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Review

Scope of Raychaudhuri equation in cosmological gravitational focusing and space-time singularities

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Indian scientist Amal Kumar Raychaudhuri established 'Raychaudhuri equation' in 1955 to describe gravitational focusing properties in cosmology. This equation is extensively used in general relativity, quantum field theory, string theory and the theory of relativistic membranes. This paper investigates the issue of the final fate of a gravitationally collapsing massive star and the associated cosmic censorship problems and space-time singularities therein with the help of Raychaudhuri equation. It is conjecture that the universe is emerged from a big bang singularity where all the known laws of physics break down. On the other hand, when the star is heavier than a few solar masses, it could undergo an endless gravitational collapse without achieving any equilibrium state. This happens when the star has exhausted its internal nuclear fuel which provides the outwards pressure against the inwards pulling gravitational forces. Then for a wide range of initial data, a space-time singularity must develop. It is conjecture that such a singularity of gravitational collapse from a regular initial surface must always be hidden behind the event horizon of gravity; this is called the cosmic censorship hypothesis. Thus cosmic censorship implies that the final outcome of gravitational collapse of a massive star must necessarily be a black hole which covers the resulting space-time singularity. So, causal message from the singularity cannot reach the external observer at infinity. Raychaudhuri equation plays a pioneer role in cosmology to describe the gravitational focusing and space-time singularities.

Key words: Cosmic censorship, Einstein equation, gravitational focusing, Raychaudhuri equation, singularities.

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INTRODUCTION

A.K. Raychaudhuri was born on September 14, 1923 in Barisal, in what is now Bangladesh. He studied at Presidency College, Kolkata, and obtained his B. Sc. and M. Sc. degrees in 1942 and 1944, respectively (Ehlers, 2007). During the 1950s, A.K. Raychaudhuri began to examine extensively the aspects of general relativity. One of his early works during this period involved the construction of a non-static solution of the Einstein equations for a cluster of radially moving particles in an otherwise empty space (Raychaudhuri, 1953).

Raychaudhuri (1955) developed an equation which contributes to find the gravitational focusing in the space-time but it was somewhat different from the way it is presented in standard textbooks and research articles at present. It describes the rate of change of the volume expansion when one moves along the timelike geodesic curves in the congruence. The Raychaudhuri equation has been discussed and analyzed in the last 60 years in a variety of fields of gravitational physics, such as, in general relativity, quantum field theory, string theory and the theory of relativistic membranes.

In this paper we describe how the matter fields with positive energy density affect the causality relations in a space-time and cause focusing in the families of non-spacelike trajectories. Here the main phenomenon is that matter focuses the non-spacelike geodesics of the space-time into pairs of focal points or conjugate points due to gravitational forces. There are null hypesurfaces such as the boundary of the future $I^{+}(p)$ for a point *p* such that no two points of such a hypersurface could be joined by a timelike curve. Thus, the null geodesic generators of such surfaces cannot contain conjugate points. Hawking and Penrose (1970) explained that singularities arise when a black hole is created due to gravitational collapse of massive bodies. A space-time singularity which cannot be observed by external observers is called a black hole. Poisson (2004) describes that when the black hole is formulated due to gravitational collapse then space-time singularities must occur. A space-time singularity is called naked if it is observable to local or distant observers. Joshi et al. (2012) discussed the genericity and stability aspects for naked singularities and black holes that arise as the final states for a complete gravitational collapse of a spherical massive matter cloud. Joshi (2009) described the naked singularities in some details. In this study we will not discuss naked singularities in space-time.

Sahu et al. (2013) studied the time delay between successive relativistic images in gravitational lensing as a possible discriminator between various collapse end states and hence as a probe of cosmic censorship. They considered both black hole and naked singularity space-times admitting photon spheres where infinite number of relativistic Einstein Rings can be formed at almost same radius and naked singularity space-times without photon sphere where multiple relativistic Einstein Rings can form almost up to the center of the lens.

The space-time singularity theorems and the related theoretical advances towards understanding the structure of space-time was first established by Hawking, Penrose and Geroch, who showed that under certain very general and physically reasonable conditions such as the positivity of energy, occurrence of trapped surface and suitable causality condition, the space-time singularities must occur as an inevitable feature, as far as a wide range of gravitation theories describing the gravitational forces are concerned (Hawking and Ellis, 1973).

The universe is not simply a random collection of irregular distributed matter, but it is a single entity, all parts of which are connected. When considering the large scale structure of the universe, the basic constituents are galaxies, which are congregation of more than 10^{10} stars bound together by their mutual gravitational attraction. The universe is the totality of galaxies which are causally connected (Joshi, 2008). A lot of black hole candidates (compact, dark, heavy objects) have been discovered observationally and most likely they are indeed black holes (Visser et al., 2009).

