ($\lim_{t\to\infty} \frac{\dot{y}_{2,t}}{y_{2,t}}$) because, in general,

$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} < \lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}$; that is, $(1-\varepsilon)\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha} < \theta_1(<\theta_2)$. If no disturbance occurs, the expansion of debts may be sustained forever, but economy 2 becomes extremely vulnerable to even a very tiny negative disturbance. If such a disturbance occurs, economy 2 will lose all its capital and will no longer be able to repay its debts. This result corresponds to the state shown by Becker (1980), and it will also be unfavorable for economy 2. Because $\lim_{t\to\infty}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t\right] = 0$, inequality (27) holds, and the transversality condition for economy 1 is satisfied. Thus, all the optimality conditions of economy 1 are satisfied if economy 2 takes the second option.

As a result, all the optimality conditions of economy 2 cannot be satisfied in any case if economy 1 takes the unilateral path. Both options to counter the unilateral behavior of economy 1 are unfavorable for economy 2. However, the expected utility of economy 2 is higher if it takes the second option rather than the first, and economy 2 will choose the second option. Hence, if economy 1 does not regard economy 2's optimality conditions, the debts owed by economy 2 to economy 1 increase indefinitely at a higher rate than consumption.

3 SUSTAINABLE HETEROGENEITY IN AN EXOGENOUS GROWTH MODEL

The multilateral paths in the endogenous growth models (heterogeneous time preference, risk aversion, and productivity models) shown in Section 2 imply that similar sustainable states exist in exogenous growth models. However, this is true only for the heterogeneous time preference model, because, in exogenous growth models, the steady state means that $\frac{\partial y_t}{\partial k_t} = \theta$; that is, the heterogeneity in risk aversion is irrelevant to the steady state, and the heterogeneous productivities do not result in permanent trade imbalances due to $\frac{\partial y_{1,t}}{\partial k_{1,t}} = \frac{\partial y_{2,t}}{\partial k_{2,t}}$. Thereby, only heterogeneous time preference is relevant to sustainable heterogeneity in exogenous growth models.

3.1 The Model

The endogenous growth model of heterogeneous time preference in Section 2 is degenerated to an exogenous growth model. If technology is exogenously given and constant ($A_t = A$), Hamiltonians for the heterogeneous time preference model shown in Section 2.2.1 degenerate to

$$H_1 = u_1(c_{1,t})\exp(-\theta_1 t) + \lambda_{1t}\left[A^{\alpha}k_{1,t}^{1-\alpha} + (1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t} - \bar{g}k_{1,t}\right]$$

for economy 1, and

$$H_2 = u_2(c_{2,t})\exp(-\theta_2 t) + \lambda_{2,t}\left[A^{\alpha}k_{2,t}^{1-\alpha} - (1-\alpha)A^{\alpha}k_{2,t}^{-\alpha}\int_0^t \tau_s ds + \tau_t - c_{2,t} + \bar{g}k_{2,t}\right] .$$

for economy 2.

3.2 Sustainable heterogeneity

First, the natures of the model when the government does not intervene, i.e., $\bar{g}=0$ are examined. The growth rate of consumption in economy 1 is

$$\frac{\dot{c}_{1,t}}{c_{1,t}}=\varepsilon^{-1}\left\{(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}+(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\frac{\partial\int_0^t\tau_s ds}{\partial k_{1,t}}-\alpha(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha-1}\int_0^t\tau_s ds-\frac{\partial\tau_t}{\partial k_{1,t}}-\theta_1\right\} .$$

Hence,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\varepsilon^{-1}\lim_{t\to\infty}\left\{(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}+(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\frac{\partial\int_0^t\tau_s ds}{\partial k_{1,t}}-\alpha(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha-1}\int_0^t\tau_s ds-\frac{\partial\tau_t}{\partial k_{1,t}}-\theta_1\right\}=0$$

and thereby

$$\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\left[1+(1-\alpha)\varPsi\right]-\varXi-\theta_1=0 ,$$

where $\varXi=\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}=\lim_{t\to\infty}\frac{\tau_t}{k_{2,t}}$ and $\varPsi=\lim_{t\to\infty}\frac{\int_0^t\tau_s ds}{k_{1,t}}=\lim_{t\to\infty}\frac{\int_0^t\tau_s ds}{k_{2,t}}$. $\lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=0$, and $\varPsi$ is constant at steady state because $k_{1,t}$ and τ_t are constant and thus $\varXi=\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}$ is constant at steady state. For $\varPsi$ to be constant at steady state, it is necessary that $\lim_{t\to\infty}\tau_t=0$ and thus $\varXi=0$. Therefore,

$$\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\left[1+(1-\alpha)\varPsi\right]-\theta_1=0 , \tag{14}$$

and

$$\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{2,t}^{-\alpha}\left[1-(1-\alpha)\varPsi\right]-\theta_2=0 \tag{15}$$

because

$$\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\varepsilon^{-1}\lim_{t\to\infty}\left\{(1-\alpha)A^{\alpha}k_{2,t}^{-\alpha}-(1-\alpha)A^{\alpha}k_{2,t}^{-\alpha}\frac{\partial\int_0^t\tau_s ds}{\partial k_{2,t}}+\alpha(1-\alpha)A^{\alpha}k_{2,t}^{-\alpha-1}\int_0^t\tau_s ds+\frac{\partial\tau_t}{\partial k_{2,t}}-\theta_2\right\}=0 . \tag{16}$$

Because $\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\left[1+(1-\alpha)\varPsi\right]=\theta_1$, $\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{2,t}^{-\alpha}\left[1-(1-\alpha)\varPsi\right]=\theta_2$, and $\frac{\partial y_{1,t}}{\partial k_{1,t}}=\frac{\partial y_{2,t}}{\partial k_{2,t}}=A^{\alpha}k_{1,t}^{-\alpha}=A^{\alpha}k_{2,t}^{-\alpha}$, then

$$\Psi = \frac{\theta_1 - \theta_2}{2(1-\alpha)\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}} \quad . \tag{17}$$

By equations (14) and (17),

$$\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}} + \lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}(1-\alpha)\Psi = \theta_1 \quad ;$$

thus,

$$\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}} = \frac{\theta_1 + \theta_2}{2} = \lim_{t\to\infty}\frac{\partial y_{2,t}}{\partial k_{2.t}} \quad . \tag{18}$$

If equation (18) holds, all the optimality conditions of both economies are indefinitely satisfied. This result is analogous to equation (29) and corresponds to the multilateral path in the endogenous growth models. The state indicated by equation (18) is called the "multilateral steady state" or "multilateral state" in the following discussion. By similar procedures as those used for the endogenous growth models in Section 2, the condition of multilateral steady state for *H* economies is shown as

$$\lim_{t\to\infty}\frac{\partial y_{i,t}}{\partial k_{i.t}} = \frac{\sum_{q=1}^{H}\theta_q}{H}$$

for any i where $i = 1, 2, \ldots, H$.

