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Abstract

The main purpose of this paper is to remark that any definable continuous path linking two regular equilibria in a regular O-minimal equilibrium manifold intersects a finite number of definable connected components locally determined. We apply the cell decomposition theorem to decompose the definable equilibrium manifold in finite connected components, the definable triviality theorem to local determinacy in each component, and the definable curve selection to have continuous paths in the manifold.

Key words: O-minimal manifold, cell decomposition, triviality, curve selection

JEL classification: D50, D51

1 Introduction

The main purpose of this paper is to remark an implicit result of Matta (2005) regarding to continuous paths in analytical equilibrium manifolds. We use the general case of O-minimal structures to remark that any definable continuous path joining two regular equilibria intersects a finite number of definable connected components of the equilibrium manifold.

This remark is important for two reasons. On one hand, it characterizes the equilibrium manifold as a composition of definable connected components locally determined. Balasko (1988) defines them as fibres. On the other hand, the definable path-connected property allows to link them by continuous definable paths.

Blume and Zame (1992) began the analysis of the general equilibrium model in O-minimal structures to prove the local determinacy of the equilibrium. These mathematical structures gives us a general framework by including commonly used cases like algebraic functions. We use them to characterize paths by changing the endowments of the economy.

2 Preliminary mathematical concepts

We take some general definitions on O-minimal structures from Van den Dries (VDD) (1998)

Definition 1 *(VDD,1998,p.2) An O-minimal structure on a dense linearly ordered non empty set is a sequence $(S_m)_{m \in N}$ such that:*

- *S_m is a boolean algebra of subsets of R^m;*
- *If $A \in S_m$ then $R \times A$ and $A \times R$ belong to S_{m+1};*
- *$\{(x_1, ..., x_n) \in R^n : x_i = x_j\} \in S_n$ for $1 \leq i < j \leq n$;*
- *If $A \in S_{m+1}$ then $\pi(A) \in S_m$ where $\pi : R^{m+1} \to R^m$ is the projection map on the first m coordinates;*
- *$\{(x, y) \in R^2 : x < y\} \in S_2$;*
- *The sets in S_1 are the finite unions of intervals and points.*

Definition 2 *(VDD,1998,p.2) A set $A \subset R^m$ is definable if it belongs to S_n.*

Algebraic and semi-algebraic sets are definable in O-minimal structures. In Bochnak, Coste and Roy (1991) there is a treatment on semi-algebraic sets. We use a common definable set called 'cell' to decompose every definable set. In chapter III of VDD (1998) there is a definition of cell, but they are basically definable sets homeomorphic to a locally closed hypercube.

Theorem 1 *(VDD, 1998, p.52) Given any definable sets $A_1, ..., A_k \subseteq R^m$ there is a decomposition of R^m partitioning each of $A_1, ..., A_k$ into finitely many cells.*

Definition 3 *(VDD,1998,p.3) A map $f : A \to B$ with definable $A \subset R^m$ and $B \subset R^n$, is definable if its graph $\Gamma(f) \subset R^{m+n}$ belongs to S_{m+n}.*

In general, the algebraic and semi-algebraic functions are definable in O-minimal structures.

Theorem 2 *(VDD, 1998, p.94) If* $a \in \overline{X} - X$ *where* X *is definable then there is a definable continuous injective map* $\gamma : (0, \epsilon) \to X$ *for some* $\epsilon > 0$ *such that* $limit_{t \to 0} \gamma(t) = a$.

Definition 4 *(VDD, 1998, p.142) Let* $f : S \to A$ *a definable map with* S, A *definable. A definable trivialization of* f *is a pair* (F, λ) *with* F *definable and* $\lambda : S \to F$ *definable such that* $(f, \lambda) : S \to A \times F$ *is a homeomorphism.*

Theorem 3 *(VDD, 1998, p.142) Let* $f : S \to A$ *be a continuous definable map. Then there is a finite partition* $A = A_1 \cup ... \cup A_M$ *of* A *into definable sets* A_i *such that* f *is definable trivial over each* A_i.