O'Connor and Ott (2011) analyzed the results of a systematic study of failing core-collapse supernovae and the formation of stellar-mass black holes. They used open-source general-relativistic 1.5D code GR1D equipped with a three-species neutrino leakage/heating scheme and over 100 pre-supernova models. They studied the effects of the choice of nuclear equation of state, zero-age main sequence mass and metallicity, rotation, and mass loss prescription on black holes formation. Sharif and Ahmad (2012) studied the effect of a positive cosmological constant on spherically symmetric dust collapse. They considered the Friedmann, Robertson-Walker (FRW) metric in the interior region whereas Schwarzschild-de Sitter in the exterior region. They also discussed the apparent horizons and their physical significance.

The existence of space-time singularities follows in the form of future or past incomplete non-spacelike geodesics in the space-time. Such a singularity would arise either in the cosmological scenarios, where it provides the origin of the universe or as the end state of the gravitational collapse of a massive star which has exhausted its nuclear fuel providing the pressure gradient against the inwards pull of gravity. The singularities forming in general gravitational collapse should always be covered by the event horizon of the gravity, and remains invisible to any external observer is called the cosmic censorship hypothesis.

Recently the kinematic quantities (expansion, shear and rotation), as well as the Raychaudhuri equations, have appeared, quite unexpectedly, in the context of quantum field theory. In the last decade, Borgman and Ford investigated gravitational effects of quantum stress tensor fluctuations (Borgman, 2004). They explained that these fluctuations produce fluctuations in the focusing of a bundle of geodesics. An explicit calculation using the Raychaudhuri equation, treated as a Langevin equation was performed to estimate angular blurring and luminosity fluctuations of the images of distant sources (Kar and SenGupta, 2007).

The holographic principle has recently played a crucial role in our understanding of quantum aspects of gravity. The principle states that the information of gravity degrees of freedom in a *D*-dimensional volume is encoded in a quantum field theory defined on the $(D-1)$-dimensional boundary of this volume. The renormalization group (RG) flow equation for the β-function of a 4-dimensional quantum gauge field theory defined on the boundary of a 5-dimensional volume can be described by geodesic congruence in a scalar coupled 5-dimensional gravitational theory. Such a gauge-gravity duality was proposed in a more general framework through the Maldacena conjecture and can be elegantly described through the RG equation with the bulk coordinate (or the holographic coordinate) as the RG parameter. It is shown that if the central charge or the *c*-function in a quantum field theory evolves monotonically under RG then the holographic principle indicates that in the corresponding dual gravity in 5-dimensions, the picture is realized through the Raychaudhuri equation governing the monotonic flow of the expansion parameter θ for the geodesic congruence in the gravity sector (Sahakian, 2000; Alvarez and Gomez, 1999; Akhmedov et al., 2011).

The paper is organized as follows. In the following second section we describe briefly the concepts of general relativity which are related to this article. Here we only included tensor, geodesic and geodesic equation, equation of geodesic deviation and Einstein's law of gravitation. Then in the next section we introduce Raychaudhuri equation and the gravitational focusing effects for the congruence of non-spacelike geodesics. Finally we describe the space-time singularities using Raychaudhuri equation.

Aims and objectives of the study

The aims and objectives of the study are to discuss space-time singularity and gravitational collapse of massive star when it collapses under its own gravity at the end of its life cycle. It is one of the most important questions in gravitation theory and relativistic astrophysics, and is in the foundation of black hole physics. The study also expresses that there was a big bang singularity at the beginning of the universe. Raychaudhuri equation supports both of the singularities in space-time. The scope of Raychaudhuri equation in gravitational focusing of non-spacelike geodesics and space-time singularities are also discussed to clarify the importance of this equation in cosmology. At present it is not known that the singularity must be either hidden within an event horizon of gravity or visible to the external universe. The quantum gravity may take over to resolve the classical space-time singularity. The objective of the study is to make easy to the common readers to understand the use of Raychaudhuri equation in classical cosmology, quantum cosmology and other branches of cosmology such as, the holographic principle.

