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An effective replicator equation for games with a continuous strategy set

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Abstract The replicator equation for a two person symmetric game, which has an interval of the real line as strategy space, is extended with a mutation term. Assuming that the distribution of the strategies has a continuous density, a partial differential equation for this density is derived. The equation is analysed for two examples. A connection is made with Adaptive Dynamics.

Keywords Evolutionary games · Replicator equation · Mutation · Dynamic stability · Partial differential equations

Mathematics Subject Classification (2000) MSC 91A22

1 Introduction

The use of continuous strategy sets in replicator dynamics introduces two new problems, compared to the situation where the set of strategies is finite. First, there are different notions of ’nearness’ possible, associated with the strong and the weak topology, respectively. The salient difference between these topologies is illustrated by the following property. Let S be the set of strategies and let δ_x with $x \in S$ denote the Dirac distribution concentrated on $\{x\}$. In the strong topology, the distance between δ_x and δ_y is equal to 2, if $x \neq y$. In the weak topology, the distance between δ_x and δ_y is small, if $|x-y|$ is small. The choice of a particular topology has implications for the concept of evolutionary stability.
The other problem is that it has not been possible to actually solve the replicator equation in the case of continuous strategy sets, except for the case that $S = \mathbb{R}$ and

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the assumption that the initial distribution is Gaussian. It was shown in Oechssler and Riedel (2002) that the distribution then retains this shape during the evolution in time and the replicator equation reduces to a coupled set of ordinary differential equations for the mean and the variance of the distribution.

The impossibility of solving the replicator equation for more general initial distributions makes it difficult to establish dynamical stability criteria for equilibrium strategies of the underlying game. In the case of a finite set of strategies, we have the concept of an Evolutionary Stable Strategy (ESS) of a game. ESS is a static concept, computable from the knowledge of the payoff function, but it is tied to dynamic stability through the theorem that an ESS of a symmetric game is an asymptotically Lyapunov stable equilibrium of the corresponding replicator equation. The definition of an ESS can be generalized to games with a continuous strategy set, but it has been shown that the ESS condition is no longer sufficient for a strategy to be a Lyapunov stable solution of the replicator equation. Various stronger static stability concepts have been introduced, which have fairly complicated interrelations, see Oechssler and Riedel (2002) and Cressman (2005).

In this paper we derive a version of the replicator equation which has the property that it can be analysed fairly deeply and can easily be solved numerically. In particular, we can find exact expressions in the limit that time goes to infinity, in the case of two important examples.

The equation we start with is the replicator equation with a mutation-term, which was introduced in Bomze and Bürger (1996). We then restrict the allowed distributions to those which have a (twice continuously differentiable) density function with full support on S. The natural topology in this case is the strong topology, which implies that the space of densities we will be working with is a subset of $L^1(S)$. We then make some, not too demanding, assumptions on the mutation kernel and apply an approximation method which is familiar from statistical physics. This then leads to a partial differential equation with boundary conditions for the density of the distribution. The equation is nonlinear and has non-local terms, .

This equation clearly does not allow for singular distributions such as δ_x as a solution, so we will not be able to make stability statements directly about δ_x. However, we will quite often find that solutions converge to a Gaussian centered at some $x \in S$ and with width going to zero as the size of the mutation term goes to zero.

The equation is analyzed in two cases. The first one corresponds to $S = \mathbb{R}$ and the payoff function

$$f(x,y) = -x^2 + 2axy\,,$$

with $a \in \mathbb{R}$. In section 3 we will show that for any initial condition, the solution will converge to a Gaussian with width ε, where ε^2 is the size of the mutation term. The mean m of this Gaussian converges to $m = 0$ if and only if $a < 1$. If $a > 1$ then m diverges to infinity.

In the second example, $S = \mathbb{R}$ and

$$f(x,y) = -x^2 + x^2y^2\,.$$

We will show in Section 4 that, also in this case, all initial distributions eventually tend to a Gaussian shape. However, the mean of this Gaussian always converges to $m = 0$, but now it is the width of the distribution that shows interesting dynamics. Depending on the initial condition, this width will either converge to ε or diverge to infinity.
In section 5 we conclude with some remarks about the connection between the results derived in these examples and local stability criteria. Also, we will consider how this version of the replicator equation relates to Adaptive Dynamics.

2 Derivation of the equation

The two-player game under consideration is symmetric and is defined through the payoff function $f(x,y)$. The domain of this function is $S \times S$, where $S \subset \mathbb{R}$ is a closed interval. We allow $S = \mathbb{R}$.
Let $\mathscr{B}$ be the Borel σ-algebra on S and $\Delta(S)$ be the subset of probability measures of the measure space $(S, \mathscr{B})$. The state of the game at time t is defined by the distribution of the strategies in the population, given by $P(t) \in \Delta(S)$. If $A \in \mathscr{B}$, then $P(t)(A) = \int_A P(t)(dx)$ is the fraction of players in the population who play a strategy $x \in A$ at time t.

There are two factors driving the evolution of $P(t)$: selection and mutation. The selection terms describes the standard assumption of replicator dynamics, namely that the fraction of strategies that have a higher payoff compared to the average payoff will increase in the population, at the expense of strategies that do worse than average.
Assume the distribution of strategies is given by $P \in \Delta(S)$. The expected payoff of a strategy $Q \in \Delta(S)$ against this population is defined as:

$$\pi(Q,P) = \int_S \int_S f(x,y)\, Q(dx)\, P(dy)\,. \tag{1}$$

In particular, the expected payoff of a pure strategy $x \in S$ against the population distribution P is given by

$$\pi(x,P) \equiv \pi(\delta_x, P) = \int_S \int_S f(x,y)\, \delta_x(dx)\, P(dy) = \int_S f(x,y)\, P(dy)\,. \tag{2}$$

We define the average payoff of the distribution $P \in \Delta(S)$ as

$$\overline{\pi}(P) = \pi(P,P)\,. \tag{3}$$

The relative fitness of strategy $x \in S$, against the population distribution P is defined as:

$$\phi(x,P) = \pi(x,P) - \overline{\pi}(P)\,. \tag{4}$$

Agents sometimes spontaneously change their strategy, by mistake or as a type of

experimentation. We assume that the probabilty that an agent mutates during a certain

time interval is the same for all agents. Let $\mu > 0$ and $\mu\, dt$ be the probability that an agent using strategy $x \in S$ mutates during a short time interval dt. Let $m(y,x)$ be the probability distribution of this mutated strategy, i.e. if $A \in \mathscr{B}$ then the probability that strategy x mutates to a strategy in A is $\int_A m(y,x)\,\lambda(dy)$, with λ the Lebesgue-measure on S.
An important assumption in the following is that $m(y,x) > 0$ for all $(x,y) \in S \times S$. This implies that all strategies have a positive probability to arise from a mutation, and in fact every strategy will be present in the population for all $t > 0$.
For a given distribution $P(t)$ of the strategies at time t, and $A \in \mathscr{B}$, the change per unit time of the fraction of strategies in A, due to mutations, is given by:

$$\mu\left(\int_A\int_S m(y,x)\,P(t)(dx)\,\lambda(dy) - \int_A\int_S m(x,y)\,\lambda(dx)\,P(t)(dy)\right). \tag{5}$$

