Judgment aggregation and agenda manipulation

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Abstract

When individual judgments ('yes' or 'no') on some propositions are aggregated into collective judgments, the agenda setter can sometimes reverse a collective judgment by changing the set of propositions under consideration (the *agenda*). I define different kinds of agenda manipulation, and axiomatically characterize the aggregation rules immune to each kind. Two axioms emerge as central for preventing agenda manipulation: the familiar *independence* axiom, requiring propositionwise aggregation, and the axiom of *implicit consensus preservation*, requiring the respect of any (possibly implicit) consensus. I prove that these axioms can almost never be satisfied together by a (non-degenerate) aggregation rule.

1 Introduction

Imagine the board of a central bank has to form collective judgments ('yes' or 'no') on some propositions about the economy, such as the proposition that prices will rise. Disagreements on a proposition are resolved by taking a majority vote. The chair of the board believes that prices will rise, but knows that the board's majority thinks otherwise. To prevent a collective 'no inflation' judgment, he (or she) removes the proposition 'prices will rise' from the agenda, while putting two new propositions on the agenda: 'GDP will grow', and 'growth implies inflation', i.e., '*if* GDP will grow, *then* prices will rise'. When it comes to voting, the two new propositions are each approved by a (different) majority. The chair is pleased, since the collective beliefs in growth and in growth implying inflation logically entail a belief in inflation. The agenda manipulation

	initial agenda	manipulated agenda	
	Inflation?	*Growth?*	*Growth implies inflation?*
Member 1	Yes	Yes	Yes
Member 2	No	No	Yes
Member 3	No	Yes	No
Majority	No	Yes	Yes

Figure 1: An agenda manipulation reversing the collective judgment on inflation

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has successfully turned an (explicit) 'no inflation' judgment into an (implicit) 'inflation' judgment. Table 1 illustrates this reversal in the case of a three-member board.

This example shows that majority voting is vulnerable to agenda manipulation. Which rules (if any) are immune to agenda manipulation? This paper defines different types of agenda manipulation, and characterizes the aggregation rules immune to each type. Two axioms on the aggregation rule turn out to play key roles in ensuring manipulation-immunity: *independence* (i.e., the analogue for judgment aggregation of Arrow's axiom of *independence of irrelevant alternatives* for preference aggregation), and *implicit consensus preservation* (i.e., the principle of respecting unanimity, in a strengthened version extended to implicit judgments). In an impossibility theorem, I prove that these two axioms can almost never be satisfied by an aggregation rule which is non-dictatorial (as well as having an unrestricted domain and generating rational collective judgments). This impossibility theorem is also of interest independently of the question of agenda manipulation, because the two axioms need not be motivated by considerations of agenda manipulation. The paper therefore has two main contributions: an analysis of agenda manipulation, and the proof of a new impossibility theorem.

The present analysis of agenda manipulation fills a gap in the literature on judgment aggregation, in which agenda manipulation is often mentioned informally, and is treated in a partially formal way by Dietrich (2006).[2] Relatedly, the judgment aggregation literature has addressed rule manipulation, specifically manipulation of the order in which a sequential aggregation rule considers the propositions (List 2004, Dietrich and List 2007c, Nehring, Pivato and Puppe 2013), and strategic voting (Dietrich and List 2007b, Dokow and Falik 2012; see also Nehring and Puppe 2002).

The paper's second contribution – a new impossibility theorem – connects to a series of impossibility results in the field; see for instance List and Pettit (2002), Pauly and van Hees (2006), Dietrich (2006), Dietrich and List (2007a), Mongin (2008), Nehring and Puppe (2008), Duddy and Piggins (2013), and papers in the *Symposium on Judgment Aggregation* in Journal of Economic Theory (C. List and B. Polak eds., 2010). Of particular interest to us is a theorem which generalizes Arrow's Theorem from preference to judgment aggregation (Dietrich and List 2007a and Dokow and Holzman 2010, both building on Nehring and Puppe 2010 and strengthening Wilson 1975). Our new theorem shows that if in the generalized Arrow theorem the Pareto-type unanimity condition is extended towards implicit agreements, then, perhaps surprisingly, the dictatorship conclusion now holds for *almost all* agendas, not just agendas of a quite special structure.

I finally mention a growing branch of the literature which constructs concrete judgment aggregation rules, and which I hope to sensitize to the problem of agenda manipulation. Many proposals have been made. Our analysis will imply that almost all proposals are vulnerable to agenda manipulation, yet in different ways and to different degrees. The proposals include premise- and conclusion-based rules (e.g., Kornhauser and Sager 1986, List and Pettit 2002, Dietrich 2006), sequential rules (e.g., List 2004, Dietrich and List 2007b), distance-based rules (e.g., Konieszni and Pino-Perez 2002, Pigozzi 2006, Miller and Osherson 2008, Eckert and Klamler 2009, Lang *et al.* 2011, Duddy and Piggins 2012a), quota rules with well-calibrated acceptance thresholds and

[2] Dietrich's (2006) analysis is different, and does not model the class of feasible agendas (from which the agenda setter chooses) and the corresponding class of aggregation rules. Nonetheless, two parallels in the findings will emerge, as pointed out below.

various degrees of collective rationality (e.g., Dietrich and List 2007b; see also Nehring and Puppe 2010), aggregation rules for restricted domains (Dietrich and List 2010, Pivato 2009), Borda-like and scoring rules (Zwicker 2011, Dietrich forthcoming, Duddy and Piggins 2012b), and rules which approximate the majority judgment set when it is inconsistent (Nehring, Pivato and Puppe 2013).

2 The framework

I now introduce the framework, following List and Pettit (2002) and more precisely Dietrich (2007 and forthcoming). A group of n individuals, labelled $i = 1, ..., n$, needs to decide which of certain interconnected propositions to collectively 'believe' or 'accept'. We assume that $n \geq 3$.[3]

The agenda. The set of propositions under consideration – the *agenda* – is subdivided into *issues*, i.e., pairs of a proposition and its negation, such as 'it will snow' and 'it won't snow'. Rationally, an agent accepts a proposition from each issue ('completeness'), and respects any logical interconnections between propositions ('consistency'). Writing '$\neg p$' for the negation of a proposition 'p', the agenda takes the form $X = \{p, \neg p, q, \neg q, ...\}$, with issues $\{p, \neg p\}$, $\{q, \neg q\}$, etc. Formally:

Definition 1 *An **agenda** is a set X (of 'propositions') with a structure of*

(a) *binary 'issues' $\{p, p'\}$ which partition X (where the members p and p' of an issue are the 'negations' of each other, written $p = \neg p'$ and $p' = \neg p$),*

(b) *interconnections, i.e., a specification of which judgment sets are rational, or formally, a system $\mathcal{J}$ of ('rational') judgment sets $J \subseteq X$, each containing exactly one member from any issue,*

where (in this paper) X is finite and $|\mathcal{J}| > 1$.[4]

The propositions could for instance be specified as sentences of formal logic. Examples are the agendas

$$X = \{p, \neg p, q, \neg q\}, \quad (1)$$
$$X = \{p, \neg p, q, \neg q, p \wedge q, \neg(p \wedge q)\}, \quad (2)$$
$$X = \{p, \neg p, p \rightarrow q, \neg(p \rightarrow q), q, \neg q\}, \quad (3)$$

where p and q are two atomic sentences (such as 'it will rain' and 'it will be hot', and $p \wedge q$ and $p \rightarrow q$ are the compound propositions 'p and q' and 'if p then q'. The structure of any of these agendas – issues and interconnections – is inherited directly from logic and so need not be specified explicitly (an advantage of logically specified agendas). It is for instance clear that for agenda (1) $\{p, \neg q\}$ is a rational judgment set, while for agenda (2) $\{p, q, \neg(p \wedge q)\}$ is not rational. For each of these agendas there are exactly

[3] All results except the 'only if' part of Theorem 1 only require that $n \geq 2$.

[4] The finiteness restriction is mainly for convenience; most results do not require it. The condition that $|\mathcal{J}| > 1$ prevents trivial agendas which offer no choice. Algebraically speaking, the agenda is the structure $X \equiv (X, \mathcal{I}, \mathcal{J})$ where $\mathcal{I}$ is the set of issues (or equivalently, the structure $X \equiv (X, \neg, \mathcal{J})$ where $\neg$ is the negation operator on X, which is indeed interdefinable with $\mathcal{I}$).

four rational judgment sets, given by the four possible combinations of truth values of p and q.

Given an agenda X, an individual's **judgment set** is the set $J \subseteq X$ of propositions he accepts. It is **complete** if it contains a member of each issue $\{p, \neg p\}$, and **consistent** if it is a subset of a rational judgment set. Note that the complete and consistent judgment sets are precisely the rational ones: $\mathcal{J} = \{J \subseteq X : J$ is complete and consistent$\}$. Typically, every proposition $p \in X$ is **contingent**, i.e., is neither a **contradiction** (for which $\{p\}$ is inconsistent), nor a **tautology** (for which $\{\neg p\}$ is inconsistent). A proposition $p \in X$ (or set $S \subseteq X$) **entails** another proposition $p' \in X$ (or set $S' \subseteq X$) if every evert rational judgment set containing p (resp. including S) also contains p' (resp. includes S').

As an important example, the **preference agenda** for a (finite non-empty) set of alternatives A is the set of sentences $X = \{xPy : x, y \in A, x \neq y\}$, where xPy reads 'x is better than y'. The negation of xPy is of course $\neg xPy = yPx$. The interconnections are given by the usual conditions defining strict linear orders, i.e., transitivity, asymmetry and connectedness. Formally, note that judgment sets $J \subseteq X$ can be identified with (irreflexive) binary relations $\succ$ over A: to any $J \subseteq X$ corresponds the relation $\succ$ satisfying $xPy \in J \Leftrightarrow x \succ y$. We can thus apply relation-theoretic notions (such as transitivity) to judgment sets, and write $\mathcal{J} = \{J \subseteq X : J$ is transitive, asymmetric and connected$\}$.

Aggregation rules. An **aggregation rule** for an agenda X is a function F which to every profile of 'individual' judgment sets $(J_1, ..., J_n)$ (from some domain, usually $\mathcal{J}^n$) assigns a 'collective' judgment set $F(J_1, ..., J_n)$. For instance, majority rule is given by

$$F(J_1, ..., J_n) = \{p \in X : \text{ more than half of } J_1, ..., J_n \text{ contain } p\}$$

and generates inconsistent collective judgment sets for many agendas and profiles. We shall be concerned with aggregation rules whose individual inputs and collective output are rational. These rules define functions $F : \mathcal{J}^n \to \mathcal{J}$.

Notation. I write $\mathcal{J}_X$ for $\mathcal{J}$ when there is ambiguity as to the agenda X in question. The negation-closure of a set Y of propositions is denoted $Y^{\pm} \equiv \{p, \neg p : p \in Y\}$. A judgment set is often abbreviated by concatenating its members in any order (e.g., $p\neg q$ is short for $\{p, \neg q\}$). We can thus concisely write the agenda (1) as $X = \{p, q\}^{\pm}$ and its system of rational judgment sets as $\mathcal{J} = \{pq, p\neg q, \neg pq, \neg p\neg q\}$.

Subagendas and superagendas. We call an agenda X a **subagenda** of another X' (and X' an **extension** or **superagenda** of X) if $X \subseteq X'$, where the structure of X – its issues $\{p, \neg p\}$ and its interconnections – is given by that of X' restricted to X, i.e., each issue of X is one of X' and the set of rational judgment sets for X is $\mathcal{J}_X = \{J \cap X : J \in \mathcal{J}_{X'}\}$. So, subagendas of an agenda X' are subsets $X \subseteq X'$ closed under negation, with structure inherited from X'. For instance, agenda (1) is a subagenda of agendas (2) and (3), and the preference agenda for a set of a alternatives A is a subagenda of the preference agenda for any larger set of alternatives $A' \supseteq A$.

3 Two axioms which limit agenda manipulation

I now state two axioms on an aggregation rule $F : \mathcal{J}^n \to \mathcal{J}$ for a given agenda X. Each of them helps to limit agenda manipulation, as formally shown in Section 6.

The first axiom is the classical condition of 'independence' or 'propositionwise aggregation'. It requires the collective judgment on any given proposition in the agenda to depend solely on the individuals' judgments on *this* proposition – the exact analogue for judgment aggregation of Arrow's axiom of *independence of irrelevant alternatives* for preference aggregation.[5]

Independence: For all propositions $p \in X$ and profiles $(J_1, ..., J_n), (J'_1, ..., J'_n) \in \mathcal{J}^n$, if, for every individual i, $p \in J_i \Leftrightarrow p \in J'_i$, then $p \in F(J_1, ..., J_n) \Leftrightarrow p \in F(J'_1, ..., J'_n)$.

This axiom is normatively as controversial as Arrow's analogous axiom. It is known to be necessary for preventing strategic voting. We here focus on its role in preventing agenda manipulation. As shown in Section 6, it is *necessary* (and under a mild coherence assumption also *sufficient*) for preventing an agenda manipulator from being able to reverse explicit collective judgments. In short, if independence is violated, then the collective judgment on a proposition $p \in X$ depends on other propositions in the agenda, and can thus be reversed by the agenda setter through adding or removing other propositions.

The second axiom can be stated in three versions. The first version draws on the *scope* of the agenda X (Dietrich 2006). Informally, the scope is the superagenda containing all propositions that can be constructed from propositions in X using Boolean connectives, such as the propositions 'p and q', 'not (p and q)', 'p or (q and r)', where $p, q, r \in X$. In other words, the scope of X is the superagenda containing any proposition whose truth value is settled by the truth values of the propositions in X. So, any rational judgment set $J \in \mathcal{J}$ settles any proposition p in the scope: either J entails p or J entails $\neg p$. Since the notion of the scope is intuitively clear, I relegate its formal definition to Section 5, and proceed immediately to the statement of the condition.

Implicit consensus preservation (version 1): For every proposition p in the scope of the agenda X, if each judgment set in a profile $(J_1, ..., J_n) \in \mathcal{J}^n$ entails p, so does the collective judgment set $F(J_1, ..., J_n)$.

