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Abstract

We introduce a new kind of representation of a not necessarily total preorder, called *strong multi-utility representation*, according to which not only the preorder itself but also its strict part is fully represented by a family of multi-objective functions. The representability by means of semicontinuous or continuous multi-objective functions is discussed, as well as the relation between the existence of a strong multi-utility representation and the existence of a Richter-Peleg utility function. We further present conditions for the existence of a semicontinuous or continuous *countable strong multi-utility representation*.

Key words: Multi-utility representation, Richter-Peleg utility, Strong multi-utility

1 Introduction

The *multi-utility representation* of a not necessarily total preorder or quasi-ordering $\precsim$ on a decision space $\mathbf{X}$ was introduced as an optimal kind of representation since it fully characterizes the preorder by means of a family $\mathbf{V}$ of real-valued (isotonic) functions, in the sense that, for all elements $x, y \in \mathbf{V}$, $x \precsim y$ is required to be equivalent to $v(x) \leq v(y)$ for all functions $v \in \mathbf{V}$. The main general contributions in this field were presented by Levin [18] and

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especially by Evren and Ok [12] (see also the more recent paper by Bosi and Herden [8]). They are concerned with the case where continuity or at least semicontinuity of the representing functions is required.
The fundamental case of a finite representing family was studied by Ok [21] and more recently by Kaminski [17]. This is the so-called *multi-objective functions* approach to the decision making problem. In particular, Kaminski [17] presented a characterization of preorders on a separable metric space which are representable by continuous multi-objective functions as well as explicit necessary conditions for the existence of such a representation. Furthermore, Kamisnki furnished conditions incompatible with the existence of a representation by continuous multi-objective functions, and showed that preferences implied by a first order stochastic dominance relation are not representable in this sense.
The more general case of countable multi-utility representations is investigated in Minguzzi [19] and Bosi and Zuanon [9]. It can be regarded as a natural extension of the multi-objective functions approach. In an apparently different line of inspection, the existence of a *Richter-Peleg utility representation* or order-preserving function (see e.g. Richter [23] and Peleg [22]) furnishes, in some sense, a synthetic and approximate description of a preorder. Although this latter kind of representation does not allow us to recover the preorder itself, it has the advantage of referring to a single function. This is enough for purposes like the search for maximal elements by maximization of an upper semicontinuous utility on a compact topological space. Here we prove that whenever there exists a *countable* multi-utility representation, a Richter-Peleg utility can be explicitly constructed that preserves the continuity properties of the extended multi-objective representation (cf., Proposition 2.5 and Remark 2.6). Therefore this particular case can be thought of as being of particular interest, which justifies our investigation of conditions for the existence of a (semicontinuous, continuous) countable multi-utility representation
In addition we introduce and discuss another specification of the ethos of multi-utility representations, called *strong multi-utility representations*. In this case, we need a family $\mathbf{V}$ of isotonic functions that not only characterizes the preorder $\precsim$ but also its *strict part* $\prec$, in the sense that now we further require, for all elements $x, y \in \mathbf{X}$, $x \prec y$ to be equivalent to $v(x) < v(y)$ for all functions $v \in \mathbf{V}$. When that family is finite we also say that $\mathbf{V}$ is a strong representation of $\precsim$ by multi-objective functions. This novel approach combines the Richter-Peleg proposal and the multi-utility approach that fully characterizes the relation. We establish a close relationship between this new kind of representation and Richter-Peleg representability, especially when no topological assumption is required: in this case, an exact equivalence holds true. Further, we show that the existence of a countable multi-utility representation is equivalent to the existence of a strong countable multi-utility representation, so that a widespread coincidence irrespective of possible topological properties arises (see Proposition 2.5).
If a strong muti-utility representation $\mathbf{V}$ of a preorder $\precsim$ exists, then maxi-

mization of *any* function $v \in \mathbf{V}$ produces maximal elements of $\precsim$. This fact is a distinctive feature of strong multi-utilities and can be thought of as one of the possible motivations of the present contribution. While it is clear that there are preorders that do not admit a strong multi-utility representation, we show that in the countable case the existence of a (continuous) multi-utility representation is equivalent to the existence of a (continuous) strong multi-utility representation. We further discuss the case when the set $\mathbf{X}$ is endowed with a second countable topology τ, by showing that in this case the existence of a continuous multi-utility representation implies the existence of a countable continuous strong multi-utility representation. Finally, to better assess the implications of our proposal we prove that a nontrivial preorder on a connected topological space is total as long as it has a continuous strong representation by multi-objective functions.