Combining (4) and (5), and suppressing in the notation the t-dependence of $P(t)$, leads to the mutation-selection equation introduced by Bürger and Bomze (1996):

$$\frac{d}{dt}P(A) = \int_A \phi(x,P)\,P(dx) + \mu\left(\int_A\int_S m(y,x)\,P(dx)\,\lambda(dy) - \int_A\int_S m(x,y)\,\lambda(dx)\,P(dy)\right). \tag{6}$$

The differential equation (6) is defined on the Banach-space $\mathscr{M} = (M(S,\mathscr{B}), ||.||_1)$. Here, $M(S,\mathscr{B})$ is the vectorspace of all signed measures on $(S,\mathscr{B})$ and $||.||_1$ is the variational norm on $M(S,\mathscr{B})$ given by:

$$||Q||_1 = \sup_{f\in F} |\int_S f(x)\,Q(dx)|.$$

The supremum is taken over the set F of all measurable functions $f : S \to \mathbb{R}$ with $\sup_{x\in S}|f(x)| \leq 1$.
The variational norm induces the strong topology. In this topology $||\delta_x - \delta_y|| = 2$ if $x \neq y$, so even though the strategies x and y can be very close, the corresponding monomorphic distributions are not. An alternative measure of closeness is the Prohorov metric, which induces the weak topology and is used in Oechssler and Riedel (2002) and Cressman and Hofbauer (2005).

Some important properties of $\mathscr{M}$, equation (6) and its solution $P(t)$ are:

- For probability measures with a continuous density w.r.t. the Lebesgue measure, the variational norm is equivalent to the L^1 norm (see Oechssler and Riedel (2001)). This means that two of these measures are close in the variational norm if and only if their densities are close in the L^1 norm.
- If the payoff function $f(x,y)$ is bounded, then equation (6) has a unique solution, defined for all $t > 0$. (see Bürger and Bomze (1996))
- If $P(0)(S) = 1$, then $P(t)(S) = 1$ for all $t > 0$, as is easily checked.

2.1 Assumption and approximation

Assume that the measure P has a density w.r.t the Lebesgue measure for all $t \geq 0$. We write:

$$P(t)(dx) = \rho(x,t)\,dx\,.$$

We will, moreover, assume that $\rho(x,t)$ is twice continuously differentiable with respect to x. Using the fact that

$$\pi(x,P) = \int_S f(x,y)\,\rho(y,t)\,dy$$

the equation (6) now reduces to:

$$\frac{\partial}{\partial t}\rho(x,t) = \left(\int_S f(x,y)\,\rho(y,t)\,dy - \int_S\int_S f(x,y)\,\rho(y,t)\rho(x,t)\,dy\,dx\right)\rho(x,t) + \mu\left(\int_S m(x,y)\,\rho(y,t)\,dy - \rho(x,t)\int_S m(y,x)\,dy\right). \tag{7}$$

To simplify (7) further, we will use an approximation that is standard in deriving the Fokker-Planck equation from the master equation in statistical physics (see van Kampen (1975)), and was already used by Kimura (1965) in a context similar to ours.
For the moment we will take $S = \mathbb{R}$. The probability distribution of the strategies $y \in S$ that arise as a mutation from a strategy $x \in S$ is assumed to be of the form $m(y,x) = \tilde{m}(|y-x|,x)$. The distribution $\tilde{m}(z,x)$ is symmetric in z, is rapidly decreasing as $z \to \pm\infty$ and has variation $\int_{-\infty}^{\infty} z^2 \tilde{m}_i(z,x)\,dz = \sigma^2(x)$. The higher order moments of $\tilde{m}(z,x)$ are at least of $\mathscr{O}(\sigma^4(x))$. A typical form for the mutation kernel is the Gaussian:

$$m(y,x) = \frac{1}{\sqrt{2\pi}\sigma(x)} e^{-(x-y)^2/2\sigma^2(x)}\,,$$

where $\sigma(x)$ is small. We can then write:

$$\begin{aligned}\int_{\infty}^{\infty} m(y,x)\,\rho(y,t)\,dy &= \int_{\infty}^{\infty} \tilde{m}(y-x,x)\,\rho(y,t)\,dy = \int_{\infty}^{\infty} \tilde{m}(z,x)\,\rho(z+x,t)\,dz \\ &= \int_{\infty}^{\infty} \tilde{m}(z,x)\,(\rho(x,t) + \frac{\partial}{\partial x}\rho(x,t)\,z + \frac{1}{2}\frac{\partial^2}{\partial x^2}\rho(x,t)\,z^2 + \ldots)\,dz \\ &= \rho(x,t) + \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}\rho(x,t) + \mathscr{O}(\sigma^4(x))\end{aligned} \tag{8}$$

Substituting (8) in (7) and neglecting the higher order terms, we find the equation that is referred to in the title of this paper:

$$\frac{\partial}{\partial t}\rho(x,t) = \left(\int_S f(x,y)\,\rho(y,t)\,dy - \int_S\int_S f(x,y)\,\rho(y,t)\rho(x,t)\,dy\,dx\right)\rho(x,t) + \frac{1}{2}\mu\sigma^2(x)\frac{\partial^2}{\partial x^2}\rho(x,t)\,. \tag{9}$$

In the case that S is a finite interval, the mutation term near the boundaries of S needs to be adapted, so that no mutations outside S are possible. This is a technically cumbersome operation, which can be solved by keeping the equation (9), but supplying it with reflecting, or Neumann, boundary conditions:

$$\frac{\partial}{\partial x}\rho(x,t)|_{\partial S} = 0\,. \tag{10}$$

2.2 Existence and properties of the solution

The equation (9) is a nonlinear, non-local, reaction diffusion equation. The function space we will be working on is that of twice continuously differentiable densities $\rho(x)$, such that ρ, ρ_x and ρ_{xx} are all in $L^1(S)$ and satisfying the Neumann boundary conditions (10).

From (9) and (10), we recover the important property that if $\int_S \rho(x,0)\,dx = 1$, then

$$\int_S \rho(x,t)\,dx = 1 \tag{11}$$

for all $t > 0$, as is easily checked.

It follows from standard positivity results for parabolic equations that if the initial value $\rho_0(x) > 0$ for all $x \in S$, then $\rho(x,t) > 0$ for all $t \geq 0$. We will from now on always assume that $\rho(x,t) > 0$, for all $x \in S$ and $t \geq 0$. Thus, the support of the measure P corresponding to ρ is the full strategy set S.

If S is compact, then existence of $\rho(x,t)$ in the above mentioned function space can be proved for all $t \geq 0$. This follows from the fact that if $f(x,y)$ is bounded on $S \times S$, the reaction term is clearly continuous in ρ and bounded:

$$|\int_S f(x,y)\,\rho(y,t)\,dy - \int_S\int_S f(x,y)\,\rho(y,t)\rho(x,t)\,dy\,dx| \leq$$
$$\int_S |f(x,y)|\rho(y,t)\,dy + \int_S\int_S |f(x,y)|\rho(y,t)\rho(x,t)\,dy\,dx \leq$$
$$\max_{(x,y)\in S\times S}|f(x,y)| \left(\int_S \rho(y,t)\,dy + \int_S\int_S \rho(y,t)\rho(x,t)\,dy\,dx\right) = 2\max_{(x,y)\in S\times S}(|f(x,y)|)\,,$$

where we have used (11) and the positivity of ρ. Comparison theorems for parabolic equations (Pao (1992)) complete the proof.