Normatively, this axiom embodies a strong form of 'respect for unanimity', as it also covers *implicit* unanimous agreements, which pertain to propositions outside the agenda. In Section 6 I give two formal arguments for the axiom, both related to the prevention of agenda manipulation. Let me here sketch each argument informally. Firstly, the axiom is invariant to redescribing ('reframing') the decision problem: the set of propositions p on which consensus must be preserved stays the same if the agenda X is replaced by a new one X' which has the same scope and is thereby equivalent – for instance by replacing two propositions p and q in X (and their negations) by the four propositions $p \wedge q$, $p \wedge \neg q$, $\neg p \wedge q$ and $\neg p \wedge \neg q$ (and their negations).[6] Secondly, the

[5] Arrow's axiom is equivalent to ours applied to the preference agenda (defined in Section 2).

[6] I thank Marcus Pivato for bringing this fact to my attention.

axiom prevents a particularly bad form of agenda manipulation, in which unanimously supported collective judgments are being reversed.

A second and equivalent version of the axiom draws on the notion of a *feature* of a judgment set. Examples are the feature of containing a given proposition $p \in X$, and the feature of containing at most two propositions from a given set $S \subseteq X$. We may identify each feature with the set $\mathcal{K} \subseteq \mathcal{J}$ of judgment sets having the feature. In its second version, our axiom requires the collective judgment set to have any feature which all individual judgment sets have:

Implicit consensus preservation (version 2): For every $\mathcal{K} \subseteq \mathcal{J}$ (every feature), if each judgment set in a profile $(J_1, ..., J_n) \in \mathcal{J}^n$ belongs to $\mathcal{K}$ (has the feature), so does the collective judgment set $F(J_1, ..., J_n)$.

Intuitively, the versions 1 and 2 are equivalent because features correspond to propositions in the scope: for instance, the feature of containing two given propositions q and r from X corresponds to the proposition 'q and r' from the scope. In its third version, the axiom requires the collective judgment set to be selected from the set of individual judgment sets:

Implicit consensus preservation (version 3): For every profile $(J_1, ..., J_n) \in \mathcal{J}^n$, the collective judgment set $F(J_1, ..., J_n)$ belongs to $\{J_1, ..., J_n\}$.

This condition has no 'dictatorship flavour' since the individual whose judgment set becomes the collective one may of course vary with the profile: he could for instance be the 'median' voter in a suitably defined sense.

Proposition 1 *The three versions of implicit consensus preservation are equivalent.*

4 The impossibility theorem

I now combine the two just-defined axioms into an impossibility result. An aggregation rule $F : \mathcal{J}^n \to \mathcal{J}$ is **dictatorial** if there is an individual i such that $F(J_1, ..., J_n) = J_i$ for all $J_1, ..., J_n \in \mathcal{J}$. As usual in the theory, the structure of the agenda matters. The agenda X is called **nested** if it takes the very special form $X = \{p_1, p_2, ..., p_m\}^{\pm}$ where m is the number of issues and where p_1 entails p_2, p_2 entails p_3, and so on (it follows that $\neg p_m$ entails $\neg p_{m-1}$, $\neg p_{m-1}$ entails $\neg p_{m-2}$, and so on). For instance, in a central bank meeting, the agenda $X = \{p_1, ..., p_{10}\}^{\pm}$ in which p_j is the proposition 'prices will grow by j percent at most' is nested, as p_j entails p_{j+1} for each j (< 10). Similarly, in a hiring committee meeting, the agenda $X = \{p_1, ..., p_5\}^{\pm}$ in which p_j is the proposition 'candidate Smith will publish fewer than j papers per year' is nested. But most relevant agendas are not nested. The agendas (1), (2) and (3) are not nested, and also the preference agenda defined in Section 2 is not nested (as long as there are more than two alternatives).

Theorem 1 *There is no independent, implicit consensus preserving and non-dictatorial aggregation rule $F : \mathcal{J}^n \to \mathcal{J}$ if and only if the agenda X is non-nested (and non-tiny[7]).*

[7] That is, X has more than four propositions (two issues). To be precise, X is **non-tiny** if it

To paraphrase the result, for almost all agendas our two axioms cannot be jointly satisfied by any non-dictatorial aggregation rule. So, far more agendas imply impossibility than in the generalized Arrow theorem mentioned in the introduction – as a consequence of requiring preservation of *implicit* consensus instead of the standard Pareto-type unanimity condition. Judgment aggregation theorists will be curious to know whether the notion of a non-nested agenda is related to a familiar kind of agenda. Non-nested agendas can be related to *non-simple* agendas.[8]

5 The scope of an agenda

To prepare the treatment of agenda manipulation, I now analyse the scope of an agenda. In principle, one may imagine two – as we shall see, equivalent – approaches to defining the scope of an agenda X, which can be stated informally as follows:

(a) the scope consists of all propositions constructible from propositions in X (by forming conjunctions for instance);
(b) the scope consists of all propositions on which the judgment ('yes' or 'no') is determined by the judgments on the propositions in X.

I choose the approach (a), but also briefly discuss the approach (b) (which is perhaps more in line with Dietrich's 2006 definition[9]). Starting with some natural terminology, I call an agenda

- **closed under conjunction** if for any propositions $p, q \in X$ there exists a proposition r (the **conjunction** of p and q) such that any rational judgment set contains r if and only if it contains both p and q,
- **closed under disjunction** if for any propositions $p, q \in X$ there exists a proposition r (the **disjunction** of p and q) such that any rational judgment set contains r if and only if it contains p or q (possibly both).

These two closure properties are in fact equivalent, as is seen shortly. In practice, an agenda is specified so as to be **redundancy-free**: no two propositions p and q are equivalent, i.e., entail each other.

Remark 1 *In any (redundancy-free) agenda X, the conjunction resp. disjunction of two propositions p and q, if existing, is unique and denoted by $p \wedge q$ resp. $p \vee q$.*

has more than four propositions (two issues), counting only contingent propositions and identifying equivalent propositions. In practice, agendas of course contain only contingent propositions and no equivalent propositions. (Propositions are equivalent if they entail each other.)

[8] An agenda X is **non-simple** if it has a subset Y of more than two elements which is *minimal inconsistent*, i.e., is inconsistent but becomes consistent if any member is removed. For instance, the preference agenda for a set of more than two alternatives (Section 2) is non-simple, since any 'cyclical' subset $Y = \{xPy, yPz, zPx\}$ is minimal inconsistent. I show in the appendix that a (non-tiny) agenda X is non-nested *if and only if* it satisfies a condition only subtly distinct from the definition of non-simplicity: X has a subset Y of more than two elements such that $(Y \backslash \{p\}) \cup \{\neg p\}$ is consistent for each $p \in Y$. Strengthening 'a subset Y' to 'an inconsistent subset Y' turns this characterization of non-nestedness into one of non-simplicity. This gives an idea of *how* non-nestedness weakens non-simplicity.

[9] In Dietrich's (2006) framework, defining the scope was a trivial exercise. That framework explicitly includes the language in which propositions are formed, so that the propositions in the scope already 'exist' and thus need not be newly introduced through an agenda extension.

Remark 2 *An agenda X is closed under conjunction if and only if it is closed under disjunction. I then call it* ***closed*** *simpliciter.*

To form the scope of an agenda, we simply 'close' the agenda. The following result ensures that this is always possible, in a unique way:

Proposition 2 *Every agenda X has a closure, i.e., a minimal closed superagenda, which is moreover unique (up to relabelling[10]).*

Definition 2 *The* ***scope*** *of an agenda X is its (up to relabelling uniquely existing) minimal closed superagenda, denoted $\overline{X}$.*

The following lemma gives a clear idea of the totality of propositions in the scope. It uses conjunctions/disjunctions of *any number* of propositions, which are defined like conjunctions/disjunctions of two propositions.[11]

Lemma 1 *Every proposition p in the scope of a (redundancy-free) agenda X can be written in disjunctive normal form, i.e., as a disjunction of conjunctions of propositions in X; for instance, $p = \vee_{J \in \mathcal{J}^p} \wedge_{q \in J} q$, where $\mathcal{J}^p := \{J \in \mathcal{J}_X : J \text{ entails } p\}$.*

I now show that any agenda X settles its scope: the judgments within X determine those within the entire scope $\overline{X}$. Formally, an agenda X is said to **settle** a superagenda X' if each rational judgment set $J \in \mathcal{J}_X$ has only one extension $J' \in \mathcal{J}_{X'}$.

Proposition 3 *Every agenda settles its scope.*

In fact, a stronger result can be shown: the scope of an agenda X is the (up to relabelling unique) *maximal* superagenda which is settled by X (and is redundancy-free outside X, i.e., contains no two equivalent propositions outside X). We could therefore have used an alternative and equivalent definition of the scope:

Definition 3 *The* ***scope*** *of an agenda X is the (up to relabelling uniquely existing) maximal superagenda settled by X (and redundancy-free outside X).*

Finally, the scope carries a familiar algebraic structure:

Remark 3 *Any closed (redundancy-free) agenda – for instance the scope of an agenda – is a Boolean algebra w.r.t. the relation of entailment between propositions, with the meet, join, and complement given by the conjunction, disjunction, and negation, respectively.*[12]

[10] Uniqueness up to relabelling means that between any two minimal closed superagendas X' and X'' there exists an (agenda) isomorphism which is constant on X. Of course, $f : X' \to X''$ is an isomorphism if it is bijective and preserves the agenda structure, i.e., the issues ($f(\neg p) = \neg f(p)$ for all $p \in X$) and the interconnections ($J \in \mathcal{J}_X \Leftrightarrow f(J) \in \mathcal{J}_{X'}$ for all $J \subseteq X$).

[11] Generalizing the earlier definition, I call a proposition r the **conjunction** (resp. **disjunction**) of a set of propositions S if any rational judgment set contains r if and only if it contains all (resp. some) $p \in S$.

[12] Without assuming redundancy-freeness, the agenda is a Boolean algebra modulo equivalence between propositions.

Recall that Boolean algebras are defined as follows. First, a **lattice** is a partially ordered set $\mathcal{L} \equiv (\mathcal{L}, \leq)$ such that any two elements $p, q \in \mathcal{L}$ have a meet $p \wedge q$ (greatest lower bound) and a join $p \vee q$ (smallest upper bound). It is **distributive** if $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ and $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ for all $p, q, r \in \mathcal{L}$. A **Boolean algebra** is a distributive lattice $(\mathcal{L}, \leq)$ such that $\mathcal{L}$ contains a greatest element $\top$ (the 'top' or 'tautology') and a bottom $\perp$ (the 'bottom' or 'contradiction'), and every element has an algebraic complement, i.e., an element whose join with p is $\top$ and whose meet with p is $\perp$. The paradigmatic Boolean algebras are the set-theoretic ones: here there exists a set Ω such that $\mathcal{L} \subseteq 2^{\Omega}$, $\leq = \subseteq$, $\top = \Omega$, $\perp = \varnothing$, and the meet, join and complement are given by the set-theoretic intersection, union and complement. By Stone's representation theorem, every Boolean algebra is isomorphic to such a set-theoretic one. Another example is the Boolean algebra generated from a logic, i.e., the set of sentences modulo logical equivalence (where the logic includes classical negation and conjunction, which induce the algebraic negation, meet and join).

6 Agenda manipulation

I now introduce and analyse different forms of (immunity to) agenda manipulation. As a by-product, the analysis shows that our two axioms – independence and implicit consensus preservation – each play a role in preventing agenda manipulation (so that Theorem 1 can be viewed as an impossibility result about preventing agenda manipulation).

In constructing an agenda, the agenda setter uses propositions from some 'universal' set of propositions: an agenda $\mathcal{L}$, which we take to be closed and redundancy-free. $\mathcal{L}$ could be as large as an entire language, or as small as a set of propositions on a relevant topic (such as a given court trial). The agenda setter thus chooses a (sub)agenda $X \subseteq \mathcal{L}$. we do not need to assume that *all* agendas $X \subseteq \mathcal{L}$ can be chosen. Rather, we consider a set $\mathcal{X}$ of agendas $X \subseteq \mathcal{L}$ deemed feasible/possible. $\mathcal{X}$ could for instance contain all agendas $X \subseteq \mathcal{L}$, or just all agendas $X \subseteq \mathcal{L}$ of size at most 6. All we assume is that $\mathcal{X}$ contains at least each binary agenda $\{p, \neg p\} \subseteq \mathcal{L}$. Since $\mathcal{L}$ is closed, the scope $\overline{X}$ of an agenda $X \in \mathcal{X}$ is again a subagenda of $\mathcal{L}$, namely the smallest closed one that includes X[13] – though quite possibly $\overline{X} \notin \mathcal{X}$ since $\overline{X}$ might be very rich and therefore infeasible.[14]

Two agendas $X, X' \in \mathcal{X}$ are **equivalent** if they have identical scope $\overline{X} = \overline{X'}$, i.e., essentially represent the same decision problem, though framed differently. Given two equivalent agendas $X, X' \in \mathcal{X}$, each judgment set for X is equivalent to one for X', and each aggregation rule for X is equivalent to one for X'. Formally, any $J \in \mathcal{J}_X$ is **equivalent** to the unique $J^* \in \mathcal{J}_{X'}$ such that J and J^* entail each other; and any aggregation rule $F : \mathcal{J}_X^n \to \mathcal{J}_X$ is **equivalent** to the unique rule $F' : \mathcal{J}_{X'}^n \to \mathcal{J}_{X'}$ defined as the image of F if any judgment set $J \in \mathcal{J}_X$ is mapped to the equivalent one $J^* \in \mathcal{J}_{X'}$, i.e.,

$$[F(J_1, ..., J_n)]^* = F'(J_1^*, ..., J_n^*) \text{ for all } J_1, ..., J_n \in \mathcal{J}_X.$$

One may argue that an axiom imposed on aggregation should ideally be **description-invariant**: that is, whenever two agendas $X, X' \in \mathcal{X}$ are equivalent (i.e., $\overline{X} = \overline{X'}$),

[13] Or so we may assume by suitably labelling the propositions in the scope.

[14] All results of this section remain true if we allow $\mathcal{L}$ and any $X \in \mathcal{X}$ to be infinite, i.e., to be agendas in a generalized sense without finiteness restriction.

then an aggregation rule $F : \mathcal{J}_X^n \to \mathcal{J}_X$ satisfies the axiom if and only if the equivalent rule $F' : \mathcal{J}_{X'}^n \to \mathcal{J}_{X'}$ does so.[15] One of our two axioms does indeed have this virtue:

Proposition 4 *Implicit consensus preservation is a description-invariant axiom.*

Description-invariance is a rare feature: it is violated by the independence axiom, and by the standard consensus axiom, which quantifies over agenda propositions rather than scope propositions. So, from the perspective of description-invariance, implicit consensus preservation is more natural than the standard consensus axiom. Two other description-invariant axioms are the anonymity axiom and the strengthened independence axiom used in Theorem 3 and Proposition 5 below.