2 Notation and definitions

Let $\mathbf{X}$ represent a *decision space* and $\precsim$ represent a preorder, also called quasi-ordering (reflexive, transitive binary relation) on $\mathbf{X}$. As usual, $\prec$ denotes the *strict part* of $\precsim$ and we use $x \precsim y$, resp. $x \prec y$, as a shorthand for $(x, y) \in \precsim$, resp. $(x, y) \in \prec$. The preorder $\precsim$ is total if for each $x, y \in \mathbf{X}$, either $x \precsim y$ or $y \precsim x$ holds true.

For every $x \in \mathbf{X}$ we set the following subsets of $\mathbf{X}$:

$$l(x) = \{y \in \mathbf{X} \mid y \prec x\}, \;\; r(x) = \{z \in \mathbf{X} \mid x \prec z\},$$

$$d(x) = \{y \in \mathbf{X} \mid y \precsim x\}, \;\; i(x) = \{z \in \mathbf{X} \mid x \precsim z\}.$$

A subset D of $\mathbf{X}$ is said to be *decreasing*, resp. *increasing*, if $d(x) \subset D$, resp. $i(x) \subset D$, for all $x \in D$.

We recall that $v : (\mathbf{X}, \precsim) \longrightarrow (\mathbb{R}, \leqslant)$ is *isotonic* or *increasing* when $x \precsim y \Rightarrow v(x) \leqslant v(y)$. Furthermore, v is *strictly isotonic* or *order preserving* if it is isotonic and in addition, $x \prec y \Rightarrow v(x) < v(y)$. Strictly isotonic functions on $(\mathbf{X}, \precsim)$ are also called *Richter-Peleg representations* of $\precsim$ (see e.g. Richter [23]). Denote by τ_{nat} the *natural topology* on the real line $\mathbb{R}$.

Following the terminology adopted by Evren and Ok [12], we say that a preorder $\precsim$ on a topological space $(\mathbf{X}, \tau)$ is *upper, resp. lower, semicontinuous* if $i(x)$, resp. $d(x)$, is a closed subset of $\mathbf{X}$ for every $x \in \mathbf{X}$. And it is continuous if it is both upper and lower semicontinuous.

A *multi-utility representation* of the preordered space $(\mathbf{X}, \precsim)$ is a family $\mathbf{V}$ of functions $v: (\mathbf{X}, \precsim) \longrightarrow (\mathbb{R}, \leqslant)$, with the property that for each $x, y \in \mathbf{X}$,

$$x \precsim y \Leftrightarrow [v(x) \leqslant v(y), \text{ for all } v \in \mathbf{V}] \tag{1}$$

We make note that each $v \in \mathbf{V}$ is an isotonic function when $\mathbf{V}$ is a multi-utility representation of $(\mathbf{X}, \precsim)$. If $\mathbf{V}$ is a countable, resp. finite, family then we say that $\mathbf{V}$ is a *countable*, resp. *finite*, *multi-utility representation* of $(\mathbf{X}, \precsim)$. In the spirit of Kaminski [17], when $\precsim$ is represented by a finite multi-utility representation we also say that it is represented by *multi-objective functions*. When there is a topology τ on $\mathbf{X}$ and $\mathbf{V}$ is a family of upper semicontinuous/lower semicontinuous/continuous functions with the property that (1) holds for each $x, y \in \mathbf{X}$, then we say that $\mathbf{V}$ is an *upper semicontinuous/lower semicontinuous/continuous multi-utility representation* of $(\mathbf{X}, \precsim)$. Combinations of these concepts (e.g., countable continuous multi-utility representation) are naturally mentioned along the paper. Kaminski [17, Section 4] gives examples of preorders for which representations by continuous multi-objective functions do not exist.

If $\mathbf{V}$ is a multi-utility representation of $(\mathbf{X}, \precsim)$ then

$$x \prec y \Leftrightarrow [v(x) \leqslant v(y) \text{ for all } v \in \mathbf{V}, \text{ and } v'(x) < v'(y) \text{ for some } v' \in \mathbf{V}] \tag{2}$$

The following result is often quoted along the paper:

Proposition 2.1 *(Evren and Ok [12, Proposition 2]) Every preorder (resp., semicontinuous preorder) on a set (resp., on a topological space) is representable by a multi-utility (resp., a semicontinuous multi-utility). If the set is countable then the preorder is representable by a countable multi-utility.*

In this regard we mention that Evren and Ok [12, Subsection 3.2] characterize the existence of *continuous* multi-utility representations for preorders on topological spaces in terms of semi-normality, a property that is not easy to check for. Then in Subsection 3.3 they deduce easier-to-check sufficient conditions for that purpose. Nachbin [20, Theorem 1] and Minguzzi [19, Proposition 5.2, Theorem 5.3] provide other sufficient conditions for the existence of continuous multi-utility representations for preorders on topological spaces. Nachbin showed that the preorder of a normally preordered space is represented by the family of continuous isotonic functions. Then Minguzzi showed that regularly preordered topological spaces that are either Lindelöff or second countable must be normally preordered.