In the case that $S = \mathbb{R}$, a solution can not be guaranteed for all time, as follows from the following example adapted from Cressman and Hofbauer (2005). Let $f(x,y) = x^2$ and $\frac{1}{2}\mu\sigma^2(x) = \varepsilon^2$ be constant. Then equation (9) becomes:

$$\frac{\partial}{\partial t}\rho(x) = \left(x^2 - \int_{\mathbb{R}} x^2\rho(x,t)\,dx\right)\rho(x,t) + \varepsilon^2\frac{\partial^2}{\partial x^2}\rho(x)\,.$$

It can be checked that $\rho(x,t) = \frac{1}{\sqrt{2\pi V(t)}}e^{-\frac{(x-m(t))^2}{2V(t)}}$ is a solution of the above equation, if $V(t)$ and $m(t)$ satisfy the diferential equations:

$$V' = 2(V^2+\varepsilon^2)\,, \quad m' = 2mV\,.$$

The solution $V(t)$ will "blow up" in finite time, for every initial value $V(0)$. Note that the corresponding $\rho(x,t)$ will "flatten out" in that time.
In the examples considered below, $S = \mathbb{R}$. However, existence of solutions for all times will be shown by construction.

3 Application to a quadratic payoff function

We will take $S = \mathbb{R}$ and consider the payoff function

$$f(x,y) = -x^2 + 2axy\,,$$

with $a \in \mathbb{R}$. The symmetric game corresponding to this payoff function has, for all $a \in \mathbb{R}$, a unique, strict, Nash equilibrium in pure strategies, namely $x = 0$.
Using the fact that $\int_S \rho(x,t)\,dx = 1$, we find that

$$\int_S f(x,y)\,\rho(y,t)\,dy - \int_S\int_S f(x,y)\,\rho(y,t)\rho(x,t)\,dy\,dx = -x^2 + 2ax\bar{x}(t) + \overline{x^2}(t) - 2a\bar{x}^2(t)\,,$$

with

$$\overline{x^n}(t) = \int_{-\infty}^{\infty} x^n \rho(x,t)\,dx\,.$$

The dependence of $\overline{x^n}$ on t will often be suppressed in the notation. We will assume that $\mu\sigma^2(x) = \varepsilon^2$ is independent of x and small. Equation (9) then becomes:

$$\rho_t = \left(-x^2 + 2ax\bar{x} + \overline{x^2} - 2a\bar{x}^2\right)\rho + \varepsilon^2\rho_{xx}\,. \tag{12}$$

In addition to this equation, we have an initial condition

$$\rho(x,0) = \rho_0(x)\,, \tag{13}$$

such that $\rho_0(x) > 0$, $\int_S \rho_0(x)\,dx = 1$ and $\rho_0(x)$ twice continuously differentiable on $\mathbb{R}$. As shown in the previous section, these conditions imply that $\int_S \rho(x,t)\,dx = 1$, $\rho(x,t) > 0$ and $\rho(x,t)$ twice continuously differentiable for all $t > 0$. This in turn implies that $\rho(x,t) \in L^1(\mathbb{R})$ and $\rho(x,t) \in L^2(\mathbb{R})$.

3.1 The Wei-Norman method

Although equation (12) is nonlinear and contains non-local terms, it can be solved explicitly. This is done by exploiting its linear appearance. We first assume that $\bar{x}$ and $\overline{x^2}$ are given functions of t. Equation (12) then becomes a linear equation with time-dependent coefficients. We note that (12) bears some resemblance to the equation for the quantum-mechanical harmonic oscillator. For these types of equations, solution methods have been devised, notably the Wei-Norman method. Using this method, equation (12) is solved yielding the solution

$$\rho(x,t;\overline{x},\overline{x^2})\,. \tag{14}$$

Solving the two equations

$$\overline{x}(t)=\int_{-\infty}^{\infty} x\rho(x,t;\overline{x},\overline{x^2})\,dx \quad , \quad \overline{x^2}(t)=\int_{-\infty}^{\infty} x^2\rho(x,t;\overline{x},\overline{x^2})\,dx$$

for $(\overline{x},\overline{x^2})$ gives a unique solution, which can be substituted in (14) to give the solution of (12). In fact, we will only need the equation for $\overline{x}$.

We recall some facts about Lie algebras, which play a role in the Wei-Norman method. A finite, real Lie algebra $\mathscr{L}$ is a vector space over the reals, spanned by a finite number of elements $\{X_1,\ldots,X_n\}$. This vector space is equipped with a Lie bracket $[\ ,\ ]:\mathscr{L}\times\mathscr{L}\to\mathscr{L}$. This bracket is bi-linear, and satisfies $[X,X]=0$, for all $X\in\mathscr{L}$ and Jacobi's identity $[X,[Y,Z]]+[Z,[X,Y]]+[Y,[Z,X]]=0$, for all $X,Y,Z\in\mathscr{L}$.
In our case, the elements of $\mathscr{L}$ are linear operators on $L^2(\mathbb{R})$ and the bracket is defined as $[X,Y]=XY-YX$.
Define $[\mathscr{L},\mathscr{L}]=\{[X,Y]\,|\,X,Y\in\mathscr{L}\}$. A Lie algebra $\mathscr{L}$ is called *solvable* if the series $\mathscr{L},[\mathscr{L},\mathscr{L}],[[\mathscr{L},\mathscr{L}],[\mathscr{L},\mathscr{L}]],\ldots$ eventually terminates in 0.

Theorem 1 *Let $\mathscr{L}$ be a finite, solvable, real Lie algebra, generated by $\{X_1,\ldots,X_n\}$. The solution to the initial value problem*

$$\frac{dU}{dt}=\left(\sum_{i=1}^{n}a_i(t)X_i\right)U \quad , \quad U(0)=U_0$$

can be written in the form

$$U(t)=exp(g_1(t)X_1)exp(g_2(t)X_2)\ldots exp(g_n(t)X_n)U_0\,.$$

Moreover, the functions $g_i(t)$ can be found as solutions of ordinary differential equations involving the coefficients $a_i(t)$.

Proof See Wei and Norman (1964).

To apply this theorem to (12), we first introduce the scalings $x=\varepsilon\xi$, $t=\varepsilon\tau$ and $\rho(x,t)=(\frac{1}{\varepsilon})\hat{\rho}(x/\varepsilon,t/\varepsilon)$, and find

$$\overline{x}(t)=\int_{\infty}^{\infty}x\,\rho(x,t)\,dx=(1/\varepsilon)\int_{\infty}^{\infty}x\,\hat{\rho}(x/\varepsilon,t/\varepsilon)\,dx=\varepsilon\int_{\infty}^{\infty}\xi\,\hat{\rho}(\xi,\tau)\,d\xi=\varepsilon\overline{\xi}(\tau)\,,$$

and similarly

$$\overline{x^2}(t)=\varepsilon^2\overline{\xi}(\tau)\,,\quad \int_{\infty}^{\infty}\hat{\rho}(\xi,\tau)\,d\xi=1\,.$$

Substituting, equation (12) becomes independent of ε:

$$\hat{\rho}_\tau = \left(-\xi^2 + 2a\xi\overline{\xi} + \overline{\xi^2} - 2a\overline{\xi}^2\right)\hat{\rho} + \hat{\rho}_{\xi\xi} \,. \tag{15}$$