BASIC CONCEPT OF GENERAL RELATIVITY

The particle must travel within the future light cone at an event *p* which satisfies the equation $g(X,X)=0$, which represents the paths of photon. No material particles and signals can travel faster than light; hence event *p* is causally related to another event *q* iff there is a non-spacelike signal between *p* and *q*. If a tensor of second rank $A_{\mu\nu}$ $(\mu,\nu=0,1,2,3)$ is symmetric then (Carroll, 2004):

$$A_{\mu\nu}=A_{\nu\mu} \tag{1}$$

and if $A_{\mu\nu}$ is anti-symmetric then:

$$A_{\mu\nu}=-A_{\nu\mu}. \tag{2}$$

Hence for a tensor with components $A_{\mu\nu}$, its symmetric and anti-symmetric parts are written respectively as:

$$A_{(\mu\nu)}=\frac{1}{2!}\left(A_{\mu\nu}+A_{\nu\mu}\right) \tag{3}$$

$$A_{[\mu\nu]}=\frac{1}{2!}\left(A_{\mu\nu}-A_{\nu\mu}\right). \tag{4}$$

The Kronecker delta is defined by:

$$g_{\mu\nu}g^{\nu\alpha}=\delta^{\alpha}_{\mu}=\begin{cases}1 \text{ if } \alpha=\mu \text{ (no summation)}\\ 0 \text{ if } \alpha\neq\mu\end{cases}.$$

In a Riemannian manifold with a positive definite metric geodesic gives the curves of shortest distance between two points *p* and *q*. The arc length between these two points on a curve $x^{\mu}=x^{\mu}(t)$ is given by:

$$S=\int_p^q g_{\mu\nu}u^{\mu}u^{\nu}\text{, where } u^{\mu}=X^{\mu}=\frac{dx^{\mu}}{dt}. \tag{5}$$

Let $\gamma(t):\Re\to M$ be a C^1-curve in *M*. If *T* is a $C^r(r\geq 0)$ tensor field on *M* then the covariant derivative of *T* along $\gamma(t)$ is defined as:

$$\frac{DT}{\partial t}=T^{a..b}_{c...d;e}X^{e}. \tag{6}$$

where *X* denotes the tangent vector field along $\gamma(t)$. Here γ is called a geodesic if the tangent vector to γ is parallel transported along it. The derivative operator ∇_X on *M* which gives the rate of change of vectors or tensor fields along the given vector field *X* at *p* for all points of *M*. If *X* denotes the tangent vector field along γ, then it is required that $\nabla_X X$ is proportional to *X*; that is, there exists a function *f* such that:

$$\nabla_X X=fX. \tag{7}$$

In the components we can express Equation 7 as:

$$\left(X^{\mu}_{;\nu}X^{\nu}\right)e_{\mu}=fXe_{\mu}, \tag{8}$$

which holds for all e_{μ}. Hence we can write Equation 8 along the curve as:

$$X^{\mu}_{;\nu}X^{\nu}=fX^{\nu}. \tag{9}$$

If $f = 0$ then from Equation 9 we can write the equation for geodesic:

$$X^{\mu}_{;\nu}X^{\nu} = 0. \quad (10)$$

Let $\{x^{\mu}\}$ be the local coordinate system, then $X^{\mu} = \frac{dx^{\mu}}{dt} = u^{\mu}$ are the components of the tangent vector to the geodesic. Here the parameter t is the affine parameter along γ and such a situation γ is called the affinely parametrized geodesic. Now the geodesic Equation 10 can be written as:

$$\frac{du^{\mu}}{dt} + \Gamma^{\mu}_{\nu\lambda}u^{\mu}u^{\nu} = 0. \quad (11)$$

The tensor"

$$R^{\alpha}_{\mu\nu\sigma} = \Gamma^{\alpha}_{\mu\sigma,\nu} - \Gamma^{\alpha}_{\mu\nu,\sigma} + \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\sigma} - \Gamma^{\alpha}_{\beta\sigma}\Gamma^{\beta}_{\mu\nu} \quad (12)$$

is a tensor of rank four and called Riemann curvature tensor. The covariant curvature tensor is defined by:

$$R_{\rho\mu\nu\sigma} = \frac{1}{2}\left(\frac{\partial^2 g_{\rho\sigma}}{\partial x^{\nu}\partial x^{\mu}} + \frac{\partial^2 g_{\mu\nu}}{\partial x^{\sigma}\partial x^{\rho}} - \frac{\partial^2 g_{\mu\sigma}}{\partial x^{\nu}\partial x^{\rho}} - \frac{\partial^2 g_{\rho\nu}}{\partial x^{\sigma}\partial x^{\mu}}\right) + g_{\alpha\lambda}\left(\Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\rho\sigma} - \Gamma^{\alpha}_{\mu\sigma}\Gamma^{\lambda}_{\rho\nu}\right) \quad (13)$$