Because,

$$\Psi = \frac{\theta_1 - \theta_2}{2(1-\alpha)\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}} = \frac{\theta_1 - \theta_2}{(1-\alpha)(\theta_1 + \theta_2)} < 0$$

by equation (18), then, by $\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}} = \Psi < 0$,

$$\lim_{t\to\infty}\int_0^t \tau_s ds < 0 \quad ;$$

that is, economy 1 possesses accumulated debts owed to economy 2 at steady state, and economy 1 has to export goods and services to economy 2 by

$$\left|(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\int_0^t \tau_s ds\right|$$

in every period to pay the debts. Nevertheless, because $\lim_{t\to\infty}\tau_t = 0$ and $\Xi = 0$, the debts do not explode but stabilize at steady state.

If both economies are not open and are isolated, $\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}=\theta_1$ and $\lim_{t\to\infty}\frac{\partial y_{2,t}}{\partial k_{2.t}}=\theta_2$ at steady state instead of the conditions shown in equation (18). Hence, at the multilateral steady state with $\theta_1<\theta_2$, the amount of capital in economy 1 is smaller than when the economy is isolated and vice versa. As a result, output and consumption in economy 1 are also smaller in the multilateral steady state with $\theta_1<\theta_2$ than when the economy is isolated.

3.3 The unilateral state

In the multilateral state, all the optimality conditions of both economies are satisfied, and heterogeneity is therefore sustainable. However, this state will be economically less preferable for economy 1 as compared with the state of Becker (1980), because consumption is smaller and debts are owed. The behaviors of the economies in the environments described in Sections 3.1 when the government does not intervene, i.e., $\bar{g}=0$. is examined in this section.

The multilateral state is not the only state on which all the optimality conditions of economy 1 are satisfied. Even if economy 1 behaves unilaterally, it can achieve optimality, but economy 2 cannot.

Lemma 5: In the heterogeneous time preference model, if each economy sets τ_t without regarding the other economy's optimality conditions, then it is not possible to satisfy all the optimality conditions of both economies.
Proof: See Harashima (2010)

Economy 1 has a path that satisfies equation (14) other than equation (17). Even if economy 1 does not consider the optimality conditions of economy 2 (i.e., economy 1 behaves unilaterally), the behavior that satisfies the following condition also makes all the optimality conditions of economy 1 satisfied:

$$\Psi=\frac{\theta_1\left[\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\right]^{-1}-1}{1-\alpha} \quad . \tag{19}$$

Equation (19) is easily obtained by transposition in equation (14). By equation (19),

$$\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}=\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}=\frac{\theta_1}{1+(1-\alpha)\Psi} \quad . \tag{20}$$

If $\Psi=0$,

$$\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}=\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}=\theta_1 \quad ,$$

which is the familiar condition for steady state in the Ramsey growth model. Economy 1 selects one of the two steady states (the multilateral state that satisfies equation [37] and the unilateral state that satisfies equation [**2]) at which all its optimality conditions are satisfied.

On the path satisfying equation (19),

$$\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} == \varepsilon^{-1}\left\langle \lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{2,t}^{-\alpha}\left\{2-\theta_1\left[\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\right]^{-1}\right\}-\theta_2\right\rangle$$

by equations (16) and (19). Because $A^{\alpha}k_{2,t}^{-\alpha} = A^{\alpha}k_{1,t}^{-\alpha}$,

$$\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1}\left[2\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}-\theta_1-\theta_2\right] .$$

By equation (20),

$$\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1}\left[\theta_1\frac{1-(1-\alpha)\varPsi}{1+(1-\alpha)\varPsi}-\theta_2\right]$$

If the economy 1 initially sets its consumption unilaterally so as to make $\varPsi = \lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}} \geq 0$, then $\frac{1-(1-\alpha)\varPsi}{1+(1-\alpha)\varPsi} \leq 1$, and therefore

$$\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1}\left[\theta_1\frac{1-(1-\alpha)\varPsi}{1+(1-\alpha)\varPsi}-\theta_2\right] < 0$$

because $\theta_1 < \theta_2$ as assumed in Section 2.1.2. Furthermore, even though $\varPsi < 0$, if

$$\frac{\theta_1-\theta_2}{(\theta_1+\theta_2)(1-\alpha)} < \varPsi \quad ,$$

$\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} < 0$. Hence, if economy 1 behaves unilaterally and sets its initial consumption so as to make $\frac{\theta_1-\theta_2}{(\theta_1+\theta_2)(1-\alpha)} < \varPsi$, the consumption of economy 2 continues to decline indefinitely, i.e., $c_{2,t}$ = 0 at steady state while $c_{1,t}$ is positive and constant at steady state. Unless economy 2 initially sets its consumption such that $c_{2,0} = \infty$, which is however impossible, the optimality condition of economy 2 is violated. This is the case Becker (1980) describes.

There are various steady states that satisfy equation (19) depending on the value of $\varPsi = \lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}}$ (i.e., the initial consumption set by economy 1). At any steady state that satisfies equation (19), all optimality conditions of economy 1 are satisfied. For economy 1, all the steady states are equally optimal. Nevertheless, economy 1 selects one of the steady states (in other words, sets a certain value of the initial consumption). For example, it may select the one that gives the highest expected utility, the highest steady state consumption, or some values of other criteria.

Unlike with the multilateral state, all the optimality conditions of economy 2 cannot be satisfied on the path satisfying equation (19) even though those of economy 1 can. Hence, economy 2 has only one path on which all its optimality conditions can be satisfied—the multilateral state. The state satisfying equation (19) is called the "unilateral steady state" or the "unilateral state" in the following discussion. Clearly, heterogeneity in time preference is not sustainable on the unilateral state.

How should economy 2 respond to the unilateral behavior of economy 1? Possibly, both economies negotiate for the trade between them, and some agreements may be reached. If no agreement is reached, however, and economy 1 never regards economy 2's optimality conditions, economy 2 generally will fall into the following unfavorable situation.

Remark 2: In the model of heterogeneous time preference, if economy 1 does not regard the optimality conditions of economy 2, the ratio of economy 2's debts (owed to economy 1) to its consumption explodes to infinity while all the optimality conditions of economy 1 are satisfied.