In order to introduce a non-trivial concept of representation in the spirit of multi-utilities, Minguzzi [19, Section 5] introduces the following notion that we call *Richter-Peleg multi-utility representation*. A preordered set $(\mathbf{X}, \precsim)$ is *represented by a Richter-Peleg multi-utility* $\mathbf{V}$ if $\mathbf{V}$ is a family of strictly isotonic functions on $(\mathbf{X}, \precsim)$ such that for each $x, y \in \mathbf{X}$, property (1) holds true. Therefore Richter-Peleg multi-utility representations are multi-utility representations. From the fact that there are preorders without a Richter-Peleg representation we deduce:

Corollary 2.2 *There are preorders that cannot be represented by Richter-Peleg multi-utilities.*

In particular, the existence of multi-utility representations does not secure existence of Richter-Peleg multi-utility representations. The class of preordered sets for which Richter-Peleg multi-utility representations exist has not been identified yet. When there is a topology τ on $\mathbf{X}$, *upper semicontinuous/lower semicontinuous/continuous Richter-Peleg multi-utility representations* of $\precsim$ can be defined in a direct manner as above. Concepts like countable continuous Richter-Peleg multi-utility representations are naturally mentioned along the paper and their meaning is inherited from the formalizations above.

Now we introduce an apparently more restrictive concept in the same vein, in order to investigate its relationship with existing concepts. A *strong multi-utility representation* of $\precsim$ is a family $\mathbf{V}$ of functions $v : (\mathbf{X}, \precsim) \longrightarrow (\mathbb{R}, \leqslant)$ such that for each $x, y \in \mathbf{X}$, both property (1) and the following property (3) hold true:

$$x \prec y \Leftrightarrow [v(x) < v(y), \text{ for all } v \in \mathbf{V}] \tag{3}$$

We make note that each $v \in \mathbf{V}$ is a strictly isotonic function or Richter-Peleg representation of $\precsim$ when $\mathbf{V}$ is a strong multi-utility representation of $\precsim$, which is not the case when the weaker concept of multi-utility is invoked. The following Proposition also helps to distinguish between the concepts of strong multi-utility and multi-utility. Its immediate proof is left to the reader.

Proposition 2.3 *Let a family $\boldsymbol{V}$ of functions $v : (\boldsymbol{X}, \precsim) \longrightarrow (\mathbb{R}, \leqslant)$ be a multi-utility representation of a preorder $\precsim$ on a set $\mathbf{X}$. Then the following conditions are equivalent:*

1. *$\boldsymbol{V}$ is a strong multi-utility representation of $\precsim$.*

2. *For every $x, y \in \mathbf{X}$ the following holds true:*

   If $v(x) \leqslant v(y)$ for all $v \in \mathbf{V}$ and there exists $v \in \mathbf{V}$ such that $v(x) < v(y)$ then $v(x) < v(y)$ for all $v \in \mathbf{V}$. (4)

Remark 2.4 *If a family $\boldsymbol{V}$ of functions $v : (\boldsymbol{X}, \precsim) \longrightarrow (\mathbb{R}, \leqslant)$ is a strong multi-utility representation of a preorder $\precsim$ on a set $\mathbf{X}$, then for all $v \in \mathbf{V}$*

denote by $\precsim_v$ the total preorder on $\mathbf{X}$ defined as follows:

$$x \precsim_v y \Leftrightarrow v(x) \leq v(y). \tag{5}$$

It is immediate to check that $\precsim = \bigcap_{v \in \mathbf{V}} \precsim_v$ and $\prec = \bigcap_{v \in \mathbf{V}} \prec_v$. The latter identity is a distinctive feature of strong multi-utilities as opposed to multi-utilities, and it has an immediate consequence in terms of the existence of maximal elements. Indeed, if x_0 is a maximal element of $\precsim_v$ for some $v \in \boldsymbol{V}$ then x_0 is a maximal element of $\precsim$ (otherwise the existence of an element $x_1 \in \mathbf{X}$ such that $x_0 \prec x_1$ would in particular imply that $x_0 \prec_v x_1$). Therefore in the case of a strong multi-utility representation $\boldsymbol{V}$ of a preorder $\precsim$, maximization of any $v \in \boldsymbol{V}$ produces maximal elements of $\precsim$. Because maximizing real-valued functions seems more natural than maximizing incomplete binary relations, this arguments brings the concept of strong multi-utility closer to prospective applications.

When there is a topology τ on $\mathbf{X}$, *upper semicontinuous/lower semicontinuous/ continuous strong multi-utility representations* of $\precsim$ can be defined in a direct manner as above. Clearly, (upper semicontinuous, lower semicontinuous, continuous) strong multi-utility representations of $\precsim$ are (upper semicontinuous, lower semicontinuous, continuous) Richter-Peleg multi-utility representations of $\precsim$. As in the previous cases, the natural meaning of concepts like countable continuous strong multi-utility representations is presumed along the paper.