We now turn to a different question: is the aggregation rule itself (rather than an axiom on it) sensitive to the agenda choice? That is, can the agenda setter reverse collective judgments by changing the agenda? In analysing this question, we consider not just agenda modifications which merely redescribe (reframe) the decision problem, but also ones which change the scope. The agenda setter could thus pick any agenda from $\mathcal{X}$. The question of whether the agenda setter can influence collective judgments obviously depends on which aggregation rules would be used for the various feasible agendas. That is, it depends on what I call the **aggregation system**, i.e., a family $(F_X)_{X\in\mathcal{X}}$ of aggregation rules $F_X : \mathcal{J}_X^n \to \mathcal{J}_X$ (where F_X represents the rule used if the agenda is $X \in \mathcal{X}$).[16]

I now introduce three conditions on the aggregation system, each one requiring immunity to agenda manipulation of a particular kind. The first condition states that the agenda setter cannot reverse any *explicit* collective judgment, i.e., any collective judgment on a proposition *in the agenda*:

Agenda-invariance: Any two feasible agendas $X, X' \in \mathcal{X}$ lead to the same collective judgment on any proposition $p \in X \cap X'$, i.e., for all $J_1, ..., J_n \in \mathcal{J}_{X\cup X'}$,

$$p \in F_X(J_1 \cap X, ..., J_n \cap X) \Leftrightarrow p \in F_{X'}(J_1 \cap X', ..., J_n \cap X').$$

Here, $J_i \cap X$ and $J_i \cap X'$ are the judgment sets submitted by individual i under the agendas X and X', respectively. By the next theorem, agenda-invariance forces each rule F_X to be independent. It also forces the rules F_X to be related to each other in a systematic way – intuitively because otherwise collective judgments are easily reversed by changing the agenda. Formally, the aggregation system $(F_X)_{X\in\mathcal{X}}$ must be **coherent**: for all feasible agendas $X, X' \in \mathcal{X}$ with $X \subseteq X'$, the rule F_X coheres with $F_{X'}$, i.e., any $J_1, ..., J_n$ in $\mathcal{J}_X$ have *at least some* extensions $J'_1 \supseteq J_1, ..., J'_n \supseteq J_n$ in $\mathcal{J}_{X'}$ such that $F_{X'}(J'_1, ..., J'_n) \supseteq F_X(J_1, ..., J_n)$. Coherence means that if the agenda is extended from X to X', then the new collective outcome *can* be compatible with the old outcome, i.e., is compatible with that outcome for *at least one* possible new profile. Coherence strikes as plausible.

[15] To be entirely precise, one can identify an axiom with the set of all aggregation rules satisfying it, i.e., the set $\mathcal{A} = \{F : F$ is an aggregation rule $\mathcal{J}_X^n \to \mathcal{J}_X$ for some agenda $X \in \mathcal{X}$ and F satisfies the axiom$\}$. An axiom is then simply a set $\mathcal{A} \subseteq \cup_{X\in\mathcal{X}}(\mathcal{J}_X)^{\mathcal{J}_X^n}$. It is called description-invariant if for all equivalent aggregation rules $F, F' \in \cup_{X\in\mathcal{X}}(\mathcal{J}_X)^{\mathcal{J}_X^n}$, $F \in \mathcal{A} \Leftrightarrow F' \in \mathcal{A}$.

[16] An aggregation system could be viewed as a single 'extended aggregation rule' with an additional parameter, the agenda.

Theorem 2 *An aggregation system* $(F_X)_{X\in\mathcal{X}}$ *is agenda-invariant if and only if it is coherent and each rule* F_X *is independent.*

Context-invariance only prevents the agenda setter from reversing *explicit* collective judgments, on proposition *in* the agenda. We now turn to a stronger requirement, which also excludes the reversal of *implicit* collective judgments, on propositions *outside* the agenda. For instance, if the agenda $X = \{p, \neg p, q, \neg q\}$ leads the collective judgment set $\{p, \neg q\}$, so that the collective implicitly accepts the proposition $p \wedge \neg q$ in the scope $\overline{X}$, then according to the new condition the acceptance of $p \wedge \neg q$ cannot be reversed by using another agenda X'.

Full agenda-invariance: Any two feasible agendas $X, X' \in \mathcal{X}$ lead to the same collective judgment on any proposition $p \in \overline{X} \cap \overline{X'}$, i.e., for all $J_1, ..., J_n \in \mathcal{J}_{X\cup X'}$,

$$F_X(J_1 \cap X, ..., J_n \cap X) \text{ entails } p \Leftrightarrow F_{X'}(J_1 \cap X', ..., J_n \cap X') \text{ entails } p.$$

Here, $J_i \cap X$ (resp. $J_i \cap X'$) is again the judgment set submitted by i under the agenda X (resp. X'). While ordinary agenda-invariance leads to independence and coherence, full agenda-invariance leads to stronger versions of independence and coherence. How are these versions defined? First, an aggregation rule F for an agenda X is called **independent on** Y ($\subseteq \overline{X}$) if the collective judgment on any proposition in Y only depends on the individuals' judgments on *this* proposition: for all propositions $p \in Y$ and all profiles $(J_1, ..., J_n)$ and $(J_1', ..., J_n')$ in the domain, if for each individual i J_i entails p if and only if J_i' entails p, then $F(J_1, ..., J_n)$ entails p if and only if, $F(J_1', ..., J_n')$ entails p. Setting $Y = X$ yields standard independence. Full agenda-invariance leads to independence on the full scope $\overline{X}$ rather than on X.

Second, I strengthen the coherence notion. Note that, for agendas $X, X' \in \mathcal{X}$ such that $X \subseteq X'$, or more generally $\overline{X} \subseteq \overline{X'}$, every judgment set $J' \in \mathcal{J}_{X'}$ entails exactly one judgment set $J \in \mathcal{J}_X$. I call an aggregation system $(F_X)_{X\in\mathcal{X}}$ **fully coherent** if, for all feasible agendas $X, X' \in \mathcal{X}$ with $\overline{X} \subseteq \overline{X'}$, the rule F_X coheres with $F_{X'}$, i.e., for any $J_1, ..., J_n$ in $\mathcal{J}_X$ there exist *at least some* $J_1', ..., J_n'$ in $\mathcal{J}_{X'}$ such that each J_i' entails J_i and $F_{X'}(J_1', ..., J_n')$ entails $F_X(J_1, ..., J_n)$. Full coherence strengthens coherence in a natural way, namely by requiring $F_{X'}$ to cohere with F_X not just when $X \subseteq X'$, but more generally when $\overline{X} \subseteq \overline{X'}$, i.e., when X' is *essentially* an extension of X. Full coherence says that if an agenda X is essentially extended to another X', then the new outcome *can* be compatible with the old one, i.e., is compatible with it for *at least one* possible new profile. Full coherence is equivalent to ordinary coherence if the scope of any feasible agenda is a feasible agenda, i.e., $X \in \mathcal{X} \Rightarrow \overline{X} \in \mathcal{X}$.

Theorem 3 *An aggregation system* $(F_X)_{X\in\mathcal{X}}$ *is fully agenda-invariant if and only if it is fully coherent and each rule* F_X *is independent on the entire scope* $\overline{X}$.

One may regard Theorems 2 and 3 as formal counterparts of claims in Dietrich (2006) about the role of independence and independence on the scope in preventing agenda manipulation, although Dietrich (2006) does not yet invoke feasible agendas, aggregation systems, and coherence or full coherence. Unfortunately, Dietrich (2006) shows that independence on the scope is not satisfied by any non-degenerate aggregation rule, provided the agenda is not trivially small:

Proposition 5 *(Dietrich 2006) For any agenda X with $|\mathcal{J}| > 2$, only dictatorial or constant aggregation rules $F : \mathcal{J}^n \to \mathcal{J}$ are independent on the entire scope $\overline{X}$.*

Combining Theorem 3 with Proposition 5, I prove in the appendix that only degenerate aggregation systems are fully agenda-invariant:

Proposition 6 *If an aggregation system $(F_X)_{X\in\mathcal{X}}$ is fully agenda-invariant, then each rule F_X with $|\mathcal{J}_X| > 2$, or more generally with $X \subseteq \cup_{Z\in\mathcal{X}:|\mathcal{J}_Z|>2}\overline{Z}$, is dictatorial or constant.*

It is thus necessary in practice to weaken the requirement of full agenda-invariance. One option is to revert to ordinary agenda-invariance. Another natural option goes as follows. Rather than requiring that *all* collective judgments are irreversible (by a change of agenda), let us merely require that the *most important* ones are irreversible, where a collective judgment counts as 'most important' if it is unanimously supported by the individuals. Indeed, reversing a unanimously supported collective judgment seems particularly bad, as it goes against ('displeases') *all* individuals. The condition that unanimously supported collective judgments cannot be reversed by agenda manipulation formally states as follows:

Conditional agenda-invariance: Any two feasible agendas $X, X' \in \mathcal{X}$ lead to the same collective judgment on any unanimously accepted proposition in $\overline{X} \cap \overline{X'}$, i.e., for all $J_1, ..., J_n \in \mathcal{J}_{X\cup X'}$ and all propositions $p \in \overline{X} \cap \overline{X'}$ entailed by each J_i,[17]

$$F_X(J_1 \cap X, ..., J_n \cap X) \text{ entails } p \Leftrightarrow F_{X'}(J_1 \cap X', ..., J_n \cap X') \text{ entails } p.$$

I now show that, as long as the aggregation system is non-degenerate, the last condition is equivalent to the requirement that each rule F_X is implicit consensus preserving. By 'non-degenerate' I mean that each rule F_X is **setwise** unanimity-preserving, i.e., $F_X(J, ..., J) = J$ for each set $J \in \mathcal{J}_X$ – a condition to be distinguished from the more demanding propositionwise unanimity condition.[18]

Theorem 4 *An aggregation system $(F_X)_{X\in\mathcal{X}}$ (whose rules F_X are setwise unanimity-preserving) is conditionally agenda-invariant if and only if each rule F_X is implicit consensus preserving.*

This result adds a second argument for implicit consensus preservation, besides the argument based on description-invariance (Proposition 4).

[17] Note that J_i entails p if and only if $J_i \cap X$ entails p (since $p \in \overline{X}$), and if and only if $J_i \cap X'$ entails p (since $p \in \overline{X'}$). So, the requirement that each J_i entails p means that p emerges as unanimously accepted, whether agenda X or agenda X' is used.

[18] The propositionwise condition states that every proposition contained in all individual judgment sets must (regardless of any disagreements on other propositions) belong to the collective judgment set. While all non-degenerate rules are setwise unanimity-preserving, not all satisfy the propositionwise condition, as is clear from premise-based, distance-based, and scoring rules.

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A Proofs

The set of individuals is denoted $N := \{1, ..., n\}$. Recall that for an agenda X the set of rational judgment sets is denoted '$\mathcal{J}$' or sometimes, to avoid ambiguity, '$\mathcal{J}_X$'.

A.1 Results of Sections 3 and 4: the impossibility

Proof of Proposition 1. For the agenda X, consider an aggregation rule $F : \mathcal{J}^n \to \mathcal{J}$. I write ICP1, ICP2 and ICP3 for the three versions of implicit consensus preservation, respectively.

'*ICP1 $\Rightarrow$ ICP3*': Assume ICP1. Consider any $(J_1, ..., J_n) \in \mathcal{J}^n$. In the scope we can form the proposition $p := (\wedge_{q \in J_1} q) \vee \cdots \vee (\wedge_{q \in J_n} q)$ (i.e., the proposition that all q in J_1 *or* all q in J_2 ... *or* all q in J_n hold). Each J_i entails $(\wedge_{q \in J_i} q)$, and hence, entails p. So, $F(J_1, ..., J_n)$ entails p by ICP1. Let J be the unique extension of $F(J_1, ..., J_n)$ to a set in $\mathcal{J}_{\overline{X}}$. Since $F(J_1, ..., J_n)$ entails p, J contains p. So, for some i, $\wedge_{q \in J_i} q \in J$, and thus $J_i \subseteq J$. It follows that $J_i = J \cap X = F(J_1, ..., J_n)$. QED

'*ICP3 $\Rightarrow$ ICP2*': Assume ICP3 and consider a feature $\mathcal{K} \subseteq \mathcal{J}$ and a profile $(J_1, ..., J_n) \in \mathcal{J}^n$ such that $J_1, ..., J_n \in \mathcal{K}$. By ICP3, $F(J_1, ..., J_n) \in \{J_1, ..., J_n\}$. So, $F(J_1, ..., J_n) \in \mathcal{K}$. QED

'*ICP2 $\Rightarrow$ ICP1*': Assume ICP2. Consider any $p \in \overline{X}$ and any profile $(J_1, ..., J_n) \in \mathcal{J}^n$ such that each J_i entails p. Since each J_i belongs to the feature $\mathcal{K} := \{J \in \mathcal{J} : J$ entails $p\}$, so does $F(J_1, ..., J_n)$ by ICP2. ■

As part of the proof of Theorem 1, I show several lemmas. For an agenda X, an aggregation rule F on $\mathcal{J}^n$ is called **systematic** if there exists a set $\mathcal{W}$ of ('winning') coalitions $C \subseteq N$ such that

$$F(J_1, ..., J_n) = \{p \in X : \{i : p \in J_i\} \in \mathcal{W}\} \text{ for all } J_1, ..., J_n \in \mathcal{J}.$$

In this case, the set $\mathcal{W}$ is uniquely determined and denoted by $\mathcal{W}_F$.

Lemma 2 *If and only if the agenda X is non-nested, every independent and implicit consensus preserving aggregation rule $F : \mathcal{J}^n \to \mathcal{J}$ is systematic.*

Proof. Let X be an agenda. We may assume without loss of generality that all $p \in X$ are contingent, because each side of the claimed equivalence remains true (or false) if the non-contingent propositions are removed from the agenda.