Contraction of curvature tensor (13) gives Ricci tensor:

$$R_{\mu\nu} = g^{\lambda\sigma}R_{\lambda\mu\sigma\nu}. \quad (14)$$

Further contraction of (14) yields Ricci scalar:

$$\hat{R} = g^{\lambda\sigma}R_{\lambda\sigma}. \quad (15)$$

The energy momentum tensor $T^{\mu\nu}$ is defined as:

$$T^{\mu\nu} = \rho_0 u^{\mu}u^{\nu} \quad (16)$$

where ρ_0 is the proper density of matter, and if there is no pressure, a perfect fluid is characterized by pressure $p = p(x^{\mu})$, then;

$$T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} + pg^{\mu\nu}. \quad (17)$$

The principle of local conservation of energy and momentum states that:

$$T^{\mu\nu}_{;\nu} = 0. \quad (18)$$

The equation:

$$D^2V^{\mu} = -R^{\mu}_{\nu\sigma\lambda}T^{\nu}V^{\lambda}T^{\sigma} \quad (19)$$

is called the equation of geodesic deviation or Jacobi equation. If $R^{\mu}_{\nu\sigma\lambda} = 0$ then $D^2V^{\mu} = 0$; if $R^{\mu}_{\nu\sigma\lambda} \neq 0$ then the neighboring non-spacelike geodesics will necessarily accelerated towards or away from each other.

According to the Newton's law of gravitation, the field equations in the presence of matter are:

$$\nabla^2\phi = 4\pi G\rho \quad (20)$$

where ϕ is the gravitational potential, ρ is the scalar density of matter, $G = 6.66\times10^{-11}m^3kg^{-1}s^{-2}$ is the gravitational constant.

If classical Equation 20 is generalized for the relative theory of gravitation then this must be expressed as a tensor equation satisfying following conditions:

(i) The tensor equation should not contain derivatives of $g_{\mu\nu}$ higher than the second order.
(ii) It must be linear in the second differential coefficients.
(iii) Its covariant divergence must vanish identically.

The covariant derivatives of $g_{\mu\nu}$ are identically zero. The only tensor which involves $g_{\mu\nu}$ up to and second order is the tensor obtained by contracting once and twice the curvature tensor $R_{\rho\mu\nu\sigma}$ of (13) and these are $R_{\mu\nu}$ and $\hat{R}$ in (14) and (15) respectively. The most appropriate tensor is the Einstein tensor which satisfies above conditions:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \quad (21)$$

where the divergence of Equation 21 is identically zero. Equation 21 is proportional to Equation 17, hence combining these two equations, Einstein's field equation can be written as (Stephani et al., 2003):

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^4}T^{\mu\nu}, \quad (22)$$

where $c = 3\times10^8$ m/s is the velocity of light and G is the gravitational constant defined earlier.

Einstein introduced a cosmological constant $\Lambda(\approx 0)$ for static universe solutions as (Mohajan, 2013a, b):

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}T^{\mu\nu}. \quad (23)$$

In relativistic units $G = c = 1$ then Equation 23 becomes:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -8\pi T^{\mu\nu}. \qquad (24)$$

which is the Einstein's law of gravitation for naturally curved material world. It is clear that divergence of both sides of Equations 23 and 24 is zero. For empty space $T_{\mu\nu} = 0$ then $R_{\mu\nu} = \Lambda g_{\mu\nu}$, therefore;

$$R_{\mu\nu} = 0 \text{ for } \Lambda = 0 \qquad (25)$$

which is Einstein's law of gravitation for empty space.

RAYCHAUDHURI EQUATION AND GRAVITATIONAL FOCUSING

Now let us consider the Raychaudhuri equation (Raychaudhuri, 1955), (for null case similar equation holds with $\frac{1}{3}$ is replaced by $\frac{1}{2}$)

$$\frac{d\theta}{dt} = -R_{\mu\nu} V^{\mu} V^{\nu} - \frac{1}{3}\theta^2 - 2\sigma^2 + 2\omega^2 \qquad (26)$$

which describes the rate of change of the volume expansion as one moves along the timelike geodesic curves in the congruence (Mohajan, 2013a). Here $\theta > 0$ is expansion, $\sigma > 0$ is shear and ω is rotation tensors which are defined as follows:

$$\theta_{\mu\nu} = V_{(\alpha;\beta)} h_{\mu}^{\alpha} h_{\nu}^{\beta}$$

$$\sigma_{\mu\nu} = \theta_{\mu\nu} - \frac{1}{3} h_{\mu\nu}\theta$$

$$\omega_{\mu\nu} = h_{\mu}^{\alpha} h_{\nu}^{\beta} V_{[\alpha;\beta]}.$$

By Einstein Equation 22 we can write (Joshi, 1993; Kar and SenGupta, 2007):

$$R_{\mu\nu} V^{\mu} V^{\nu} = 8\pi\left(T_{\mu\nu} V^{\mu} V^{\nu} + \frac{1}{2} T\right). \qquad (27)$$

