A dynamic limit order market with fast and slow traders

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July 2012

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This Version: October 2013
First Version: June 2011

Abstract

We study the role of high-frequency trading in a dynamic limit order market. Fast traders' ability to revise their quotes quickly after news arrivals helps to reduce the inefficiency that is rooted in the risk of being "picked off", which increases trade. However, their presence induces slow traders to strategically submit limit orders with a lower execution probability, thereby reducing trade. Because speed is a source of market power, it enables fast traders to extract rents from others and triggers a costly arms race that reduces social welfare. The model generates a number of testable implications concerning the effects of high-frequency trading in limit order markets.

Keywords: High-frequency trading, Limit Order Market

JEL Classification: G19, C72, D62

*This paper is a substantially revised version of the second chapter of my doctoral dissertation at Universitat Pompeu Fabra. Financial support from the Generalitat de Catalunya is gratefully acknowledged. I would like to thank Álvaro Cartea, Jean-Edouard Colliard, Hans Degryse, Başak Güneş, Roberto Pascual, Francisco Peñaranda, and Augusto Rupérez Micola for helpful discussions at various stages of the project. I am particularly grateful to Thierry Foucault, whose detailed comments on an earlier draft have greatly improved the paper. All remaining errors are mine. The views expressed in this paper are those of the author and do not necessarily reflect those of the European Central Bank or the Eurosystem.

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1 Introduction

High-frequency trading (HFT), a variant of algorithmic trading, relies on sophisticated computer programs for the implementation of trading strategies that involve a vast amount of orders in very small time intervals. The fact that market participants are spending considerable resources in an effort to gain speed advantages of a few milliseconds suggests that there are large payoffs to being faster than others.[1] Accordingly, HFT has grown tremendously over the past decade and by now is estimated to account for 70% of trading in U.S. equities as well as 40% of spot FX volume.[2] This development has ignited a heated debate among financial economists, practitioners, and regulators about the benefits and concerns related to HFT. While its advocates argue that technology increases market efficiency through improved liquidity and price discovery[3], others claim that faster market participants use their speed advantage to extract rents and are a threat to market stability and integrity.[4]

This paper contributes to this debate by presenting a stylized model of trading in a limit order market where agents differ in their trading speed, which is thought to capture the difference between (fast) HFTs and (slow) human market participants. We build on the model of Foucault (1999), in which limit orders cannot be revised after submission and may therefore become stale upon the arrival of new value-relevant information. The resulting risk of being "picked off" gives rise to an inefficiency, because a high level of asset price volatility leads agents to choose limit orders with a low execution probability and thus reduces the likelihood that gains from trade are ultimately realized. We extend Foucault's model by endowing a proportion α of the trading crowd with a *relative* speed advantage that improves their ability to manage outstanding limit orders compared to the remaining market participants. More specifically, we assume that fast traders (FTs) are able to revise their limit orders after news arrivals, but only in case the next agent is a slow trader (ST).

We analyze the stationary equilibrium of this dynamic limit order market and compare it to the baseline case of identical traders studied by Foucault (1999). Overall, the presence of FTs has two opposing effects on the probability that gains from trade are shared. On the one hand, their ability to revise some of their quotes after news arrivals reduces the existing inefficiency due to the risk of being picked off and therefore increases trade. On the other hand, FTs' speed advantage introduces a new inefficiency because it can induce STs to strategically submit limit orders with a lower execution probability, which reduces trade.

In order to understand this second effect, it is useful to interpret the limit order market as a sequential bargaining process, where traders may either accept outstanding offers (via a market

[1] see "Time is money when it comes to microwaves", Financial Times, May 10, 2013.

[2] See "EBS take new step to rein in high-frequency traders", Reuters Newswire, August 23, 2013.

[3] See e.g. Optiver, "High Frequency Trading", Position Paper, 2011, http://fragmentation.fidessa.com/wp-content/uploads/High-Frequency-Trading-Optiver-Position-Paper.pdf

[4] See SEC Chairman Mary Schapiro's speech "Remarks Before the Security Traders Association", www.sec.gov/news/speech/2010/spch092210mls.htm.

order) or alternatively make an offer to the next agent (via a limit order). As usual in these situations, agents' bargaining power is determined by their outside option which here is given endogenously by the expected payoff earned from submitting a limit order. Now because they face a lower risk of being picked off, the alternative of posting quotes becomes relatively more attractive to FTs. This implies that they need to be offered a higher share of the surplus in order to be convinced to accept an existing offer by using a market order. This situation creates a dilemma for STs: They can either keep their chances of execution constant by increasing the aggressiveness of their limit orders in order to attract both STs and FTs, or alternatively accept a decrease in execution probability by only targeting STs. While the value of their outside option decreases in either case, the latter choice (which is optimal if α is small) is socially inefficient as gains from trade are realized less often.

Aside from affecting trading volume and STs' limit order execution probabilities, the shift in market power between STs and FTs yields a number of additional testable implications concerning the effects of HFT in limit order markets. For example, one can show that FTs are more likely to act as makers than as takers in equilibrium, and that their market orders execute at more favourable prices that those of STs. While FTs' limit orders face a reduced risk of being picked off, the risk of adverse selection simultaneously increases for STs. In addition, the presence of FTs pushes quotes closer to the asset's fundamental value if volatility is sufficiently high (the opposite holds for low volatility). These predictions are consistent with the growing body of empirical research on HFT.

Even though the presence of FTs may ultimately allow more gains from trade to be reaped, this increase in efficiency does not benefit STs because their reduced bargaining power ensures that they are left with a smaller share of the total surplus and consequently *always* worse off in equilibrium. This has important consequences for social welfare once one discards the assumption that speed is given exogenously but rather considers the possibility that agents may become fast upon investing into trading technology at a fixed cost. Because STs and FTs must earn the same net profits in equilibrium, the equilibrium level of investment always leads to a social welfare loss compared to the benchmark situation with only STs. Consequenty, policy interventions that aim at reducing the rents associated with being fast can possibly improve upon the market outcome by preventing a costly arms race. Based on this intuition, we suggest that regulators consider mandating pro-rata matching for the most liquid stocks as well as randomized "speed bumps" that have been recently adopted in several FX markets.

The literature on algorithmic trading and HFT has grown substantially in recent years [see e.g. the surveys by Biais and Woolley (2012) and Foucault (2012)]. Most closely related to our work is the paper by Biais, Foucault, and Moinas (2013) which studies the impact of HFT in a Glosten and Milgrom (1985) framework. In their model, FTs have a higher chance of finding trading opportunities than slow market participants and therefore help to increase the likelihood that gains from trade are realized. But at the same time, they are a source of adverse selection due to private information, which raises the bid-ask spread payable by everyone and therefore reduces trade. Just like here, FTs exert a negative externality and investment into HFT can be excessive in equilibrium. However, the underlying mechanism is different. In our model, FTs are actually able to *avoid* being adversely selected. However, because agents trade directly with each other in a dynamic setting, this effectively increases their market power and allows them to extract rents from slower market participants.

Also closely related, Jovanovic and Menkveld (2012) study competitive middlemen who intermediate between early limit order traders and late market order traders. Similar to our model, HFTs may reduce adverse selection by updating quotes quickly and therefore increase trade. Yet, HFTs' ability to process (hard) information quickly can also introduce a new adverse selection problem that lowers trade. Based on the entry of a new trading venues for Dutch stocks, they conduct a calibration exercise which reveals a slight increase in welfare.