1. In this part we assume that X is non-nested and consider an independent and implicit consensus preserving rule $F : \mathcal{J}^n \to \mathcal{J}$. I show that F is systematic (drawing on Dietrich and List 2013). For any $p, q \in X$, I define $p \sim q$ to mean that there exists a finite sequence $p_1, ..., p_k \in X$ with $p_1 = p$ and $p_k = q$ such that any neighbours p_l, p_{l+1} are not exclusive (i.e., $\{p_l, p_{l+1}\}$ is consistent) and not exhaustive (i.e., $\{\neg p_l, \neg p_{l+1}\}$ is consistent). I prove five claims: the first four gradually establish that $p \sim q$ for *all* $p, q \in X$, and the last shows that F is systematic.

Claim 1: For all $p, q \in X$, $p \sim q \Leftrightarrow \neg p \sim \neg q$.

It suffices to show one direction of implication, as $\neg\neg p = p$ for all $p \in X$. Let $p, q \in X$ with $p \sim q$. Then there is a path $p_1, ..., p_k \in X$ between p to q where any neighbours p_j, q_{j+1} are non-exclusive and non-exhaustive. To see why $\neg p \sim \neg q$, note that $\neg p_1, ..., \neg p_k$ is a path between $\neg p$ and $\neg q$ where any neighbours $\neg p_j, \neg p_{j+1}$ are non-exclusive (as p_j, p_{j+1} are non-exhaustive) and non-exhaustive (as p_j, p_{j+1} are non-exclusive). QED

Claim 2: If $p \in X$ entails $q \in X$, then $p \sim q$.

If $p \in X$ entails $q \in X$, then $p \sim q$ in virtue of a direct connection: p, q are neither exclusive nor exhaustive (for instance, $\{p, q\}$ is consistent because p is not a contradiction and entails q. QED

Claim 3: $\sim$ is an equivalence relation on X, and for all $p, q \in X$, $p \sim q$ or $p \sim \neg q$. (So each equivalence class contains at least one member of each issue $\{q, \neg q\}$.)

Reflexivity, symmetry and transitivity are all obvious (where reflexivity uses that every $p \in X$ is contingent). Now consider $p, q \in X$ such that $p \not\sim q$; we have to show that $p \sim \neg q$. Since $p \not\sim q$, $\{p, q\}$ or $\{\neg p, \neg q\}$ is inconsistent. In either case, one of p and $\neg q$ entails the other, so that $p \sim \neg q$ by Claim 2. QED

Claim 4: $p \sim q$ for all $p, q \in X$.

Let X_+ be an equivalence class w.r.t. $\sim$ and suppose for a contradiction that $X_+ \neq X$. Then, by Claim 3, X_+ must contain exactly one member of each issue $\{r, \neg r\}$. We show that X_+ is weakly ordered by the entailment relation between propositions – implying that X is nested, a contradiction. As the entailment relation on X_+ is of course transitive, it remains to show that it is complete on X_+. So we consider $p, q \in X_+$, and have to show that p entails q or q entails p. We have $p \not\sim \neg q$, since otherwise X_+ would include the entire issue $\{q, \neg q\}$. So $\{p, \neg q\}$ or $\{\neg p, q\}$ is inconsistent. Hence, p entails q or q entails p. QED

Claim 5: F is systematic.

Since F is independent, there exists a family $(\mathcal{W}_p)_{p\in X}$ of sets of coalitions such that

$$F(J_1, ..., J_n) = \{p \in X : \{i : p \in J_i\} \in \mathcal{W}_p\} \text{ for all } J_1, ..., J_n \in \mathcal{J}. \quad (4)$$

It suffices to show that $\mathcal{W}_p$ is the same for all $p \in X$. By Claim 4 and the definition of $\sim$, it suffices to show that $\mathcal{W}_p = \mathcal{W}_q$ for all $p, q \in X$ which are non-exclusive and non-exhaustive. Consider such $p, q \in X$. Consider any $C \subseteq N$ and let us show that $C \in \mathcal{W}_q \Leftrightarrow C \in \mathcal{W}_q$. As $\{p, q\}$ and $\{\neg p, \neg q\}$ are consistent, there exist $J_1, ..., J_n \in \mathcal{J}$ such that $p, q \in J_i$ for all $i \in C$ and $\neg p, \neg q \in J_i$ for all $i \in N \backslash C$. We now apply implicit consensus preservation, in any of its three variants. Using either variant 1 (and the fact that each J_i entails the proposition $(p \wedge q) \vee (\neg p \wedge \neg q)$ in the scope), or variant 2 (and the fact that each J_i belongs to the feature $\mathcal{K} := \{J \in \mathcal{J} : p \in J \Leftrightarrow q \in J\}$), or variant 3, it follows that $p \in F(J_1, ..., J_n) \Leftrightarrow q \in F(J_1, ..., J_n)$. By (4), the left side of this equivalence holds if and only if $C \in \mathcal{W}_p$ and the right side holds if and only if $C \in \mathcal{W}_q$. So $C \in \mathcal{W}_p \Leftrightarrow C \in \mathcal{W}_q$. QED

2. Now assume that X is nested, i.e., of the form $X = \{p_1, ..., p_m\}^{\pm}$ where m is the number of issues and where p_1 entails p_2, p_2 entails p_3, etc. I consider the aggregation rule F on $\mathcal{J}^n$ defined as follows: for all $J_1, ..., J_n \in \mathcal{J}$, $F(J_1, ..., J_n)$ consists of each p_j contained in *all* J_i and each $\neg p_j$ contained in *some* J_i. We have to show that F (i) maps into $\mathcal{J}$, (ii) is independent, (iii) is implicit consensus preserving, and (iv) is not systematic. The properties (ii) and (iv) are obvious (where (iv) uses that $n > 1$ and that X contains a pair of contingent propositions $p, \neg p$ because $|\mathcal{J}| > 1$). It remains to prove (i) and (iii). Now (i) follows from (iii) by version 3 of implicit consensus preservation. To see why (iii) holds, note that for each $J \in \mathcal{J}$ there is a cut-off level $t \in \{1, ..., m+1\}$ such that $J = \{\neg p_1, ..., \neg p_{t-1}, p_t, ..., p_m\}$, and that therefore for all $J_1, ..., J_n \in \mathcal{J}$ we have $F(J_1, ..., J_n) = J_i$ where i is the (or an) individual with highest cut-off level. ■

The next lemma is the main technical step towards Theorem 1 and provides two alternative characterizations of non-nested agendas. (Compare the characterization in (b) with the definition of *non-simple* agendas mentioned in Section 4: the only difference is that (b) allows Y to be consistent.)

Lemma 3 *For any agenda X, the following are equivalent:*

(a) *X is non-nested (and non-tiny).*
(b) *X has a subset Y such that $|Y| \geq 3$ and $(Y\backslash\{p\}) \cup \{\neg p\}$ is consistent for all $p \in Y$.*
(c) *X has a subset Y such that $|Y| = 3$ and $(Y\backslash\{p\}) \cup \{\neg p\}$ is consistent for all $p \in Y$.*

Proof. Let X be an agenda. I write $p \vdash q$ to mean that p ($\in X$) entails q ($\in X$), and $S \vdash q$ to mean that S ($\subseteq X$) entails q. We may assume without loss of generality. that X contains only contingent propositions, and is redundancy-free, i.e., contains no two equivalent propositions. The reason is that otherwise it suffices to do the proof for any redundancy-free subagenda containing only contingent propositions, because each of the conditions (a), (b) and (c) holds for X if and only if it holds for that subagenda; to see for instance why (b) holds for X if and only if it holds for the subagenda, note that $(Y\backslash\{p\}) \cup \{\neg p\}$ can only be consistent for all $p \in Y$ if Y contains no two equivalent propositions and no non-contingent propositions.

The equivalence between (b) and (c) is straightforward (to see why (b) implies (c), simply replace the set Y in (b) by a three-member subset of it). It is also relatively easy to see why (c) implies (a). Indeed, whenever (a) is violated, so is (c), by the following argument. First, if X is tiny, then (c) is violated since every three-element set $Y \subseteq X$ takes the form $Y = \{q, \neg q, p\}$ for some $p, q \in X$, and thus $Y \backslash \{p\} \cup \{\neg p\}$ fails to be consistent. Second, if X is nested, say $X = \{r, \neg r : r \in Z\}$ for some subset $Z \subseteq X$ linearly ordered by entailment, condition (c) is violated since any three-element set $Y \subseteq X$ has elements $p \neq q$ which both belong to Z or both belong to $\{\neg p : p \in Z\}$, so that (by the linear orderedness of Z and of $\{\neg r : r \in Z\}$ w.r.t. entailment) $p \vdash q$ or $q \vdash p$, which implies that $(Y \backslash \{q\}) \cup \{\neg q\}$ or $(Y \backslash \{p\}) \cup \{\neg p\}$ is inconsistent.

It remains to show that (a) implies (c). Let X be non-nested and non-tiny; we show (c). We distinguish between two cases.

Case 1: no $p, q \in X$ are logically independent, i.e., for no $p, q \in X$ each of the sets $\{p, q\}$, $\{p, \neg q\}$, $\{\neg p, q\}$ and $\{\neg p, \neg q\}$ is consistent.

Claim 1.1. There exists a (with respect to set-inclusion) maximal nested (sub)agenda $X^* \subseteq X$.

This follows from the fact that the set of nested subagendas $V \subseteq X$ is non-empty (because it contains any one-issue subagenda $\{p, \neg p\}$) and finite (because X is finite). QED

Since X^* is nested, we may write it as $X^* = \{p, \neg p : p \in X^*_+\}$ where X^*_+ is a subset of X^* which contains exactly one member of each issue $\{p, \neg p\} \subseteq X^*$ and is linearly ordered w.r.t. set-inclusion.

Claim 1.2. There exists an $s \in X \backslash X^*$ such that $\{s, p\}$ is consistent for all $p \in X^*_+$.

Since X^* is nested but X is not, we have $X^* \subsetneq X$, and thus there are $r, \neg r \in X \backslash X^*$. It suffices to show that at least one of r and $\neg r$ is consistent with each $p \in X^*_+$. This is true because otherwise there would exist $p, p' \in X^*_+$ such that $\{r, p\}$ and $\{\neg r, p'\}$ are inconsistent, which (recalling that $p \vdash p'$ or $p' \vdash p$, and writing p'' for the logically stronger one of p and p') implies that $\{r, p''\}$ and $\{\neg r, p''\}$ are inconsistent, so that $\{p''\}$ is inconsistent, a contradiction since p'' is contingent. QED

I define

$$\begin{aligned} Y_1 \;\; : & \;\; = \{p \in X^*_+ : p \vdash s\}, \\ Y_2 \;\; : & \;\; = \{p \in X^*_+ : \neg p \vdash s\}. \end{aligned}$$

Claim 1.3. $Y_1 \cap Y_2 = \varnothing$, and $Y_1 \cup Y_2 = X^*_+$.

First, $Y_1 \cap Y_2 = \varnothing$, because otherwise there would be a $p \in X^*_+$ such that $p \vdash s$ and $\neg p \vdash s$, a contradiction as s is not a tautology. Second, suppose for a contradiction that $p \in X^*_+ \backslash (Y_1 \cup Y_2)$. I ultimately show that the agenda $X^* \cup \{s, \neg s\}$ is nested, a contradiction as X^* is a *maximal* nested subagenda of X.

Since p and s are not logically independent (by assumption of Case 1), and since $\{p, s\}$ is consistent (by Claim 1.2), $\{p, \neg s\}$ is consistent (as $p \notin Y_1$) and $\{\neg p, \neg s\}$ is consistent (as $p \notin Y_2$), it follows that $\{\neg p, s\}$ is inconsistent, so that $s \vdash p$. We next show that s entails not just in p, but also all other propositions in $X^*_+ \backslash Y_1$:

$$s \vdash p' \text{ for all } p' \in X^*_+ \backslash Y_1. \tag{5}$$

To show this, let $p' \in X^*_+ \backslash Y_1$, and note first that $\neg p'$ and $\neg s$ are entailed by $\{\neg p', \neg p, \neg s\}$. Hence (as $s \vdash p$, i.e., $\neg p \vdash \neg s$), $\neg p'$ and $\neg s$ are entailed by $\{\neg p', \neg p\}$. So, since the set

$\{\neg p', \neg p\}$ is consistent (as either $\neg p' \vdash \neg p$ or $\neg p \vdash \neg p'$), the set $\{\neg p', \neg s\}$ is also consistent. Since p' and s are not logically independent (by assumption of Case 1), and since $\{p', s\}$ is consistent (by Claim 1.2), $\{p', \neg s\}$ is consistent (as $p' \notin Y_1$) and $\{\neg p', \neg s\}$ is consistent (as just shown), it follows that $\{\neg p', s\}$ is inconsistent, so that $s \vdash p'$. This proves (5).

Note that for every event p' in X_+^*, either $p' \vdash s$ (if $p' \in Y_1$) or $s \vdash p'$ (if $Y \notin Y_1$, by (5)). So the augmented (sub-)agenda $X^* \cup \{s, \neg s\}$ is nested, a contradiction as X^* is a maximal nested subagenda of X. QED

Claim 1.4. $Y_1, Y_2 \neq \varnothing$.

By Claim 1.3 we may equivalently show that $Y_1, Y_2 \neq X_+^*$. Suppose for a contradiction that $Y_1 = X_+^*$ or $Y_2 = X_+^*$. Then $X^* \cup \{s, \neg s\}$ is a nested agenda, a contradiction since X^* was defined as a maximal nested subagenda of X. QED

The proof of condition (c) is completed by combining Claim 1.4 with the following observation:

Claim 1.5. For all $q \in Y_1$ and $r \in Y_2$, the set $Y := \{\neg q, r, s\}$ satisfies the requirements of condition (c), i.e., $|Y| = 3$ and $(Y \backslash \{p\}) \cup \{\neg p\}$ is consistent for each $p \in Y$.