3 The remark

There is a finit number of commodities $\mathcal{L} = \{1, ..., L\}$ and consumers $\mathcal{I} = \{1, ..., I\}$. Let $X = \mathbb{R}^L$ be the space of commodities. Blume and Zame (1992) assume the preference order $\succsim \subseteq X \times X$ is a definable set of $\mathbb{R}^{2L}$. Let $\mathbb{R}^L$ be the space of the initial endowments $\omega_i = (\omega_i^l)_{i \in \mathcal{I}}^{l \in \mathcal{L}}$. Let $\Omega = \mathbb{R}^{LI}$ be the space of economies.

By using the standard numeraire, let $\mathbb{R}_{++}^{L-1}$ be the space of the prices $p^1, ..., p^L$ such that $p^{l \in \mathcal{L}} > 0$ and $p^1 = 1$. The solution of the consumer problem defines the marshallian demand function $f_i : \mathbb{R}_{++}^{L-1} \times \mathbb{R} \to \mathbb{R}^L$.

Theorem 4 *(Blume and Zame, 1992) If the consumption set* X *and the preference order belong to* S*, then the graph of the demand correspondence belong to* S.

Definition 5 *The tuple* $(\boldsymbol{p}, \omega) \in S \times \Omega$ *is a walrasian equilibrium if* $\Sigma_{i=1}^I f_i(\boldsymbol{p}, \boldsymbol{p}\omega_i) = \Sigma_{i=1}^I \omega_i$.

We say that $\mathbf{p} \in S$ is an equilibrium price vector of $\omega \in \Omega$ if $(\mathbf{p}, \omega)$ is a walrasian equilibrium in $S \times \Omega$. Let $E \subseteq S \times \Omega$ be the set of walrasian equilibrium tuples $(\mathbf{p}, \omega)$. Balasko (1988) shows that E is an smooth manifold. Blume and Zame (1994) and Matta (2005) proves that it is real analytic.

Example 1 *Let us use a pure exchange regular definable CES-economy with* $\rho = 0$, $\mathcal{I} = \{1, 2\}$ *and* $\mathcal{L} = \{x, y\}$*. Solving the general equilibrium problem, we find the equilibrium manifold is the hyperbolic paraboloid:*

$$E = \{(\boldsymbol{p}, \omega) \in S \times \Omega : p_y \Sigma_{i=1}^2 \omega_i^y = p_x \Sigma_{i=1}^2 \omega_i^x \wedge p_y = 1\}$$

The manifold E *is a real algebraic manifold composed of algebraic demand and supply functions. It implies* E *is a definable set in O-minimal structures. See technical note 1 to the intuition.* $\square$

The manifold E has many interesting properties. We are particularly interested in the path-connectedness. It implies there always exists a continuous path joining any two regular equilibria in E.

Proposition 1 *The set E is a definable path-connected manifold.*

Proof: Balasko (1988) proves that E is path-connected. It implies directly that E is a definable path-connected manifold $\square$

Example 2 *Parametrizing the surface of E in example 1 with $p_x(s)$ and $\Sigma_{i=1}^2 \omega_i^x(s)$, being s the line element, we arrive to the following definable geodesic equations:*

$$\frac{d^2 p_x}{ds^2} = -\frac{2\Sigma_{i=1}^2 \omega_i^x}{1 + p_x^2 + (\Sigma_{i=1}^2 \omega_i^x)^2} \frac{dp_x}{ds} \frac{d\Sigma_{i=1}^2 \omega_i^x}{ds}$$

$$\frac{d^2 \Sigma_{i=1}^2 \omega_i^x}{ds^2} = -\frac{2p_x}{1 + p_x^2 + (\Sigma_{i=1}^2 \omega_i^x)^2} \frac{dp_x}{ds} \frac{d\Sigma_{i=1}^2 \omega_i^x}{ds}$$

They are continuous curves joining regular equilibria in E by minimizing the distance between them. See technical note 2 to the procedure. $\square$

The natural projection $\pi : S \times \Omega \to \Omega$ is a definable map. Balasko (1988) proves that E is a manifold diffeomorphic to $\mathbb{R}^{LI}$ by restricting π to $\pi \mid_{E \subseteq S \times \Omega}: E \to \Omega$.