A number of other papers also study HFT from a theoretical perspective. Cartea and Penalva (2013) propose a model where their increased speed allows HFTs to impose a haircut on liquidity traders, which raises trading volume and price volatility, but reduces the welfare of liquidity traders. Foucault, Hombert and Rosu (2013) study the trading strategy of an informed trader who is able to react faster than others to news. They conclude that this speed advantage makes the informed trader's order flow more volatile and increases his relative share in trading volume. Rosu and Martinez (2013) study HFTs as strategic informed traders who instantaneously react to new information and ensure that it is reflected in prices immediately. Pagnotta and Philippon (2012) provide a model where competing exchanges invest into speed and compete for investors. They provide conditions under which competition, fragmentation and speed can improve or reduce welfare. Finally, Bernales (2013) studies a setup similar to the one considered here, but the additional degree of complexity forces him to resort to numerical techniques.

Empirical research on HFT and/or algorithmic trading can be roughly split into two categories. The first strand of the literature, which includes Hendershott, Jones and Menkveld (2011), Boehmer, Fong and Wu (2012) and Hasbrouck and Saar (2013), uses empirical proxies based on anonymized data to examine the impact of computerized trading activity on market quality. In short, these papers conclude that computer-based trading causally improves market liquidity (e.g. lowers bid-ask spreads), while the results concerning the impact on price volatility are rather mixed. The second, growing strand of the literature uses proprietary datasets that enable the direct identification of trading by HFTs, either as a group of even down to the level of individual market participants. This approach has the benefit that it offers deeper insights by allowing researchers to study the behaviour of different traders types and the associated effects on market outcomes. The results emerging from these research efforts suggest that being fast is very profitable [e.g. Menkveld (2012), Baron, Brogaard and Kirilenko (2012)], facilitates price discovery [e.g. Chaboud, Chiquoine, Hjalmarsson and Vega (2013), Brogaard, Hendershott, and Riordan (2013)], is frequently, but not exclusively used to trade passively [e.g. Menkveld (2012), Hagströmer and Norden (2013), Malinova, Park and

Riordan (2013)], and helps to reduce trading costs relative to slower market participants [Hendershott and Riordan (2013), Malinova, Park and Riordan (2013), Carrion (2013)]. Moreover, consistent with the present model and other theoretical work, recent evidence by Brogaard, Hagströmer, Norden and Riordan (2013) suggests that FTs face a reduced risk of being picked off but at the same time expose others to adverse selection.

This paper is organized as follows. Section 2 provides an outline of the model, whose equilibrium is presented subsequently in Section 3. Section 4 develops a number of empirical predictions concerning the impact of HFT in limit order markets, while social welfare and policy implications are discussed in Section 5, followed by the Conclusion. All proofs are relegated to the Appendix A, while Appendix B contains some figures that help to illustrate the model.

2 The model

2.1 The limit order market

We consider an infinite-horizon[5] version of Foucault's (1999) dynamic limit order market. There is a single risky asset whose fundamental value follows a random walk, i.e.

$$v_t = v_{t-1} + \varepsilon_t$$

where the innovations are i.i.d. and can take values of $+\sigma$ and $-\sigma$ with equal probability. Traders arrive sequentially at time points $t = 1, 2, \ldots$ and are risk-neutral. Trading arises due to differences in traders' private values for the asset. Specifically, we assume that at time t', a trader arriving at time $t \leq t'$ values the asset at

$$R_{t'} = v_{t'} + y_t$$

which is the sum of the asset's current fundamental value and the time-invariant private valuation $y_t \in \{+L, -L\}$, where both realizations occur with equal probability and $L > 0$. Moreover, the private valuations are i.i.d. across traders and independent of the asset value innovations. We call agents with a high (low) private valuation buyers (sellers).

Upon their arrival, traders choose between submitting a limit order or a market order for one unit of the asset. Sell (buy) market orders execute at the currently best bid (ask), which we denote by B_t^m (A_t^m). Limit orders are stored in the order book for one period, after which they are assumed to expire for tractability. This implies that at each point in time, the limit order book either a) contains a bid quote b) an ask quote or c) is empty.[6]

[5] Foucault (1999) assumes that the terminal date is stochastic, as the trading process stops after each period with constant probability $1 - \rho > 0$. An infinite horizon may be interpreted as the limiting case where $\rho \to 1$.

[6] We write $B_t^m = -\infty$ ($A_t^m = \infty$) if there is no bid (ask) quote available.

One crucial assumption in the Foucault (1999) model is that traders cannot revise their limit orders once they are submitted, which exposes their quotes to the risk of being picked off after news arrivals as in Copeland and Galai (1983). Here, we depart from the original setting by assuming that there are two types of agents, fast traders (FTs) and slow traders (STs), where α denotes the probability that an agent is fast (assumed i.i.d. across traders and independent from y and ε). Speed is valuable because it enables agents to adjust their quotes quickly when news hit the market. Accordingly, we assume that FTs are able to cancel their limit order and resubmit a new one after the realization of ε_{t+1}, yet before the arrival of the next trader provided that he is slow. If, in contrast, the next agents turns out to be fast as well, the quote cannot be revised.[7] STs can never cancel their outstanding orders. Figure 1 in Appendix B illustrates the timing of events in between trader arrivals. Notice that being fast is a purely *relative* advantage and therefore only valuable if at least some market participants are slow. Accordingly, the model collapses to the Foucault model for both $\alpha = 0$ and $\alpha = 1$. While we take α as exogenous for most of our analysis, Section 5 endogenizes investment into trading speed, similar in spirit to Biais, Foucault, and Moinas (2013).

2.2 Payoffs and strategies

Consider a given period and assume that a seller[8] enters the market. As in Foucault (1999), the focus is on Markov-perfect equilibria so that we may drop the time subscripts in order to avoid notational clutter. Let $v + \varepsilon$ be the current value of the asset, where $\varepsilon \in \{+\sigma, -\sigma\}$ is the most recent innovation and v denotes the asset value in the previous period. Now assume that the seller's expected profit when choosing to post a limit order is equal to V_k^{LO}, where $k \in \{ST, FT\}$ refers to his type. Clearly, he will opt for a market sell order if the currently best available bid price B^m is such that[9]

$$B^m - (v + \varepsilon - L) \geq V_k^{LO} \qquad k \in \{ST, FT\} \tag{1}$$

Here, the expected profit obtained from posting a limit order constitutes an agent's endogenous outside option when deciding upon whether or not to submit a market order. This implies that the seller's order choice is entirely determined by whether or not the best available bid is above his sell cutoff price, $\hat{B}_k^{v+\varepsilon}$, which is the bid price that makes him indifferent between submitting a limit

[7] Importantly, this assumption does not require that a FT knows the next agent's type. Rather, news arrivals trigger a race between limit and market order traders to react first, and the assumption here is that the market order trader wins the race unless the limit order trader is *faster*. This is also consistent with the Foucault (1999) model, where all agents are equally fast.

[8] Due to symmetry, it suffices to consider one side of the market for each decision. We detail the order choice (market vs. limit order) for a seller and the quotation problem for a buyer.