Consider any $q \in Y_1$ and $r \in Y_2$ and let $Y := \{\neg q, r, s\}$. To see why $|Y| = 3$, note that $\neg q \neq r$ since $r \in X_+^*$ while $\neg q \notin X_+^*$, and that $s \neq \neg q, r$ since $\neg q, r \in X^*$ while $s \notin X^*$. Further:

- $\{q, r, s\}$ is consistent, because, firstly, $\{q, s\}$ is consistent by Claim 1.2, and, secondly, $q \vdash r$, as q and r belong to the linearly ordered set X_+^* and as $r \nvdash q$ (by the fact that $q \in Y_1$ and $r \notin Y_1$).
- $\{\neg q, \neg r, s\}$ is consistent, because, firstly, $\neg r \vdash \neg q$ (since $q \vdash r$, as just shown), and, secondly, $\neg r \vdash s$ (since $r \in Y_2$).
- $\{\neg q, r, \neg s\}$ is consistent, because, firstly, $\neg s \vdash \neg q$ (since $q \vdash s$, as $q \in Y_1$), and, secondly, $\neg s \vdash r$ (since $\neg r \vdash s$, as $r \in Y_2$). QED

Case 2: $p, q \in X$ are logically independent, i.e., all of $\{p, q\}$, $\{p, \neg q\}$, $\{\neg p, q\}$ and $\{\neg p, \neg q\}$ are consistent. Consider such $p, q \in X$. Since $|X| > 4$ there is an $r \in X \backslash \{p, \neg p, q, \neg q\}$. As r is non-contradictory, it can be consistently added to at least one of the (consistent) sets $\{p, q\}$, $\{p, \neg q\}$, $\{\neg p, q\}$ and $\{\neg p, \neg q\}$. We may assume without loss of generality that $\{p, \neg q, r\}$ is consistent (otherwise, simply interchange p with $\neg p$ and/or q with $\neg q$). The argument distinguishes between two subcases.

Subcase 2.1: $\{\neg p, \neg q, \neg r\}$ and $\{p, q, \neg r\}$ are both consistent. In this case, condition (c) holds for $Y := \{p, \neg q, \neg r\}$, since each of the sets $\{\neg p, \neg q, \neg r\}$, $\{p, q, \neg r\}$ and $\{p, \neg q, r\}$ is consistent.

Subcase 2.2: $\{\neg p, \neg q, \neg r\}$ or $\{p, q, \neg r\}$ is inconsistent (perhaps both are). We assume without loss of generality that $\{p, q, \neg r\}$ is inconsistent, i.e., $\{p, q\} \vdash r$. (The proof is analogous in the other case.) There are three subsubcases.

Subsubcase 2.2.1: $\{\neg p, q, \neg r\}$ and $\{p, \neg q, \neg r\}$ are both consistent. Here, condition (c) holds for $Y := \{p, q, \neg r\}$, since each of the sets $\{\neg p, q, \neg r\}$, $\{p, \neg q, \neg r\}$ and $\{p, q, r\}$ is consistent (the latter set being consistent because $\{p, q\}$ is consistent and entails r).

Subsubcase 2.2.2: $\{\neg p, q, \neg r\}$ is inconsistent. So $\{\neg p, q\} \vdash r$. As also $\{p, q\} \vdash r$, we have $q \vdash r$. We once again distinguish between cases:

- First assume $\{\neg p, \neg q, \neg r\}$ is inconsistent. Then condition (c) holds with $Y =$

$\{\neg p, \neg q, r\}$, because $\{p, \neg q, r\}$, $\{\neg p, q, r\}$ and $\{\neg p, \neg q, \neg r\}$ are consistent (where $\{\neg p, q, r\}$ is consistent as $\{\neg p, q\}$ is consistent and $q \vdash r$).

- Second assume $\{\neg p, \neg q, \neg r\}$ is inconsistent, i.e., $\{\neg p, \neg q\} \vdash r$. Since also $q \vdash r$, we have $\neg r \vdash \neg q, p$. Condition (c) holds with $Y = \{p, \neg q, r\}$, because $\{\neg p, \neg q, r\}$ is consistent (as $\{\neg p, \neg q\}$ is consistent and entails r), $\{p, q, r\}$ is consistent (as $\{p, q\}$ is consistent and entails r) and $\{p, \neg q, \neg r\}$ is consistent (as $\neg r \vdash \neg q, p$).

Subsubcase 2.2.3: $\{p, \neg q, \neg r\}$ is inconsistent. (If in the following proof for the current subsubcase we interchange p and q, then we obtain an alternative, but longer, proof for Subsubcase 2.2.2.) Since $\{p, \neg q, \neg r\}$ is inconsistent, $\{p, \neg q\} \vdash r$. As also $\{p, q\} \vdash r$, it follows that $p \vdash r$. We now show that

$$\begin{array}{ll} & (*)\ \{\neg p, q, r\} \text{ and } \{\neg p, \neg q, \neg r\} \text{ are consistent} \\ \textit{or} & (**)\ \{\neg p, \neg q, r\} \text{ and } \{\neg p, q, \neg r\} \text{ are consistent.} \end{array} \quad (6)$$

To show this, we assume that (*) is violated and show that (**) holds, by distinguishing between two cases:

- First, let $\{\neg p, q, r\}$ be inconsistent. It follows, on the one hand, that $\{\neg p, q, \neg r\}$ is consistent (as $\{\neg p, q\}$ is consistent), and, on the other hand, that $\{\neg p, \neg q, r\}$ is consistent (as otherwise, by the inconsistency of $\{\neg p, q, r\}$, $\{\neg p, r\}$ would be inconsistent, i.e., $r \vdash p$, a contradiction since $p \vdash r$ and $p \neq r$). This proves (**).
- Second, let $\{\neg p, q, r\}$ be consistent. Then $\{\neg p, \neg q, \neg r\}$ is inconsistent as (*) is violated. It follows, one the one hand, that $\{\neg p, \neg q, r\}$ is consistent (as $\{\neg p, \neg q\}$ is consistent), and, on the other hand, that $\{\neg p, q, \neg r\}$ is consistent (as otherwise $\{\neg p, \neg r\}$ would be inconsistent, i.e., $\neg r \vdash p$, a contradiction since $p \vdash r$). This proves (**).

We can now prove condition (c). In the case of (*), (c) holds with $Y = \{\neg p, \neg q, r\}$, since $\{p, \neg q, r\}$ is consistent (as $\{p, \neg q\}$ is consistent and $p \vdash r$), $\{\neg p, q, r\}$ is consistent (by (*)) and $\{\neg p, \neg q, \neg r\}$ is consistent (by (*)). In the case of (**), (c) holds with $Y = \{\neg p, q, r\}$, since $\{p, q, r\}$ is consistent (as $\{p, q\}$ is consistent and $p \vdash r$), $\{\neg p, \neg q, r\}$ is consistent (by (**)) and $\{\neg p, q, \neg r\}$ is consistent (by (**)). ■

Drawing on Lemma 3, I next show that for *almost every* agenda the set of winning coalitions of a systematic and implicit consensus preserving aggregation rule defines an ultrafilter (which would not be true if implicit consensus preservation were replaced by the standard unanimity condition).

Lemma 4 *Consider a systematic and implicit consensus preserving aggregation rule $F : \mathcal{J}^n \rightarrow \mathcal{J}$ for an agenda X, and coalitions $C, C' \subseteq N$.*

(a) If X satisfies $|\mathcal{J}| > 2$, then [$C \in \mathcal{W}_F$ and $C \subseteq C'$] $\Rightarrow C' \in \mathcal{W}_F$.
(b) If X is non-nested and non-tiny, then $C, C' \in \mathcal{W}_F \Rightarrow C \cap C' \in \mathcal{W}_F$.
(c) $C \in \mathcal{W}_F \Leftrightarrow N \backslash C \notin \mathcal{W}_F$.

Note that $|\mathcal{J}| > 2$ if and only if X has more than two propositions (one issue),[19] a very mild assumption, satisfied notably by non-tiny agendas.

Proof. Let X, F, C and C' be as specified.

[19] counting only contingent propositions and counting equivalent propositions (if any) only once

(a) Suppose $|\mathcal{J}| > 2$, $C \in \mathcal{W}_F$ and $C \subseteq C'$. We show that $C' \in \mathcal{W}_F$. As $|\mathcal{J}| > 2$, there exist contingent and pairwise non-equivalent propositions $p, \neg p, q, \neg q \in X$. There must exist a member of $\{p, \neg p\}$ which entails neither q nor $\neg q$, as can be shown using that the propositions $p, \neg p, q, \neg q$ are contingent and pairwise non-equivalent. Without loss of generality. we assume that p entails neither q nor $\neg q$ (otherwise simply interchange p and $\neg p$). So $\{p, q\}$ and $\{p, \neg q\}$ are each consistent. Note that at least one of $\{\neg p, q\}$ and $\{\neg p, \neg q\}$ is consistent, as $\neg p$ is not a contradiction. Without loss of generality. we assume the latter (otherwise interchange q and $\neg q$). To summarize, each of the sets $\{p, q\}$, $\{p, \neg q\}$ and $\{\neg p, \neg q\}$ is consistent. We may therefore consider a profile $(J_1, ..., J_n) \in \mathcal{J}^n$ such that

$$J_i \supseteq \begin{cases} \{p, q\} & \text{for all } i \in C, \\ \{p, \neg q\} & \text{for all } i \in C' \backslash C, \\ \{\neg p, \neg q\} & \text{for all } i \in N \backslash C'. \end{cases}$$

First, since each J_i contains p or $\neg q$, so does $F(J_1, ..., J_n)$ by implicit consensus preservation (version 2). Second, $q \in F(J_1, ..., J_n)$ since $\{i \in N : q \in J_i\} = C \in \mathcal{W}_F$. These two facts imply that $p \in F(J_1, ..., J_n)$. So, as $\{i : p \in J_i\} = C'$, we have $C' \in \mathcal{W}_F$.

(b) Suppose X is non-nested and non-tiny, and assume $C, C^* \in \mathcal{W}_F$. We show that $C \cap C^* \in \mathcal{W}_F$. By assumption on X and Lemma 3, there is a three-element set $Y = \{p, q, r\} \subseteq X$ such that each of $\{\neg p, q, r\}$, $\{p, \neg q, r\}$ and $\{p, q, \neg r\}$ is consistent. This allows us to construct a profile $(J_1, ..., J_n) \in \mathcal{J}^n$ such that

$$J_i \supseteq \begin{cases} \{\neg p, q, r\} & \text{if } i \in C \cap C^* \\ \{p, q, \neg r\} & \text{if } i \in C^* \backslash C \\ \{p, \neg q, r\} & \text{if } i \in N \backslash C^*. \end{cases}$$

First, $q \in F(J_1, ..., J_n)$ as $\{i : q \in J_i\} = C^* \in \mathcal{W}_F$. Second, as $C \in \mathcal{W}_F$ and $C \subseteq C \cup (N \backslash C^*)$, we have $C \cup (N \backslash C^*) \in \mathcal{W}_F$ by part (a); hence $r \in F(J_1, ..., J_n)$ as $\{i : r \in J_i\} = C \cup (N \backslash C^*)$. Third, as each J_i contains $\neg p$ or $\neg q$ or $\neg r$, so does $F(J_1, ..., J_n)$ by implicit consensus preservation (version 2). These three facts imply that $\neg p \in F(J_1, ..., J_n)$. Hence, as $\{i : \neg p \in J_i\} = C \cap C^*$, we have $C \cap C^* \in \mathcal{W}_F$.

(c) This claim is obvious, as (by $|\mathcal{J}| > 1$) we can choose a contingent proposition $p \in X$ and construct a profile in $\mathcal{J}^n$ in which all $i \in C$ accept p and all $i \in N \backslash C$ accept $\neg p$. ■

I can now prove Theorem 1, whose 'if' part will follow from the above lemmas.

Proof of Theorem 1. 1. In this part of the proof, let the agenda X be non-nested and non-tiny, and let $F : \mathcal{J}^n \to \mathcal{J}$ be independent and implicit consensus preserving. I need to show that F is dictatorial. By Lemma 2, F is systematic. By Lemma 4, the set of winning coalitions $\mathcal{W}_F$ is an ultrafilter over the set of individuals N. As is well-known, every ultrafilter over a finite set is principal, i.e., there is an individual $j \in N$ such that $\mathcal{W}_F = \{C \subseteq N : j \in C\}$. Clearly, j is a dictator.

2. Conversely, assume the agenda X is nested or tiny. I need to construct a non-dictatorial rule $F : \mathcal{J}^n \to \mathcal{J}$ which is independent and implicit consensus preserving. As $n \geq 3$, we may choose an odd-sized subgroup $M \subseteq N$ containing at least three

individuals. (For instance $M = N$ if n is odd, or $M = \{1, 2, 3\}$.) Define F as the aggregation rule on $\mathcal{J}^n$ given by majority voting among M, i.e.,

$$F(J_1, ..., J_n) = \{p \in X : |\{i \in M : p \in J_i\}| > |M|/2\} \text{ for all } J_1, ..., J_n \in \mathcal{J}.$$

I have to show that F (i) maps into $\mathcal{J}$, (ii) is independent, (iii) is implicit consensus preserving, and (iv) is not dictatorial. Properties (ii) and (iv) hold obviously; regarding (ii), F is in fact even systematic, and regarding (iv) it matters that $|M| > 1$ and $|\mathcal{J}| > 1$. Properties (i) and (iii) both follow as soon as we have shown version 3 of implicit consensus preservation. Consider $J_1, ..., J_n \in \mathcal{J}$. To show that $F(J_1, ..., J_n) \in \{J_1, ..., J_n\}$, I distinguish between two cases.

Case 1: X is nested, i.e., of the form $X = \{p_1, ..., p_m\}^{\pm}$ where m is the number of issues and where p_1 entails p_2, p_2 entails p_3, etc. Notice that for each $J \in \mathcal{J}$ there is a cut-off level $t = t_J \in \{1, ..., m+1\}$ such that $J = \{\neg p_1, ..., \neg p_{t-1}, p_t, ..., p_m\}$, and that

$$F(J_1, ..., J_n) = J_i = \{\neg p_1, ..., \neg p_{t_{J_i}-1}, p_{t_{J_i}}, ..., p_m\},$$

where i is the *median individual in* M, i.e., the (or an) individual i in M such that more than half of the individuals j in M have a cut-off level $t_{J_j} \leq t_{J_i}$, and more than half of the individuals j in M have a cut-off level $t_{J_j} \geq t_{J_i}$.