The reasoning behind Remark 2 is as follows. When economy 1 selects the unilateral state and sets $c_{1,0}$ so as to achieve this path, there are two options for economy 2. The first option is for economy 2 to also pursue its own optimality without regarding economy 1: that is, to select its own unilateral state. The second option is to adapt to the behavior of economy 1 as a follower. If economy 2 takes the first option, it sets $c_{2,0}$ without regarding $c_{1,0}$. As the proof of Lemma 5 indicates, unilaterally optimal growth rates are different between the two economies and $\frac{\dot{c}_{1,t}}{c_{1,t}} > \frac{\dot{c}_{2,t}}{c_{2,t}}$; thus, the initial consumption should be set as $c_{1,0} < c_{2,0}$. Because $\frac{\partial y_{1,t}}{\partial k_{1,t}} = \frac{\varpi}{2mv} \frac{\partial (y_{1,t} + y_{2,t})}{\partial A_t} = \frac{\partial y_{2,t}}{\partial k_{2,t}}$ and $k_{1,t} = k_{2,t}$ must be kept, capital and technology are equal and grow at the same rate in both economies. Hence, because $c_{1,0} < c_{2,0}$, more capital is initially produced in economy 1 than in economy 2 and some of it will need to be exported to economy 2. As a result, $\frac{\dot{c}_{1,t}}{c_{1,t}} > \frac{\dot{k}_{1,t}}{k_{1,t}} = \frac{\dot{k}_{2,t}}{k_{2,t}} > \frac{\dot{c}_{2,t}}{c_{2,t}}$, which means that all the optimality conditions of both economies cannot be satisfied. Since $\lim_{t\to\infty} \frac{\dot{c}_{1,t}}{c_{1,t}} > \lim_{t\to\infty} \frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty} \frac{\dot{k}_{2,t}}{k_{2,t}} > \lim_{t\to\infty} \frac{\dot{c}_{2,t}}{c_{2,t}}$, capital soon becomes abundant in economy 2, and excess goods and services are produced in that economy. These excess products are exported to and utilized in economy 1. This process escalates as time passes because $\lim_{t\to\infty} \frac{\dot{c}_{1,t}}{c_{1,t}} > \lim_{t\to\infty} \frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty} \frac{\dot{k}_{2,t}}{k_{2,t}} > \lim_{t\to\infty} \frac{\dot{c}_{2,t}}{c_{2,t}}$, and eventually almost all consumer goods and services produced in economy 2 are consumed by households in economy 1. These consequences will be unfavorable for economy 2.

If economy 2 takes the second option, it should set $c_{2,0} = \infty$ to satisfy all its optimality conditions, as the proof of Lemma 5 indicates. Setting $c_{2,0} = \infty$ is impossible, but economy 2 as the follower will initially set $c_{2,t}$ as large as possible. This action gives economy 2 a higher expected utility than that of the first option, because consumption in economy 2 in the second case is always higher. As a result, economy 2 imports as many goods and services as possible from economy 1, and the trade deficit of economy 2 continues until $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha} (1-\alpha)^{1-\alpha} \int_0^t \tau_s ds = \tau_t$ is achieved; this is, $\frac{\dot{\tau}_t}{\tau_t} = \frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}$ is achieved. The current account deficits and the accumulated debts of economy 2 will continue to increase indefinitely. Furthermore, they will increase more rapidly than the growth rate of outputs ($\lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}}$) because, in general, $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} < \lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}$; that is, $(1-\varepsilon)\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha} < \theta_1(<\theta_2)$. If no disturbance occurs, the expansion of debts may be sustained forever, but economy 2 becomes extremely vulnerable to even a very tiny negative disturbance. If such a disturbance occurs, economy 2 will lose all its capital and will no longer be able to repay its debts. This result corresponds to the state shown by Becker (1980), and it will also be unfavorable for economy 2. Because $\lim_{t\to\infty}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t\right] = 0$, inequality (27) holds, and the transversality condition for economy 1 is satisfied. Thus, all the optimality conditions of economy 1 are satisfied if economy 2 takes the second option.

As a result, all the optimality conditions of economy 2 cannot be satisfied in any case if economy 1 takes the unilateral state. Both options to counter the unilateral behavior of economy 1 are unfavorable for economy 2. However, the expected utility of economy 2 is higher if it takes the second option rather than the first, and economy 2 will choose the second option. Hence, if economy 1 does not regard economy 2's optimality conditions, the debts owed by economy 2 to economy 1 increase indefinitely at a higher rate than consumption.

3.4 Doom of the less advantaged economies

Remark 2 indicate that economy 2's ratio of debt to consumption continues to increase indefinitely on the unilateral state. Such an indefinitely increasing ratio may not matter if there is no shock or disturbance. However, if even a very tribunal negative shock occurs, economy 2 will be ruined because the huge amount of accumulated debts cannot be refinanced. In this case, "ruin" means that economy 2 will go bankrupt or be exterminated because its consumption has to be zero unless the authority intervenes to some extent (e.g., debt relief after personal bankruptcy). Even if economy 2 continues to exist by the mercy of economy 1, it will fall into a slave-like state indefinitely without the authority's intervention.

4 SUSTAINABLE HETEROGENEITY WITH GOVERNMENT INTERVENTION

Sustainable heterogeneity, as described in this paper, is a very different state from what Becker (1980) described. The difference emerges because, on a multilateral state, economy 1 behaves fully considering economy 2's situation. The multilateral state therefore will not be naturally selected by economy 1, and the path selection may have to be decided politically (Harashima, 2010). On the other hand, when economy 1 behaves unilaterally, the government may intervene in economic activities so as to achieve, for example, social justice.

In this section, I show that even if economy 1 behaves unilaterally, sustainable heterogeneity can always be achieved with appropriate government intervention.