[9] We assume that agents submit a market order in case they are indifferent. This choice is arbitrary and does not affect our conclusions.

order or a market order given the current asset value $v+\varepsilon$

$$\hat{B}_k^{v+\varepsilon} = V_k^{LO} + (v+\varepsilon-L) \qquad\qquad k \in \{ST, FT\} \tag{2}$$

Now consider the quote-setting problem of a buyer that arrives one period earlier. When choosing his bid price, he faces the trade-off that a more aggressive limit order implies a higher probability of execution but at the same time yields a lower profit conditional on execution. Importantly, the execution of a limit order does not only depend on the type of trader that follows in the next period but also on the forthcoming realization of ε. Now let $p(B)$ denote the execution probability of a ST's buy limit order with bid price B. Given the discreteness of innovations (ε) and trader types (k), this probability is an increasing step function, and the "jumps" occur at the points where B crosses one of the possible sell cutoff prices that the agent arriving in the next period may possibly have. Clearly, optimality implies that the set of bid quotes agents effectively choose from is equal to the set of sellers' sell cutoff prices.[10] The objective function of a slow buyer that decides to submit a limit order can be written as

$$V_{ST}^{LO} = \max_{B_{ST}} \{ p(B_{ST})(v + E_{Ex}[\varepsilon] + L - B_{ST}) \} \tag{3}$$

where $E_{Ex}[\cdot]$ denotes expectation conditional on execution. The decision problem of a fast buyer opting for limit orders is slightly more complex because he may revise his quote upon the realization of ε conditional on the next trader being a ST. Hence he effectively chooses a tuple of three bid prices $(B_{FT}, B_{FT}^{+\sigma}, B_{FT}^{-\sigma})$. Now let $q_{|k,\varepsilon}(B)$ denote the execution probability of a FT's limit order with bid price B conditional on the next period's trader type and asset value innovation.[11] Then, the fast buyer's objective function is

$$V_{FT}^{LO} = \max_{B_{FT}, B_{FT}^{+\sigma}, B_{FT}^{-\sigma}} \left\{ \begin{array}{c} \alpha q_{|FT}(B_{FT})(v + E_{Ex}[\varepsilon] + L - B_{FT}) \\ +\frac{(1-\alpha)}{2} q_{|ST,+\sigma}(B_{FT}^{+\sigma})(v + \sigma + L - B_{FT}^{+\sigma}) \\ +\frac{(1-\alpha)}{2} q_{|ST,-\sigma}(B_{FT}^{-\sigma})(v - \sigma + L - B_{FT}^{-\sigma}) \end{array} \right\} \tag{4}$$

It is easy to see that our assumption on FTs' ability to revise their limit orders implies that $B_{FT}^{+\sigma}$ and $B_{FT}^{-\sigma}$ are set in perfect knowledge about both the next trader's type and the forthcoming realization of ε. Hence they must be optimally chosen to be equal to a ST's cutoff price in the next period, i.e. $B_{FT}^{+\sigma} = \hat{B}_{ST}^{v+\sigma}$ and $B_{FT}^{-\sigma} = \hat{B}_{ST}^{v-\sigma}$, so that the maximization problem simplifies to

$$\max_{B_{FT}} \left\{ \alpha q_{|FT}(B_{FT})(v + E_{Ex}[\varepsilon] + L - B_{FT}) \right\} \tag{5}$$

[10] A bid price strictly above a cutoff price is suboptimal (it has the same execution probability as a slightly lower price). As there are no gains from trade between two buyers, bid quotes will only aim at sellers in equilibrium.

[11] The conditional execution probabilities for STs' limit orders are accordingly denoted by $p_{|k,\varepsilon}(B)$.

Clearly, FTs' ability to revise some of their limit orders ensures that the endogenous outside option of posting limit orders is more valuable to them than to STs in equilibrium. Because the relative advantage of being fast is directly tied to the risk of being picked off[12], the following ordering of sell cutoff prices must always obtain in equilibrium.

Lemma 1 *In equilibrium,* $\hat{B}_{ST}^{v-\sigma*} \leq \hat{B}_{FT}^{v-\sigma*} \leq \hat{B}_{ST}^{v+\sigma*} \leq \hat{B}_{FT}^{v+\sigma*}$

The equilibrium is computed by simultaneously solving equations (2), (3) and (5) and verifying the absence of profitable deviations.

3 Equilibrium

3.1 Optimal quotes

Similar to Colliard and Foucault (2012), we categorize agents' strategies according to their respective (conditional) limit order execution probabilities. The following definition will be helpful in characterizing the equilibrium.

Definition 1 *A ST (FT) uses a high fill-rate strategy if his (initial) limit order is such that* $p^*_{|\varepsilon} > 0$ $(q^*_{|FT,\varepsilon} > 0)$ *for all* ε. *All other quotes constitute a low fill-rate strategy. Moreover, a ST uses a specialized strategy if* $p^*_{|ST,\varepsilon} > p^*_{|FT,\varepsilon}$ *for some* ε. *Otherwise, the strategy is unspecialized.*

Intuitively, a high fill-rate strategy corresponds to posting a limit order that has a strictly positive execution probability for every possible future realization of ε and is therefore exposed to the risk of being picked off. In contrast, low fill-rate strategies do not face this risk and consequently offer lower chances of execution. Importantly, both types of traders face this choice because FTs' initial quotes can still be picked off by other FTs. Furthermore, a specialized strategy refers to submitting a limit order that, for some values of ε, only attracts execution by STs, but not by FTs. If, for a given ε, the order attracts either both types of traders or no trader at all, it is called unspecialized. Notice that this distinction is only relevant for STs because FTs de-facto only make a choice regarding the initial quote and this one is fully characterized by its fill-rate. Equipped with this typology of strategies, we can now state the following.

Proposition 1 *For fixed parameters* (α, σ, L), *there exists a unique Markov-perfect equilibrium in the limit order market. In equilibrium*

a) STs use a high fill-rate strategy for $\sigma < \sigma^*_{ST}(\alpha)$ *and a low fill-rate strategy otherwise.*

[12] In the limit when $\sigma \to 0$, being fast is not beneficial anymore because limit orders are no further exposed to the risk of being picked off.

b) STs use a specialized strategy for $\alpha < \alpha_S^*(\sigma)$ and an unspecialized strategy otherwise.
c) FTs use a high fill-rate strategy for $\sigma < \sigma_{FT}^*(\alpha) \leq \sigma_{ST}^*(\alpha)$ and a low fill-rate strategy otherwise. The definitions of $\sigma_{FT}^*(\alpha)$, $\sigma_{ST}^*(\alpha)$ and $\alpha_S^*(\sigma)$ are provided in the Appendix.

The trade-off between execution probability and expected profit conditional on execution helps to understand the intuition that underlies Proposition 1. In line with Foucault (1999), parts a) and c) state that a high (low) level of volatility induces agents to post limit orders with low (high) chances of execution. Intuitively, larger innovations imply a more severe adverse selection risk for limit order traders, and the natural reaction is then to protect oneself from unfavourable price movements by posting less aggressive limit orders. Part b) is similarly intuitive and rests on the fact that, in equilibrium, the outside option of posting limit orders is more valuable for FTs than for STs. If α is relatively low, it is not very attractive for STs to target FTs with their limit orders because this leads only to a small increase in execution probability but at the same time requires considerably more aggressive quotes. Hence they only use an unspecialized strategy when α is sufficiently large. Figure 2 in Appendix B depicts the functions $\sigma_{FT}^*(\alpha)$, $\sigma_{ST}^*(\alpha)$ and $\alpha_S^*(\sigma)$ in the (α, σ)-space, where we have set $L = 1$ (this is without loss of generality as only the ratio of σ and L is relevant).

The assumption that ε is both discrete and bounded allows us to obtain a closed-form solution. However, it restricts traders to choose among extremes (high and low fill-rate strategies), and small parameter changes may lead to large changes in outcomes that are not robust to alternative distributional assumptions. In order to mitigate such distortions, we define $\underline{\sigma} \equiv 8L/13$ and $\overline{\sigma} \equiv 4L/(7-\sqrt{5})$ and henceforth apply the following parameter restriction.[13]

Assumption 1 (technical) $\sigma \in \Sigma \equiv \underline{\Sigma} \cup \overline{\Sigma}$ where $\underline{\Sigma} = [0, \underline{\sigma})$ and $\overline{\Sigma} = [\overline{\sigma}, \infty)$.

The excluded interval $[\underline{\sigma}, \overline{\sigma})$ is depicted in Figure 2. Effectively, this assumption rules out an equilibrium where STs use a high fill-rate strategy and FTs use a low fill-rate strategy, such that we are left with 4 distinct types of equilibria. This also helps to simplify the exposition considerably, because it implies that all traders use the same strategy in terms of fill-rate for all possible values of α.

