Are Real Options "Real"? Isolating Uncertainty from Risk in Real Options Analysis

So, Leh-chyan

National Tsing Hua University

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Isolating Uncertainty from Risk in Real Options Analysis*

Leh-chyan So
Department of Quantitative Finance
National Tsing Hua University
Taiwan

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Abstract

This paper derives an adjusted Black-Scholes pricing formula. In separating risk and uncertainty using the robust control technique, we find that both uncertainty and risk raise management's subjective evaluation of real options. We suggest a simple method to filter the risk of the project and to acquire a more reliable value of real options without the influence of uncertainty. In addition, we propose that an investment opportunity may be postponed inappropriately, as under uncertainty the exercise of investment may be delayed by the project manager. To our knowledge, any similar quantitative methods have not hitherto been mentioned in terms of isolating uncertainty from risk in real options analysis that we consider here.

Keywords: Option to defer, investment opportunity, uncertainty, Black-Scholes pricing formula, volatility.

JEL: G11, G12, G13.

1. Introduction

Many approaches have proposed methods to evaluate investment opportunities. Among them, traditional discounted cash flow (DCF) approaches, such as the standard net present value (NPV), are easy to apply but have been criticized for their neglect of a project manager's flexibility to adjust earlier decisions according to uncertainties that are revealed later (for example, Trigeorgis, 1996). On the contrary, real options have enjoyed great popularity recently (for example, Dixit and Pindyck, 1994; Trigeorgis, 1996; Amram and Kulatilaka, 1999) since, in the real world, managers have the right to undertake investment opportunities and realize positive profits. This flexibility not only protects managers against downside risk but also provides upside potential.

As a manager has the flexibility to retract initial planning, it is risk and uncertainty in the future that triggers an upgrade in the investment opportunity. Hence, in the light of real options, the investment opportunity should be worth more when its volatility is high. Real options extend financial options into an investment opportunity analysis of real assets and often assign a higher value to the investment opportunity because of the value of time.

In practice, the Black-Scholes pricing formula is commonly used in real options analysis. Many studies have reported the influence and estimation of the six factors affecting the price of an option (for example, Leslie and Michaels, 1997; Davis, 1998; Fernandez, 2013). Among the six key factors, volatility is especially notorious for the difficulty in estimation (for example, Lander and Pinches, 1998). Furthermore, option prices are very sensitive to estimation of the volatility of the underlying assets. As noted in Trigeorgis (1990), a 50 percent increase in volatility raises the option value by about 40 percent.

In the Black-Scholes model, volatility, σ, is defined as the standard deviation of the continuously compounded rate of return provided by the underlying asset within a year. The binomial tree valuation approach proposed by Cox, Ross, and Rubinstein (CRR, 1979), which also enjoys great popularity, preserves the notation and definition of σ to depict future price movements. When we use the Black-Scholes model or the CRR model, we have to use historical data to estimate the value of σ and substitute it into the valuation model to find an option price.

If a project manager knows the true distribution process governing future price movements, then there is less uncertainty. However, in most cases, during real options analysis, the manager relies on imperfect knowledge of the model and parameters to make decisions. For example, to highlight the importance of uncertainty in real options analysis, Bräutigam, Esche, and Mehler-Bicher (2003) particularized nearly every kind of uncertainty, such as project uncertainty, uncertainty about intangibles, financial uncertainty, product uncertainty, market uncertainty, region-specific uncertainty, and unknown uncertainties, which shows how much uncertainty can be present.

One simple way of dealing with uncertainty is to raise the original volatility slightly, which has been estimated to show that future price movement could be more volatile due to uncertainty arising from personal judgment. The idea behind this rule of thumb is that risk and uncertainty share reasonably similar meanings, and they are used interchangeably in many cases.

Although the terms *risk* and *uncertainty* are often used interchangeably in the literature in links to volatility, they have different meanings. Risk, an objective term, represents a probability distribution of potential outcomes. Risk aversion is an attitude that penalizes the expected return of a risky investment. Consider the following situation. An investment pays off either \$1 or \$0 with equal probability. Another pays off \$0.5 with certainty. Although the two investments offer the same expected return, a risk-averse investor will reject the investment in the former.

Uncertainty is a subjective term and represents a lack of confidence in probability estimates. If people are uncertain, they worry about a worst-case scenario. The following example is adapted from Ellsberg's (1961) experiment. There are two urns, each of which contains several red and green balls. The total number of the balls in each urn is the same. The balls in one urn are half red and half green. However, the ratio in the other urn is unknown to the player. The rule is that \$1 can be obtained when a red ball is drawn out. Although the two urns offer the same expected return, an uncertainty-averse investor will not draw a ball from the latter because the investor tends to think pessimistically, and so expects that the urn offers lower odds.