Case 2: X is tiny. As one easily checks, we may assume without loss of generality. that X is redundancy-free and contains only contingent propositions. Then, as X is tiny, it is either a one-issue agenda or a two-issue agenda. In the first case, $F(J_1, ..., J_n)$ is a singleton $\{p\}$, which equals J_i for any individual i accepting p. In the second case, $F(J_1, ..., J_n)$ is a binary set $\{p, q\}$; since the subgroups $\{i \in M : p \in J_i\}$ and $\{i \in M : q \in J_i\}$ each contain a majority of the individuals in M, these subgroups share at least one individual i, whose judgment set is therefore $J_i = \{p, q\} = F(J_1, ..., J_n)$. ■

A.2 Results of Section 5: the scope of an agenda

I prove these results in a slightly different order, and draw on additional lemmas.

Proof of Remark 1. The conjunction (disjunction) of elements p, q of a redundancy-free agenda X is unique because any two conjunctions (disjunctions) of p and q entail each other, hence coincide as X is redundancy-free. The simple argument is spelled out in more detail in the proof of Remark 3. ■

Proof of Remark 2. Suppose an agenda X is closed under conjunction. Let $p, q \in X$. Let $r \in X$ be the (possibly not unique) conjunction of $\neg p$ and $\neg q$. Then $\neg r$ is the (possibly not unique) disjunction of p and q. Indeed, any $J \in \mathcal{J}$ contains p or q if and only if it is not the case that $\neg p, \neg q \in J$; which is equivalent to $r \notin J$, i.e., to $\neg r \in J$. Analogously, one can show that if X is closed under disjunction then any $p, q \in X$ have a conjunction in X, namely the proposition $\neg r$ where r is a disjunction of $\neg p$ and $\neg q$. ■

Lemma 5 *The notions of consistency, entailment, conjunction and disjunction are preserved by any extension of the agenda (and thus can be used without referring explicitly to an agenda). Formally, for any agenda X and any superagenda X' (e.g., the scope of X),*

(a) *a set $S \subseteq X$ is consistent w.r.t. X if and only if it is so w.r.t. X',*
(b) *a proposition $p \in X$ (or set $S \subseteq X$) entails a proposition $p' \in X$ (or set $S' \subseteq X$) w.r.t. X if and only if it does so w.r.t. X',*
(c) a proposition $r \in X$ is the (or a) conjunction/disjunction of certain propositions in X w.r.t. X if and only if it is so w.r.t. X'.

Proof. Part (b) follows from part (a), since the entailment notion is reducible to the consistency notion (e.g., p entails p' if and only if $\{p, \neg p'\}$ is inconsistent). Further, part (c) follows from part (b), since the notions of conjunction and disjunction are reducible to the entailment notion: r is a conjunction of a set of propositions S if and only if r and S entail each other, and r is a disjunction of the set of propositions S if and only if $\neg r$ and $\{\neg p : p \in S\}$ entail each other. To prove part (a), recall that (*) $\mathcal{J}_X = \{J' \cap X : J' \in \mathcal{J}_{X'}\}$. Consider any $S \subseteq X$. First, let S be consistent w.r.t. X. Then there is a $J \in \mathcal{J}_X$ such that $S \subseteq J$. By (*), we may write $J = J' \cap X$ for some $J' \in \mathcal{J}_{X'}$. Clearly, $S \subseteq J'$, whence S is consistent w.r.t. X'. Conversely, assume S is consistent w.r.t. X'. Then we may choose a $J' \in \mathcal{J}_{X'}$ such that $S \subseteq J'$. By (*), $\mathcal{J}_X$ contains $J := J' \cap X$. Note that $S \subseteq J$. So S is consistent w.r.t. X. ■

Proof or Remark 3. Let X be a closed redundancy-free agenda and $\vdash$ the relation of entailment between propositions. The proof proceeds in four claims.

Claim 1: $(X, \vdash)$ is a lattice whose meet and join are given by the operations of conjunction $\wedge$ and disjunction $\vee$, respectively.

First, $\vdash$ is a partial order: it is clearly reflexive and transitive, and it is also anti-symmetric as X is redundancy-free. Next, for any $p, q \in X$, the conjunction $p \wedge q$ is the greatest lower bound of p and q because, firstly, it is a lower bound (i.e., $p \wedge q \vdash p, q$), and, secondly, if r is also a lower bound, then $r \vdash p \wedge q$, as $r \vdash p, q$ and $\{p, q\} \vdash p \wedge q$. Analogously, for any $p \in X$, the disjunction $p \vee q$ is the smallest upper bound of p and q. QED

Claim 2: The lattice $(X, \vdash)$ is distributive.

Let $p, q, r \in X$. Since $p \vdash p \vee q$ and $p \vdash p \vee r$, we have (*) $p \vdash (p \vee q) \wedge (p \vee r)$. Since $q \wedge r$ entails q (which entails $p \vee q$) and entails r (which entails $p \vee r$), (**) $q \wedge r \vdash (p \vee q) \wedge (p \vee r)$. By (*) and (**),

$$p \vee (q \wedge r) \vdash (p \vee q) \wedge (p \vee r). \tag{7}$$

We next show the converse implication,

$$(p \vee q) \wedge (p \vee r) \vdash p \vee (q \wedge r). \tag{8}$$

Consider any $J \in \mathcal{J}$ containing $(p \vee q) \wedge (p \vee r)$, and let us show that $p \vee (q \wedge r) \in J$. As $(p \vee q) \wedge (p \vee r)$ entails $p \vee q$ and also $p \vee r$, we have $p \vee q, p \vee r \in J$. So, J contains p or q (or both), and contains p or r (or both). So, J contains p or contains both q and r; in the latter case, $q \wedge r \in J$. Since, as we have shown, $p \in J$ or $q \wedge r \in J$, we have $p \vee (q \wedge r) \in J$, as desired. By (7) and (8), and by the asymmetry of $\vdash$, $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$. By analogous arguments, $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$. QED

Claim 3: X has a smallest element $\bot$ and a greatest element $\top$, namely the contradiction $\wedge_{p \in X} p$ and the tautology $\vee_{p \in X} p$, respectively.

It is obvious that $\wedge_{p \in X} p$ entails each $q \in X$ and that each $q \in X$ entails $\vee_{p \in X} p$. QED

Claim 4: For each $p \in X$, $p \wedge \neg p = \bot$ and $p \vee \neg p = \top$ (i.e., $\neg p$ is the algebraic complement of p).

Let $p \in X$. Since $\{p, \neg p\}$ is inconsistent, $p \wedge \neg p = \bot$. Since every $J \in \mathcal{J}$ contains p or $\neg p$, every $J \in \mathcal{J}$ contains $p \vee \neg p$, whence $p \vee \neg p = \top$. ■

Lemma 6 *For any agenda X and any closed (redundancy-free) superagenda X' – possible X itself or the scope of X – a set $A \subseteq X$ is consistent if and only if, in X', $\wedge_{p \in A} p \neq \bot$.*

Proof. Let X and X' be as specified. By Lemma 5, we need not distinguish between consistency w.r.t. X and w.r.t. X'. We proceed by showing three claims.

Claim 1: $\bot$ is the only element of X' which is not contained in any rational judgment set $J \in \mathcal{J}_{X'}$.

This follows from four facts (some of which draw on Remark 3): (i) $\bot$ is the only element of X' which entails its own algebraic complement (a basic fact about Boolean algebras); (ii) the algebraic complement of an element p is its (agenda-theoretic) negation $\neg p$; (iii) an element p entails another q if and only if no $J \in \mathcal{J}_{X'}$ contains both p and $\neg q$; (iv) every $J \in \mathcal{J}_{X'}$ contains exactly one of member of each pair $p, \neg p \in X$. QED

Claim 2: For any $J \in \mathcal{J}_{X'}$ and any $A \subseteq J$, we have $\wedge_{p \in A} p \in J$.

Let $J \in \mathcal{J}_{X'}$ and $A \subseteq J$. By Remark 3 we can think of '$\wedge$' alternatively as the conjunction operator (defined agenda-theoretically) or the meet (defined Boolean-algebraically). The claim holds by induction on the size of A. If $A = \varnothing$, the claims holds because then $\wedge_{p \in A} p = \top$ and $\top \in J$ (as $\top = \neg \bot$, where $\bot \notin J$ by Claim 1). Now assume A has size $m \geq 1$ and suppose the claim holds for any smaller size. We may write $A = A' \cup \{q\}$ with $q \notin A'$. By induction hypothesis, $\wedge_{p \in A'} p \in J$. Since J contains both $\wedge_{p \in A'} p$ and q, J contains their conjunction $(\wedge_{p \in A'} p) \wedge q = \wedge_{p \in A} p$ by definition of conjunction. QED

Claim 3: A set $A \subseteq X'$ is consistent if and only if $\wedge_{p \in A} p \neq \bot$.

First, let $A \subseteq X'$ be consistent. Then it has an extension $J \in \mathcal{J}_{X'}$, which by Claim 2 contains $\wedge_{p \in A} p$. So by Claim 1 $\wedge_{p \in A} p \neq \bot$. Conversely, assume $\wedge_{p \in A} p \neq \bot$. Then by Claim 1 there is a $J \in \mathcal{J}_{X'}$ containing $\wedge_{p \in A} p$. So, as $\wedge_{p \in A} p$ entails each $p \in A$, J contains each $p \in A$, i.e., $A \subseteq J$. ■

Proof of Proposition 2. Let X be an agenda. As one easily checks, we may assume without loss of generality. that X is redundancy-free.

1. In this part we show that we may assume without loss of generality. that X is a 'semantic' agenda given as follows: there exists a finite set of 'worlds' $\Omega \neq \varnothing$ such that $X \subseteq 2^{\Omega}$, where (i) each issue takes the form $\{A, \overline{A}\}$ (I write $\overline{A}$ for the complement $\Omega \backslash A$ of any set $A \subseteq \Omega$), (ii) the set $\mathcal{J}_X$ of rational judgment sets consists of those sets $J \subseteq X$ which contain exactly one member of each issue and satisfy $\cap_{A \in J} A \neq \varnothing$, and (iii) rational judgment sets in $\mathcal{J}_X$ correspond to worlds in Ω, in the sense that the assignment $J \mapsto \cap_{A \in J} A$ defines a bijection from $\mathcal{J}_X$ to $\{\{\omega\} : \omega \in \Omega\}$.

To show this, we consider any agenda V and construct a semantic agenda X of the given sort to which V is isomorphic. Let the set of worlds be $\Omega := \mathcal{J}_V$. To each $p \in V$ corresponds a set of worlds, the 'extension' of p, given by $[p] := \{\omega \in \Omega : p \in \omega\}$. Note that the assignment $p \mapsto [p]$ defines a bijection from V to the set $X := \{[p] : p \in V\}$. I define an agenda by the set X, endowed with

- issues defined as the sets $\{[p], [\neg p]\}$ (which indeed partition X into pairs, since the sets $\{p, \neg p\}$ partition V into pairs and since $p \mapsto [p]$ maps V bijectively to X),
- rational judgment sets defined as the sets $J \subseteq X$ containing exactly one member of each issue and satisfying $\cap_{A \in J} A \neq \varnothing$.

This agenda X satisfies (i) since $[\neg p] = \overline{[p]}$ for all $p \in V$, and satisfies (ii) immediately by definition. To show that it satisfies (iii), we first show that for each $J \in \mathcal{J}_X$ the intersection $\cap_{A \in J} A$ $(\neq \varnothing)$ is indeed a singleton. Assume for a condition that it contains distinct $\omega, \omega' \in \Omega$. Since $\omega \neq \omega'$, there is a $p \in V$ such that $p \in \omega' \backslash \omega$ and $\neg p \in \omega \backslash \omega'$. So, $\omega \notin [p]$ and $\omega' \notin [\neg p]$. Since J contains either $[p]$ or $[\neg p]$, it follows that either $\omega \notin \cap_{A \in J} A$ or $\omega' \notin \cap_{A \in J} A$, a contradiction. Second, one has to check injectivity and surjectivity of the mapping from $\mathcal{J}_X$ to $\{\{\omega\} : \omega \in \Omega\}$; we leave this to the reader.

Finally, to show that V and X are isomorphic (as agendas), it suffices to show that $p \mapsto [p]$ defines an (agenda) isomorphism. This is so because the assignment $p \mapsto [p]$ is bijective, and bijectively maps the issues $\{p, \neg p\}$ of V to those of X, and the rational judgment sets of V to those of X (the latter can be shown by verifying that the assignment $J \mapsto \{[p] : p \in J\}$ defines a bijection from $\mathcal{J}_V$ to $\mathcal{J}_X$).

2. From now on we assume that X takes the semantic form defined in part 1. In the current part, we show the existence claim. As one can check, X is a subagenda of the agenda $X' := 2^\Omega$ whose issues are the pairs $\{A, \overline{A}\}$ $(A \subseteq \Omega)$ and whose rational judgment sets are the sets of the form $\{A \subseteq \Omega : \omega \in A\}$ $(\omega \in \Omega)$. It suffices to show that X' is a *minimal closed* extension of X. First, X' is closed, where the conjunction is given by the intersection, and the disjunction by the union. Second, we have to show minimality. Consider any superagenda X'' of X which is a strict subagenda of X'. We have to show that X'' is not closed. As X'' is a subagenda of X', it inherits its issues from X', and thus X'' is closed under complement: $A \in X'' \Rightarrow \overline{A} \in X''$. Since X' $(= 2^\Omega)$ is the *only* subset of 2^Ω which includes X and is closed under intersection and complement, and since X'' is closed under complement, X'' cannot be closed under intersection. It follows that X'' is not closed (i.e., not closed under conjunction), by the following argument. Choose any $A, B \in X''$ such that $A \cap B \notin X''$. Suppose for a contradiction that X'' contains a C which (relative to agenda X'') is the conjunction of A and B, i.e., is equivalent to $\{A, B\}$. Sine $A \cap B \notin X''$, $C \neq A \cap B$. So, since also $C \subseteq A$ and $C \subseteq B$ (as C entails A and B relative to the agenda X''), we have $C \subsetneq A \cap B$. Choose any $\omega \in (A \cap B) \backslash C$. Note that $J'' := \{D \in X'' : \omega \in D\}$ belongs to $\mathcal{J}_{X''}$, and contains A and B but not C. So (still relative to agenda X'') $\{A, B\}$ does not entail C, a contradiction since C is the conjunction of A and B.