4.1 Heterogeneous time preference model

Government intervention is first considered in the two-economy model constructed in Section 3. If the government intervenes (i.e., $\bar{g} > 0$),

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} \neq \lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} \quad .$$

Because $\bar{g} > 0$, equations (14) and (15) are changed to ,

$$\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\left[1+(1-\alpha)\Psi\right]-\theta_1-\bar{g}=0 \quad , \tag{21}$$

and

$$\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{2,t}^{-\alpha}\left[1-(1-\alpha)\Psi\right]-\theta_2+\bar{g}=0 \tag{22}$$

If economy 1 behaves unilaterally such that equation (21) is satisfied, then

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = 0$$

and

$$\Psi = \frac{(\theta_1+\bar{g})\left[\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\right]^{-1}-1}{1-\alpha} \quad .$$

At the same time, if economy 2 behaves unilaterally such that equation (22) is satisfied, then

$$\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = 0 \quad .$$

By equations (21) and (22)

$$\bar{g} = \frac{\theta_2-\theta_1}{2}+\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}(1-\alpha)\Psi$$

because $k_{1,t} = k_{2,t}$. In addition,

$$\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha} = \frac{\theta_1+\theta_2}{2} = \lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}} = \lim_{t\to\infty}\frac{\partial y_{2,t}}{\partial k_{2.t}} \quad .$$

This equation is identical to equation (18) and is satisfied at the multilateral steady state. Therefore,

$$\bar{g} = \frac{\theta_2-\theta_1}{2}+(1-\alpha)\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}\Psi = \frac{\theta_2-\theta_1}{2}+(1-\alpha)\frac{\theta_1+\theta_2}{2}\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}} \quad . \tag{23}$$

If $\bar{g}$ is set equal to equation (23), all optimality conditions of both economies 1 and 2 are satisfied even though economy 1 behaves unilaterally.

There are various values of Ψ depending on the initial consumption economy 1 sets. If economy 1 behaves in such a way as to make $\lim_{t\to\infty}\int_0^t \tau_s ds < 0$, particularly, make $\bar{g}$ = 0 such that

$$\bar{g} = \frac{\theta_2 - \theta_1}{2} + (1-\alpha)\frac{\theta_1 + \theta_2}{2}\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}} = 0 \quad ,$$

then

$$\Psi = \frac{\theta_1 - \theta_2}{2(1-\alpha)\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1,t}}} \tag{24}$$

by equation (23). Equation (24) is identical to equation (17), that is, the state where equation (24) is satisfied is identical to the multilateral state (with no government intervention, i.e., $\bar{g}$ = 0). On the other hand, if economy 1 behaves in such a way as to make $\lim_{t\to\infty}\int_0^t \tau_s ds = 0$,

$$\bar{g} = \frac{\theta_2 - \theta_1}{2} > 0 \quad .$$

This condition is identical to that for sustainable heterogeneity with government intervention in the endogenous growth model shown in Harashima (2012). Furthermore, if economy 1 behaves in such a way as to make $\lim_{t\to\infty}\int_0^t \tau_s ds > 0$, $\bar{g}$ is positive and given by equation (23).

There are various steady states depending on the values of $\Psi = \lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}}$ and thus the initial consumption set by economy 1. Nevertheless, at any steady state that satisfies equation (24), all optimality conditions of economy 1 are satisfied (by government's intervention, all optimality conditions of economy 2 are also satisfied). For economy 1, all steady states are equally optimal. Economy 1 selects one of steady states (in other words, set the initial consumption), for example, it may select the one that gives the highest expected utility, the highest steady state consumption, or some values of other criteria. Note however that too large positive Ψ requires zero initial consumption and thus a certain upper bound of Ψ will exist.

4.2 Multi-economy models

4.2.1 Heterogeneous time preference model

In this section, only the case of $\Psi = \lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}} = 0$ is considered for simplicity. As was assumed in Section 2, there are H economies that are identical except for time preference. If H = 2, when sustainable heterogeneity is achieved, economies 1 and 2 consist of a combined economy (economy 1+2) with twice the population and a rate of time preference of $\frac{\theta_1 + \theta_2}{2}$. Suppose there is a third economy with a time preference of θ_3. Because economy 1+2 has twice the population of economy 3, if

$$\bar{g} = \frac{\theta_3 - \frac{\theta_1 + \theta_2}{2}}{3} \quad ,$$

then

$$\lim_{t \to \infty} \frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t \to \infty} \frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t \to \infty} \frac{\dot{c}_{3,t}}{c_{3,t}} = 0 \quad .$$

By iterating similar procedures, if the government's transfers between economy H and economy $1+2+\cdots+(H-1)$ is such that

$$\bar{g} = \frac{\theta_H - \frac{\sum_{q=1}^{H-1} \theta_q}{H-1}}{H} \quad ,$$

then

$$\lim_{t \to \infty} \frac{\dot{c}_{i,t}}{c_{i,t}} = 0$$

for any i (= 1, 2, $\cdots$, H).

5 EVOLUTIONARY ORIGIN OF UTILITY

5.1 Genes and utility

The gene-centered view of evolution indicates that evolution is the result of the differential survival of competing genes (see, e.g., Hamilton 1964a; b, Williams, 1966), and the gene is the unit of selection. Genes compete to survive, and only genes that "won" the competition have survived the evolutionary process by fully utilizing their phenotypic effects. The gene-centered view implies that species are governed by an extremely strong desire for the indefinite continuation of their genes. Although some mutations may have existed that made an individual lack such a desire, such mutations must eventually be exterminated through natural selection. A strong desire to survive as a phenotypic effect indicates that humans are extremely motivated to avoid of being exterminated.

Altruistic behaviors of individuals in a group that shares a common pool of genes may be observed, but the gene-centered view implies that the group as a whole will demonstrate an extremely strong desire to escape the possibility of being exterminated. Some individuals may even die to save the group, but the group will never willingly choose to be destroyed because the common pool of genes would be lost.

The concept of utility should be consistent with the theory of evolution, and the above arguments indicate that the prospect of being exterminated should produce extreme fear (i.e., extreme disutility) in human beings. As Becker (1980) showed, unless sustainable heterogeneity is achieved, less advantaged households will perish when even a very small negative shock occurs, so the possibility of extinction does occur in dynamic models with a heterogeneous population. The possibility of extinction should result in a situation of extreme disutility for households in an economy, and human beings are "programmed" to take extreme actions to try to escape this result, thereby enabling the common pool of genes to survive. The gene-centered view of evolution indicates that the extreme disutility experienced in this situation is a natural outcome of evolution.

5.2 Extreme disutility to unsustainable heterogeneity

Sections 3.3 and 3.4 indicate that, on a unilateral state without government intervention, less advantaged economies are exterminated or, at best, fall into a slave-like state. The slave-like state can be seen as equivalent to being exterminated in the sense that the members of those economies are treated more like disposable materials. As discussed in Section 5.1, either extermination or living in a slave-like state should generate extreme fear and disutility in residents of these economies. Hence, the unilateral state without government intervention will generate extreme disutility in the less advantaged economies.

Note that households are assumed to live infinitely long in this paper; thus, extermination does not mean the death of an individual with a finite lifespan. It is the extinction of a dynasty, and in biological terms, indicates that all group members who share a common pool of genes perish.