Recent studies have found that risk and uncertainty have different influences on decision makers. Alessandri's (2003) empirical findings show that managers treat risk and uncertainty separately, and use different decision rules to respond to each. Alessandri et al. (2004) emphasize the importance of identifying the risks and uncertainties inherent in the decision-making process. They suggested that qualitative approaches should be used instead of quantitative ones to evaluate capital projects with higher uncertainty.

Nishimura and Ozaki (2006) showed that uncertainty and risk have different effects on the value of an investment opportunity. Miao and Wang (2011) emphasized the effect of distinguishing risk from uncertainty for an option exercise or in the optimal exit problem. Trojanowska and Kort (2010) focused on how uncertainty affects investment timing. They claimed that uncertainty aversion causes a firm to consider a project to have higher risk and to overprice the risk when there is uncertainty. They predicted that the probability of investment monotonically decreases according to the level of uncertainty in long-term projects. By using the Choquet-Brownian motions to describe uncertainty, Roubaud, Lapied, and Kast (2010) suggested that a decision maker's pro and con attitudes toward ambiguity might influence the decision to exercise options to invest.

Owing to the above arguments, it is possible that such a rule of thumb offers room for the manager to manipulate the parameter, σ, in the Black-Scholes pricing formula to exaggerate the value of an investment opportunity. Should we not consider risk purely in the uncertainty-absent Black-Scholes world if we are unanimous regarding the objectivity of the Black-Scholes model?

The purpose of this paper is to separate uncertainty from risk in the commonly used Black-Scholes pricing formula and to examine how uncertainty affects option prices. We show that the value of real options obtained by the Black-Scholes pricing formula may not be real if the concepts of risk and uncertainty are vague. In addition, we want to show that uncertainty can affect the timing of investments.

First, we describe the basic framework. The approach is most closely associated with the robust control approach to uncertainty, which depicts model uncertainty through a set of priors and introduces a penalty function to a general utility function in order to capture investor uncertainty (for example, Anderson, Hansen, and Sargent, 2000; Kogan and Wang, 2002; Boyle, Uppal, and Wang, 2003; Uppal and Wang, 2003; Maenhout, 2004). Investors under high uncertainty are concerned about a worst-case scenario. Consequently, the investor will choose alternative models that are distant from the reference model. Hence, the robust control approach assigns a lower penalty to distant perturbations. If the level of uncertainty is low, the investor will choose alternative models that are similar to the reference model. Hence, the robust control approach assigns a higher penalty to more distant perturbations. The penalty is inversely relative to the investor's uncertainty.

The organization of the remainder of the paper is as follows. The model and theoretical results are described in Section 2, a numerical example is given in Section 3, and the conclusions are presented in the final section.

2. Model

2.1 Basic concepts

We have already shown that risk and uncertainty have different influences on decision makers and on the value of an investment opportunity. However, according to the definition of the original Black-Scholes model, only risk is taken into consideration when pricing contingent claims. By using the robust control technique, we added an extra parameter to depict a decision maker's attitude of uncertainty and derived an adjusted Black-Scholes pricing formula. By denoting risk and uncertainty as two parameters in the pricing formula, we can avoid the problem arising from using these two terms interchangeably, and can assess their influences separately. As a result, the evaluation of real options, as well as the optimal investment timing, may be more reliable.

2.2 Theoretical model

Throughout the paper, we denote the risk-free interest rate by a constant, r. We assume that the gross project value, S, follows the geometric Brownian motion with expected return, v, and volatility, σ:

$$\frac{dS}{S} = vdt + \sigma dB. \tag{1}$$

Taking the non-traded property of the underlying asset, we assume that the below-equilibrium return shortfall is q (for example, McDonald and Siegel, 1986). Hence, the dynamic process of the gross project value is adjusted to:

$$\frac{dS}{S} = (v - q)dt + \sigma dB. \tag{2}$$

In the literature, q functions as a dividend yield. The expected return of S with dividend is v.

2.2.1 Decision process for the manager under no uncertainty

Suppose that the manager knows the true probability law of the project return, given the probability measure Q. The total wealth dynamic process of the manager is:

$$\frac{dW}{W} = (r + \omega(v - r) - \frac{C}{W})dt + \omega\sigma dB, \tag{3}$$

where ω represents the proportion of wealth allocated to the investment project.

The expected utility of the manager in continuous time is:

$$J_t = E^Q\left[\int_t^T e^{-\rho s}u(C_s)ds + e^{-\rho(T-t)}J_T\right],$$

where $u(C_s)$ takes the form of the power utility, $u(C_s) = \frac{C_s^{1-\gamma}}{1-\gamma}$, where γ is the risk aversion coefficient.