We can write equation (15) in the form

$$\hat{\rho}_\tau = \left(Z + 0.Y + 2a\overline{\xi}X + (\overline{\xi^2} - 2a\overline{\xi}^2 - 1)I\right)\hat{\rho} \,, \tag{16}$$

where for $f \in L^2(\mathbb{R})$:

$$\begin{aligned} Zf &= \left(\frac{d^2}{d\xi^2} - \xi^2 + 1\right) f \\ Yf &= \frac{d}{d\xi} f \\ Xf &= \xi f \\ If &= f \,. \end{aligned}$$

The elements of $\{X,Y,Z,I\}$ are the generators of a Lie algebra with the following commutation relations:

$$[Z,X] = 2Y \quad , \quad [Z,Y] = 2X \quad , \quad [Y,X] = I \,, \tag{17}$$

and $[A,I] = 0$ for all $A \in \{X,Y,Z,I\}$.
It is easy to check that this Lie-algebra is solvable, so that the theorem can be applied. We write the solution of the initial value problem (15) in the form:

$$\hat{\rho}(\xi,\tau) = e^{g_I(\tau)I} e^{g_X(\tau)X} e^{g_Y(\tau)Y} e^{g_Z(\tau)Z} \hat{\rho}_0(\xi) \,, \tag{18}$$

where

$$\hat{\rho}_0(\xi) = \varepsilon\rho_0(\varepsilon\xi) \,,$$

the rescaled initial value. The order of the operators could have been chosen differently, but the sequel will show that the form (18) is very practical.
To derive the differential equations for the functions $g_i(\tau)$, we will repeatedly use the formula:

$$e^{\lambda A} B e^{-\lambda A} = B + \lambda [A,B] + \frac{\lambda^2}{2!}[A,[A,B]] + \frac{\lambda^3}{3!}[A,[A,[A,B]]] + \dots \,, \tag{19}$$

where A and B are elements of a Lie-algebra and $\lambda \in \mathbb{C}$. As a special case we note that:

$$e^{\lambda A} A = A e^{\lambda A} \,,$$

for all A in the Lie algebra. In our case, the series on the right side of (19) always terminates, which makes it possible to write for any $A, B \in \{Z, Y, X, I\}$:

$$e^{\lambda A} B = L(Z,Y,X,I) e^{\lambda A} ,$$

where $L(Z,Y,X,I)$ is some linear expression in its arguments.
Differentiating (18) with respect to τ yields:

$$\begin{aligned}\hat{\rho}_\tau = \big(& g_I' I \, e^{g_I I} e^{g_X X} e^{g_Y Y} e^{g_Z Z} + g_X' e^{g_I I} X \, e^{g_X X} e^{g_Y Y} e^{g_Z Z} + g_Y' e^{g_I I} e^{g_X X} Y \, e^{g_Y Y} e^{g_Z Z} \\ & + g_Z' e^{g_I I} e^{g_X X} e^{g_Y Y} Z \, e^{g_Z Z} \big) \hat{\rho}_0(\xi) ,\end{aligned} \tag{20}$$

where $g_i' = \dfrac{d}{d\tau} g_i(\tau)$.
Using (19) and the commutation relations, we find:

$$\begin{aligned} e^{g_X X} Y &= (Y - g_X I) \, e^{g_X X} \\ e^{g_Y Y} Z &= \big(Z - 2 g_Y X - g_Y^2 I \big) \, e^{g_Y Y} \\ e^{g_X X} Z &= \big(Z - 2 g_X Y + g_X^2 I \big) \, e^{g_X X} . \end{aligned}$$

Substituting these expressions in (20) and collecting the coefficients, we find:

$$\begin{aligned}\hat{\rho}_\tau &= \big(g_I' I + g_X' X + g_Y' (Y - g_X I) + g_Z' (Z - 2 g_X Y + g_X^2 I - 2 g_Y X - g_y^2 I) \big) \hat{\rho} \\ &= \big(g_Z' Z + (g_Y' - 2 g_X g_Z') Y + (g_X' - 2 g_Y g_Z') X + (g_I' - g_Y' g_X + (g_X^2 - g_Y^2) g_Z') I \big) \hat{\rho} .\end{aligned} \tag{21}$$

Comparing (21) with (16) yields the system of equations:

$$\begin{aligned} g_Z' &= 1 \\ g_Y' - 2 g_X g_Z' &= 0 \\ g_X' - 2 g_Y g_Z' &= 2a\overline{\xi} \\ g_I' - g_Y' g_X + (g_X^2 - g_Y^2) g_Z' &= \overline{\xi^2} - 2a\overline{\xi}^2 - 1 , \end{aligned} \tag{22}$$

with intial conditions $g_I(0) = g_X(0) = g_Y(0) = g_Z(0) = 0$. We will ignore the last equation, since in the expression

$$\hat{\rho} = e^{g_I I} e^{g_X X} e^{g_Y Y} e^{g_Z Z} \hat{\rho}_0(\xi)$$

the factor $e^{g_I I}$ is simply a normalization term, which can also be calculated from the condition $\int_{\mathbb{R}} \hat{\rho}(\xi, t) \, d\xi = 1$.
The other three equations can be easily integrated and we find in particular that

$$g_Z(\tau) = \tau .$$

For $g_X(\tau)$ and $g_Y(\tau)$ we can find explicit expressions which involve $\overline{\xi}$, however their exact form is not relevant for what follows.

3.2 Solution for large values of τ

Consider first the result of the operator $e^{g_Z(\tau)Z}$ acting on an initial function $\hat{\rho}_0(\xi)$. Let

$$e^{sZ}\hat{\rho}_0(\xi) = f(\xi,s)\,,$$

(with $s \in \mathbb{R}$) then $f(\xi,s)$ is the solution of the partial differential equation

$$\frac{\partial f}{\partial s} = Zf = (-\xi^2+1)f + \frac{\partial^2 f}{\partial \xi^2} \quad , \quad f(\xi,0) = \hat{\rho}_0(\xi)\,. \tag{23}$$

It is well known that the eigenfunctions of Z are the Hermite functions $\{\phi_n(\xi)\}$, $n = 0,1,\ldots$, with corresponding eigenvalues $\lambda_n = -2n$. The Hermite functions form an orthonormal base of $L^2(\mathbb{R})$, with

$$\phi_0(\xi) = (2\pi)^{-1/4}e^{-\xi^2/2}$$

Since $\hat{\rho}_0(\xi) \in L^2(\mathbb{R})$, we can write

$$\hat{\rho}_0(\xi) = \sum_{n=0}^{n=\infty} a_n\phi_n(\xi) \quad , \quad a_n = \int_{\mathbb{R}} \phi_n(\xi)\hat{\rho}_0(\xi)d\xi\,.$$

We note that $a_0 > 0$, because $\phi_0(\xi)\hat{\rho}_0(\xi) > 0$. The solution of (23) can now be written as:

$$e^{sZ}\hat{\rho}_0(\xi) = f(\xi,s) = \sum_{n=0}^{n=\infty} e^{-2ns}a_n\phi_n(\xi)\,. \tag{24}$$