The term $T_{\mu\nu} V^{\mu} V^{\nu}$ is the energy density measured by a timelike observer with the unit tangent four velocity of the observer, V^{μ}. In classical physics:

$$T_{\mu\nu} V^{\mu} V^{\nu} \geq 0. \qquad (28)$$

Such an assumption is called the weak energy condition (the matter density observed by the corresponding observers is always non-negative; that is, $\rho \geq 0$ and $\rho + p \geq 0$). Now let us consider (Joshi, 2013):

$$T_{\mu\nu} V^{\mu} V^{\nu} \geq -\frac{1}{2} T. \qquad (29)$$

Such an assumption is called the strong energy condition (the trace of the tidal tensor measured by he corresponding observers is always non-negative; that is, $\rho + p \geq 0$ and $\rho + 3p \geq 0$) which implies from Equation 27 for all timelike vectors V^{μ},

$$R_{\mu\nu} V^{\mu} V^{\nu} \geq 0. \qquad (30)$$

Both the strong and weak energy condition will be valid for perfect fluid provided energy density $\rho \geq 0$ and there are no large negative pressures. An additional energy condition required often by the singularity theorems is the dominant energy condition which states that in addition to the weak energy condition, the pressure of the medium must not exceed the energy density (that is, $\rho \geq |p|$). The dominant energy condition also states that $T_{\nu}^{\mu} V^{\nu}$ is non-spacelike and future-directed. Equation 30 implies that the effect of matter on space-time curvature causes a focusing effect in the congruence of timelike geodesics due to gravitational attraction.

Let us suppose γ is a timelike geodesic. Then two points *p* and *q* along γ are called conjugate points if there exists Jacobi field along γ which is not identically zero but vanishes at *p* and *q*. If infinitesimally nearby null geodesics of the congruence meet again at some other point *q* in future, then *p* and *q* are called conjugate to each other, where $\theta \to -\infty$ at *q* (Figure 1). We can define conjugate point another way as follows:

Let *S* be a smooth spacelike hypersurface in *M* which is an embedded three dimensional sub-manifold. Consider a congruence of timelike geodesics orthogonal to *S*. Then a point *p* along a timelike geodesic γ of the congruence is called conjugate to *S* along γ if there exists a Jacobi vector field along γ which is non-zero at *S* but vanishes at *p*, which means that there are two infinitesimally nearby geodesics orthogonal to *S* which intersects at *p* (Figure 2). Again we face equivalent condition that the expansion for the congruence orthogonal to *S* tends to $\theta \to -\infty$ at *p*. If V^{μ} denotes the normal to *S*, then the extrinsic curvature $\chi_{\mu\nu}$ of *S* is defined as:

$$\chi_{\mu\nu} = \nabla_{\mu} V_{\nu} \qquad (31)$$

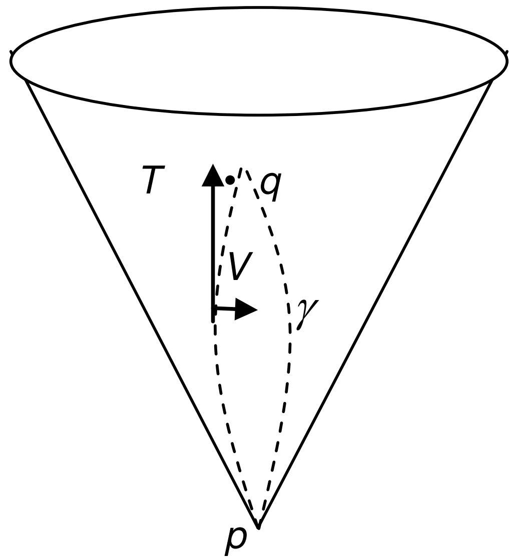

Figure 1. Infinitesimally separated null geodesics cross at *p* and *q*, which are conjugate points along the curve γ.