Suppose that the manager has to choose optimal consumption and investment weights to maximize utility:

$$\max_{C,\omega}\left\{u(c) - \rho J_t + WJ_w\left[r + \omega(v - r) - \frac{C}{W}\right] + \frac{1}{2}W^2 J_{ww}\omega^2\sigma^2\right\} = 0. \tag{4}$$

By taking derivatives of (4) with respect to C and ω, we have the optimal consumption:

$$u_c = J_W, \tag{5}$$

and investment choice:

$$\omega = -\frac{J_w}{WJ_{ww}\sigma^2}(v-r), \tag{6}$$

implied by the first-order conditions.

In order to derive the exact formula for the optimal investment in the project, we have to specify the form of the value function. As it is assumed that the manager has power utility $u(C_s) = \frac{C_s^{1-\gamma}}{1-\gamma}$, the value function takes the form:

$$J(W) = \kappa\frac{W^{1-\gamma}}{1-\gamma}, \tag{7}$$

where κ is a constant depending on the parameter of the environment.

Substituting (7) into (6), the optimal portfolio weight in the investment opportunity for the manager is given by $\omega = \frac{(v-r)}{\gamma\sigma^2}$.

2.2.2 Decision process for the manager under uncertainty

When the manager is under uncertainty and is not confident about the probability estimates, we have to apply the robust control technique to deal with this issue. Moreover, when the manager has to evaluate the investment opportunity using real options analysis under uncertainty, the misspecification problem should be taken into account.

Suppose that the manager considers the alternative model, Q^{ξ}, rather than the reference model, Q. Applying Uppal and Wang's (2003) method, the optimization problem should be adjusted to:

$$\sup_{C,\omega}\inf_{\delta}\left\{\begin{array}{l} u(C)-\rho J_t+WJ_w\left[r+\omega(v-r)-\dfrac{C}{W}\right]+\dfrac{1}{2}W^2J_{ww}\omega^2\sigma^2 \\ +J_wW\omega\sigma^2\delta+\dfrac{\Theta(J)}{2}\delta^2\varphi\sigma^2 \end{array}\right\}=0, \qquad (8)$$

where the second line of the brace is additional to (4), and the former term of the second line reflects the adjusted drift term resulting from the change of measure from Q to Q^{ξ}. The term δ reflects the difference between the adjusted drift term and the original one. The latter term of the second line is associated with the penalty function, where $\Theta(J)$ converts the penalty to units of the utility. In addition, we use φ, the penalty parameter, to measure the manager's subjective confidence about the reference model.

As mentioned above, the manager under uncertainty worries about a worst-case scenario. The manager will then choose alternative models that are more distant from the reference model. As a result, the robust control approach assigns a lower penalty to more distant perturbations. If the level of uncertainty is low, the investor will choose alternative models that are much the same as the reference model. As a result, the robust control approach assigns a higher penalty to more distant perturbations. That is, the lower is the value of φ, the higher is the level of the manager's uncertainty, and the penalty is inversely related to the investor's uncertainty. Hence, the reciprocal of φ can be treated as the level of the manager's uncertainty. We have used a similar framework with an extension of inserting multi-dimensional Lagrange multipliers to discuss issues on evaluation of employee stock options in So (2009), where the techniques to solve this kind of optimization problem are given in greater detail.

The optimal consumption can be obtained by solving the first-order condition $u_c = J_W$. After taking derivatives of (8) with respect to ω, we obtain:

$$\omega=-\frac{J_w}{WJ_{ww}\sigma^2}(v-r)-\frac{J_w\delta}{WJ_{ww}}. \qquad (9)$$

Differentiating (8) with respect to δ, we obtain:

$$\delta = -\frac{J_w W \omega}{\Theta(J)\varphi}. \tag{10}$$

Substituting (10) into (9) leads to:

$$\omega = -\frac{J_w}{W J_{ww} \sigma^2}(v-r) + \frac{(J_w)^2 \omega}{J_{ww}\Theta(J)\varphi}. \tag{11}$$

As $J(W) = \kappa \frac{W^{1-\gamma}}{1-\gamma}$, we obtain $J_w = \kappa W^{-\gamma}$. Following Maenhout (2004) and Uppal and Wang (2003), we assume that $\Theta(J) = \frac{1-\gamma}{\gamma} J(W) = \frac{W^{1-\gamma}}{\gamma}\kappa$. The optimal investment opportunity for the uncertainty-averse manager is:

$$\omega^* = \frac{1}{(1+\frac{1}{\varphi})} \frac{(v-r)}{\gamma\sigma^2} = \frac{\hat{\chi}}{\gamma\sigma^2}, \tag{12}$$

where $\hat{\chi} \equiv \left(\frac{v-r}{1+\frac{1}{\varphi}} \right)$

The term $\hat{\chi}$ in (12) could be explained as the manager's subjective estimation of the risk premium for the project. The manager's uncertainty leads to a downgrade in the risk premium for the project. Hence, the optimal investment opportunity would be lower than if the manager knew the data generating process of the project value exactly.