Proposition 2 *If the graph of the demand correspondence is definable in O-minimal structures, then the graph of the map $\pi \mid_{E \subseteq S \times \Omega}: E \to \Omega$ is also definable in O-minimal structures.*

Proof: The set E is the zeros of the definable function $\Sigma_{i=1}^I f_i(\mathbf{p}, \mathbf{p}\omega_i) - \Sigma_{i=1}^I \omega_i$. It defines E as a real algebraic manifold which is indeed definable in O-minimal structures. The Tarski-Seidenberg principle implies $\pi \mid_{E \subseteq S \times \Omega}: E \to \Omega$ is a definable map $\square$

Theorem 5 *If the equilibrium manifold E is definable in O-minimal structures, then any continuous definable path joining two arbitrary regular equilibria intersects a finite number of connected and locally determined definable components of E.*

Proof: Let $\{E_{p_i}\}_{p_i \in S}$ for $i = \{1, 2, ..., n\}$ a family of definable subsets of E with $E_{p_i} = \{\omega \in \Omega : (p_i, \omega) \in E\}$ Apply the cell decomposition theorem (Theorem 1) to have a partition of each E_{p_i} in a finite number of definable components (fibres) $E_{p_{ij}} = \{\omega \in \Omega : (p_{ij}, \omega) \in E\}$ for $j = \{1, 2, ..., m\}$ Let $E_{p_i} = \cup_j E_{p_{ij}}$ with $E_{p_{i(j)}} \cap \overline{E}_{p_{i(j+1)}} \neq \emptyset$ or $\overline{E}_{p_{i(j)}} \cap E_{p_{i(j+1)}} \neq \emptyset$.

Without loss of generality consider the case $E_{p_{i(j)}} \cap \overline{E}_{p_{i(j+1)}} \neq \emptyset$. Suppose by contradiction that $E_{p_{i(j)}} \cup E_{p_{i(j+1)}} = A \cup B$ with A, B open in E, $E_{p_{i(j)}} \subseteq A$, $E_{p_{i(j+1)}} \subseteq B$, $A \cap B = \emptyset$ and $A \cap \overline{B} \neq \emptyset$. Take $e^* \in A \cap \overline{B}$ then $e^* \in C \subseteq A$ with C open in A. It implies $C \cap B \neq \emptyset$ contradicting $A \cap B = \emptyset$ so $E_{p_{i(j)}} \cup E_{p_{i(j+1)}}$ is definably connected.

The map $\pi \mid_{E_{p_{ij}} \subseteq E}: E_{p_{ij}} \to \Omega$ is definably continuous. Take a closed definable subset Ω' with $\Omega \supseteq \Omega' = \cup_i \Omega'_i$ being Ω'_i a connected component of Ω'. To local determinacy we follow Blume and Zame (1992) and apply definable triviality theorem (Theorem 3) to have $\pi \mid_{E_{p_{ij}}} (h(e_{ij})) = \omega'_i$ for $e_{ij} \in E_{p_{ij}}$, $\omega'_i \in \Omega'_i$ and definable homeomorphism $h : E_{p_{ij}} \to \pi \mid_{E_{p_{ij}}}^{-1} (\Omega'_i)$

Apply the definable curve selection theorem (Theorem 2) to find a continuous definable path γ in the connected definable set $E_{p_{i(j)}} \cup E_{p_{i(j+1)}}$ connecting one equilibrium in $E_{p_{i(j)}}$ with another one in $E_{p_{i(j+1)}}$. The local determinacy in each fibre and the continuity of γ implies the claimed result $\square$

Example 3 *If we simplify the curves in example 2 by assuming $p_x = \Sigma_{i=1}^2 \omega_i^x$ then we have:*

$$\frac{d^2 p_x}{ds^2} = \frac{d^2 \Sigma_{i=1}^2 \omega_i^x}{ds^2} = -\frac{2p_x}{1+2p_x^2}(dp_x/ds)^2$$

This second order differential equation gives us the different curves connecting two arbitrary regular equilibria minimizing the distance between them.