In the following, we refer to the different types of equilibria by the strategy chosen by STs for parsimony. We abbreviate the unspecialized low (high) fill-rate equilibrium as ULFR (UHFR) and the specialized low (high) fill-rate equilibrium as SLFR (SHFR). Outcomes for $\alpha = 0$ (i.e. the Foucault (1999) model) are denoted by the subscript 0, e.g. V_0^{LO*} refers to the equilibrium expected profits from limit orders in a market with only STs.

[13] We numerically verify that all results presented in this paper are also obtained under normally distributed innovations.

3.2 Market events and bargaining power

In order to compute (expected) equilibrium outcomes we will frequently require knowledge about the likelihood of a particular event to occur on the equilibrium path. To this end, we define the following four mutually exclusive events. The arriving agent can be 1) a ST submitting a limit order, 2) a ST submitting a market order, 3) a FT submitting a limit order, or 4) a FT submitting a market order. Now let $\varphi^* = (\varphi_{ST}^{LO*}, \varphi_{ST}^{MO*}, \varphi_{FT}^{LO*}, \varphi_{FT}^{MO*})$ denote the stationary probability distribution of these events in equilibrium, where we naturally have $\varphi_{ST}^{LO*} + \varphi_{ST}^{MO*} = 1 - \alpha$ and $\varphi_{FT}^{LO*} + \varphi_{FT}^{MO*} = \alpha$. Then one can show the following.

Proposition 2 *For fixed parameters (α, σ, L), the stationary probability distribution of equilibrium events is given by*

$$\begin{aligned}
\varphi_{ST}^{LO*} &= \frac{1-\alpha}{\chi}[1 - \alpha(\frac{1}{2} - q^*_{|FT})] \\
\varphi_{ST}^{MO*} &= \frac{1-\alpha}{\chi}[\chi - 1 + \alpha(\frac{1}{2} - q^*_{|FT})] \\
\varphi_{FT}^{LO*} &= \frac{\alpha}{\chi}[1 + (1-\alpha)(p^*_{|ST} - p^*_{|FT})] \\
\varphi_{FT}^{MO*} &= \frac{\alpha}{\chi}[\chi - 1 - (1-\alpha)(p^*_{|ST} - p^*_{|FT})]
\end{aligned}$$

where $\chi = (1 + \alpha q^*_{|FT})(1 + (1-\alpha)p^*_{|ST}) - \frac{\alpha(1-\alpha)}{2} p^*_{|FT}$

In the following, it will often be helpful for understanding the model's intuition to interpret the limit order market as a sequential bargaining game over a surplus of $2L$. Agents post take-it-or-leave-it offers (limit orders), which the following trader may either accept (via a market order) or reject and instead make another offer (via a limit order) to the agent arriving one period after him. As usual in these situations, agents' bargaining power is determined by the value of their outside option, which here is equal to the expected profit from posting a limit order. As already mentioned, the alternative of posting a limit order is more valuable to FTs than to STs because timely quote revisions reduce the risk of being picked off. However, it turns out that this directly implies that STs bargaining power deteriorates compared to the situation without FTs.

Corollary 1 $V_{FT}^{LO*} > V_0^{LO*} > V_{ST}^{LO*}$ *for all* $\alpha \in (0, 1)$.

4 Empirical Implications

Based on agents' equilbrium quotation strategies and the resulting stationary distribution of market events, one may derive a number of implications concerning the possible effects of HFT in limit order markets. Whenever possible, we relate these predictions to existing theoretical and empirical research.

4.1 Trading volume and limit order execution probabilities

We start by examining the trading rate (or per-period expected trading volume) which is defined as the unconditional probability of observing a trade in a given period, that is

$$TR^* \equiv \varphi_{ST}^{MO*} + \varphi_{FT}^{MO*} \tag{6}$$

It is important to stress that the trading rate is a measure of efficiency (and thus welfare) in this model as it effectively states how frequently gains from trade are realized. We discuss issues related to social welfare in Section 5. Based on Propositions 1 and 2, we deduce the following.

Corollary 2 *The presence of FTs (weakly) increases the trading rate except if both α and σ are low, that is $TR^* < TR_0^*$ for $\sigma \in \underline{\Sigma}$ and $0 < \alpha < \alpha_S^*(\sigma)$ and $TR^* \geq TR_0^*$ otherwise.*

Compared to a market with only STs, the presence of FTs affects the trading rate in two ways. First, they help to reduce the inefficiency that usually arises when σ is large and some of the potential gains from trade between buyers and sellers are not realized because the risk of being picked off induces agents to post limit orders with low execution probabilities. In these situations, their ability to revise quotes quickly when trading with STs protects them against adverse price movements and accordingly reduces the need for quoting cautiously, thereby increasing trade. Second, the presence of FTs introduces an additional inefficiency because their larger outside option effectively creates a dilemma for STs when deciding about their quotation strategy. They may either *i*) increase the aggressiveness of their quotes in order to aim at execution from both FTs and STs (thus maintaining a constant probability of execution compared to the case where $\alpha = 0$) or *ii*) accept a lower execution probability by using a specialized strategy. While we know from Corollary 1 that their expected profits from limit orders decline in either case, the second choice [which is optimal for $\alpha < \alpha_S^*(\sigma)$] is inefficient from a social point of view because it reduces the likelihood that the total surplus of $2L$ is shared. For $\sigma \in \underline{\Sigma}$, this effect yields a net decrease in the trading rate because the efficiency gain of fast cancellations (which dominates otherwise) only arises for a sufficiently high level of volatility. In addition, it gives rise to the following empirical prediction.

Corollary 3 *The presence of FTs induces STs to submit limit orders with lower execution probabilities, that is we have* $p^* \leq p_0^*$ *for all* $\alpha \in (0,1)$.

Notice that FTs may submit more than one limit order per arrival due to their ability to revise quotes, such that it need not be the case that their orders have a higher execution probability than p_0^*. However, they obviously have a higher chance of generating a trade conditional on deciding to post market orders.

While Corollary 3 is, to our knowledge, novel to the literature, several other theoretical papers make predictions about the impact of HFT on trading volume. In Biais, Foucault, and Moinas (2013), FTs are more likely to locate trading opportunities (which increases trading volume) but at the same time possess private information and therefore create adverse selection (which reduces trade). In Jovanovich and Menkveld (2012), the introduction of competitive HFT middlemen raises trading activity by reducing an existing winner's curse problem (similar in spirit to this paper), but at the same time may also lower volume by creating a new adverse selection problem via their superior ability of processing hard information quickly (e.g. real-time datafeeds on index futures). In Cartea and Penalva (2011), HFTs intermediate between end-investors and therefore increase trading volume mechanically. While the accelerated increase in trading activity over the past decades [see e.g. Chordia et a. (2011)] has been accompanied by the advent of HFT, direct empirical evidence on the impact of HFT on trading volume is rather scarce. Jovanovich and Menkveld (2012) report an increase in trade frequency for Dutch equities vis-à-vis a Belgian control group following the introduction of a HFT-friendly trading venue. Brogaard, Hagströmer, Norden and Riordan (2013) report a small, but statistically insignificant increase in volume for Swedish stocks following the introduction of a new co-location service on Nasdaq OMX in Sweden. Clearly, further work based on natural experiments is desirable to shed light on this issue. However, Corollary 2 suggests that future studies should control for the level of volatility because the impact of FTs on trading volume varies with the severity of the adverse selection risk faced by limit orders.