In the following steps, we compute the manager's marginal utility function, which will serve as the stochastic discount factor for the subjective evaluation of the real option to replace the commonly-used objective stochastic discount factor. By using Ito's Lemma, we obtain:

$$
\begin{aligned}
dJ_w &= \frac{\partial J_w}{\partial t}dt + \frac{\partial J_w}{\partial W}dW + \frac{1}{2}\frac{\partial^2 J_w}{\partial W^2}(dW)^2 \\
&= \kappa(-\gamma)W^{-\gamma-1}\left\{W\left[r+\omega^*(v-r)+(\omega^*)^2\sigma^2\delta^* - \frac{C^*}{W}\right]dt + W\omega^*\sigma dB^{\xi}\right\} \\
&\quad + \frac{1}{2}\kappa(-\gamma)(-1-\gamma)W^{-\gamma-2}W^2(\omega^*)^2\sigma^2 dt \\
&= (-\gamma)\kappa W^{-\gamma}\left\{\left[r+\omega^*(v-r)+(\omega^*)^2\sigma^2\delta^* - \frac{C^*}{W}\right]dt + \omega^*\sigma dB^{\xi}\right\} \\
&\quad + \frac{1}{2}\kappa(-\gamma)(-1-\gamma)W^{-\gamma}(\omega^*)^2\sigma^2 dt \\
&= \kappa W^{-\gamma}\left\{(-\gamma)\left[r+\omega^*(v-r)+(\omega^*)^2\sigma^2\delta^* - \frac{C^*}{W}\right] + \frac{1}{2}\gamma(1+\gamma)(\omega^*)^2\sigma^2\right\}dt \\
&\quad + (-\gamma)\kappa W^{-\gamma}\omega^*\sigma dB^{\xi},
\end{aligned}
\tag{13}
$$

where dB^{ξ} is a Brownian motion under the new measure Q^{ξ}.

By using $J(W) = \kappa\frac{W^{1-\gamma}}{1-\gamma}$, we can show that

$$
J_w = \kappa W^{-\gamma}, \tag{14}
$$

and substitution of (14) into (13) leads to:

$$
\begin{aligned}
dJ_w &= J_w\left\{(-\gamma)\left[r+\omega^*(v-r)+\omega^*\sigma^2\delta^* - \frac{C^*}{W}\right] + \frac{1}{2}\gamma(1+\gamma)(\omega^*)^2\sigma^2\right\}dt \\
&\quad + J_w(-\gamma)\omega^*\sigma dB^{\xi}.
\end{aligned}
\tag{15}
$$

Using (15), the dynamic process of the manager's marginal utility function is given by:

$$
\begin{aligned}
\frac{dJ_w}{J_w} &= \left\{(-\gamma)\left[r+\omega^*(v-r)+\omega^*\sigma^2\delta^* - \frac{C^*}{W}\right] + \frac{1}{2}\gamma(1+\gamma)(\omega^*)^2\sigma^2\right\}dt \\
&\quad - \gamma\omega^*\sigma dB^{\xi}.
\end{aligned}
\tag{16}
$$

As $u_c = J_W$ is implied by the first-order condition, we have $u_c = C^{-\gamma} = J_w = \kappa W^{-\gamma}$, so that:

$$\frac{C^*}{W} = \kappa^{\frac{-1}{\gamma}}. \tag{17}$$

Substituting (17) into (16) leads to:

$$\frac{dJ_w}{J_w} = \left\{ (-\gamma)\left[r + \omega^*(v-r) + \omega^*\sigma^2\delta^* + \gamma\kappa^{\frac{-1}{\gamma}} \right] + \frac{1}{2}\gamma(1+\gamma)(\omega^*)^2\sigma^2 \right\} dt \tag{18}$$
$$-\gamma\omega^*\sigma dB^{\xi}$$