2.1 Equivalences among variations of the multi-utility concept

The previous concept of multi-utility representation of a preorder is empty, in the sense that it does not generate a proper subclass of preorders representable by a multi-utility. In fact the proof that every semicontinuous preorder is representable by a semicontinuous multi-utility is virtually trivial. Only continuous multi-utility representations permit to discriminate among preorders (Evren and Ok [12, Section 3]; Bosi and Herden [8]).
It is interesting to notice that if we restrict our attention to the class of preordered sets representable by *countable* (resp., and continuous) multi-utilities the following Proposition 2.5 proves that a widespread coincidence arises among the different specifications of multi-utility representations [1].

Proposition 2.5 *Let $\precsim$ be a preorder on a topological space $\boldsymbol{X}$. The following conditions are equivalent:*

[1] In Section 4 we give further results in this line when we restrict our attention to second countable topological spaces.

1. $\precsim$ can be represented by a countable continuous strong multi-utility.

2. $\precsim$ can be represented by a countable continuous Richter-Peleg multi-utility.

3. $\precsim$ can be represented by a countable continuous multi-utility.

The equivalence remains true if the term 'continuous' is deleted from each clause, or replaced with 'upper/lower semicontinuous'.

Proof. As is the case of Proposition 2.10, we only need to prove the implication $3 \Rightarrow 1$. If $\precsim$ is represented by a countable family $\mathbf{V} = \{v_n : n \in \mathbb{N}\}$ of continous isotonic functions $v_n : (\mathbf{X}, \precsim, \tau) \longrightarrow (\mathbb{R}, \leqslant, \tau_{nat})$, without loss of generality we can assume that the values of each $v_n \in \mathbf{V}$ lie in $[0,1]$. Then $f = \sum_{n \in \mathbb{N}} 2^{-n} v_n$ is a continuous Richter-Peleg representation of $\precsim$, and therefore $\mathbf{U} = \{v_n + \alpha f : n \in \mathbb{N}, \alpha \in \mathbb{Q}, \alpha > 0\}$ is a countable continuous strong multi-utility representation of $\precsim$.

If the term 'continuous' is deleted from each clause, or replaced with 'upper/lower semicontinuous', the argument above can be mimicked to prove the equivalence of the conditions arising. □

Remark 2.6 *Minguzzi [19, Lemma 5.4] proves the equivalence of statements 2 and 3 (under the continuity assumption).*

Remark 2.7 *The argument in Proposition 2.5 proves that when $\precsim$ can be represented by a countable (resp., continuous) multi-utility then it has a (resp., continuous) Richter-Peleg representation.*

Furthermore, in view of Corollary 2.2, the properties in Proposition 2.5 are not universally applicable to all preorders.

Banerjee and Dubey [3] recall the concept of an ethical social welfare relation on $[0,1]^{\mathbb{N}}$. They prove in their Proposition 1 that no SWR admits a Richter-Peleg representation. Then in their Theorem 2 they prove that no SWR admits a countable multi-utility representation. An appeal to Remark 2.7 permits to derive the latter result from the former immediately.

Similarly, in this Section we explore the relationships between the concept of 'having strong multi-utilities' and the apparently weaker concept of 'having Richter-Peleg multi-utilities', in the possible presence of additional requirements. We find a generalized coincidence that we proceed to make explicit.

Theorem 2.8 *Let $\precsim$ be a preorder on a set $\mathbf{X}$. The following conditions are equivalent:*

1. $\precsim$ can be represented by a strong multi-utility.

2. $\precsim$ can be represented by a Richter-Peleg multi-utility.

3. There is a Richter-Peleg representation of $\precsim$.

Proof. Because $1 \Rightarrow 2 \Rightarrow 3$ has been established, we only need to prove $3 \Rightarrow 1$. We use a technique from Evren and Ok [12, Corollary 1]. Let $\mathbf{V}$ be a multi-utility representation of $\precsim$, and let f be a Richter-Peleg representation of $\precsim$. Then $\mathbf{U} = \{v + \alpha f : v \in \mathbf{V}, \alpha \in \mathbb{Q}, \alpha > 0\}$ is a strong multi-utility representation of $\precsim$. $\square$

Remark 2.9 *Theorem 2.8 remains valid if there is a topology on $\boldsymbol{X}$ and we insert the term 'upper/lower semicontinuous' in each of the clauses of the statement.*

However, introducing continuity into the picture requires the following variation of the statement of Theorem 2.8:

Proposition 2.10 *Let $\precsim$ be a preorder on a topological space $\boldsymbol{X}$. The following conditions are equivalent:*

1. $\precsim$ can be represented by a continuous strong multi-utility.