It could be argued that being forced to live in a slave-like state does not generate extreme disutility because the common pool of genes is preserved. However, the members of these economies can be exterminated at will at any time by the most advantaged economy. Therefore, such states merely mean that extermination is postponed, and the expectations of either being exterminated or falling into in a slave-like state will equally generate extreme disutility.

5.3 The utility of being exterminated

The utility function $u_i(c_{i,t})$ is modified to

$$u_i(\sigma_{i,t}, c_{i,t}) \ ,$$

where $\sigma_{i,t}$ takes two values, 1 and 0. $\sigma_{i,t} = 0$ if economy i is exterminated, and $\sigma_{i,t} = 1$ if economy i is not exterminated (extermination includes falling in a slave-like state). The utility function allows negative values of utility. Being exterminated (i.e., $\sigma_{i,t} = 0$) generates extreme disutility such that

$$u_i(0, c_{i,t}) = -\infty$$

for any $c_{i,t}$; that is, extreme disutility is expressed as infinite disutility. If economy i expects to be exterminated in some future period t' such that $E\,(\sigma_{i,t}) = 0$ for $t > t'$, then

$$Eu_i(\sigma_t, c_{i,t}) = -\infty$$

for $t > t'$. If economy i does not expect to be exterminated in the future such that $E\,(\sigma_{i,t}) = 1$ for any t, then

$$Eu_i(\sigma_t, c_{i,t}) = Eu_i(1, c_{i,t}) \ .$$

Note that infinite disutility may indicate that utility is cardinal. Nevertheless, the infinite disutility of $u_i(0, c_{i,t})$ expressed here as $u_i(0, c_{i,t}) = -\infty$ can be defined by an ordinal expression such that $u_i(0, c_{i,t})$ is identical for any $c_{i,t}$, and

$$u_i(1,0) \succ u_i(0, c_{i,t})$$

for any $c_{i,t}$ [e.g., $u_i(1,0) \succ u_i(0,\infty)$], where $u_i(1, c_{i,t,1}) \succ u_i(1, c_{i,t,2})$ when $c_{i,t,1} > c_{i,t,2}$.

6 SUSTAINABLE HETEROGENEITY AS THE UNIQUE SOCIALLY OPTIMAL ALLOCATION

6.1 *The utility possibility frontier*

A modified utility possibility frontier is needed for analyses using dynamic models with a heterogeneous population.

6.1.1 The utility possibility frontier for an endogenous growth model

Because the model used in this paper is a dynamic one, streams of utilities have to be compared. The utility possibility frontier, therefore, does not consist of period-utilities but of discounted sums of expected utilities. For simplicity, the two-economy model is used where economy 1 has a lower rate of time preference than economy 2. Let

$$\tilde{U}\left[E\int_{t=0}^{\infty} u_1(\sigma_{1,t}, c_{1,t}) \exp(-\theta_1 t) dt \ , \ E\int_{t=0}^{\infty} u_2(\sigma_{2,t}, c_{2,t}) \exp(-\theta_2 t) dt \right] = 0$$

be the utility possibility frontier of economies 1 and 2, where $\sigma_{i,t}$ is σ of economy i (= 1, 2) in period t and $\tilde{U}(\bullet)$ is a two-dimensional function.

The summation of expected period-utilities indicates that period-utilities are cardinal over time in an economy. Nevertheless, the discounted sums of expected utilities derived from different future paths are not required to be cardinal. They merely express ordinal rankings; for example, a higher value of $E\int_{t=0}^{\infty} u_i(1, c_{i,t}) \exp(-\theta_i t) dt$ simply means that economy i prefers the path that leads to the higher value over another path with a lower value, and

$$E\int_{t=0}^{\infty} u_i(1,0) \exp(-\theta_i t) dt \succ E\int_{t=0}^{\infty} u_i(\sigma_{i,t}, c_{i,t}) \exp(-\theta_i t) dt$$

for any $c_{i,t}$ if $E(\sigma_{i,t}) = 0$ for $t > t'$ ($E\int_{t=0}^{\infty} u_i(\sigma_{i,t}, c_{i,t}) \exp(-\theta_i t) dt$ is expressed here as $-\infty$ in this case). In addition, comparability of utilities among different economies is not required; that is, the utilities of economies 1 and 2 do need not to be comparable in this model. Note however that although an ordinal expression is possible, a cardinal expression is used for simplicity in the following examinations.

As shown in Section 3, the value of ψ indicates the degree of unilateral behavior of economy 1. $\psi = \dfrac{\theta_1 - \theta_2}{2(1-\alpha)\lim_{t\to\infty} \dfrac{\partial y_{1,t}}{\partial k_{1,t}}}$ indicates that economy 1 selects the multilateral state. Conversely, if economy 1 selects the most extreme unilateral state, ψ takes its upper bound value. Let $\bar{c}_{i,\psi,t}$ be the consumption of economy i (= 1, 2) corresponding to a given degree of unilateral behavior of economy 1 (ψ). The points on the utility possibility frontier that achieve sustainable heterogeneity are expressed by

$$\left[E\int_{t=0}^{\infty} u_1\left(1, \bar{c}_{1,\psi,t}\right) \exp\left(-\theta_1 t\right) dt \ ,\ E\int_{t=0}^{\infty} u_2\left(1, \bar{c}_{2,\psi,t}\right) \exp\left(-\theta_2 t\right) dt \right] \ .$$

6.1.2 The shape of the utility possibility frontier

The analyses in Sections 2 and 3 indicate that the points on the utility possibility frontier that achieve sustainable heterogeneity consist only of the curve segment *AB* in Figure 1. Point *A* indicates the multilateral state, and point *B* indicates the upper bound of the unilateral state with appropriate government intervention. As the degree of unilateral behavior of economy 1 (ψ) continuously moves, unilateral states with government's intervention continuously move and thus curve segment *AB* is continuous. Whether the curve segment *AB* slopes downward or upward is not important for the results shown below. The results depend not on the direction but on the monotonicity of the curve segment, that is, the monotonous relationship between ψ and $E\int_{t=0}^{\infty} u_i\left(1, \bar{c}_{i,\psi,t}\right) \exp\left(-\theta_i t\right) dt$.

The government's responses to the unilateral behaviors of economy 1, by which sustainable heterogeneity is achieved, are very limited—only responses corresponding to unilateral states that are chosen. Given a degree of unilateral behavior of economy 1 (i.e., given a value of ψ), only one government response, which is indicated by equation (25), correspondingly can successfully achieve sustainable heterogeneity. Therefore only one point on curve segment *AB* consists of the utility possibility frontier for any given value of ψ.