From this expression, it follows that:

$$\lim_{s\to\infty} ||e^{sZ}\hat{\rho}_0(\xi) - a_0\phi_0(\xi)||_2 = 0\,.$$

In other words, whatever the initial distribution $\hat{\rho}_0(\xi)$, the expression $e^{sZ}\hat{\rho}_0(\xi)$ as s goes to infinity, will tend to a normal distribution (multiplied by a positive factor), with mean equal to zero and variance equal to one.
The action of the operator e^{sY} on functions $g \in L^2(\mathbb{R})$ is that of the shift operator:

$$\left(e^{sY}g\right)(\xi) = \left(e^{s\frac{d}{dx}}g\right)(x) = g(\xi+s)\,,$$

for all $s \in \mathbb{R}$.
Finally, e^{sX} acts as a simple multiplication:

$$\left(e^{sX}g\right)(\xi) = e^{s\xi}g(\xi)\,.$$

We are now in a position to write down, for every given $\overline{\xi}(\tau)$ and initial function $\hat{\rho}_0(\xi)$, the solution $\hat{\rho}(\xi,\tau;\overline{\xi})$ of (15). Using this solution, we could then solve the

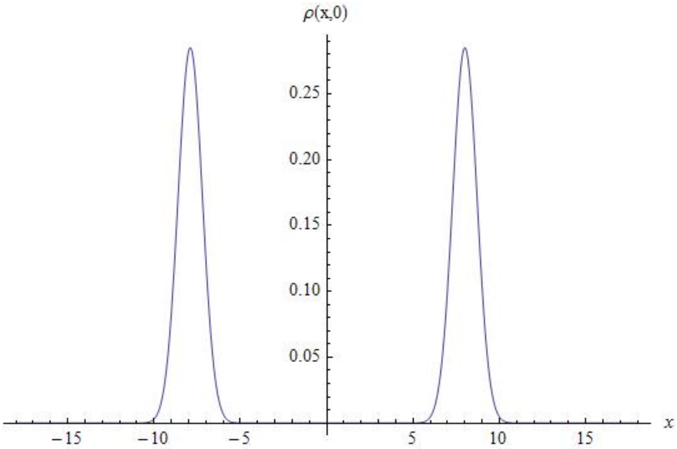

Fig. 1 Initial condition for equation (15) is a sum of Gaussians, both with variance $\sigma^2 = 0.49$, centered at $x = -7.9$ and $x = 8.0$.

equation

$$\overline{\xi}(\tau) = \int_{\mathbb{R}} \xi \hat{\rho}(\xi,\tau;\overline{\xi})\, d\xi \,,$$

which would give a complete solution of the initial problem (12). Deriving an explicit expression for $\overline{\xi}$ for a general initial distribution $\hat{\rho}_0(\xi)$ is, however, not possible. This is not a major problem, as the early development of the distribution is less interesting and can, for a specific initial distribution, be found numerically. What is much more important is the eventual fate of the solution. As it happens, the asymptotic behaviour of the solution for large τ can be found, and it is independent of the initial distribution.

We approximate $\hat{\rho}(\xi,\tau)$ for large τ as follows:

$$\begin{aligned}
\hat{\rho}(\xi,\tau) &= e^{g_I(\tau)I} e^{g_X(\tau)X} e^{g_Y(\tau)Y} e^{g_Z(\tau)Z} \hat{\rho}_0(\xi) \\
&= n(\tau) e^{g_X(\tau)X} e^{g_Y(\tau)Y} e^{\tau Z} \hat{\rho}_0(\xi) \\
&\approx n(\tau) e^{g_X(\tau)X} e^{g_Y(\tau)Y} e^{-\xi^2/2} \\
&= n(\tau) e^{g_X(\tau)\xi} e^{-(\xi+g_Y(\tau))^2/2} = n(\tau) e^{-(\xi-g_X(\tau)+g_Y(\tau))^2/2}, \qquad (25)
\end{aligned}$$

where $n(\tau)$ is the aforementioned normalisation factor, which, with some abuse of notation, has absorbed all τ-dependent terms in each step of the derivation.
This approximation is a Gaussian, with variance $= 1$ and mean $\overline{x} = g_X(\tau) - g_Y(\tau))$.
Using (22), we then have

$$\frac{d\overline{x}}{d\tau} = g'_X(\tau) - g'_Y(\tau) = -2(g_X(\tau) - g_Y(\tau)) + 2a\overline{x}(\tau) = (-2+2a)\overline{x}, \qquad (26)$$

with solution

$$\overline{x} = e^{2(a-1)\tau} .$$

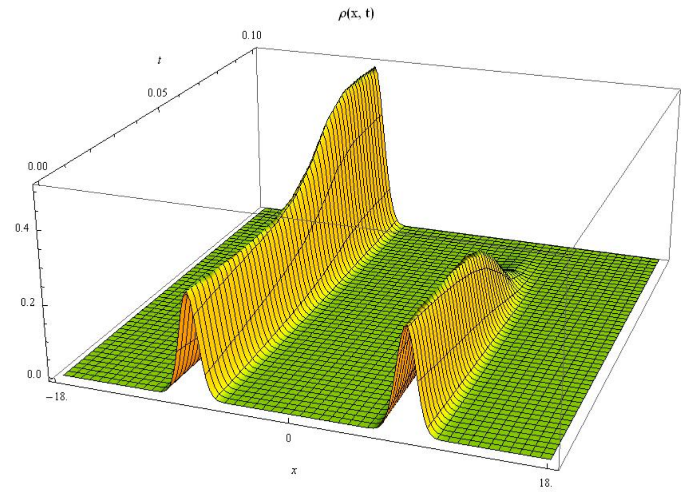

Fig. 2 Short-time evolution of initial condition of Figure 1. $a = 0.9$.

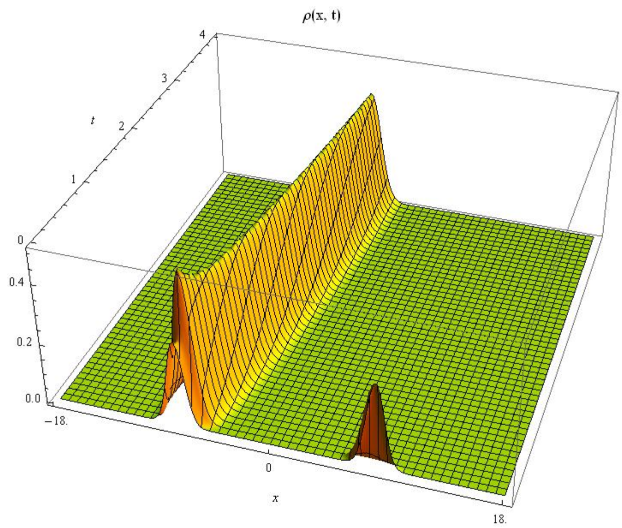

Fig. 3 Long-time evolution of initial condition of Figure 1. $a = 0.9$.

From this expression we see that, asymptotically for $\tau \to \infty$, the solution of (12) shows one of two possible behaviours. If $a < 1$, the solution converges to a Gaussian with width equal to one, and a mean that converges exponentially to zero. If $a > 1$, the solution still converges to a Gaussian with width one, but now the mean grows unboundedly.

In terms of the original variable, for $a < 1$ the distribution $\rho(x,t)$ converges to a

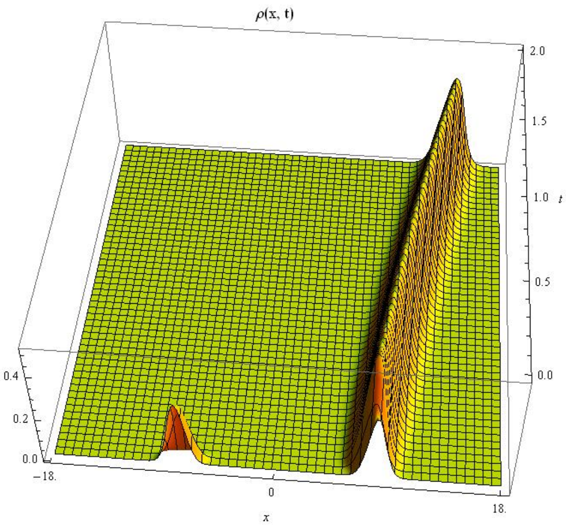

Fig. 4 Long-time evolution of initial condition of Figure 1. $a = 1.1$.