We need to arrange the terms in the brace of (18). Using $J(W) = \kappa\frac{W^{1-\gamma}}{1-\gamma}$ and substituting the optimal values into (8), we have the following relationship:

$$u(C^*) + \kappa W^{-\gamma}C^* + \kappa W^{1-\gamma}\left[r + \omega^*(v-r)\right] + \frac{1}{2}(-\gamma)\kappa W^{1-\gamma}(\omega^*)^2\sigma^2$$
$$+\kappa W^{1-\gamma}\omega^*\sigma^2\delta^* + \frac{\Theta(J)}{2}(\delta^*)^2\varphi\sigma^2 = 0. \tag{19}$$

Combined with (17), (19) becomes

$$-\gamma\kappa^{-\frac{1}{\gamma}} + (1-\gamma)\left[r + \omega^*(v-r)\right] + \frac{1}{2}(1-\gamma)(-\gamma)(\omega^*)^2\sigma^2$$
$$+(1-\gamma)\omega^*\sigma^2\delta^* + \frac{\Theta(J)}{2}(1-\gamma)\frac{(\delta^*)^2\varphi\sigma^2}{W^{1-\gamma}\kappa} = 0. \tag{20}$$

From (20), we obtain a simplified form of (18):

$$\frac{dJ_w}{J_w} = \begin{Bmatrix} -r - \left[\omega^*(v-r)\right] + \gamma(\omega^*)^2\sigma^2 \\ -\omega^*\sigma^2\delta^* - \dfrac{\Theta(J)}{2}(1-\gamma)\dfrac{(\delta^*)^2}{W^{1-\gamma}\kappa} \end{Bmatrix} dt - \gamma\omega^*\sigma dB.^{\xi} \tag{21}$$

Substituting $\Theta(J)=\frac{1-\gamma}{\gamma}J(W)=\frac{W^{1-\gamma}}{\gamma}\kappa$ and $\delta=-\frac{J_w W\omega}{\Theta(J)\varphi}=\frac{-\gamma\omega}{\varphi}$ into (21), we have:

$$\frac{dJ_w}{J_w}=\left\{-r-\omega^*(v-r)+\frac{\gamma(1+2\varphi+\gamma)}{2\varphi}(\omega^*)^2\sigma^2\right\}dt-\gamma\omega^*\sigma dB^{\xi}. \tag{22}$$

Finally, when we substitute (12) into (22), we can derive the dynamic process of the manager's subjective stochastic discount factor under uncertainty:

$$\frac{dJ_w}{J_w}=\left\{-r-\frac{(v-r)^2\varphi(1-\gamma)}{2(1+\varphi)^2\gamma\sigma^2}\right\}dt-\frac{(v-r)}{(1+\frac{1}{\varphi})\sigma}dB^{\xi}\equiv-\hat{r}dt-\frac{(v-r)}{(1+\frac{1}{\varphi})\sigma}dB^{\xi}, \tag{23}$$

where $\hat{r}\equiv r+\frac{(v-r)^2\varphi(1-\gamma)}{2(1+\varphi)^2\gamma\sigma^2}$.

In the extreme case where φ approaches infinity, our result would be $\hat{r}=r$, which means that when the manager knows the true probability law of the project, the subjective risk-free interest rate is exactly that of the real world. However, as the manager here is risk-averse with $\gamma>1$,

$$\hat{r}=r+\frac{(v-r)^2\varphi(1-\gamma)}{2(1+\varphi)^2\gamma\sigma^2}<r,$$

in general, implying that the manager's uncertainty about the true probability law of project returns may lead to the consideration that one dollar invested today has a lower future value than its market value.

We explain the intuition behind this outcome. As the manager's uncertainty would lead to a downgrade in the risk premium for the investment opportunity, the investment in the project shrinks. In other words, the manager over-invests in the risk-free asset, providing a worse payoff than the investment opportunity. Owing to this suboptimal allocation in the mind of the manager, one dollar invested today provides a lower future value than its market value. This is the reason for observing an inverse relationship between the subjective risk-free interest rate, $\hat{r}$, and the subjective measure of confidence, φ.

2.2.3 Evaluation of real options for the manager under uncertainty

We derive the subjective value of the real option for the manager under uncertainty. Let $R(S,t)$ be the subjective price of the real option with the non-traded asset as its underlying asset. Applying the martingale approach:

$$
\begin{aligned}
& E^{Q^{\xi}}\left[d(J_w R(S,t))\right]=0 \\
& \Rightarrow E^{Q^{\xi}}\left[J_w dR(S,t)+R(S,t)dJ_w+dJ_w dR(S,t)\right]=0 \\
& \Rightarrow J_w E^{Q^{\xi}}\left[R_s dS+R_t dt+\frac{1}{2}R_{ss}(dS)^2+R\frac{dJ_w}{J_w}+\frac{dJ_w}{J_w}R_s dS\right]=0. \qquad (24)
\end{aligned}
$$