2. $\precsim$ can be represented by a continuous Richter-Peleg multi-utility.

3. $\precsim$ can be represented by a continuous multi-utility, and there are continuous Richter-Peleg representations of $\precsim$.

Proof. The implications $1 \Rightarrow 2 \Rightarrow 3$ are trivial. The implication $3 \Rightarrow 1$ can be proven by mimicking the proof of Theorem 2.8. $\square$

Evren and Ok [12, Remark 2] show that the notions of continuous Richter-Peleg representation and continuous multi-utility representation are logically independent. Coupled with Proposition 2.10, this means that *the notion of continuous strong (or Richter-Peleg) multi-utility representation is strictly stronger than the notion of continuous multi-utility representation.*

In the sequel $\chi_{\mathbf{A}}$ will stand for the *indicator function* of any subset $\mathbf{A}$ of $\mathbf{X}$.

Corollary 2.11 *Let $\precsim$ be a preorder on a set $\boldsymbol{X}$ and assume that there exists a countable subset $\boldsymbol{D}$ of $\boldsymbol{X}$ such that for all $x, y \in \boldsymbol{X}$ the following separability condition holds:*

$$x \prec y \Rightarrow \exists d \in \boldsymbol{D} : x \prec d \precsim y \tag{6}$$

Then the following conditions are equivalent:

1. $\precsim$ can be represented by an upper semicontinuous strong multi-utility.

2. $\precsim$ *is upper semicontinuous.*

Proof. Since the implication $1 \Rightarrow 2$ is clear, we only need to show that if $\precsim$ is upper semicontinuous and condition (6) holds, then there exists an upper semicontinuous Richter-Peleg representation of $\precsim$. Let $\mathbf{D} = \{d_n : n \in \mathbb{N}^+\}$ be a countable subset of $\mathbf{X}$ satisfying condition (6) and define $f := \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{i(d_n)}$ in order to immediately verify that f is an upper semicontinuous Richter-Peleg representation of $\precsim$. $\square$

2.2 Other characterizations

A preorder $\precsim$ on a topological space $(\mathbf{X}, \tau)$ is said to be *weakly upper semicontinuous* if for every pair $(x, y) \in \prec$ there exists some open decreasing subset O of $\mathbf{X}$ such that $x \in O$ and $y \in \mathbf{X} \setminus O$. One verifies immediately that a preorder $\precsim$ on $(\mathbf{X}, \tau)$ is weakly upper semicontinuous if and only if for every pair $(x, y) \in \prec$ there exists an upper semicontinuous isotone (increasing) function $f : (\mathbf{X}, \precsim, \tau) \longmapsto (\mathbb{R}, \leq, \tau_{nat})$ such that $f(x) < f(y)$. Indeed, let O be some open decreasing subset of $\mathbf{X}$ such that $x \in O$ and $y \in \mathbf{X} \setminus O$. Then one may define the desired upper semicontinuous isotone function $f : (\mathbf{X}, \precsim, \tau) \longmapsto (\mathbb{R}, \leq, \tau_{nat})$ by setting for all $z \in \mathbf{X}$:

$$f(z) := \begin{cases} 0 \; if \;\; z \in O \\ 1 \; if \;\; z \in \mathbf{X} \setminus O \end{cases}.$$

It is easily seen that an upper semicontinuous preorder is also weakly upper semicontinuous (see Herden and Levin [14]). Weak upper semicontinuity of a preorder is a necessary condition for the existence of upper semicontinuous Richter-Peleg utility representations, and therefore for the existence of upper semicontinuous (Richter-Peleg) multi-utility representations (cf., Theorem 2.8). We proceed to recall that even upper semicontinuity is necessary for the existence of upper semicontinuous multi-utility representations, which improves an implication by Evren and Ok (cf., Proposition 2.1):

Proposition 2.12 (Bosi and Zuanon [9]) *Let* $\precsim$ *be a preorder on a topological space* $(\boldsymbol{X}, \tau)$*. Then the following conditions are equivalent.*

(i) $\precsim$ *has an upper semicontinuous multi-utility representation.*

(ii) $\precsim$ *is upper semicontinuous.*

(iii) *For every* $x, y \in \boldsymbol{X}$ *such that not*$(y \precsim x)$ *there exists an open decreasing subset* O_{xy} *of* $\boldsymbol{X}$ *such that* $x \in O_{xy}$, $y \notin O_{xy}$.

(iv) *For every* $x, y \in \boldsymbol{X}$ *such that not*$(y \precsim x)$ *there exists an upper semicontinuous increasing function* $f_{xy} : (\boldsymbol{X}, \precsim, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$ *such that* $f_{xy}(x) < f_{xy}(y)$.