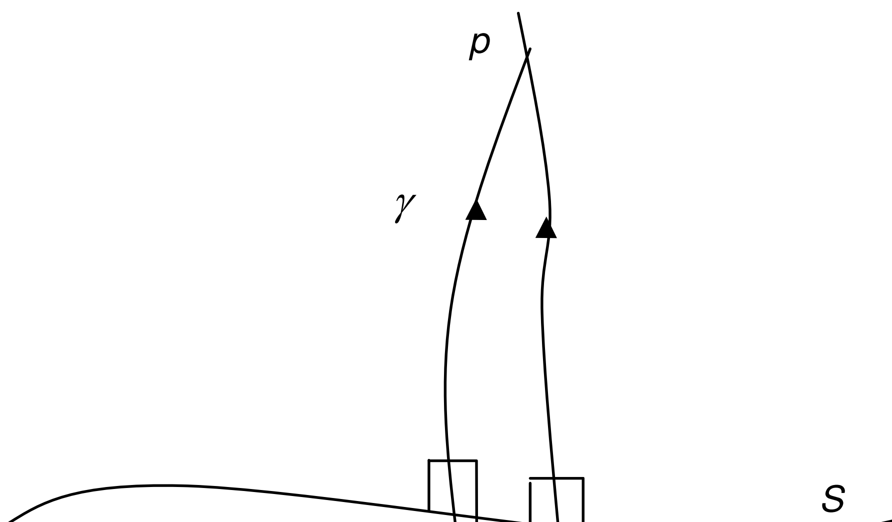

Figure 2. A point *p* conjugate to the spacelike hypersurface *S*. The timelike geodesic is orthogonal to *S*, which is intersected by another infinitesimally nearby timelike geodesic.

which is evaluated at *S*. So, $\chi_{\mu\nu}V^{\mu} = \chi_{\mu\nu}V^{\nu} = 0$. Since *S* is orthogonal to the congruence this implies $\omega_{\mu\nu} = 0$, hence $\chi_{\mu\nu} = \chi_{\nu\mu}$. The trace of the extrinsic curvature, is denoted by χ, and is given by:

$$\chi = \chi^{\mu}_{\mu} = h^{\mu\nu}\chi_{\mu\nu} = \theta \qquad (32)$$

where θ is expansion of the congruence orthogonal to *S*.

Let us consider the situation when the space-time satisfies the strong energy condition and the congruence of timelike geodesics is hypersurface orthogonal, then $\omega_{\mu\nu} = 0$ implies $\omega^2 = 0$ then Equation 26 gives:

$$\frac{d\theta}{d\tau} \le -\frac{\theta^2}{3} \qquad (33)$$

which means that the volume expansion parameter must be necessarily decreasing along the timelike geodesics. Let us denote θ_0 as initial expansion then integrating (33) we get:

$$\frac{1}{\theta} \geq \frac{\tau}{3} + c. \quad (34)$$

Initially $\theta = \theta_0$ then (34) becomes:

$$\frac{1}{\theta} \geq \frac{\tau}{3} + \frac{1}{\theta_0}. \quad (35)$$

By (35) we confirm that if the congruence is initially converging and θ_0 is negative then $\theta \to -\infty$ within a proper time distance $\tau \leq \frac{3}{\theta_0}$, provided γ can be extended to that value of the proper time.

Now suppose $\theta = \chi = 0$, and further, it is bounded above by a negative value $\theta_{\max}$, so all the timelike curves of the congruence orthogonal to *S* will contain a point conjugate to *S* within a proper time distance $\tau \leq \frac{3}{|\theta_{\max}|}$, provided the geodesics can be extended to that value of the proper time.

By the above results the existence of space-time singularities exist in the form of geodesic incompleteness. Now we introduce the gravitational focusing effect for the congruence of null geodesics orthogonal to a spacelike two surfaces as follows (Joshi, 1993):

Let *M* be a space-time satisfying $R_{\mu\nu}K^{\mu}K^{\nu} \geq 0$ for all null vectors K^{μ} and γ be a null geodesic of the congruence. If the convergence θ of null geodesic from some point *p* is $\theta = \theta_0 < 0$ at some point *q* along γ, then within an affine distance less than or equal to $\frac{2}{|\theta_0|}$ from *q* the null geodesic γ will contain a point conjugate to *p*, provided that it can be extended to that affine distance.

SPACE-TIME SINGULARITIES

Now let $\gamma(s)$ be any past directed null geodesic in *M* then,

$$\underset{s \to k}{Lim} \inf T_{\mu\nu}K^{\mu}K^{\nu} > 0 \quad (36)$$

must hold along γ where *k* is the limit of the affine parameter in the past. Such a condition arises when matter and radiation are present, for example, and the microwave background radiation, which should have higher densities in the past in view of the observed expansion of the universe (Hawking and Ellis, 1973).

Violation of any one of the higher-order causality condition in *M* implies that *M* is null geodesically incomplete, provided:

(1) The weak energy condition holds on *M*; that is, $T_{\mu\nu}V^{\mu}V^{\nu} \geq 0$ for any timelike vector V^{μ}.