Using (23), (24) can be rearranged as:

$$
J_w\left[R_s S(v-q)+R_t+\frac{1}{2}R_{ss}S^2\sigma^2-R\hat{r}-R_s S\frac{(v-r)}{1+\frac{1}{\varphi}}\right]dt=0. \qquad (25)
$$

As a result, the partial differential equation is given by:

$$
\begin{aligned}
& R_s S(v-q)+R_t+\frac{1}{2}R_{ss}S^2\sigma^2-R\hat{r}-R_s S\frac{(v-r)}{1+\frac{1}{\varphi}}=0 \\
& \Rightarrow \frac{1}{2}R_{ss}S^2\sigma^2+SR_s(\hat{r}-\hat{q})-\hat{r}R+R_t=0, \qquad (26)
\end{aligned}
$$

where

$$
\hat{r}=r+\frac{(v-r)^2\varphi(1-\gamma)}{2(1+\varphi)^2\gamma\sigma^2}
$$

and

$$\hat{q} = q + \left(\hat{r} - \frac{r}{1+\frac{1}{\varphi}} \right) - \frac{v}{1+\varphi}.$$

When φ approaches infinity, $\hat{r} = r$, $\hat{q}$ degenerates to q, and our result collapses to the original Black-Scholes partial differential equation.

As we can consider equation (26) to be the typical Black-Scholes partial differential equation with the subjective rate of return, $\hat{r}$, and the subjective below-equilibrium return shortfall, $\hat{q}$, the subjective value of the real option on the non-traded asset can be obtained immediately as:

$$\hat{C} = Se^{-\hat{q}(T-t)} N(d_1) - Ie^{-\hat{r}(T-t)} N(d_2), \tag{27}$$

where I denotes that investment cost,

$$\hat{r} = r + \frac{(v-r)^2 \varphi(1-\gamma)}{2(1+\varphi)^2 \gamma\sigma^2}, \quad \hat{q} = q + \left(\hat{r} - \frac{r}{1+\frac{1}{\varphi}} \right) - \frac{v}{1+\varphi},$$

$$d_1 = \frac{\ln(\frac{S}{I}) + (\hat{r} - \hat{q} + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \text{ and } d_2 = d_1 - \sigma\sqrt{T-t}.$$

When φ approaches infinity, $\hat{r}$ equals r and $\hat{q}$ degenerates to q, which is the result of the Black-Scholes pricing formula. However, when φ is more distant from infinite, we have $\hat{r} < r$ and $\hat{q} < q$. As the manager's uncertainty decreases the subjective below-equilibrium return shortfall more than it decreases the subjective interest rate, the manager boosts the subjective value of the real option, compared with the case where the manager is aware of the true probability law of project returns. It can be argued that both uncertainty and risk have a positive effect on the value of real options.

We now consider the optimal timing to exercise the option. As equation (26) can be treated as the typical Black-Scholes partial differential equation with the subjective required rate of return, $\hat{r}$, and subjective below-equilibrium return shortfall, $\hat{q}$, we know the optimal timing to exercise the option by applying McDonald and Siegel's (1986) results. The project value must be as large as S^* before the manager decides to invest:

$$S^* = \frac{\beta_1}{\beta_1 - 1} I > I, \tag{28}$$

where I denotes that investment cost,

$$\beta_1 = \left(\tfrac{1}{2} - (\hat{r} - \hat{q})/\sigma^2\right) + \sqrt{\left((\hat{r} - \hat{q})/\sigma^2 - \tfrac{1}{2}\right)^2 + 2\hat{r}/\sigma^2} \ ,$$

$$\hat{r} = r + \frac{(v - r)^2 \varphi(1-\gamma)}{2(1+\varphi)^2 \gamma\sigma^2}, \quad \hat{q} = q + \left(\hat{r} - \frac{r}{1 + \dfrac{1}{\varphi}}\right) - \frac{v}{1+\varphi}.$$

2.3 Main results

Our main results are summarized as follows:

<u>Result 1</u>

We introduced an extra parameter, φ, in the model to depict the manager's attitude of uncertainty. The lower is the value of φ, the higher is the level of the manager's uncertainty. The reciprocal of φ can be treated as the level of the manager's uncertainty.

The manager under uncertainty is not confident about the probability estimates or the reference model. As a result, the manager worries about the worst-case scenario. The manager will then choose alternative models that are more distant from the reference model. The robust control approach assigns a ***lower*** penalty to more distant perturbations. If the level of the investor's uncertainty is low to zero, the investor will choose alternative models that are similar the reference model. As a result, the robust control approach assigns a ***higher*** penalty to more distant perturbations. That is, the lower is the value of φ, the higher is the level of the manager's uncertainty.