4.2 The risk of being picked off

In this model, agents face the risk of being picked off when they are not sufficiently fast to react to new information by adjusting their quotes. As a consequence, market order traders will sometimes obtain additional windfall profits on top of their outside option value because favourable price movements (from their perspective) render outstanding limit orders stale. These situations arise in the high fill-rate equilibria, where agents post bid quotes that attract execution after both price increases and decreases. While such limit orders guarantee a higher chance of execution, a price decrease induces regret in the sense that the trader would prefer lowering his bid after observing the latest news. Accordingly, the risk of being picked off due to price changes for a buy limit order submitted by a type$-k$ trader is given by

$$\pi^*_{ST} = \begin{cases} \frac{\frac{1}{2}p^*_{|-\sigma}}{p^*} & \text{for } \sigma \in \underline{\Sigma} \\ 0 & \text{for } \sigma \in \overline{\Sigma} \end{cases} \tag{7}$$

$$\pi^*_{FT} = \begin{cases} \frac{\frac{\alpha}{2}q^*_{|FT,-\sigma}}{\alpha q^*_{|FT}+(1-\alpha)\frac{1}{2}} & \text{for } \sigma \in \underline{\Sigma} \\ 0 & \text{for } \sigma \in \overline{\Sigma} \end{cases} \tag{8}$$

As already implied previously, FTs' face a reduced adverse selection risk due to their ability to revise outstanding orders when being followed by a STs. However, their presence simultaneously increases the risk of being picked off for STs despite the fact that those are not even able to adjust their quotes in a market without FTs. The reason for this is once again the difference in traders' bargaining power and can be understood by considering the SHFR equilibrium where $B^*_{ST} = \hat{B}^{v+\sigma*}_{ST}$. Due to their higher outside option, FTs are only willing to submit sell market orders after asset value decreases, that is they only accept the most profitable trading opportunities. Accordingly, the relative odds that slow limit order traders have their quotes hit under unfavourable conditions increase.

Corollary 4 *Compared to the case where $\alpha = 0$, STs (FTs) face a higher (lower) risk of being picked off, i.e. $\pi^*_{ST} \geq \pi^*_0 \geq \pi^*_{FT}$ for all $\alpha \in (0, 1)$.*

Unlike in Bais, Foucault, and Moinas (2013), differences in speed give rise to an asymmetry in the distribution of adverse selection risks across traders. While FTs are picked off less frequently, similar to Jovanovich and Menkveld (2012), this directly translates into a more severe winners' curse problem for STs. Empirically, the π^*_k can be proxied by the price impacts faced by different trader types' limit orders. Brogaard, Hagströmer, Norden and Riordan (2013) study trading on Nasdaq OMX in Sweden and are able to identify a shock to the speed hierarchy by exploiting the introduction of an enhanced co-location service. Consistent with the above prediction, market participants subscribing to the update are able to reduce the risk of being picked off. In addition, some of their results also indicate that FTs impose adverse selection on slow market participants, which is also consistent with the higher permanent price impacts of FTs' market orders documented by Brogaard, Hendershott, and Riordan (2013) and Hendershott and Riordan (2011) for Nasdaq and the German stock exchange, respectively. Furthermore, Carrion (2013) provides additional evidence that FTs' limit (market) orders are associated with lower (higher) permanent price impacts than those of other traders.

4.3 The cost of immediacy for market orders

Following Foucault (1999), we may define the expected trading cost, $E(\tau_k^*)$, as the signed difference between the transaction price and the asset's fundamental value. Now because traders differ in their outside options and moreover face different quotes when arriving to the market (some of the quotes faced by STs have been revised), the trading costs for a particular transaction do not only depend on ε but also on the previous agent's type. Let $\tau_{j,k}^{\varepsilon}$ denote the trading cost incurred by a type-k trader whose market order executes against the limit order of a type-j trader conditional on the latest realization of ε, where $j, k \in \{ST, FT\}$, and let $\omega_{j,k}^{\varepsilon *}$ be the associated probability of this event in equilibrium. Then,

$$E(\tau_k^*) = \sum_{j,\varepsilon} \frac{\omega_{j,k}^{\varepsilon *}}{\sum\limits_{j,\varepsilon} \omega_{j,k}^{\varepsilon *}} \tau_{j,k}^{\varepsilon *} \tag{9}$$

Now because agents' profits from market orders (and hence their expected trading costs) are directly tied to their bargaining power [see equation (1)], it is immediate that FTs trade at more favourable prices than STs.

Corollary 5 *STs incur higher trading costs than FTs, that is* $E(\tau_{ST}^*) > E(\tau_{FT}^*)$ *for all* $\alpha \in (0, 1)$.

Notice that STs do not only pay higher trading costs than FTs as a consequence of their relatively worse outside option. In addition, they also face a decreased likelihood of being able to pick off stale quotes and reap the associated windfall profits in excess of V_{ST}^{LO*}. Given that FTs are sufficiently quick in adjusting their quotes, STs will only encounter these particularly profitable trading opportunities when hitting the quotes of other STs.[14]

Several empirical papers provide evidence that is consistent with Corollary 5. Hasbrouck and Saar (2009) document a considerable increase in the proportion of "fleeting" limit orders from 1990 to 2005, which is suggestive of an increasingly large fraction of quotes not being acessible by slow market participants. Garvey and Wu (2010) show that geographical distance to the market center is negatively related to execution speed and positively related to transactions costs. More direct evidence is presented by Hendershott and Riordan (2013), who study transactions data on the German stock exchange and find that "algorithmic traders consume liquidity when it is cheap" in the sense that they pay lower effective spreads than slower human traders. Similarly, Malinova, Park and Riordan (2013), Carrion (2013), and Brogaard, Hagströmer, Norden and Riordan (2013) show that FTs pay lower effective spreads than STs for Candian, U.S., and Swedish stock exchange, respectively.

[14] Notice that Corollary 5 does not imply that $E(\tau_{ST}^*) > E(\tau_0^*)$ because, for intermediate values of α, the increased aggressiveness of STs' quotes in the unspecialized equilibria can push STs' trading costs below those prevailing in the absence of FTs. However, this effect merely constitutes a re-distribution of gains from trade among STs and is consequently neutral for the breakdown of total trading profits across trader types.

4.4 Make-take decisions

In most other theoretical models that study high-frequency trading, FTs are assumed to be specialized on one side of the market, i.e. they always trade via limit orders or always use market orders. Departing from this assumption allows us to analyze how differences in speed affect the agents' probability of being makers or takers. To this end, we define the maker-taker ratio for a given trader type as the probability of trading via a limit order divided by the probability of trading via a market order. Using the notation introduced in the previous subsection, we have

$$MT_k^* = \frac{\omega_{k,ST}^* + \omega_{k,FT}^*}{\omega_{ST,k}^* + \omega_{FT,k}^*} \tag{10}$$

Now it is easy to see that FTs will only display a make-take ratio of more than 1 if and only if they are more likely to supply liquidity to STs than to consume liquidity from them. It turns out that this is always the case, which is the combined result of two effects. First, FTs enjoy the maximal execution probability of 1/2 conditional on the arrival of a ST because their ability to revise quotes in the light of new information has eliminated the risk of being picked off. Second, FTs are relatively less likely than STs to submit market orders when arriving to the market because they reject some of the quotes that STs would find worth accepting (due to their higher outside option). This is easily verified by looking at the stationary probability distribution in Proposition 2.

In addition, it is also interesting to explore how volatility affects the make-take breakdown for each trader type. To this end, we may hold (α, L) fixed and compare outcomes across equilibria for different parameter values for σ. Intuitively, a higher level of volatility induces traders to submit less aggressive limit orders because of adverse selection. However, FTs will be able to continue providing liquidity to STs without incurring any additional risk because the latter are only able to hit the revised quotes. Consequently, a higher level of σ increases the relative odds of FTs being makers.