3. Finally, we show the uniqueness claim. Since the agenda X' defined in part 2 is a minimal closed extension of X, it suffices to show that any other such extension of X is equal to X' up to relabelling. Let Z be an arbitrary minimal closed superagenda of X. We need to define an agenda isomorphism $f : X' \to Z$ which is constant on X. For all $\omega \in \Omega$ and all $Y \subseteq X'$ $(= 2^\Omega)$, let $Y_\omega := \{A \in Y : \omega \in A\}$, and for all $B \in X'$ $(= 2^\Omega)$ let

$$p_B := \vee_{\omega \in B} (\wedge X_\omega) \ \ (\in Z). \tag{9}$$

Here and in what follows, let '$\vee$', '$\wedge$' and '$\neg$' refer to the disjunction, conjunction and negation operators of Z (rather than of X or X'). By Remark 3, '$\vee$', '$\wedge$' and '$\neg$' can alternatively be viewed as the algebraic operations of join, meet and complement in the Boolean algebra Z. So, standard algebraic rules apply, such as associativity, commutativity and distributivity of $\vee$ and $\wedge$. Also, let $\top$ and $\bot$ be the greatest and smallest elements of the Boolean algebra Z, respectively; clearly, $\top$ is the (only) tautology and $\bot$ the (only) contradiction of the agenda Z.

Claim 1: For all $Y \subseteq X$, $A \in X \backslash Y$ and $\omega \in \Omega$ we write $Y_\omega^A := Y_\omega \cup \{A\}$. For every subagenda Y of X, $A \in X \backslash Y$, and $\omega \in \overline{A}$, either $\wedge Y_\omega^A = \bot$ or $Y_\omega^A = Y_{\omega'}^A$ for some $\omega' \in A$.

Consider any subagenda Y of X, $A \in X \backslash Y$, and $\omega \in \overline{A}$. First assume Y_ω^A is inconsistent w.r.t. agenda X. Then $\wedge Y_\omega^A = \bot$ by Lemma 6. Now assume Y_ω^A is consistent w.r.t. agenda X. So there is an $\omega' \in \cap Y_\omega^A$. In particular, $\omega' \in \cap Y_\omega$. So, for each $B \in Y$, $\omega \in B \Rightarrow \omega' \in B$. In fact, the '$\Rightarrow$' can be replaced by '$\Leftrightarrow$', since ω and ω' belong to the same number of sets B in Y (i.e., to half the these sets, as $B \in Y \Leftrightarrow \overline{B} \in Y$). So, $Y_\omega = Y_{\omega'}$, and hence, $Y_\omega^A = Y_{\omega'}^A$. QED

Claim 2: For all $B \in X'$, $\neg p_B = p_{\overline{B}}$.

Let $B \in X'$. Since $\neg$ coincides with the algebraic complement operation in Z, it suffices to show that $p_B \vee p_{\overline{B}} = \top$ and $p_B \wedge p_{\overline{B}} = \bot$.

We first prove that $p_B \vee p_{\overline{B}} = \top$. Since

$$p_B \vee p_{\overline{B}} = [\vee_{\omega \in B} \wedge X_\omega] \vee \left[\vee_{\omega \in \overline{B}} \wedge X_\omega\right] = \vee_{\omega \in \Omega} \wedge X_\omega,$$

we have to prove that $\vee_{\omega \in \Omega} \wedge X_\omega = \top$. We first show that

$$\vee_{\omega \in \Omega} \wedge X_\omega = \vee_{\omega \in \Omega} \wedge Y_\omega, \tag{10}$$

where Y is any set of the form $X \backslash \{A, \overline{A}\}$ with $A \in X$. Note that

$$\vee_{\omega \in \Omega} \wedge X_\omega = [\vee_{\omega \in A} \wedge X_\omega] \vee \left[\vee_{\omega \in \overline{A}} \wedge X_\omega\right] = \left[\vee_{\omega \in A} \wedge Y_\omega^A\right] \vee \left[\vee_{\omega \in \overline{A}} \wedge Y_\omega^{\overline{A}}\right],$$

where the last expression uses notation introduced in Claim 1. This expression is a disjunction of terms (disjuncts) of two types: any $\wedge Y_\omega^A$ with $\omega \in A$ (type 1) and any $\wedge Y_\omega^{\overline{A}}$ with $\omega \in \overline{A}$ (type 2). The result is not affected by adding the following new disjuncts: any $\wedge Y_\omega^A$ with $\omega \in \overline{A}$ (type 3) and any $\wedge Y_\omega^{\overline{A}}$ with $\omega \in A$ (type 4). Indeed, by Claim 1 each new disjunct of type 3 is either $\bot$ or coincides with a disjunct of type 1, and any new disjunct of type 4 is either $\bot$ or coincides with a disjunct of type 2. After adding these new disjuncts and re-grouping, the expression becomes

$$\left[\vee_{\omega \in \Omega} \wedge Y_\omega^A\right] \vee \left[\vee_{\omega \in \Omega} \wedge Y_\omega^{\overline{A}}\right].$$

Noting that each Y_ω^A equals $\{A\} \cup Y_\omega$ and each $Y_\omega^{\overline{A}}$ equals $\{\overline{A}\} \cup Y_\omega$, and then using distributivity twice, the last expression reduces to

$$[A \wedge (\vee_{\omega \in \Omega} \wedge Y_\omega)] \vee \left[\overline{A} \wedge (\vee_{\omega \in \Omega} \wedge Y_\omega)\right] = \left[A \vee \overline{A}\right] \wedge (\vee_{\omega \in \Omega} \wedge Y_\omega) = \vee_{\omega \in \Omega} \wedge Y_\omega.$$

This proves (10). By an analogous argument, one can show that (unless $Y = \varnothing$), we have $\vee_{\omega \in \Omega} \wedge Y_\omega = \vee_{\omega \in \Omega} \wedge Y'_\omega$ for a set Y' of the form $Y \backslash \{A, \overline{A}\}$ with $A \in Y$; which together with (10) yields that $\vee_{\omega \in \Omega} \wedge X_\omega = \vee_{\omega \in \Omega} \wedge Y'_\omega$. Continuing in this fashion, we ultimately obtain that $\vee_{\omega \in \Omega} \wedge X_\omega = \vee_{\omega \in \Omega} \wedge \varnothing_\omega = \top$, as desired.

We finally have to prove that $p_B \wedge p_{\overline{B}} = \bot$. Using distributivity twice,

$$\begin{aligned} p_B \wedge p_{\overline{B}} &= [\vee_{\omega \in B} \wedge X_\omega] \wedge \left[\vee_{\omega' \in \overline{B}} \wedge X_{\omega'}\right] \\ &= \vee_{\omega \in B} \left([\wedge X_\omega] \wedge \left[\vee_{\omega' \in \overline{B}} \wedge X_{\omega'}\right]\right) \\ &= \vee_{\omega \in B} \left(\vee_{\omega' \in \overline{B}} \left([\wedge X_\omega] \wedge [\wedge X_{\omega'}]\right)\right). \end{aligned}$$

It thus suffices to show that for all $\omega \in B$ and $\omega \in \overline{B}$ we have $[\wedge X_\omega] \wedge [\wedge X_{\omega'}] = \bot$. Let $\omega \in B$ and $\omega \in \overline{B}$. Since $\omega \neq \omega'$, there is an $A \in X$ such that $\omega \in A$ and $\omega' \in \overline{A}$. Since $A \in X_\omega$, $\wedge X_\omega$ entails A. Analogously, since $\overline{A} \in X_{\omega'}$, $\wedge X_{\omega'}$ entails $\overline{A}$. It follows that $[\wedge X_\omega] \wedge [\wedge X_{\omega'}]$ entails $A \wedge \overline{A}$. As $A \wedge \overline{A} = \bot$ (since A and $\overline{A}$ are complements in the algebra Z), it follows that $[\wedge X_\omega] \wedge [\wedge X_{\omega'}]$ entails $\bot$, hence, equals $\bot$. QED

Claim 3: $p_B = B$ for all $B \in X$.

Consider any $B \in X$. We regard B as an element of the extended agenda $Z \supseteq X$. Since Z is redundancy-free, it suffices to show that p_B and B entail each other. We first show that p_B entails B. Since p_B is the *least* upper bound of all $\wedge X_\omega$ with $\omega \in B$, it suffices to show that B is an upper bound, i.e., that each of these $\wedge X_\omega$ entails B. This is so because for each $\omega \in B$ the set X_ω contains B. Second, we show that B entails p_B, or equivalently, that $\neg p_B$ entails $\neg B$. This follows from the previous argument applied to $\overline{B}$ rather than B, because $\neg p_B = p_{\overline{B}}$ by Claim 2 and because $\neg B = \overline{B}$ (as Z is a superagenda of X, so that B's Z-relative negation $\neg p_B$ coincides with B's X-relative negation $\overline{B}$). QED

Claim 4: $Z = \{p_B : B \in X'\}$.

The set $S := \{p_B : B \in X'\}$ $(\subseteq Z)$ is closed under negation by Claim 2, hence defines a subagenda of Z. The agenda S is closed because for any $B, C \in X'$ the disjunction of p_B and p_C (relative to the agenda Z) equals $p_{B \cup C}$, hence belongs to the agenda S (relative to which it of course still defines the disjunction of p_B and p_C). Moreover, the agenda S includes X by Claim 3, hence is a superagenda of X. Since Z is by definition a *minimal* closed superagenda of X, it follows that $S = Z$. QED

Claim 5: For all $A, B \in X'$, $A \subseteq B$ if and only if p_A entails p_B.

For each $\omega \in \Omega$ we have $p_{\{\omega\}} \neq \bot$; this is because the set X_ω is consistent with respect to agenda X, and hence $p_{\{\omega\}} = \wedge X_\omega \neq \bot$ by Lemma 6. Now consider any $A, B \in X'$. First, if $A \subseteq B$, then p_A clearly entails p_B since p_B is a disjunction of *at least* those terms of which p_A is a disjunction. Conversely, now assume that p_A entails p_B. As $A \backslash B \subseteq \overline{B}$, $p_{A \backslash B}$ entails $p_{\overline{B}}$; and so, as $p_{\overline{B}} = \neg p_B$ by Claim 2, $p_{A \backslash B}$ entails $\neg p_B$. Also, as $A \backslash B \subseteq A$, $p_{A \backslash B}$ entails p_A; and so, as p_A entails p_B, $p_{A \backslash B}$ entails p_B. Since, as we have shown, $p_{A \backslash B}$ entails both $\neg p_B$ and p_B, it entails $\neg p_B \wedge p_B = \bot$. Hence, $p_{A \backslash B} = \bot$. It follows that $A \backslash B = \varnothing$, i.e., $A \subseteq B$, since if there were an $\omega \in A \backslash B$, then $p_{\{\omega\}}$ would entail $p_{A \backslash B}$, whence $p_{\{\omega\}} = \bot$, in contradiction with what was shown at the start of the proof of the claim. QED

Claim 6: For all $A, B \in X'$, $p_{A \cup B} = p_A \vee p_B$ and $p_{A \cap B} = p_A \wedge p_B$.

Let $A, B \in X'$. The first identity holds immediately by definition of p_A and p_B. As for the second identity, using de Morgan's Law (valid in Boolean algebras) and then Claim 2, $p_A \wedge p_B = \neg(\neg p_A \vee \neg p_B) = \neg(p_{\overline{A}} \vee p_{\overline{B}})$. Now using the first identity, it follows that $p_{A \wedge B} = \neg p_{\overline{A} \cup \overline{B}}$, which reduces to $p_{A \cap B}$ by $\overline{A} \cup \overline{B} = \overline{A \cap B}$ and Claim 2. QED

Claim 7: The assignment $B \mapsto p_B$ defines an agenda isomorphism between X' and Z which is constant on X. (This completes the proof.)

This assignment – call it f – is constant on X by Claim 3, and surjective by Claim 4. To show injectivity, consider distinct $A, B \in X'$. We may assume without loss of generality. that $A \not\subseteq B$ (since otherwise the roles of A and B can be interchanged). By Claim 5, p_A does not entail p_B, and so $p_A \neq p_B$. It remains to show that f preserves the agenda structure: the issues (resp. negation operator) and the interconnections. This could be deduced from Claim 5 since, firstly, by Claim 5 the (bijective) function f is a Boolean-algebra isomorphism, and, secondly, for a closed agenda, the agenda structure and the Boolean-algebra structure are interdefinable, as can be verified.[20] But let me give a direct proof. First, f preserves the issues structure, since for each $A \in X'$ we have $\neg p_A = p_{\overline{A}}$ and $\overline{A}$ is the X'-relative negation of A. Second, consider a set $S \subseteq X'$; we show that S is consistent (in the sense of X') if and only if its image $\{p_B : B \in S\}$ is consistent (in the sense of Z). This holds for the following reasons. S is consistent if and only if $\cap S \neq \varnothing$, which is in turn equivalent to $p_{\cap S} \neq p_\varnothing$, i.e., to $p_{\cap S} \neq \bot$. By Claim 6, the latter is equivalent to $\wedge_{B \in S} p_B \neq \bot$, which is in turn equivalent to the consistency of $\{p_B : B \in S\}$ by Lemma 6. ■

Proof of Lemma 1. This lemma follows from the proof of Proposition 2. ■

Proof of Proposition 3. Let X be an agenda. It suffices to show that for each $J \in \mathcal{J}_X$ and $p \in \overline{X}$, J entails p or entails $\neg p$, or equivalently, $\wedge_{q \in J} q$ entails p or entails $\neg p$. This follows from the fact that, by Lemma 1, $\wedge_{q \in J} q$ is an atom of $\overline{X}$, i.e., a logically strongest element of $\overline{X} \backslash \{\bot\}$. ■

A.3 Results of Section 6: agenda manipulation

Proof of Theorem 2. We consider any aggregation system $(F_X)_{X \in \mathcal{X}}$.

1. First, suppose $(F_X)_{X \in \mathcal{X}}$ is agenda-invariant.

Claim 1: $(F_X)_{X \in \mathcal{X}}$ is coherent.