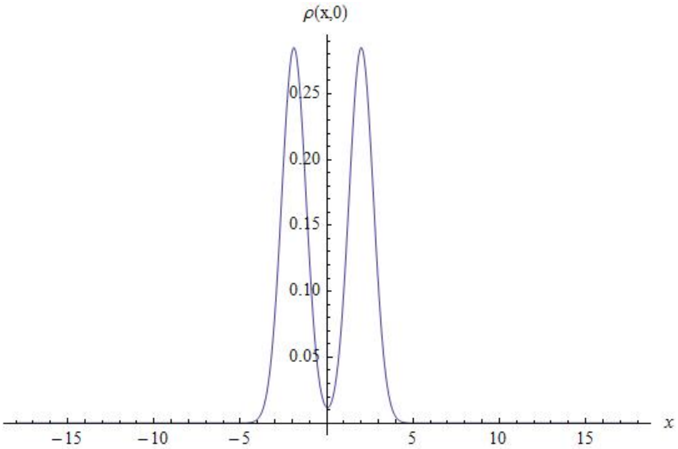

Fig. 5 Initial condition for equation (15) is a sum of Gaussians, both with variance $\sigma^2 = 0.49$, centered at $x = -1.9$ and $x = 2.0$.

normal distribution centered at $x = 0$ and a width of ε, for all initial distributions. This convergence happens on a time scale of $1/\varepsilon$. For $a > 1$ the distribution does not converge.

3.3 Numerical simulations

The figures 1 through 6 were made by discretizing the x-variable on the interval $[-20, 20]$ in $N = 150$ points, yielding N functions $x_i(t)$, $i = 1, \ldots, N$. In the equation

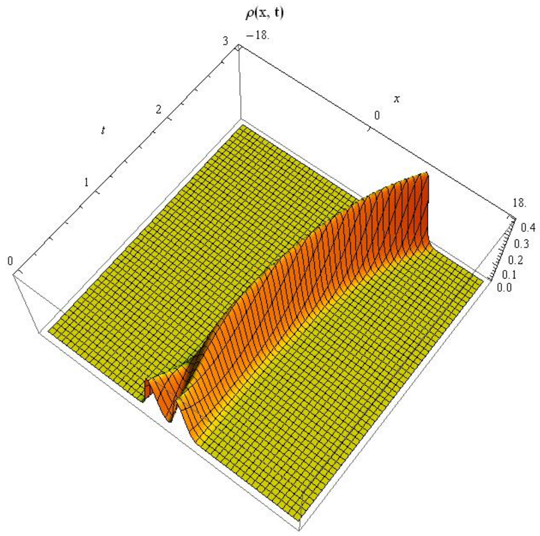

Fig. 6 Long-time evolution of initial condition of Figure 5. $a = 1.5$.

(9), the second derivative was replaced by the standard approximation and integration by a simple summation. The resulting system of ordinary differential equations for $x_i(t)$ was then solved, using Mathematica routines. Although this is a very unsophisticated method, the results agree completely with the analysis of the previous section. In the Figures 1 and 2 we have taken $a = 0.9$ and as initial condition a sum of two sharp peaks, with equal mass and width and almost symmetrically placed. In Figure 2, the evolution is shown on a short time-scale. Initially, the two peaks co-exist until at about $t = 0.05$ the peak at $x = 8$ collapses and all the mass of the distribution becomes concentrated near the peak at $x = -7.9$. On this time scale, the location of the peaks has hardly moved. In Figure 3 the further evolution is shown. After the collapse of the right peak, the now single-peaked distribution takes on a Gaussian shape and the mean of this distribution moves towards $x = 0$. Convergence to the steady state, approximated by (25) is virtually complete at about $t = 15$.
In Figure 4 we have taken the same initial condition as in Figure 1, but now with $a = 1.1$. We see that it is now the left peak which collapses after a short period. The surviving peak at the right again assumes a Gaussian shape and starts to move towards infinity, as predicted by equation (26).
Figure 5 shows an initial condition which is again the sum of two peaks, but now close together. Figure 6 shows that these peaks first merge to one peak, centered at approximately $x = 0$, then this peak starts to move away from this position, because $a = 1.5 > 1$.

4 Payoff function with fourth order term

We now apply the method of the previous section to the payoff function

$$f(x,y) = -x^2 + x^2y^2\,,$$

and the strategy set $S = \mathbb{R}$. Substituting this function in (9) and using the scalings $x = \varepsilon\xi$, $t = \varepsilon\tau$ and $\rho(x,t) = (\frac{1}{\varepsilon})\hat{\rho}(x/\varepsilon, t/\varepsilon)$, we find

$$\hat{\rho}_\tau = \Big(-\xi^2(1-\varepsilon^2\overline{\xi^2}) + \overline{\xi^2}(1-\varepsilon^2\overline{\xi^2})\Big)\,\hat{\rho} + \hat{\rho}_{\xi\xi}\,. \tag{27}$$

We can write equation (27) in the form

$$\hat{\rho}_\tau = \Big(Z + 0.V + \varepsilon^2\overline{\xi^2}W + (\overline{\xi^2} - \varepsilon^2\overline{\xi^2}^2 - 1)I\Big)\,\hat{\rho}\,, \tag{28}$$

where for $f \in L^2(\mathbb{R})$:

$$\begin{aligned} Zf &= \left(\frac{d^2}{d\xi^2} - \xi^2 + 1\right) f \\ Vf &= \xi\,\frac{d}{d\xi} f \\ Wf &= \xi^2 f \\ If &= f\,. \end{aligned}$$

The operator V was chosen to make $\{Z,V,W,I\}$ a closed Lie-algebra. The commutation relations are:

$$[Z,W] = 4V + 2I \quad , \quad [Z,V] = 2Z + 4W - 2I \quad , \quad [W,V] = -2W\,, \tag{29}$$

and $[A,I] = 0$ for all $A \in \{X,Y,Z,I\}$.
This Lie-algebra is not solvable, and the series on the rightside of (19) does not terminate for all elements of the algebra. However, it is still possible to sum the series, as we shall show later. Because of this, we believe that the conclusion of the theorem of Wei and Norman still holds, although the condition of solvability is not met.
Assume therefore that the solution of (28) has the form

$$\hat{\rho}(\xi,\tau) = e^{g_I(\tau)I}e^{g_V(\tau)V}e^{g_W(\tau)W}e^{g_Z(\tau)Z}\hat{\rho}_0(\xi)\,. \tag{30}$$

To find the equations for g_V, g_W and g_Z, we differentiate (30) with respect to τ:

$$\begin{aligned} \hat{\rho}_\tau = \big(&g_I' I\, e^{g_I I}e^{g_V V}e^{g_W W}e^{g_Z Z} + g_V' e^{g_I I} V\, e^{g_V V}e^{g_W W}e^{g_Z Z} + g_W' e^{g_I I}e^{g_V V} W\, e^{g_W W}e^{g_Z Z} \\ &+ g_Z' e^{g_I I}e^{g_V V}e^{g_W W} Z\, e^{g_Z Z}\big)\hat{\rho}_0(\xi)\,. \end{aligned} \tag{31}$$