For simplicity, the possibility of too much government intervention is not considered, and $\bar{g}$ never exceeds the value for sustainable heterogeneity. Hence, all other responses result in a disutility of $-\infty$ for economy 2 because it expects to be exterminated in future such that

$$Eu_2\left(0, c_{2,t}\right) = -\infty$$

after a finite period of time; thus,

$$E\int_{t=0}^{\infty} u_2\left(\sigma_{2,t}, c_{2,t}\right) \exp\left(-\theta_2 t\right) dt = -\infty \ .$$

The utility possibilities of such unsustainable heterogeneity for all values of ψ are depicted by the line *CD* in Figure 1. Given a value of ψ, a part of the line *CD* correspondingly consists of the utility possibility frontier of unsustainable heterogeneity. Let such part of the line *CD* be "the line $C(\psi)D(\psi)$," where point $C(\psi)$ indicates the insufficient intervention that gives the smallest discounted sum of expected utility of economy 1 and point $D(\psi)$ indicates the insufficient intervention that gives the largest. Each point on the line $C(\psi)D(\psi)$ has a corresponding value of $\bar{g}$, all of which are insufficient to achieve sustainable heterogeneity for the given ψ.

As a result, given a degree of unilateral behavior of economy 1 (i.e., given a value of ψ), the utility possibility frontier is composed of the two parts: a point on the curve segment *AB* and the line $C(\psi)D(\psi)$.

6.2 The social welfare function

Here, a social welfare function is assumed to be adopted by the society consisting of the all economies. The assumptions in Arrow (1951) are modified (e.g., the assumption that every individual has a single-peaked preference is added). The social welfare function that is defined on the same space as the utility possibility frontier is

$$\tilde{W}\left[E\int_{t=0}^{\infty} u_1(\sigma_{1,t}, c_{1,t})\exp(-\theta_1 t)dt \ , \ E\int_{t=0}^{\infty} u_2(\sigma_{2,t}, c_{2,t})\exp(-\theta_2 t)dt \right] = W \quad ,$$

where $\tilde{W}(\bullet)$ is a two-dimensional function and W is a variable. Its shape is not specified but it at least satisfies the following typical features: completeness, transitivity, and continuity. Thus, its indifference curves do not cross and are sloping downward to the right. The social welfare function's indifference curves are either convex or concave to the origin. In addition, on any indifference curve, as $c_{i,t} \rightarrow 0$ for any t, $c_{i,t} \rightarrow \infty$ for any t ($i \neq j$). I call this type of social welfare function a "general type social welfare function."

Next, suppose a continuous function such that

$$\tilde{V}\left[E\int_{t=0}^{\infty} u_1(\sigma_{1,t}, c_{1,t})\exp(-\theta_1 t)dt \ , \ E\int_{t=0}^{\infty} u_2(\sigma_{2,t}, c_{2,t})\exp(-\theta_2 t)dt \right] = 0$$

defined on the same space as the utility possibility frontier is. Points satisfying this function are indicated by (v_1, v_2), where $\frac{dv_2}{dv_1} > 0$ and $v_2 = 0$ when $v_1 = 0$, as shown as the dotted line in Figure 2. The indifference curve that crosses the function $\tilde{V}(\bullet) = 0$ at point (v_1, v_2) is

$$\tilde{W}(v_1, v_2) = \tilde{W}\left[E\int_{t=0}^{\infty} u_1(\sigma_{1,t}, c_{1,t})\exp(-\theta_1 t)dt \ , \ E\int_{t=0}^{\infty} u_2(\sigma_{2,t}, c_{2,t})\exp(-\theta_2 t)dt \right] \ .$$

Suppose another type of social welfare function such that, for any point (v_1, v_2), $\tilde{W}(v_1, v_2) = \tilde{W}\left[v_1 \ , \ E\int_{t=0}^{\infty} u_2(\sigma_{2,t}, c_{2,t})\exp(-\theta_2 t)dt \right]$ for any $E\int_{t=0}^{\infty} u_2(\sigma_{2,t}, c_{2,t})\exp(-\theta_2 t)dt \leq v_2$. That is, the indifference curves are vertical if $E\int_{t=0}^{\infty} u_2(\sigma_{2,t}, c_{2,t})\exp(-\theta_2 t)dt \leq v_2$, as shown as the solid lines in Figure 2. I call this type of social welfare function a "Nietzsche type social welfare function." This type of social welfare function is completely different from the general type social welfare function because it does not possess the nature that as $c_{i,t} \rightarrow 0$ for any t, $c_{i,t} \rightarrow \infty$ for any t ($i \neq j$) on any indifference curve. The Nietzsche type social welfare function may be loathed by many people because it indicates that a society should not care about its members being exterminated and does not exclude the social preference that only the strongest should prevail. Although a few people may support the Nietzsche type social welfare function, the probability of violent political conflicts will become extremely high if a society adopts it (see Harashima, 2010).

6.3 The almost unique socially optimal allocation

The socially optimal state is given by the point where the utility possibility frontier and an indifference curve of the social welfare function come in contact with each other. As shown in Section 6.1, however, the utility possibility frontier's shape is not simple. Given a degree of unilateral behavior of economy 1, it is composed of a point on the curve segment AB and the line $C(\psi)D(\psi)$.

Given a value of ψ, let the corresponding point on the curve segment AB be indicated by $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$. Let also $W(\varsigma)$ be W of the indifference curve that crosses the point $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$, and $(\gamma_{1,W(\varsigma)}, \gamma_{2,W(\varsigma)})$ indicate points on the indifference curve $W(\varsigma)$. In addition, let the point $D(\psi)$ be indicated by (δ_1, δ_2), $W(\delta)$ be W of the indifference curve that crosses the point $D(\psi)$, and $(\gamma_{1,W(\delta)}, \gamma_{2,W(\delta)})$ indicate points on the indifference curve $W(\delta)$. As argued in Sections 5.3 and 6.1.1, δ_2 is expressed as $-\infty$.

Because of the nature of the point of sustainable heterogeneity $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$, the following proposition is self-evident.

Proposition 4: If the social welfare function is a general type and its indifference curves are convex to the origin, then only the point $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$ is optimal.

Because it is highly likely that social welfare functions in most societies are general type functions and their indifferent curves are convex to the origin, Proposition 4 indicates that generally the point of sustainable heterogeneity $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$ is uniquely socially optimal.