Result 2

For the manager with power utility and with the attitude of uncertainty described in Result 1, the subjective value of the real option on the non-traded asset is

$$\hat{C} = Se^{-\hat{q}(T-t)}N(d_1) - Ie^{-\hat{r}(T-t)}N(d_2),$$

where I ***denotes that investment cost,***

$$\hat{r} = r + \frac{(v-r)^2\varphi(1-\gamma)}{2(1+\varphi)^2\gamma\sigma^2}, \quad \hat{q} = q + \left(\hat{r} - \frac{r}{1+\frac{1}{\varphi}} \right) - \frac{v}{1+\varphi},$$

$$d_1 = \frac{\ln(\frac{S}{I}) + (\hat{r} - \hat{q} + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \quad ,$$

and the reciprocal of φ ***is the level of the manager's uncertainty.***

Result 3

The project value must be as large as S^* ***before the manager with power utility and with the attitude of uncertainty described in Result 1 decides to invest:***

$$S^* = \frac{\beta_1}{\beta_1 - 1} I > I,$$

where I ***denotes the investment cost,***

$$\beta_1 = \left(\tfrac{1}{2} - (\hat{r} - \hat{q})/\sigma^2\right) + \sqrt{\left((\hat{r} - \hat{q})/\sigma^2 - \tfrac{1}{2}\right)^2 + 2\hat{r}/\sigma^2},$$

$$\hat{r} = r + \frac{(v-r)^2\varphi(1-\gamma)}{2(1+\varphi)^2\gamma\sigma^2}, \quad \hat{q} = q + \left(\hat{r} - \frac{r}{1+\frac{1}{\varphi}} \right) - \frac{v}{1+\varphi} \quad ,$$

and the reciprocal of φ ***is the level of the manager's uncertainty.***

3. Numerical Example

Table 1 summarizes the values of all of the parameters used in calibration, namely the risk aversion coefficient, γ ; the rate of return of the investment opportunity, v; the riskless interest rate, r; the dividend yield (shortfall), q; the time to maturity, $T-t$; the volatility of the investment opportunity, σ; and the subjective measure of confidence about the probability law of project returns, φ.

Without further information, we assume that $r = q = 0.04$ (for example, Dixit and Pindyck, 1994). In order to illustrate the effects of risk, we set volatility to lie in the range (0.2, 0.5), that is, 20% to 50%. It is difficult to assign a value to the subjective parameter, φ. However, Maenhout (2004) made a recommendation to choose a suitable value of φ, namely φ should be chosen to make the difference between the objective and subjective risk premium for the project less than 3% for a 95% confidence interval. We eliminate unreasonable values of φ less than 6, and let the values of φ increase from 6 to infinity to examine how the subjective measure of confidence about the probability law of the project return, or uncertainty, affects the valuation of the real option.

Table 2 shows the values of real options after separating the effects of risk and uncertainty. The results of neglect of uncertainty are shown in the column $\varphi = \infty$. We find that the manager's uncertainty, like risk, raises the subjective value of the real options. Based on these findings, if we mistake uncertainty for risk and overestimate the value of the parameter σ, we would conclude wrongly that the value of the real option is high.

What is the more reliable value of the real option? We now provide a simple method to filter the risk of the project and to acquire a more reliable value of real options without the influence of uncertainty. As the Black-Scholes option pricing formula does not take account of uncertainty, under the guise of risk, uncertainty is often hidden in the Black-Scholes pricing formula. Suppose that the estimated volatility is 0.4, which could contain information about both risk and uncertainty. Substituting this value into the pricing formula, we find that the value of the real option is 31.75. However, the manager may not be aware of the situation and may be operating under high uncertainty, say $\varphi = 6$.

We interpolate between 25.10 and 32.34, as presented in Table 2, to obtain an implied volatility of 0.292. It should be noted that approximately 25% of the estimated volatility may come from the manager's uncertain about the future environment (i.e., 1-0.292/0.4). After filtering the "true" risk through our model, the value of the real option is actually only 23.84. This more reliable value is indeed much less than 31.75, which was obtained by the Black-Scholes formula. That is, the value of an investment opportunity may be exaggerated by 25% (i.e., 1-23.84/31.75) in this case.

For external supervision, we suggest the following steps be taken to determine a more reliable value of real options:

1. Determine the reported σ of the investment opportunity, which could contain information about both risk and uncertainty.

2. Substitute the value of σ and the estimated S_0, I, r, q, and $T-t$ into the original Black-Scholes formula to obtain the value of the real option, C.