It is fairly straightforward to derive that:

$$\begin{aligned} &e^{g_V V}W = e^{2g_V}We^{g_V V} \\ &e^{g_W W}Z = \big(Z - g_W(2I+4V) + 4g_W^2 W\big)\,e^{g_W W}\,. \end{aligned}$$

Define $[V,Z]_{(n)} = [V,[V,Z]_{(n-1)}]$ for $n \geq 1$ and $[V,Z]_{(0)} = Z$. Then

$$e^{g_V V} Z e^{-g_V V} = \sum_{n=0}^{\infty} \frac{g_V^n}{n!} [V,Z]_{(n)} \tag{32}$$

After calculation of a few iterations, it becomes clear that:

$$[V,Z]_{(n)} = a_n [V,Z] + b_n W\,, \quad a_1 = 1\,, b_1 = 0 \tag{33}$$

Using the commutation rules we find the recursions:

$$a_n = -2a_{n-1} \quad b_n = 2b_{n-1} - 2(-2)^n\,,$$

with solutions:

$$a_n = (-2)^{(n-1)}\,, b_n = -(2^n + (-2)^n) \tag{34}$$

Substituting (34) and (33) in (32) leads, after some manipulation, to

$$e^{g_V V} Z = (e^{-2g_V} Z - 2\sinh(2g_V) W - e^{-2g_V} I) e^{g_V V}$$

We can now calculate:

$$\begin{aligned}
&e^{g_V V} e^{g_W W} Z e^{g_Z Z} = e^{g_V V} (Z - g_W(2I + 4V) + 4g_W^2 W) e^{g_W W} e^{g_Z Z} = \\
&(e^{-2g_V} Z - 2\sinh(2g_V) W - e^{-2g_V} I - 2g_W I - 4g_W V + 4g_W^2 e^{2g_V} W) e^{g_V V} e^{g_W W} e^{g_Z Z}
\end{aligned}$$

After substitution and collecting terms, we find that

$$\begin{aligned}
\hat{\rho}_\tau = \big(&(g_I' - 2g_Z' g_W) I + (g_V' - 4g_Z' g_W) V + \\
&(g_W' e^{2g_V} + g_Z'(-2\sinh(2g_V) + 4g_W^2 e^{2g_V})) W + g_Z' e^{-2g_V} Z\big)\hat{\rho}\,.
\end{aligned} \tag{35}$$

Comparing the terms of (31) and (35), we find the set of differential equations

$$\begin{aligned}
&e^{-2g_V} g_Z' = 1 \\
&g_V' - 4g_W g_Z' = 0 \\
&e^{2g_V} g_W' + (-2\sinh(2g_V) + 4g_W^2 e^{2g_V}) g_Z' = \varepsilon^2 \overline{\xi^2}\,.
\end{aligned} \tag{36}$$

As in the previous section, we ignore the equation for g_I.
The equation for g_Z cannot be solved directly. However, if we assume that $g_V(\tau) > \alpha$ for some $\alpha \in \mathbb{R}$ and for all $\tau > 0$, then $e^{2g_V(\tau)} > e^{\alpha} > 0$ and therefore $g_Z(\tau) = \int_0^\tau e^{2g_V(\tau')} d\tau'$ is an increasing function such that $\lim_{\tau\to\infty} g_Z(\tau) = \infty$. We will show later that the assumption $g_V(\tau) > \alpha$ for all $\tau > 0$ is justified.

4.1 Solution for large τ

From (24) it follows that

$$\lim_{\tau\to\infty} e^{g_Z(\tau)Z}\hat{\rho}_0(\xi,\tau) = \lim_{\tau\to\infty}\sum_{n=0}^{n=\infty} e^{-2n\,g_Z(\tau)} a_n\phi_n(\xi) = a_0\phi_0(\xi)\,,$$

where convergence is in the L^2 norm. We will take this multiple of a Gaussian of variance one and mean zero as the approximation of $e^{g_Z(\tau)Z}\hat{\rho}_0(\xi,\tau)$ for large τ.
The operator V is the generator of scalings, as follows from the fact that $e^{\lambda V}f(x) := g(x,\lambda)$ is the solution of

$$\frac{\partial}{\partial\lambda}g(x,\lambda) = V\,g(x,\lambda) = x\frac{\partial}{\partial x}g(x,\lambda)\,,\quad g(x,0) = f(x)$$

It is easily checked that $g(x,\lambda) = f(e^{\lambda}x)$ is the solution of the above equation. Therefore, $e^{g_V V}\phi_0(\xi) = \phi_0(e^{g_V}\xi)$, a Gaussian with mean zero, but width now stretched by a factor e^{g_V}.
Finally, $e^{g_W W}$ is simply a multiplication by $e^{g_W\xi^2}$.
Combining these elements we have that for large τ, an approximation is given by

$$\begin{aligned}\hat{\rho}(\xi,\tau) &= n(\tau)e^{g_V(\tau)V}e^{g_W(\tau)W}e^{g_Z(\tau)Z}\hat{\rho}_0(\xi) \approx n(\tau)e^{g_V(\tau)V}e^{(g_W-\frac{1}{2})\xi^2}\\ &= n(\tau)e^{-\frac{1}{2}(1-2g_W)e^{2g_V}\xi^2}\,.\end{aligned}$$

In other words, for every initial condition, the solution converges to a Gaussian with mean equal to zero, but with variance

$$\sigma^2 = (1-2g_W)^{-1}e^{-2g_V} \tag{37}$$

This approximation closes the set of differential equations (36), since for large τ we know that $\overline{\xi^2}$ can be approximated by σ^2. This then yields an autonomous set of ordinary differential equations, which should be studied for large values of τ.
However, we are mainly interested in the evolution of σ^2, for which it is possible to derive a simple equation. For this, σ^2 is substituted for $\overline{\xi^2}$ in (36), yielding:

$$e^{2g_V}g_W' = \varepsilon^2\sigma^2 - 1 + (1-4g_W^2)e^{4g_V}\,.$$

Then:

$$\begin{aligned}\frac{d}{d\tau}\sigma &= (1-2g_W)^{-3/2}g_W'e^{-g_V} - (1-2g_W)^{-1/2}e^{-g_V}g_V' = \sigma((1-2g_W)^{-1}g_w' - g_V')\\ &= \sigma(\sigma^2e^{2g_V}g_W' - g_V') = \sigma(\sigma^2(\varepsilon^2\sigma^2 - 1 + (1-4g_W^2)e^{4g_V}) - 4g_We^{2g_V})\\ &= \sigma(\sigma^2(\varepsilon^2\sigma^2 - 1 + 2\sigma^{-2}e^{2g_V} - \sigma^{-4}) - 2(e^{2g_V} - \sigma^{-2}))\\ &= \varepsilon^2\sigma^5 - \sigma^3 - \sigma^{-1} + 2\sigma^{-1} = \varepsilon^2\sigma^5 - \sigma^3 + \sigma^{-1}\,.\end{aligned}$$

Or, in terms of σ^2:

$$\frac{1}{2}\frac{d}{d\tau}\sigma^2 = \varepsilon^2(\sigma^2)^3 - (\sigma^2)^2 + 1\,. \tag{38}$$

This equation has two fixed points, which for small ε have the form:

$$\sigma_a^2 = 1 + \frac{1}{2}\varepsilon^2 + \mathscr{O}(\varepsilon^4) \quad \sigma_r^2 = \frac{1}{\varepsilon^2} + \mathscr{O}(1)$$

The fixed point σ_A^2 is an attractor for equation (38) which attracts all solutions with $0 < \sigma^2(0) < \sigma_r^2$, while σ_R^2 is a repellor and all solutions with $\sigma^2(0) > \sigma_r^2$ diverge to infinity.