(v) $\precsim$ *is weakly upper semicontinuous and for every* $x, y \in \boldsymbol{X}$ *such that not*$(y \precsim x)$ *and not*$(x \precsim y)$ *there exist two upper semicontinuous increasing functions* f_{xy}, $g_{xy} : (\boldsymbol{X}, \precsim, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$ *such that* $f_{xy}(x) < f_{xy}(y)$ *and* $g_{xy}(x) > g_{xy}(y)$.

Proposition 2.12 admits an improvement when countability is introduced in the following form:

Proposition 2.13 *Let* $\precsim$ *be a preorder on a topological space* $(\boldsymbol{X}, \tau)$. *Then the following conditions are equivalent.*

(i) $\precsim$ *has a countable upper semicontinuous multi-utility representation.*

(ii) *There exists a countable family* $\mathbf{O} = \{O_n\}_{n \in \mathbb{N}}$ *of open decreasing subsets of* $\mathbf{X}$ *such that for every* $x, y \in \boldsymbol{X}$ *such that not*$(y \precsim x)$ *there exists* $n \in \mathbb{N}$ *such that* $x \in O_n$, $y \notin O_n$.

(iii) *There exists a countable family* $\mathbf{V} = \{v_n\}_{n \in \mathbb{N}}$ *of upper semicontinuous isotonic functions such that for every* $x, y \in \boldsymbol{X}$ *such that not*$(y \precsim x)$ *there exists* $n \in \mathbb{N}$ *such that* $v_n(x) < v_n(y)$.

(iv) *There exists a topology* τ' *on* $\mathbf{X}$ *coarser than* τ *with a countable basis consisting of decreasing subsets of* $\mathbf{X}$ *such that* $\precsim$ *is upper semicontinuous with respect to* τ'.

Proof. (i) $\Rightarrow$ (ii). Let $\mathbf{V} = \{v_n\}_{n \in \mathbb{N}}$ be a countable upper semicontinuous multi-utility representation of a preorder $\precsim$ on a topological space $(\mathbf{X}, \tau)$. Define, for every $n \in \mathbb{N}$ and $q \in \mathbb{Q}$, the open decresing set $O_{n,q} = v_n^{-1}(]-\infty, q[)$. Then for every $x, y \in \mathbf{X}$ such that *not*$(y \precsim x)$ there exists $n \in \mathbb{N}$ and $q \in \mathbb{Q}$ such that $v_n(x) < q < v_n(y)$ and therefore we have that $x \in O_{n,q}$, $y \notin O_{n,q}$.
(ii) $\Rightarrow$ (iii). Let $\mathbf{O} = \{O_n\}_{n \in \mathbb{N}}$ be a countable family of open decreasing subsets of X with the indicated property. Then define, for all $n \in \mathbb{N}$,

$$v_n(z) := \begin{cases} 0 \; if \;\; z \in O_n \\ 1 \; if \;\; z \in \mathbf{X} \backslash O_n \end{cases},$$

in order to immediately verify that the countable family $\mathbf{V} = \{v_n\}_{n \in \mathbb{N}}$ of upper semicontinuous isotonic functions satisfies condition (iii).
(iii) $\Rightarrow$ (iv). Let $\mathbf{V} = \{v_n\}_{n \in \mathbb{N}}$ be a countable family of upper semicontinous isotonic functions on $(X, \precsim, \tau)$ with the indicated property, and define, for every $n \in \mathbb{N}$ and $q \in \mathbb{Q}$, the open decreasing set $O_{n,q} = v_n^{-1}(] - \infty, q[)$. Then

consider the topology τ' on $\mathbf{X}$ whose basis is $\{O_{n,q}\}_{n\in\mathbb{N},q\in\mathbb{Q}}$. It is clear that τ' is a second countable topology on $\mathbf{X}$ which is coarser that τ. In order to show that $\precsim$ is upper semicontinuous with respect to τ', consider two elements $x, z \in \mathbf{X}$ such that $z \notin i(x)$. Then there exist $n \in \mathbb{N}$ and $q \in \mathbb{Q}$ such that $v_n(z) < q < v_n(x)$ and therefore $O_{n,q}$ is a τ'-open subset of $\mathbf{X}$ containing z such that $z' \notin i(x)$ for all $z' \in O_{n,q}$. Hence, $i(x)$ is τ'-closed for all $x \in \mathbf{X}$.

(iv) $\Rightarrow$ (i). Let $\mathbf{B} = \{B_n\}_{n\in\mathbb{N}}$ be a countable basis of $(\mathbf{X}, \tau')$, where τ' is a topology coarser that τ and every set B_n is decreasing. Since the preorder $\precsim$ is upper semicontinuous with respect to τ', there exists a countable τ'-upper semicontinuous multi-utility representation by the above implication "(ii) $\Rightarrow$ (iii)". It is clear that such a representation is also τ-upper semicontinuous. $\square$

3 Countability restrictions

When $\mathbf{X}$ is countable, the following Corollary 3.1 proves that all preorders defined on it can be represented by countable strong multi-utilities.