Consider $X, X' \in \mathcal{X}$ with $X \subseteq X'$ and $J_1, ..., J_n \in \mathcal{J}_X$. Each J_i is consistent, and thus extendible to a $J_i' \in \mathcal{J}_{X'}$. I show that $F_X(J_1, ..., J_n) \subseteq F_{X'}(J_1', ..., J_n')$. Consider any $p \in F_X(J_1, ..., J_n)$. Applying agenda-invariance to the agendas X and X', the proposition p ($\in X = X \cap X'$) and the judgment sets J_i' ($\in \mathcal{J}_{X'} = \mathcal{J}_{X \cup X'}$), and noting that each J_i' satisfies $J_i' \cap X = J_i$ and $J_i' \cap X' = J_i'$, we obtain that

$$p \in F_X(J_1, ..., J_n) \Leftrightarrow p \in F_{X'}(J_1', ..., J_n').$$

So, as $p \in F_X(J_1, ..., J_n)$ by assumption, $p \in F_{X'}(J_1', ..., J_n')$. QED

Claim 2: Each F_X is independent.

Consider any $X \in \mathcal{X}$, $p \in X$, and $(J_1, ..., J_n), (J_1', ..., J_n') \in \mathcal{J}_X^n$ such that, for all i, $p \in J_i \Leftrightarrow p \in J_i'$. Define Z as the agenda $\{p, \neg p\} \in \mathcal{X}$. For each i, let K_i be $\{p\}$ if $p \in J_i$ (or equivalently, $p \in J_i'$), and as $\{\neg p\}$ otherwise. Applying agenda-invariance to the agendas X, Z and the judgment sets J_i ($\in \mathcal{J}_X = \mathcal{J}_{X \cup X'}$), and noting that $J_i \cap X = J_i$ and $J_i \cap Z = K_i$, we obtain

$$p \in F_X(J_1, ..., J_n) \Leftrightarrow p \in F_Z(K_1, ..., K_n). \tag{11}$$

[20] For instance, a subset A is consistent in the agenda sense if and only if its algebraic meet is not $\bot$, by Lemma 6.

Applying agenda-invariance again, this time to the agendas X, Z and the judgment sets J_i' ($\in \mathcal{J}_X = \mathcal{J}_{X \cup Z}$), and noting that $J_i' \cap X = J_i'$ and $L_i' \cap Z = K_i$, we obtain

$$p \in F_X(J_1', ..., J_n') \Leftrightarrow p \in F_Z(K_1, ..., K_n). \tag{12}$$

By (11) and (12), $p \in F_X(J_1, ..., J_n) \Leftrightarrow p \in F_Z(J_1', ..., J_n')$. QED

2. Now suppose $(F_X)_{X \in \mathcal{X}}$ is coherent and each F_X is independent. I prove agenda-invariance. Consider any $X, X' \in \mathcal{X}$, $p \in X \cap X'$ and $J_1, ..., J_n \in \mathcal{J}_{X \cup X'}$, and let us show that $p \in F_X(J_1 \cap X, ..., J_n \cap X)$ if and only if $p \in F_{X'}(J_1 \cap X', ..., J_n \cap X')$. Consider the agenda $Z := \{p, \neg p\} \in \mathcal{X}$, and for each i let K_i be $\{p\}$ if $p \in J_i$ and $\{\neg p\}$ otherwise. Note that $p \in K_i$ is equivalent to $p \in J_i \cap X$ and also to $p \in J_i \cap X'$, because each of these three statements is equivalent to $p \in J_i$. By coherence applied to the agendas Z and X, the judgment sets $K_1, ..., K_n \in \mathcal{J}_Z$ have extensions $L_1 \supseteq K_1, ..., L_n \supseteq K_n$ in $\mathcal{J}_X$ such that $F_Z(K_1, ..., K_n) \subseteq F_X(L_1, ..., L_n)$. It follows that

$$p \in F_X(L_1, ..., L_n) \Leftrightarrow p \in F_Z(K_1, ..., K_n). \tag{13}$$

Further, for any i, $p \in L_i$ is equivalent to $p \in K_i$ (as $L_i \supseteq K_i$), which is in turn equivalent to $p \in J_i \cap X$ (as shown above). Hence, as F_X is independent, $p \in F_X(L_1, ..., L_n) \Leftrightarrow p \in F_X(J_1 \cap X, ..., J_n \cap X)$. By (13) it follows that

$$p \in F_X(J_1 \cap X, ..., J_n \cap X) \Leftrightarrow p \in F_Z(K_1, ..., K_n). \tag{14}$$

By a similar argument for the agenda X',

$$p \in F_{X'}(J_1 \cap X', ..., J_n \cap X') \Leftrightarrow p \in F_Z(K_1, ..., K_n). \tag{15}$$

By (14) and (15), $p \in F_X(J_1 \cap X, ..., J_n \cap X) \Leftrightarrow p \in F_{X'}(J_1 \cap X', ..., J_n \cap X')$. ■

Proof of Theorem 3. Let $(F_X)_{X \in \mathcal{X}}$ be any aggregation system.

1. First let $(F_X)_{X \in \mathcal{X}}$ be fully agenda-invariant.

Claim 1: $(F_X)_{X \in \mathcal{X}}$ is fully coherent.

Consider $X, X' \in \mathcal{X}$ with $\overline{X} \subseteq \overline{X'}$ and $J_1, ..., J_n$ in $\mathcal{J}_X$. For any individual i, since J_i is consistent, it is extendible to a $\widehat{J}_i \in \mathcal{J}_{X \cup X'}$; we let $J_i' := \widehat{J}_i \cap X'$ ($\in \mathcal{J}_{X'}$). We have to show that (i) each J_i' entails J_i, and (ii) $F_{X'}(J_1', ..., J_n')$ entails $F_X(J_1, ..., J_n)$. Regarding (i), for any i, as $J_i' \in \mathcal{J}_{X'}$ and $\overline{X} \subseteq \overline{X'}$, the set J_i' entails exactly one set in $\mathcal{J}_X$; so, by the consistency of $J_i' \cup J_i$ ($= \widehat{J}_i$), J_i' entails J_i. Regarding (ii), consider any $p \in F_X(J_1, ..., J_n)$ and let us show that $F_{X'}(J_1', ..., J_n')$ entails p. Applying full agenda-invariance to the agendas X, X', the proposition p ($\in \overline{X} = \overline{X} \cap \overline{X'}$) and the sets $\widehat{J}_i$ (which satisfy $\widehat{J}_i \cap X = J_i$ and $\widehat{J}_i \cap X' = J_i'$), we obtain

$$F_X(J_1, ..., J_n) \text{ entails } p \Leftrightarrow F_{X'}(J_1', ..., J_n') \text{ entails } p.$$

The left-hand side holds as $p \in F_X(J_1, ..., J_n)$. So, $F_{X'}(J_1', ..., J_n')$ entails p. QED

Claim 2: Each F_X is independent on $\overline{X}$.

Consider any $X \in \mathcal{X}$, any $p \in \overline{X}$, and any $(J_1, ..., J_n), (J_1', ..., J_n') \in \mathcal{J}_X$ such that, for all i, J_i entails p if and only if J_i' does so. Let Z be the agenda $\{p, \neg p\} \in \mathcal{X}$. For each i, I define K_i as $\{p\}$ if J_i (or equivalently J_i') entails p, and as $\{\neg p\}$ otherwise, and I define $L_i := J_i \cup K_i$ and $L_i' := J_i' \cup K_i$. Applying full agenda-invariance to the agendas X, Z and the judgment sets L_i (which belong to $\mathcal{J}_{X\cup Z}$ and satisfy $L_i \cap X = J_i$ and $L_i \cap Z = K_i$), we obtain

$$F_X(J_1, ..., J_n) \text{ entails } p \Leftrightarrow F_Z(K_1, ..., K_n) \text{ entails } p. \tag{16}$$

Now applying full agenda-invariance to the agendas X, Z and the judgment sets L_i' (which belong to $\mathcal{J}_{X\cup Z}$ and satisfy $L_i' \cap X = J_i'$ and $L_i' \cap Z = K_i$), we obtain

$$F_X(J_1', ..., J_n') \text{ entails } p \Leftrightarrow F_Z(K_1, ..., K_n) \text{ entails } p. \tag{17}$$

The relations (16) and (17) jointly imply that $F_X(J_1, ..., J_n)$ entails p if and only if $F_Z(J_1', ..., J_n')$ entails p. QED

2. Conversely, assume that $(F_X)_{X\in\mathcal{X}}$ is fully coherent and each F_X is independent on the scope $\overline{X}$. To show full agenda-invariance, we consider any $X, X' \in \mathcal{X}$, $p \in \overline{X} \cap \overline{X'}$ and $J_1, ..., J_n \in \mathcal{J}_{X\cup X'}$, and show that $F_X(J_1 \cap X, ..., J_n \cap X)$ entails p if and only if $F_{X'}(J_1 \cap X', ..., J_n \cap X')$ entails p. Consider the agenda $Z := \{p, \neg p\} \in \mathcal{X}$. For each i, define K_i as $\{p\}$ if J_i entails p and $\{\neg p\}$ otherwise. By construction, J_i entails K_i. So, $J_i \cap X$ also entails K_i (as $J_i \cap X \in \mathcal{J}_X$ and $p \in \overline{X}$), and so $J_i \cap X$ entails p if and only if $p \in K_i$. For analogous reasons, $J_i \cap X'$ entails K_i, and so $J_i \cap X'$ entails p if and only if $p \in K_i$. By full coherence applied to the agendas Z, X (which indeed satisfy $\overline{Z} \subseteq \overline{X}$ as $p \in \overline{X}$) and the judgment sets $K_i \in \mathcal{J}_Z$, there exist some $L_1, ..., L_n \in \mathcal{J}_X$ such that each L_i entails K_i and $F_X(L_1, ..., L_n)$ entails $F_Z(K_1, ..., K_n)$. As $F_X(L_1, ..., L_n)$ entails $F_Z(K_1, ..., K_n)$ (and as $F_Z(K_1, ..., K_n)$ is $\{p\}$ or $\{\neg p\}$),

$$F_X(L_1, ..., L_n) \text{ entails } p \Leftrightarrow p \in F_Z(K_1, ..., K_n). \tag{18}$$

Similarly, for any i, as L_i entails K_i (and as K_i is $\{p\}$ or $\{\neg p\}$), L_i entails p if and only if $p \in K_i$, which was shown to hold if and only if $J_i \cap X$ entails p. So, as F_X is independent on $\overline{X}$, $F_X(L_1, ..., L_n)$ entails p if and only if $F_X(J_1 \cap X, ..., J_n \cap X)$ entails p. By (18) it follows that

$$F_X(J_1 \cap X, ..., J_n \cap X) \text{ entails } p \Leftrightarrow p \in F_Z(K_1, ..., K_n). \tag{19}$$

By an analogous argument for the agenda X',

$$F_{X'}(J_1 \cap X', ..., J_n \cap X') \text{ entails } p \Leftrightarrow p \in F_Z(K_1, ..., K_n). \tag{20}$$

The relations (19) and (20) imply that $F_X(J_1 \cap X, ..., J_n \cap X)$ entails p if and only if $F_{X'}(J_1 \cap X', ..., J_n \cap X')$ entails p. ■

Proof of Proposition 6. Let $(F_X)_{X\in\mathcal{X}}$ be a fully agenda-invariant aggregation system, and consider an $X \in \mathcal{X}$ such that $X \subseteq \cup_{Z\in\mathcal{X}:|\mathcal{J}_Z|>2}\overline{Z}$. If $|\mathcal{J}_X| > 2$, then by Theorem 3 F_X is independent on $\overline{X}$, whence by Proposition 5 dictatorial or constant. Now suppose $|\mathcal{J}_X| \leq 2$, i.e., $|\mathcal{J}_X| = 2$. So X contains only one pair $p, \neg p$ of contingent propositions (given that X is redundancy-free because $\mathcal{L}$ is redundancy-free). Since $X \subseteq \cup_{Z\in\mathcal{X}:|\mathcal{J}_Z|>2}\overline{Z}$, we may choose a $Z \in \mathcal{X}$ such that $|\mathcal{J}_Z| > 2$ and $\overline{Z}$ contains p (and thus $\neg p$). So, since p and $\neg p$ are the only (contingent) propositions in X, $\overline{X} \subseteq \overline{Z}$. By Theorem 3, F_Z is independent on $\overline{Z}$, hence dictatorial or constant by Proposition 5. By full coherence and the fact that $\overline{X} \subseteq \overline{Z}$, the rule F_X inherits from F_Z the property of being dictatorial or constant. ■

Proof of Theorem 4. Consider an aggregation system $(F_X)_{X\in\mathcal{X}}$.

1. First let this system be conditionally agenda-invariant with each F_X setwise unanimity-preserving. Fix any $X \in \mathcal{X}$. Let $p \in \overline{X}$ be entailed by each of $J_1, ..., J_n \in \mathcal{J}_X$. We have to show that $F_X(J_1, ..., J_n)$ entails p. Consider the agenda $X' := \{p, \neg p\} \in \mathcal{X}$. Applying conditional agenda-invariance to the judgment sets $J'_i := J_i \cup \{p\} \in \mathcal{J}_{X\cup X'}$ (each of which entails $p \in \overline{X} \cap \overline{X'}$) and noting that each J'_i satisfies $J'_i \cap X = J_i$ and $J'_i \cap X' = \{p\}$, we obtain that

$$F_X(J_1, ..., J_n) \text{ entails } p \Leftrightarrow F_{X'}(\{p\}, ..., \{p\}) \text{ entails } p.$$

The right-hand side (in which 'entails' can be replaced by 'contains') holds since $F_{X'}$ is setwise unanimity-preserving. So the left-hand side holds, as desired.

2. Conversely, assume each F_X is implicit consensus preserving (and thus also setwise unanimity-preserving). To show conditional agenda-invariance, we consider any $X, X' \in \mathcal{X}$ and $p \in \overline{X} \cap \overline{X'}$, and any $J_1, ..., J_n \in \mathcal{J}_{X\cup X'}$ each of which entails p. We assume that $F_{X'}(J_1 \cap X', ..., J_n \cap X')$ entails p and show that $F_X(J_1 \cap X, ..., J_n \cap X)$ entails p (the converse implication being analogous). Fix an individual i. Since J_i is consistent and entails p, J_i is consistent with p. So $J_i \cap X$ is also consistent with p, and therefore cannot entail $\neg p$. Now, as $p \in \overline{X}$, every judgment set in $\mathcal{J}_X$ (such as $J_i \cap X$) entails either p or $\neg p$. So $J_i \cap X$ entails p. As this is true for all i and as F_X is implicit consensus preserving, $F_X(J_1 \cap X, ..., J_n \cap X)$ entails p. ■