Additional evidence for the correctness of the above analysis comes from the fact that

$$\hat{\rho}(\xi,t) = \frac{1}{\sqrt{2\pi}\alpha(t)} e^{-\frac{\xi^2}{2\alpha^2(t)}},$$

is a solution of (27) and the equation for α^2 is exactly equal to the equation (38), with σ^2 replaced by α^2.

In terms of the original variables x and t, all solutions of the unscaled equation for $\rho(x,t)$ converge to a Gaussian shape with mean zero. The variance either converges to a fixed point of $\mathscr{O}(\varepsilon^2)$, or it diverges to infinity. We will denote the distribution corresponding to σ_a^2 as $\overline{\rho}(x)$. The equation has another stationary solution, namely a Gaussian with mean zero and variance close to one, which corresponds with σ_r^2. This stationary solution is unstable, since any Gaussian with mean zero and variance slightly different from σ_r^2 will not remain close to this solution.

It would be tempting from the above to conclude that $\overline{\rho}(x)$ is a stable solution of (27). This is, however, not the case, as follows from the following counterexample which is adapted from Oechssler and Riedel (2002). Consider an initial condition

$$\rho_0(x) = (1-\nu)p_0(x) + \nu p_a(x),$$

with $p_0(x)$ and $p_a(x)$ Gaussians centered at $x = 0$ and $x = a > 0$, respectively, both with variance equal to ε^2 and $\nu > 0$ small. It is clear that for a large enough compared to ε, $||\rho_0 - \overline{\rho}||_1 = \mathscr{O}(\nu)$, so the measures corresponding to $\rho_0(x)$ and $\overline{\rho}(x)$ are close in the variational norm. The unscaled version of equation (27) is

$$\hat{\rho}_t = (-x^2 + \overline{x^2})(1 - \overline{x^2})\rho + \varepsilon^2 \rho_{xx}. \tag{39}$$

At $t = 0$, we have $\overline{x^2} \to \nu a^2$ as $\varepsilon \to 0$. Therefore, for ε sufficiently small, at $x = a$ and $t = 0$ the term $(-x^2 + \overline{x^2})(1 - \overline{x^2}) \approx (1-\nu)(\nu a^2 - 1)a^2 > 0$ if $a^2 > \frac{1}{\nu}$. Because ε is small, the influence of the term $\varepsilon^2 \rho_{xx}$ can be ignored initially, so $\rho(a,t)$ will initially increase, thereby increasing $||\rho_0 - \overline{\rho}||_1$. In graphical terms, the mass at $x = a$ will increase, at the expense of the mass at $x = 0$. Therefore, $\overline{\rho}(x)$ is not stable.

5 Conclusion

In this paper, we have derived a partial differential equation which approximates the replicator equation with mutations of Bomze and Bürger (1998) for symmetric games with a one-dimensional continuous strategy set S. We showed for two examples that the asymptotic behaviour for large time of the solution can be given, for all initial conditions.
This approach has a price and a reward. The price is that we assume that the measures describing the distribution of strategies have a continuous density and a full support. This makes it impossible to consider distributions such as δ_x, where the whole population plays the same strategy $x \in S$. Under our assumption, all strategies will always be present in the population, although some only in minute fractions. Questions about the dynamical stability of such distributions can therefore not be asked in this set-up, let alone answered.
The reward is that the dynamics of the replicator equation can be studied explicitly, both analytically and numerically. For the example $S = \mathbb{R}$ and

$$f(x,y) = -x^2 + 2axy$$

we find convergence to a Gaussian with mean zero and variance of order ε^2, where ε^2 is the size of the mutation term (the product of the frequency of mutations and their average size), if and only if $a < 1$. For $a > 1$ the solution converges to a Gaussian whose mean then diverges to infinity. For $a < 1$ we therefore have a globally attracting solution, which converges weakly to δ_0 as $\varepsilon \to 0$. In the case of this payoff function, $x = 0$ is a Continuously Stable Strategy (CSS) also only if $a < 1$, see Oechssler and Riedel (2002). We have therefore established a partial dynamical foundation for the CSS for quadratic payoff functions. It is only partial because of the limitations on the perturbations that are considered.
There are interesting connections to Adaptive Dynamics (AD) (see Diekmann 2004) here. In it's simplest form, AD studies the evolution of one trait, modelled by a real parameter. AD assumes that the population is monomorphic in the trait space and the location of the resident trait evolves according to the canonical equation. This equation reflects the idea that some mutants with traits close to that of the resident can invade the population and replace the resident. The change is such that (local) increase in fitness is optimal. In our context, the fitness of a mutant with trait x against a resident with trait y is exactly the payoff function $f(x,y)$, and the canonical equation has the form:

$$\frac{d}{dt}x(t) = \mu \frac{\partial}{\partial x'} f(x',x)|_{x'=x}.$$

where $x(t)$ is the trait of the resident and $\mu > 0$ is a constant reflecting various properties of the mutation process.
Replicator dynamics, at least for the case of a one-dimensional traits (or strategies) can be seen as similar to AD, however without the assumption of monomorphism, see Cressman and Hofbauer (2005). The results of this paper show that, for $f(x,y) = -x^2 + 2axy$, the assumption that the population is monomorphic can be said to be satisfied. Independent of initial condition, the solution of the replicator equation (9) will converge to a Gaussian with width ε. This can be interpreted as a practically

monomorphic population, as ε is assumed to be small. Moreover, the equation for the mean of the distribution (26) is the same as the canonical equation. From this it follows that the steady state Gaussian centered around $x = 0$ is stable if and only if the solution $x = 0$ of the canonical equation is stable, which is the CSS condition.

The analysis of the examples in this paper depends heavily on the fact that we took $S = \mathbb{R}$. It is not easily adapted to the case where S is bounded. Nevertheless, we believe that in the example $f(x,y) = -x^2 + 2axy$, the results carry over to the bounded case. This is supported by the numerical results, which for obvious reasons were done with a bounded strategy set. The only difference is that, where in the case $S = \mathbb{R}$ solutions can diverge to infinity when $a > 1$, for bounded S the solution will converge to a distribution concentrated on an ε neighbourhood of the right boundary of S.
For the example $f(x,y) = -x^2 + x^2y^2$, there may be a qualitative difference between the bounded and the unbounded case. In particular, the statement that the steady state distribution centered near $x = 0$ is not Lyapunov stable, even though $x = 0$ satisfies the CSS condition for this function, relies on a counterexample that only works for $S = \mathbb{R}$. It may well be that for bounded S, we still have the equivalence of Lyapunov stability of a distribution near a Nash equilibrium $\bar{x}$ and the CSS condition for $\bar{x}$.
Future work on equation (9) may include answering the above questions for bounded strategy sets. Also, the work of Champagnat et. al. (2006) shows that it may be possible to base the approximation of the mutation term by the Laplacian on a more rigorous base, starting from individual stochastic processes.

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