4 Technical notes

Technical note from Example 1 *By using the demand function:*

$$x_i = \frac{p_y^{\frac{1}{1-\rho}}(p_x \omega_i^x + p_y \omega_i^y)}{p_y^{\frac{1}{1-\rho}} p_x + p_y p_x^{\frac{1}{1-\rho}}}; i \in \{1,2\}$$

Excess demand function is:

$$Z_x = \frac{p_y^{\frac{1}{1-\rho}}(p_x(\omega_i^x + \omega_j^x) + p_y(\omega_i^y + \omega_j^y))}{p_y^{\frac{1}{1-\rho}} p_x + p_y p_x^{\frac{1}{1-\rho}}} - (\omega_i^x + \omega_j^x); j \neq i \in \{1,2\}$$

Using the Walras law we obtain the equilibrium manifold of example 1.

Technical note from Example 2 *Let us use the general case $\rho \neq 0$. For the parametric regular curve*

$$r(u,v) = [u, v, u^{\frac{1}{1-\rho}} v]$$

The coefficients of the first fundamental form are

$$E = 1 + \left(\frac{1}{1-\rho}\right)^2 u^{\frac{2\rho}{1-\rho}} v^2; F = \left(\frac{1}{1-\rho}\right) u^{\frac{1+\rho}{1-\rho}} v; G = 1 + u^{\frac{2}{1-\rho}}$$

The Christoffel symbols are

$$\Gamma_{11}^1 = \frac{\rho v^2 u^{\frac{3\rho-1}{1-\rho}}}{(1-\rho)^3\left[1 + u^{\frac{2}{1-\rho}} + \left(\frac{1}{1-\rho}\right)^2 u^{\frac{2\rho}{1-\rho}} v^2\right]}$$

$$\Gamma_{11}^2 = \frac{\rho v u^{\frac{2\rho}{1-\rho}}}{(1-\rho)^2\left[1 + u^{\frac{2}{1-\rho}} + \left(\frac{1}{1-\rho}\right)^2 u^{\frac{2\rho}{1-\rho}} v^2\right]}$$

$$\Gamma_{12}^1 = \frac{v u^{\frac{2\rho}{1-\rho}}}{(1-\rho)^2\left[1 + u^{\frac{2}{1-\rho}} + \left(\frac{1}{1-\rho}\right)^2 u^{\frac{2\rho}{1-\rho}} v^2\right]}$$

$$\Gamma_{12}^2 = \frac{u^{\frac{1+\rho}{1-\rho}}}{(1-\rho)\left[1 + u^{\frac{2}{1-\rho}} + \left(\frac{1}{1-\rho}\right)^2 u^{\frac{2\rho}{1-\rho}} v^2\right]}$$

$$\Gamma_{22}^1 = 0$$

$$\Gamma_{22}^2 = 0$$

The geodesic equations are:

$$\frac{d^2u}{ds^2} = -\frac{\rho v^2 u^{\frac{3\rho-1}{1-\rho}}}{(1-\rho)^3\left[1+u^{\frac{2}{1-\rho}}+\left(\frac{1}{1-\rho}\right)^2 u^{\frac{2\rho}{1-\rho}}v^2\right]}\left(\frac{du}{ds}\right)^2 - 2\frac{vu^{\frac{2\rho}{1-\rho}}}{(1-\rho)^2\left[1+u^{\frac{2}{1-\rho}}+\left(\frac{1}{1-\rho}\right)^2 u^{\frac{2\rho}{1-\rho}}v^2\right]}\frac{du}{ds}\frac{dv}{ds}$$

$$\frac{d^2v}{ds^2} = -\frac{\rho v u^{\frac{2\rho}{1-\rho}}}{(1-\rho)^2\left[1+u^{\frac{2}{1-\rho}}+\left(\frac{1}{1-\rho}\right)^2 u^{\frac{2\rho}{1-\rho}}v^2\right]}\left(\frac{du}{ds}\right)^2 - 2\frac{u^{\frac{1+\rho}{1-\rho}}}{(1-\rho)\left[1+u^{\frac{2}{1-\rho}}+\left(\frac{1}{1-\rho}\right)^2 u^{\frac{2\rho}{1-\rho}}v^2\right]}\frac{du}{ds}\frac{dv}{ds}$$

If we use $\rho = 0$ we find the geodesics from example 2.

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