Corollary 3.1 *Let $\precsim$ be a preorder on a countable set $\mathbf{X}$. Then there are countable strong multi-utility representations of $\precsim$.*

Proof. We just need to use Proposition 2.1 coupled with Proposition 2.5. Or equivalently, we can appeal to Proposition 2.1 and the fact that every preorder on a countable set $\mathbf{X}$ has a Richter-Peleg representation, in combination with the argument of Theorem 2.8. $\square$

Conversely, we proceed to derive a restriction on the preorders for which finite continuous strong multi-utility representations exist under the assumption that the topology is connected. We obtain the following result:

Proposition 3.2 *If a nontrivial preorder $\precsim$ on a connected topological space $(\mathbf{X}, \tau)$ has a strong representation by continuous multi-objective functions $\mathbf{V} = \{v_1, ..., v_n\}$ then $\precsim$ is total.*

Proof. It is immediate to check that if a preorder $\precsim$ on a topological space $(\mathbf{X}, \tau)$ has a continuous multi-utility representation then $\precsim$ is continuous, i.e., both $d(x)$ and $i(x)$ are closed subsets of $\mathbf{X}$ for all $x \in \mathbf{X}$ (see e.g. Proposition 5 in Bosi and Herden [8] or Theorem 3.1 in Kaminski [17] for a restricted version). To conclude we appeal to the following well known result:

Lemma 3.3 (Schmeidler [24]) *Let $\precsim$ be a nontrivial preorder on a connected topological space (X, τ). If for every $x \in X$ the sets $d(x)$ and $i(x)$ are closed and the sets $l(x)$ and $r(x)$ are open, then the preorder $\precsim$ is total.*

Therefore it suffices to show that under our assumptions, both $l(x)$ and $r(x)$ are open subsets of $\mathbf{X}$ for all $x \in \mathbf{X}$. To prove this fact we observe that

$$\begin{aligned} l(x) &= \{y \in \mathbf{X} \mid y \prec x\} = \{y \in \mathbf{X} \mid v_i(y) < v_i(x), \text{ for all } i \in \{1, ..., n\}\} = \\ &= \bigcap_{i=1}^{n} v_i^{-1}(]-\infty, v_i(x)[), \text{ and} \\ r(x) &= \{y \in \mathbf{X} \mid x \prec y\} = \{y \in \mathbf{X} \mid v_i(x) < v_i(y), \text{ for all } i \in \{1, ..., n\}\} = \\ &= \bigcap_{i=1}^{n} v_i^{-1}(]v_i(x), +\infty[) \end{aligned}$$

for each $x \in \mathbf{X}$. From these equalities and continuity of the v_i functions, the conclusion follows immediately. □

4 The Second Countability axiom

In this Section we explore the implications of the Second Countability axiom with regard to the construction of continuous and semicontinuous multi-utility representations.

When $(\mathbf{X}, \tau)$ is a second countable topological space, the problem of the existence of continuous strong multi-utility representations of the preorder $\precsim$ on $\mathbf{X}$ is related to the continuous analogue of the Dushnik-Miller theorem as described in the following proposition. We recall that a preorder $\precsim$ on a topological space $(\mathbf{X}, \tau)$ is said to be *weakly continuous* if for every pair $(x, y) \in \prec$ there exists a continuous increasing real-valued function f_{xy} on $(\mathbf{X}, \tau)$ such that $f_{xy}(x) < f_{xy}(y)$. Further, a preorder $\precsim$ on a set $\mathbf{X}$ is said to satisfy the *continuous analogue of the Dushnik and Miller theorem* (see Bosi and Herden [6,7]) if it is the intersection of all the continuous total preorders $\leq$ extending it (i.e., all the continuous total preorders $\leq$ such that $\precsim \subset \leq$).

Proposition 4.1 *Let $\precsim$ be a weakly continuous preorder on a second countable topological space $(\boldsymbol{X}, \tau)$. If $\precsim$ satisfies the continuous analogue of the Duhnik-Miller theorem, then $\precsim$ has a continuous strong multi-utility representation.*

Proof. By Bosi and Herden [8, Proposition 3.4], there is a continuous multi-utility representation of $\precsim$. In addition, there is a continuous Richter-Peleg representation of $\precsim$ by Bosi et al. [5, Theorem 3.1]. Therefore the conclusion follows from Proposition 2.10. □

Minguzzi [19] has provided a sufficient condition for a preorder to have a countable continuous Richter-Peleg multi-utility, which in combination with

Proposition 2.5 produces the following Corollary:

Corollary 4.2 *Every second countable regularly preordered space can be represented by a countable continuous strong multi-utility.*

Proof. Minguzzi [19, Theorem 5.5] proves that such a preordered space can be represented by a countable continuous Richter-Peleg multi-utility. We just need to apply Proposition 2.5 to produce our thesis. $\square$

The Second Countability axiom facilitates the existence of countable upper semicontinuous multi-utility representations too. As an immediate consequence of Proposition 2.13 we get Corollary 4.3 below. It requires to introduce the following concept: a preorder $\precsim$ on a topological space (X, t) is *d-compliant* if for every open subset O of $\mathbf{X}$ also the set $d(O)$ is open (we denote by $d(A)$ the smallest decreasing superset of a subset A of $\mathbf{X}$).

Corollary 4.3 *Let $\precsim$ be a preorder on a second countable topological space $(\mathbf{X}, \tau)$. Then $\precsim$ has a countable upper semicontinuous multi-utility representation provided that $\precsim$ is upper semicontinuous and d-compliant.*

Proof. Let $\mathbf{B} = \{B_n\}_{n\in\mathbb{N}}$ be a countable basis of $(\mathbf{X}, \tau)$, and define $O_n = d(B_n)$ for all $n \in \mathbb{N}$. Then it is easy to check that the countable family $\mathbf{O} = \{O_n\}_{n\in\mathbb{N}}$ of open decreasing subsets of $\mathbf{X}$ satisfies the assumptions of Proposition 2.13 (ii). Hence the thesis follows. $\square$

The Second Countability axiom has important implications when the existence of continuous multi-utility is granted. Our next result gives arguments to that purpose.

Proposition 4.4 *Suppose that a preorder $\precsim$ on a second countable topological space (X, τ) has a continuous multi-utility representation $\boldsymbol{V}$. Then $\precsim$ has a countable continuous strong multi-utility representation $\boldsymbol{V}$.*

Proof. We benefit from a technique in Minguzzi [19, Theorem 5.5]. Define $G(\precsim) = \{(x, y) \in X \times X : x \precsim y\}$ and $G_v = \{(x, y) \in X \times X : v(x) \precsim v(y)\}$ for each $v \in \mathbf{V}$. Then $G(\precsim) = \bigcap_{v\in\mathbf{V}} G_v$ and each G_v is closed by continuity of v. The product space $X \times X$ is second countable (Willard [25, 16E]) hence hereditary Lindelöff (Hocking and Young [15, Exercise 2-17]), which ensures the existence of a countable family $\mathbf{V}' \subseteq \mathbf{V}$ such that $G(\precsim) = \bigcap_{v\in\mathbf{V}'} G_v$. This means that $\mathbf{V}'$ is a countable continuous multi-utility representation of $\precsim$. In order to conclude we invoke Proposition 2.5. $\square$

In particular, Proposition 4.4 says that even though the notion of continuous strong multi-utility representation is strictly stronger than the notion of continuous multi-utility representation (as emphasized earlier), they are equivalent when we restrict ourselves to second countable topological spaces.

We recall that a preorder $\precsim$ on a topological space $(\mathbf{X}, \tau)$ is said to be *closed* if $\precsim$ is a closed subset of $X \times X$ with respect to the product topology $\tau \times \tau$ on $X \times X$ that is induced by τ. If a preorder $\precsim$ is total, then it is closed if and only if it is *continuous* (i.e., for every point $x \in X$ both sets $d(x) = \{y \in X \mid y \precsim x\}$ and $i(x) = \{z \in X \mid x \precsim z\}$ are closed subsets of X). The following corollary of Proposition 9 is based both upon a result presented by Evren and Ok [12] and a well known theorem in mathematical utility theory proved by Estévez and Hervés [11].

Corollary 4.5 *Let $(\mathbf{X}, \tau)$ be a locally compact metrizable topological space. Then the following conditions are equivalent:*

(i) *The topology τ on X is separable.*

(ii) *Every closed preorder $\precsim$ on $(\mathbf{X}, \tau)$ has a countable continuous strong multi-utility representation $\boldsymbol{V}$.*

Proof. In order to prove the implication (i) $\Rightarrow$ (ii), just consider the fact that from Evren and Ok [12, Corollary 1] every closed preorder on a locally compact separable metric space has a continuous multi-utility representation and then apply the above Proposition 4.4 (recall the well known fact that a metric space is separable if and only if it is second countable). In order to show that also the implication (ii) $\Rightarrow$ (i) is valid, assume that the topology τ on X is not separable. Then from the Theorem proved by Estévez and Hervés [11], there exists a continuous (and therefore closed) total preorder that does not admit a continuous (Richter-Peleg) utility representation. Then such a preorder cannot admit a countable continuous (strong) multi-utility representation $\mathbf{V}$ (see Remark 4). $\square$

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