Corollary 6 *FTs are always more likely to trade via a limit order than STs, that is we have* $MT_{FT}^* \geq 1 \geq MT_{ST}^*$ *for all* $\alpha \in (0,1)$. *Moreover,* MT_{ST} (MT_{FT}) *is decreasing (increasing) in* σ *for all* $\alpha \in (0,1)$.

The prediction that FTs are more likely to act as makers than as takers is consistent with the extensive market-making activities of large HFT firms (e.g. Getco, Knight, Citadel, Optiver, etc.). In addition, several empirical papers provide evidence that is consistent with this view. Menkveld (2013) studies a large HFT that participated in almost 15% of all transactions in Dutch stocks traded on Euronext and Chi-X and used limit orders roughly 80% of the time. Hagströmer and Norden (2013) and Malinova, Park and Riordan (2013) provide evidence that is consistent with HFT mainly acting as makers in Swedish and Canadian equities, respectively. Chaboud, Chiquoine, Hjalmarsson and Vega (2013) study trading in three different FX pairs and find that in the two most liquid pairs (EUR/USD and JPY/USD) humans are more likely to consume liquidity from computers than vice versa. However, this result does not obtain for the third pair (JPY/EUR), which is likely due to the possibility that most of these trades stem from agents exploiting triangular arbitrage opportunities by market orders. The reason for the finding in Brogaard, Hendershott, and Riordan (2013) that FTs are roughly equally likely to provide or consume liquidity may be similarly founded in arbitrage activities across trading venues in the heavily fragmentated U.S. equity market. To our knowledge, the prediction concerning the impact of volatility on the make-take breakdown for STs and FTs has not yet been tested formally in the literature.

4.5 The pricing error of quotes

Price efficiency is a frequently studied concept in market microstructure and it usually arises in models of asymmetric information. Although there is no uncertainty (and thus no learning) about the true asset value in the present model, quoted prices will still deviate from fundamentals for two reasons. First, information arrivals render existing quotes stale unless they can be updated sufficiently fast. Second, limit order traders have market power and their attempt to extract rents will push quotes away from fundamentals. We define the pricing error of posted quotes (a measure of inefficiency) as the expectation of the absolute difference between the true asset value and the best available quote, that is

$$PE^* = E\left[|v - B^m|\right] \tag{11}$$

It turns out that the presence of FTs may have both a positive and a negative effect on the pricing error. On the one hand, their ability to revise quotes quickly reduces the discrepancy between quotes and fundamentals. On the other hand, speed is a source of market power and may therefore increase the pricing error. Consequently, if fundamentals are sufficiently volatile, pricing errors due to market power are of second order and FTs help to make quotes more efficient. If, however, σ is small relative to L, the shift in bargaining power across trader types can result in larger absolute deviations.

Corollary 7 *Compared to the case where $\alpha = 0$, the presence of FTs decreases (increases) the pricing error if volatility is high (low), that is $PE^* < PE_0^*$ ($PE^* > PE_0^*$) for $\sigma \in \overline{\Sigma}$ ($\sigma \in \widehat{\Sigma} \subset \underline{\Sigma}$) and $\alpha \in (0, 1)$.*

Chaboud, Chiquoine, Hjalmarsson and Vega (2013) examine the rise of algorithmic trading in the foreign exchange market and conclude that FTs increase price efficiency. In line with the first part of the above Corollary, part of this improvement is due to the revision of outstanding quotes. However, they also find that FTs correct short-term mispricing by using market orders, which is more consistent with the models put forward by Biais, Foucault, and Moinas (2013) and Martinez and Rosu (2013). Similarly, Brogaard, Hendershott, and Riordan (2013) find that FTs contribute positively to price efficiency both with limit and with market orders using data on HFT activity from Nasdaq. To our knowledge there is to date no empirical research that is consistent with FTs increasing pricing errors.

5 Welfare and Policy implications

5.1 Welfare

We now turn to a discussion of welfare. Each agent's expected utility is given by his trading profits, which may be expressed as a weighted average of the expected profits from posting limit and market orders

$$W_k^* = \frac{\varphi_k^{LO*}}{\varphi_k^{LO*} + \varphi_k^{MO*}} V_k^{LO*} + \frac{\varphi_k^{MO*}}{\varphi_k^{LO*} + \varphi_k^{MO*}} (L - E(\tau_k^*)) \tag{12}$$

Total welfare can then be computed by weighting the profits for each type by its share in the overall trader population. Because gains from trade are only realized upon the submission of a market order, this is equal to $2L$ times the trading rate.

$$W^* = \alpha W_{FT}^* + (1-\alpha) W_{ST}^* = 2L \times TR^* \tag{13}$$

It turns out that although the presence of FTs actually increases W^* relative to its level for $\alpha = 0$ if σ is sufficiently large (see Corollary 2), the associated redistribution in bargaining power allows them to capture all the extra surplus. As a consequence, STs are *always* worse off, not only in the instances where differences in speed lead to less trading.

Corollary 8 *We have $W_{FT}^* > W_0^* > W_{ST}^*$ for all $\alpha \in (0,1)$.*

This result has profound implications for social welfare once we discard the assumption that α is exogenous. To see this, assume now that all traders are born slow but may decide to becom fast prior to the start of trading at a cost c as in Biais, Foucault, and Moinas (2013). Let α^* denote the proportion of traders who decide to become fast in equilibrium. Then, an interior solution[15] implies that both trader types earn the same net profits

$$W_{FT}^*(\alpha^*) - c = W_{ST}^*(\alpha^*) \tag{14}$$

[15] Existence may fail for some c because the difference $W_{FT}^*(\alpha^*) - W_{ST}^*(\alpha^*)$ is not continuous at the points where we move from one equilibrium to another.

and social welfare with endogenous investment in trading speed is given by

$$\hat{W}^*(\alpha^*) = 2L \times TR^*(\alpha^*) - \alpha^* c \tag{15}$$

Obviously, corner equilibria may arise in cases where c is either prohibitively high or alternatively where the incremental benefit of becoming fast is always strictly positive. However, the fact that speed is a purely relative advantage implies $\hat{W}^*(1) = \hat{W}^*(0) - c$, so that we conclude the following.

Corollary 9 *Equilibrium investment in relative trading speed always lowers social welfare relative to the situation where $\alpha = 0$, that is $\hat{W}^*(\alpha^*) < \hat{W}^*(0)$ for all $\alpha^* > 0$.*

Together with the equilibrium condition (14), the decrease in STs' trading profits (Corollary 8) implies that the *equilibrium* level of investment always leads to a welfare loss relative to a market where all traders are slow. This effect arises because speed helps agents to exert market power on STs, which creates incentives for becoming fast even in situations where it is socially wasteful and accordingly there is too much investment from a social welfare perspective. STs are hurt because their lower bargaining power leads to a reduced profitability of limit orders and fewer attractive trading opportunities as they are less likely to encounter stale quotes. Consequently, the negative externality here is different from the one in Biais, Foucault, and Moinas (2013). Also, unlike in their model, full investment ($\alpha^* = 1$) is always detrimental to welfare because the resulting outcomes are identical to the ones arising when all traders are slow. In this context, it is important to highlight that our model is entirely focused on the role of relative speed and does not consider other, potentially beneficial effects related to the use of computer technology (e.g. reduced search costs).

5.2 Policy implications

Because investment in trading speed generates a negative externality for slow market participants, regulatory intervention can possibly improve on the market outcome that entails a social welfare loss. However, it is important to note that some investment into speed can be beneficial as FTs allow more gains from trade to be reaped for $\sigma \in \overline{\Sigma}$, such that the optimal level of α is strictly positive in this case for c sufficiently small. Thus, in an ideal world, the policies aiming at mitigating the negative effects of HFT should be designed carefully in order to allow society to enjoy some of the possible benefits.