3. Assign the manager's level of uncertainty, φ. When the manager faces higher uncertainty, this value should be very low.

4. Filter the true risk. Consider the adjusted Black-Scholes pricing formula to obtain the implied σ which, upon substitution with S_0, I, r, q, $T-t$, γ, and v into the pricing formula, gives C.

5. Substitute the implied σ and the estimated S_0, I, r, q, and $T-t$ into the original Black-Scholes formula to obtain a more reliable value of the real option.

Table 3 shows the critical values to invest as the subjective measure of confidence about the probability law of asset returns (φ) and the volatility of the project (σ) vary. The results of neglecting uncertainty are shown in column $\varphi = \infty$. We find that the manager's uncertainty, like risk, raises the critical value to invest. An increase in uncertainty, or a decrease in the manager's confidence about the probability law, will increase S^*, and hence postpone the investment project.

4. Conclusions

In this paper, we established a framework that separated risk and uncertainty in order to evaluate real options. In addition to risk, uncertainty also raised the value of real options. By using risk and uncertainty interchangeably, we would overestimate the value of real options. Therefore, we cannot trust the value of real options unless we clarify and identify risk and uncertainty. The proposed theoretical model responded well to Alessandri's (2003) empirical findings. Although the Black-Scholes pricing formula is user friendly, it nevertheless has drawbacks when applied to the evaluation of capital projects with higher uncertainty. To our knowledge, any similar quantitative methods have not hitherto been mentioned in terms of isolating uncertainty from risk in real options analysis that we consider here.

There are at least two practical benefits of our approach. For internal management, by parameterizing uncertainty and associated attitudes, our approach provides a more credible valuation of real options than a rule of thumb. For external supervisors, our approach helps to detect whether the valuation of real options is exaggerated.

The separation of ownership and management is a feature of modern corporate governance. As the owners and managers of the firm belong to different parties, the conflicts of interest, called agency problems, may arise. Managers could take some plans to maximize their own profit by sacrificing the owners' interests. We caution external supervision, as a solution to mitigate agency problems, about the results from real options analysis. If risk and uncertainty are unidentifiable, there would be opportunities for the manager to manipulate the parameter, σ, in the Black-Scholes pricing formula to exaggerate the value of an investment opportunity. Moreover, it is possible that an investment opportunity may be postponed inappropriately.

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Table 1

Parameter Values Used in Calibration

Parameters	S_0	I	γ	v	r	q	$T-t$	σ	φ
Values	100	100	5	0.13	0.04	0.04	10	0.2~0.5	$6\sim\infty$

Notes: The parameters are the risk aversion coefficient, rate of return of the investment opportunity, riskless interest rate, dividend yield (shortfall), time to maturity, volatility of the investment opportunity, and subjective measure of confidence about the probability law of asset returns, respectively. We let the values of σ and φ vary in the specified ranges to see their influence on the evaluation of real options.

Table 2

Subjective Values of Real Options

	φ=6	φ=8	φ=10	φ=12	φ=14	φ=16	φ=18	φ=20	φ=100	$\varphi=\infty$
σ=0.2	25.10	23.02	21.75	20.90	20.29	19.84	19.48	19.20	17.15	16.69
σ=0.3	32.34	30.46	29.30	28.51	27.94	27.52	27.18	26.91	24.95	24.50
σ=0.4	39.60	37.73	36.58	35.79	35.23	34.80	34.46	34.19	32.21	31.75
σ=0.5	46.32	44.43	43.25	42.46	41.88	41.44	41.09	40.82	38.78	38.31

Notes: The table displays the subjective values of real options as the subjective measure of confidence about the probability law of asset returns (φ) and the volatility of the project (σ) vary. Other parameter values are given in Table 1.

Table 3

Critical Values of Project Value

	φ=6	φ=8	φ=10	φ=12	φ=14	φ=16	φ=18	φ=20	φ=100	$\varphi=\infty$
σ=0.2	339.28	285.40	261.47	247.99	239.34	233.33	228.91	225.52	204.48	200.44
σ=0.3	416.94	370.82	347.44	333.32	323.87	317.11	312.02	308.07	282.15	276.88
σ=0.4	548.78	493.95	465.20	447.50	435.51	426.85	420.30	415.18	381.05	373.98
σ=0.5	718.51	649.29	612.48	589.64	574.09	562.81	554.27	547.57	502.60	493.21

Notes: The table gives the critical values of the project value as the subjective measure of confidence about the probability law of asset returns (φ) and the volatility of the project (σ) vary. The manager has to defer the investment project until the project value is higher than the critical value. Other parameter values are given in Table 1.