I next examine social optimality when the social welfare function's indifference curves are concave to the origin.

Lemma 6: If the social welfare function is a general type and its indifference curves are concave to the origin, then only the point $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$ is optimal.

Proof: Because the social welfare function is a general type and its indifference curves are concave to the origin, then $\gamma_{1,W(\varsigma)} > \varsigma_{1,\psi}$ if $\gamma_{2,W(\varsigma)} < \varsigma_{2,\psi}$, and as $\gamma_{1,W(\varsigma)}$ becomes larger, $\gamma_{2,W(\varsigma)}$ becomes smaller. Let $\gamma_{2,W(\varsigma),D}$ be $\gamma_{2,W(\varsigma)}$ when $\gamma_{1,W(\varsigma)} = \delta_1$. Because the social welfare function is not a Nietzsche type, then $\gamma_{2,W(\varsigma),D} > \delta_2 = -\infty$. Therefore, $W(\varsigma) > W(\delta)$. Because the values of W of the indifference curves that cross any other point on the line $C(\psi)D(\psi)$ than the point $D(\psi)$ are less than $W(\delta)$, then only the point $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$ is optimal. ■

Lemma 6 shows that even though the social welfare function's indifference curves are concave to the origin, the point of sustainable heterogeneity $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$ is uniquely determined to be socially optimal if the social welfare function is a general type.

Next, I examine social optimality when the social welfare function is a Nietzsche type. Let $(v_{1,W(\varsigma)}, v_{2,W(\varsigma)})$ be (v_1, v_2) on the indifference curve $W(\varsigma)$. When the social welfare function is Nietzsche type, then $\gamma_{1,W(\varsigma)} \leq v_{1,W(\varsigma)}$, where $\gamma_{1,W(\varsigma)} < v_{1,W(\varsigma)}$ if $\gamma_{2,W(\varsigma)} > v_{2,W(\varsigma)}$ and $\gamma_{1,W(\varsigma)} = v_{1,W(\varsigma)}$ if $\gamma_{2,W(\varsigma)} \leq v_{2,W(\varsigma)}$.

Lemma 7: If the social welfare function is a Nietzsche type, and
(a) if $v_{1,W(\varsigma)} < \delta_1$, then only the point (δ_1, δ_2) is optimal,
(b) if $v_{1,W(\varsigma)} > \delta_1$, then only the point $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$ is optimal, and
(c) if $v_{1,W(\varsigma)} = \delta_1$, then only the points $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$ and (δ_1, δ_2) are optimal.

Proof: Because the social welfare function is a Nietzsche type and thus its indifference curves are concave to the origin, then $\delta_1 = \gamma_{1,W(\delta)}$ and if $\gamma_{2,W(\varsigma)} = \delta_2 = -\infty$, then $v_{1,W(\varsigma)} = \gamma_{1,W(\varsigma)}$. Hence, the following statements apply.
(a) If $v_{1,W(\varsigma)} < \delta_1$, then $\gamma_{1,W(\varsigma)} < \gamma_{1,W(\delta)}$ for $\gamma_{2,W(\varsigma)} = \delta_2 = -\infty$, and thus $W(\varsigma) < W(\delta)$. Because the values of W of the indifference curves that cross any other point on the line $C(\psi)D(\psi)$ than the point $D(\psi)$ are less than $W(\delta)$, then only the point (δ_1, δ_2) is optimal.
(b) If $v_{1,W(\varsigma)} > \delta_1$, then $\gamma_{1,W(\varsigma)} > \gamma_{1,W(\delta)}$ for $\gamma_{2,W(\varsigma)} = \delta_2 = -\infty$, and thus $W(\varsigma) > W(\delta)$. By the same reason as the latter part of (a), only the point $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$ is optimal.
(c) If $v_{1,W(\varsigma)} = \delta_1$, then $\gamma_{1,W(\varsigma)} = \gamma_{1,W(\delta)}$ for $\gamma_{2,W(\varsigma)} = \delta_2 = -\infty$, and thus $W(\varsigma) = W(\delta)$. Again, by the same reason as the latter part of (a), only points $(\varsigma_{1,\psi}, \varsigma_{2,\psi})$ and (δ_1, δ_2) are optimal. ■

Lemma 7 indicates that Nietzsche type social welfare functions are distinguished into the following three categories.
Category (i): only point (δ_1, δ_2) is socially optimal (corresponding to the case $v_{1,W(\varsigma)} < \delta_1$).
Category (ii): only point ($\varsigma_{1,\psi}$, $\varsigma_{2,\psi}$) is only socially optimal (corresponding to the case $v_{1,W(\varsigma)} > \delta_1$).
Category (iii): only points ($\varsigma_{1,\psi}$, $\varsigma_{2,\psi}$) and (δ_1, δ_2) are socially optimal (corresponding to the case $v_{1,W(\varsigma)} = \delta_1$).

Proposition 5: If the social welfare function is either a general or Nietzsche type, the point ($\varsigma_{1,\psi}$, $\varsigma_{2,\psi}$) is only socially optimal allocation for any social welfare function except categories (i) and (iii) Nietzsche type social welfare functions,
Proof: First, by Proposition 4, if the social welfare function is a general type and its indifferent curves are convex to the origin, the point ($\varsigma_{1,\psi}$, $\varsigma_{2,\psi}$) is optimal. Second, by Lemma 6, if the social welfare function is a general type and its indifferent curves are concave to the origin, the point ($\varsigma_{1,\psi}$, $\varsigma_{2,\psi}$) is optimal. Finally, by Lemma 7
, if the social welfare function is a category (ii) Nietzsche type, the point ($\varsigma_{1,\psi}$, $\varsigma_{2,\psi}$) is optimal, whereas if it is either a category (i) or (iii) Nietzsche type, the point (δ_1, δ_2) can be socially optimal. ■

Proposition 5 is important because it indicates that, for almost all generally usable (i.e., preferences are complete, transitive, and continuous) social welfare functions, the point of sustainable heterogeneity ($\varsigma_{1,\psi}$, $\varsigma_{2,\psi}$) is the only socially optimal allocation. In addition, it is highly likely that very few people actually support category (i) or (iii) Nietzsche type social welfare functions because they will generate violent political conflicts (see Harashima, 2010), and they will almost certainly always be in the minority. Hence these types of welfare functions will be rarely adopted in democratic societies where policies are decided by majority.[6] In other words, category (i) or (iii) Nietzsche type social welfare functions would only be adopted by a democratic society when its economic and social situations were extraordinary abnormal. If the situation is not extraordinarily abnormal, category (i) and (iii) Nietzsche type social welfare functions can be excluded, and we can assert that for any generally usable social welfare function, the point of sustainable heterogeneity ($\varsigma_{1,\psi}$, $\varsigma_{2,\psi}$) is uniquely socially optimal.