In the context of our model, a natural way of achieving such an outcome would be the introduction of a fee on order cancellations. Effectively, this measure would lower FTs' profits and therefore reduce the incentives to become fast in equilibrium. In line with this idea, France and Italy have recently introduced surcharges for HFTs that submit more than five messages per trade and do not act as market makers as part of their transactions taxes on equities. However, it is important to keep in mind that such a policy would not affect market participants that use their speed advantage to pick off others through market orders, a mechanism absent in our theory.

A realistic feature of our model is that differences in speed lead to a re-distribution in market power across traders. In today's markets, standard rules such as continuous trading and price-time priority allow agents to extract considerable rents by being first in a queue of limit orders or seizing fleeting trading opportunities ahead of slower competitors. Because of the "winner takes it all" nature of many trading situations, agents are willing to spend large sums on only marginal speed improvements. These developments call for policies that aim at reducing the market power associated with being fast in order to lower the incentives for market participants to engage in a costly arms race. Suitable measures could be the adoption of pro-rata matching for the most liquid assets and the introduction of tiny "speed bumps" such as the ones that are have been recently introduced in several FX trading platforms.[16] While such rather small changes are unlikely to distort market functioning, they would certainly help to eliminate trading strategies that solely rely on relative speed (e.g. latency arbitrage).

6 Conclusion

This paper contributes to the ongoing controversy about the benefits and concerns related to HFT by presenting a stylized model of a limit order market where investors differ in their trading speed. We show that FTs help to reap more gains from trade because their speed advantage reduces the risk of being "picked off" and therefore eliminates the need for posting cautious limit orders. However, a new inefficiency arises as STs strategically submit limit orders with a lower execution probability as a response to FTs' increased outside option. Because differences in speed lead to a redistribution of market power across the trader population, STs are always worse off than in a market with identical traders. This directly translates into a social welfare loss if the proportion of FTs is not given exogenously but rather determined in equilibrium through costly investment. We discuss some potential policy implications that may help to stop the ongoing arms race in the financial industry. In addition, the model delivers a number of empirical predictions concerning the possible effects of HFT in limit order markets, which are either novel to the literature or consistent with existing empirical work.

[16] In April 2013, eleven major banks launched ParFX, a trading platform for currencies where each order is delayed randomly by 20-80 milliseconds. Similarly, EBS started to batch incoming messages and randomizing their order of arrival in August. See "EBS take new step to rein in high-frequency traders", Reuters Newswire, August 23rd, 2013.

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Appendix A - Proofs

Proof of Lemma 1

Clearly, the ability to revise limit orders can never be a disadvantage such that we trivially have $V_{FT}^{LO*} \geq V_{ST}^{LO*}$, and equation (2) then implies that $\hat{B}_{FT}^{v+\varepsilon*} \geq \hat{B}_{ST}^{v+\varepsilon*}$ for all ε.

It remains to show that $\hat{B}_{ST}^{v+\sigma*} \geq \hat{B}_{FT}^{v-\sigma*}$. First, notice that L is the maximum *expected* gains from trade that per period (if two agents with different private valuations trade they share a surplus of $2L$, but this occurs at most with probability 1/2) we must have $L \geq V_k^{LO*} \geq 0$ for $k \in \{ST, FT\}$. Now assume that $\sigma \geq L/2$. Using equation (2), we have $v_t + \sigma \geq \hat{B}_{ST}^{v+\sigma*} \geq v_t + \sigma - L$ and $v_t - \sigma \geq \hat{B}_{FT}^{v-\sigma*} \geq v_t - \sigma - L$, which directly implies $\hat{B}_{ST}^{v+\sigma*} \geq \hat{B}_{FT}^{v-\sigma*}$. Instead, suppose that $\sigma < L/2$ and consider a fast buyer submitting a buy limit order. It is easy to see that in this case we have $\frac{\alpha}{4}[v - \sigma + L - \hat{B}_{FT}^{v+\sigma*}] + \frac{\alpha}{4}[v + \sigma + L - \hat{B}_{FT}^{v+\sigma*}] \geq \frac{\alpha}{4}[v - \sigma + L - \hat{B}_{FT}^{v-\sigma*}]$ such that his optimal choice is $B_{FT}^* = \hat{B}_{FT}^{v+\sigma*}$. Notice that a buyer arriving at one period later will never execute this order because $v - \sigma + L > \hat{B}_{FT}^{v+\sigma*}$, that is the bid price B_{FT}^* is below his lowest possible valuation. Now consider a slow buyer and suppose he posts a buy limit order with $B_{ST}^* = \hat{B}_{FT}^{v+\sigma*}$. As this is not necessarily his equilibrium strategy we have that $V_{ST}^{LO*} \geq \frac{1}{2}[v + L - \hat{B}_{FT}^{v+\sigma*}]$. But we just concluded that $V_{FT}^{LO*} = \frac{\alpha}{2}[v + L - \hat{B}_{FT}^{v+\sigma*}] + \frac{1-\alpha}{4}[v - \sigma + L - \hat{B}_{ST}^{v-\sigma*}] + \frac{1-\alpha}{4}[v + \sigma + L - \hat{B}_{ST}^{v+\sigma*}]$, and therefore $V_{FT}^{LO*} - V_{ST}^{LO*} \leq \frac{1-\alpha}{4}[\hat{B}_{FT}^{v+\sigma*} - \hat{B}_{ST}^{v-\sigma*}] + \frac{1-\alpha}{4}[\hat{B}_{FT}^{v+\sigma*} - \hat{B}_{ST}^{v+\sigma*}] = \frac{1-\alpha}{2}[V_{FT}^{LO*} - V_{ST}^{LO*}] + \frac{1-\alpha}{2}\sigma$. Then, using equation (2), we obtain $V_{FT}^{LO*} - V_{ST}^{LO*} \leq \frac{1-\alpha}{1+\alpha}\sigma$, which lets us conclude that $\hat{B}_{ST}^{v+\sigma*} > \hat{B}_{FT}^{v-\sigma*}$.

Proof of Proposition 1

Finding an equilibrium involves 2 steps:
1) Conjecture equilibrium strategies and solve for the resulting outside option values / equilibrium cutoff prices.
2) Find parameter restrictions for which the assumed strategies are best replies (i.e. deviations are not profitable)

Type 1 equilibrium:
Step 1: Assume $B_{ST}^* = \hat{B}_{ST}^{v-\sigma*}$ and $B_{FT}^* = \hat{B}_{FT}^{v-\sigma*}$ which implies that $p^* = \frac{1-\alpha}{4}$ and $q_{|FT}^* = \frac{1}{4}$. Using the the optimally revised FT quotes with conditional execution probabilities $q_{|ST,+\sigma} = q_{|ST,-\sigma} = \frac{1}{2}$, the equilibrium expected profits from posting limit orders are given by

$$\begin{aligned}
V_{ST}^{LO*} &= \frac{1-\alpha}{4}\left[v-\sigma+L-\hat{B}_{ST}^{v-\sigma*}\right] && (16)\\
V_{FT}^{LO*} &= \frac{\alpha}{4}\left[v-\sigma+L-\hat{B}_{FT}^{v-\sigma*}\right] && (17)\\
&\quad+\frac{1-\alpha}{4}\left[v+\sigma+L-\hat{B}_{ST}^{v+\sigma*}\right]\\
&\quad+\frac{1-\alpha}{4}\left[v-\sigma+L-\hat{B}_{ST}^{v-\sigma*}\right]
\end{aligned}$$

Using the fact that the cutoff prices are linear in the asset value, we can use equation (2) to obtain a system of 2 equations in the 2 unknowns which can be solved for $V_{ST}^{LO*} = (2L)\frac{1-\alpha}{5-\alpha}$ and $V_{FT}^{LO*} = (2L)\frac{8-\alpha(3+\alpha)}{(5-\alpha)(4+\alpha)}$.