(2) The matter tensor satisfies $\underset{s \to k}{Lim} \inf T_{\mu\nu}K^{\mu}K^{\nu} > 0$ on all null geodesics in *M*.

The Raychaudhuri Equation 26 for null geodesic is:

$$\frac{d\theta}{dt} = -R_{\mu\nu}K^{\mu}K^{\nu} - \frac{1}{2}\theta^2 - 2\sigma^2 \quad (\omega^2 = 0), \quad (37)$$

where K^{μ} is the tangent to the geodesic. The conjugacy of *p* and *q* for a function *y* is defined by

$$\theta \equiv \frac{1}{y} \cdot \frac{dy}{ds}$$

and is given as follows:

A new function *z* is defined by $z^2 = y$, then $\theta \equiv \frac{2}{z} \cdot \frac{dz}{ds}$, so (37) becomes:

$$\frac{d^2 z}{ds^2} + H(s)\, z = 0 \quad (38)$$

where $H(s) = \frac{1}{2}\left(R_{\mu\nu}K^{\mu}K^{\nu} + 2\sigma^2\right)$.

Thus *y* will be zero at *p* and *q* iff *z* is zero at *p* and *q*. Now let γ be past complete, then condition (1) above implies $R_{\mu\nu}K^{\mu}K^{\nu} \geq 0$ for all null vector K^{μ}. Since $\sigma^2 > 0$ then Equation 38 gives:

$$\int_0^{\infty} \frac{1}{2}\left(R_{\mu\nu}K^{\mu}K^{\nu} + 2\sigma^2\right) ds = \int_0^{\infty} H(s)\, ds = \infty. \quad (39)$$

In this case the null trajectory γ must contain infinitely many points (Tipler, 1977) and any two of such points can be timelike related. Let us define a length scale *y* associated with the volume *V*(*t*) defined by $y^3 = V$. Now taking second derivative of Raychaudhuri Equation 26 we get:

$$\frac{d^2y}{dt^2}+\frac{1}{3}\left(R_{\mu\nu}V^{\mu}V^{\nu}+2\sigma^2\right)y=0. \quad (40)$$

Let us choose $F(t)=\frac{1}{3}\left(R_{\mu\nu}V^{\mu}V^{\nu}+2\sigma^2\right)=\frac{A}{t^2}$ for $A > 0$ then Equation 40 becomes:

$$\frac{d^2y}{dt^2}+\frac{A}{t^2}y=0. \quad (41)$$

Here we will choose a trial solution to solve Equation 41 because trial solution of differential equation results precisely and concisely so that common readers can understand the solution easily. This type of differential equation also has analytical solution but in that case we have to use related boundary conditions and that case the solution would be complicated and tedious to the readers. So then trial solution seems to be preferable to us and we have avoided analytical solution here. Let $y=t^{\alpha}$ be a trial solution of Equation 41, then we get:

$$\alpha(\alpha-1)t^{\alpha-2}+At^{\alpha-2}=0 \quad (42)$$

which is true for all $t^{\alpha-2}$, so that:

$$A=\alpha-\alpha^2. \quad (43)$$

Since $V(t)\to 0$ at $t = 0$ we must have $\alpha>0$. Again since $A > 0$ so that from Equation 43 for we get:

$$\alpha=\frac{1\pm\sqrt{1-4A}}{2}.$$

Since for $0<\alpha<1$, we have $\sqrt{1-4A}\geq 0$ $\Rightarrow A\leq\frac{1}{4}$.

Hence solution for y is given by:

$$y=t^{\left[1\pm\sqrt{(1-4A)}\right]/2}. \quad (44)$$

The volume $V(t)\to 0$ is near the singularity at least as fast as $y\to t^{3/2}$. Hence at a strong curvature singularity, the gravitational tidal forces associated with this singularity are so strong that any object trying to cross it must be destroyed to zero size.

DISCUSSION

We have discussed Raychaudhuri equation which expresses the gravitational focusing of geodesics and conjugate points in a space-time manifold. The discussion implies that there was a big bang singularity at the beginning of the universe and also there will be a big crunch in the future within a fixed time at the end state of the universe and then another singularity will create in space-time manifold. It also supports the space-time singularities in the black hole regions. The aim of the study was to discuss the application of Raychaudhuri equation in different fields of physics and to indicate the importance of it to the future researchers. Finally we can remark that the study is very interesting and it is most advanced area of relativity and cosmology “which is the space-time puzzle and results the black holes formation.”

CONCLUSION

In this paper we have described gravitational focusing of geodesics and conjugate points in a space-time manifold. Raychaudhuri equation predicts that there may occur singularities in the beginning and at the end state of gravitational focusing. We have used Raychaudhuri equation to explain space-time singularities and show the procedures with detail mathematical calculations and diagrams where necessary.