Proposition 5 provides a clue to solve an important problem in studies of social welfare, that is, the unspecifiability of socially optimal allocation resulting from the difficulty in specifying the shape of the social welfare function. Proposition 5 escapes this problem because the socially optimal allocation is uniquely determined no matter the shape of the social welfare function. Therefore, it is no longer necessary to form a specific social ordering to determine the socially optimal growth path in a heterogeneous population.

7 CONCLUDING REMARKS

Historically, it has been difficult to universally agree upon a criterion for socially optimal allocation because of utility's interpersonal incomparability, Arrow's general possibility theorem, and other factors. This paper examined social optimality in dynamic models with a heterogeneous population and showed that a state exists in which all of the optimality conditions of a heterogeneous population are satisfied. The existence of such a state provides us with additional meaningful information for studying social optimality.

The endogenous growth model in Harashima (2012) shows that sustainable heterogeneity, which is defined as the state at which all optimality conditions of all

[6] As shown in Section 6.2, it is assumed that the assumptions in Arrow (1951) are modified.

heterogeneous households are satisfied, is uniquely determined to be the socially optimal allocation for almost all generally usable social welfare functions. The exogenous growth model in this paper shows the same result. The only exceptions are some variants of a Nietzsche type social welfare function, which will rarely be adopted in democratic societies unless the economic and social situations are extraordinarily abnormal. Sustainable heterogeneity is achievable even if the most advantaged household behaves unilaterally if the government appropriately intervenes. The uniquely determined socially optimal allocation in a heterogeneous population can be accomplished without specifying the shape of the social welfare function, and therefore, the problem of unspecifiability of social optimality can be solved.

Sustainable heterogeneity as the unique socially optimal allocation will have important implications to currently passionately disputed issues such as the Occupy Wall Street movement, anti-globalization (e.g., Klein, 2000; Stiglitz, 2002), anti-market fundamentalism (e.g., Gray, 1998; Stiglitz, 2002, 2009; Soros, 2008), and true measures of happiness (e.g., Sen, 1976; Arrow et al., 1995). In addition, sustainable heterogeneity will provide additional theoretical foundations for debt relief, wealth taxes, progressive taxation, and international aid. On the other hand, sustainable heterogeneity also indicates that there is a unique sustainable level of inequality in consumption.

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Figure 1 ***The utility possibility frontiers of sustainable and unsustainable heterogeneity***

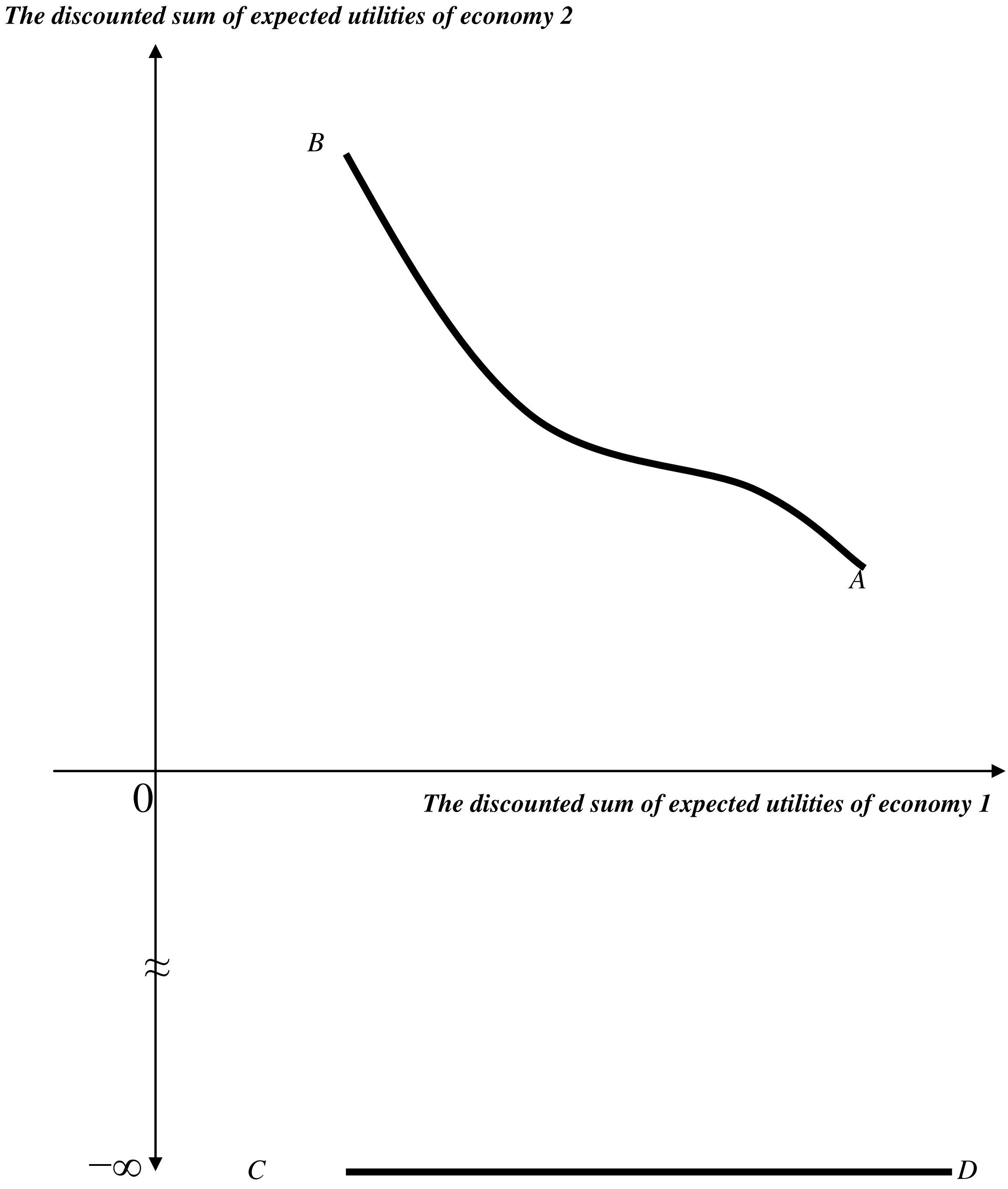

Figure 2 ***Indifference curves of a Nietzsche type social welfare function***