This paper is based on theoretical discussion and mathematical calculations. Experimental attempts are also taken to know the answer of the questions: What is the universe made of? How did it start? etc. Since 1952 CERN (Conseil Européen pour la Recherche Nucléaire), the European Organization for Nuclear Research, physicists and engineers are probing the fundamental structure of the universe. They use the world’s largest and most complex scientific instruments to study the basic constituents of matter; the fundamental particles. The particles are made to collide together at close to the speed of light. The process gives the physicists clues about how the particles interact, and provides insights into the fundamental laws of nature. The Large Hadron Collider (LHC) is the world’s largest and most powerful particle accelerator and remains the latest addition to CERN’s accelerator complex. We hope in near future the scientists can provide us the true information about the universe in some details.

In this paper we have avoided the complicated mathematical calculations thinking for the common readers. We hope all the readers who have elementary ideas in mathematics and physics can realize the paper with full interest.

REFERENCES

Akhmedov ET, Gahramanov IB, Musaev ET (2011). Hints on integrability in the Wilsonian/ holographic renormalization group, JETP Lett. 93(9):545–550.

Alvarez E, Gomez C (1999). Holography and the c-theorem, Nucl. Phys. B541, 441 ibid, hep-th/0009203.

Borgman J (2004). Fluctuations of the expansion: the Langevin–Raychaudhuri equation, PhD. Thesis, Tufts University, USA.
Carroll SM (2004). Spacetime and geometry: an introduction to general relativity, San Francisco: Addison-Wesley.
Ehlers J (2007). A K Raychaudhuri and his equation, PRAMANA J. Phys. Ind. Acad. Sci. 69(1):7-14.
Hawking SW, Ellis GFR (1973). The large scale structure of space-time, Cambridge Univ. Press, Cambridge.
Hawking SW, Penrose R (1970). The singularities of gravitational collapse and cosmology, Proc. Roy. Soc. Lon. A314:529-548.
Joshi PS (1993). Global aspects in gravitation and cosmology, Claren. Press, Oxford.
Joshi PS (2008). Gravitational collapse and spacetime singularities, Cambridge Univ. Press, Cambridge.
Joshi PS (2009). Naked singularities, Sci. Am. 300(2):36-43.
Joshi PS (2013). Spacetime singularities, arXiv:1311.0449v1 [gr-qc] 3 November.
Joshi PS, Malafarina D, Saraykar RV (2012). Genericity aspects in gravitational collapse to black holes and naked singularities, arXiv:1107.3749v2 [gr-qc] 4 July.
Kar S, SenGupta S (2007). The Raychaudhuri equations: a brief review, PRAMANA. J. Phy. Ind. Acad. Sci. 69(1):49-76.
Mohajan HK (2013a). Singularity theorems in general relativity, M. Phil. Dissert., Lamb. Acad. Pub., Germany.
Mohajan HK (2013b). Schwarzschild geometry from exact solution of Einstein equation, J. Env. Treat. Tech. 1(2):69-75.
O'Connor E, Ott CD (2011). Black hole formation in failing core-collapse supernovae, arXiv:1010.5550v2 [astro-ph.HE] 10 March.
Poisson E (2004). A relativist's toolkit: the mathematics of black hole mechanics, Cambridge Univ. Press, Cambridge.
Raychaudhuri AK (1953). Arbitrary concentration of matter and the Schwarzschild singularity, Phys. Rev. 89:417-432.
Raychaudhuri AK (1955). Relativistic cosmology. Phys. Rev. 98(4):1123-1126.
Sahakian V (2000). Holography, a covariant c-function, and the geometry of the renormalization group. Phys. Rev. D62:126011-126018.
Sahu S, Patil M, Narasimha D, Joshi PS (2013). Time delay between relativistic images as a probe of cosmic censorship, arXiv:1310.5350v1 [gr-qc] 20 October.
Sharif M, Ahmad *Z* (2012). Gravitational dust collapse with cosmological constant. World Appl. Sci. J. 16(11):1516-1520.
Stephani H, Kramer D, MacCallum M, Hoenselaers C, Hertl E (2003). Exact solutions to Einstein's field equations (2nd Ed.), Cambridge Univ. Press, Cambridge.
Tipler F (1977). Singularities and causality violation. Ann. Phys. 108(1):1-36.
Visser M, Barcelo C, Liberati S, Sonego S (2009). Small, dark, and heavy: but is it a black hole?, arXiv:0902.0346v2 [gr-qc] 9 February.