Step 2: Given that optimality implies that any equilibrium quote must be chosen from the set of sellers' sell cutoff prices, there are no profitable deviations for slow buyers iff

$$\begin{aligned}
\frac{1-\alpha}{4}\left[v-\sigma+L-\hat{B}_{ST}^{v-\sigma*}\right] &\geq \frac{1}{4}\left[v-\sigma+L-\hat{B}_{FT}^{v-\sigma*}\right] && (18)\\
\frac{1-\alpha}{4}\left[v-\sigma+L-\hat{B}_{ST}^{v-\sigma*}\right] &\geq \frac{1}{4}\left[v-\sigma+L-\hat{B}_{ST}^{v+\sigma*}\right] && (19)\\
&\quad+\frac{1-\alpha}{4}\left[v+\sigma+L-\hat{B}_{ST}^{v+\sigma*}\right]\\
\frac{1-\alpha}{4}\left[v-\sigma+L-\hat{B}_{ST}^{v-\sigma*}\right] &\geq \frac{1}{4}\left[v-\sigma+L-\hat{B}_{FT}^{v+\sigma*}\right] && (20)\\
&\quad+\frac{1}{4}\left[v+\sigma+L-\hat{B}_{FT}^{v+\sigma*}\right]
\end{aligned}$$

Similarly, fast buyers have no incentives to deviate iff:

$$\begin{aligned}
\frac{\alpha}{4}\left[v-\sigma+L-\hat{B}_{FT}^{v-\sigma*}\right] &\geq \frac{\alpha}{4}\left[v-\sigma+L-\hat{B}_{FT}^{v+\sigma*}\right] && (21)\\
&\quad+\frac{\alpha}{4}\left[v+\sigma+L-\hat{B}_{FT}^{v+\sigma*}\right]
\end{aligned}$$

Using again equation (2), it can be easily verified that this is the case for $\alpha \leq \sqrt{5}-2$ and $\sigma \geq L\frac{4}{5-\alpha}$, such that the above quotation strategies constitute an equilibrium for this range of parameters. The proof for the remaining equilibrium types follows exactly the same logic, such that we omit them for brevity. Now define the functions $\sigma^*_{ST}(\alpha)$, $\sigma^*_{FT}(\alpha)$ and $\alpha^*_S(\sigma)$ as

$$\sigma^*_{ST}(\alpha) \equiv \begin{cases} \sigma^*_1(\alpha) & \text{if } \alpha \le \alpha^*_1 \\ \sigma^*_4(\alpha) & \text{if } \alpha^*_1 < \alpha \le \alpha^*_2 \\ \sigma^*_5(\alpha) & \text{if } \alpha^*_2 < \alpha \end{cases} \tag{22a}$$

$$\sigma^*_{FT}(\alpha) \equiv \begin{cases} \sigma^*_3(\alpha) & \text{if } \alpha \le \alpha^*_2 \\ \sigma^*_5(\alpha) & \text{if } \alpha^*_2 < \alpha \end{cases} \tag{22b}$$

$$\alpha^*_S(\sigma) \equiv \begin{cases} \alpha^*_1 & \text{if } \sigma \ge \sigma^*_1(\alpha^*_1) \\ \sigma^{*-1}_4(\sigma) & \text{if } \sigma^*_1(\alpha^*_1) > \sigma \ge \sigma^*_3(\alpha^*_2) \\ \sigma^{*-1}_2(\sigma) & \text{if } \sigma^*_3(\alpha^*_2) > \sigma \end{cases} \tag{22c}$$

where $\alpha^*_1 \equiv \sqrt{5} - 2$, $\alpha^*_2 \equiv \frac{\sqrt{33}-5}{2}$, $\sigma^*_1(\alpha) \equiv L\frac{4}{5-\alpha}$, $\sigma^*_2(\alpha) \equiv L\frac{2\alpha(1+\alpha)}{3-4\alpha}$, $\sigma^*_3(\alpha) \equiv L\frac{4(4+\alpha)}{26-\alpha^2}$, $\sigma^*_4(\alpha) \equiv L\frac{2(1-\alpha)(4+\alpha)}{7+3\alpha}$, and $\sigma^*_5(\alpha) \equiv L\frac{4(1+\alpha)}{7+3\alpha}$ and the superscript -1 denotes an inverse function. Then, the following table contains the equilibrium strategies, execution probabilities and parameter conditions for each type of equilibrium.

Equilibrium	B^*_{ST}	B^*_{FT}	p^*	$q^*_{\|FT}$	Condition α	Condition σ
1 (SLFR)	$\hat{B}^{v-\sigma*}_{ST}$	$\hat{B}^{v-\sigma*}_{FT}$	$\frac{1-\alpha}{4}$	$\frac{1}{4}$	$\alpha \le \alpha^*_S(\sigma)$	$\sigma \ge \sigma^*_{ST}(\alpha)$
2 (ULFR)	$\hat{B}^{v-\sigma*}_{FT}$	$\hat{B}^{v-\sigma*}_{FT}$	$\frac{1}{4}$	$\frac{1}{4}$	$\alpha > \alpha^*_S(\sigma)$	$\sigma \ge \sigma^*_{ST}(\alpha)$
3	$\hat{B}^{v+\sigma*}_{ST}$	$\hat{B}^{v-\sigma*}_{FT}$	$\frac{2-\alpha}{4}$	$\frac{1}{4}$	$\alpha \le \alpha^*_S(\sigma)$	$\sigma^*_{ST}(\alpha) > \sigma \ge \sigma^*_{FT}(\alpha)$
4 (SHFR)	$\hat{B}^{v+\sigma*}_{ST}$	$\hat{B}^{v+\sigma*}_{FT}$	$\frac{2-\alpha}{4}$	$\frac{1}{2}$	$\alpha \le \alpha^*_S(\sigma)$	$\sigma < \sigma^*_{FT}(\alpha)$
5 (UHFR)	$\hat{B}^{v+\sigma*}_{FT}$	$\hat{B}^{v+\sigma*}_{FT}$	$\frac{1}{2}$	$\frac{1}{2}$	$\alpha > \alpha^*_S(\sigma)$	$\sigma < \sigma^*_{FT}(\alpha)$

and the associated outside option values are given by

Equilibrium	V^{LO*}_{ST}	V^{LO*}_{FT}
1 (SLFR)	$(2L)\frac{1-\alpha}{5-\alpha}$	$(2L)\frac{8-\alpha(3+\alpha)}{(5-\alpha)(4+\alpha)}$
2 (ULFR)	$(2L)\frac{1+\alpha}{7+3\alpha}$	$(2L)\frac{3-\alpha}{7+3\alpha}$
3	$(2L)\frac{2-\alpha}{6-\alpha} - \frac{2}{6-\alpha}\sigma$	$(2L)\frac{8-\alpha(2+\alpha)}{(6-\alpha)(4+\alpha)} + \frac{4(1-\alpha)}{(6-\alpha)(4+\alpha)}\sigma$
4 (SHFR)	$(2L)\frac{2-\alpha}{6-\alpha} - \frac{2}{6-\alpha}\sigma$	$(2L)\frac{4+\alpha(2-\alpha)}{(6-\alpha)(2+\alpha)} + \frac{2-\alpha(8-\alpha)}{(6-\alpha)(2+\alpha)}\sigma$
5 (UHFR)	$(2L)\frac{1}{3} - \frac{2}{3(1+\alpha)}\sigma$	$(2L)\frac{1}{3} + \frac{1-3\alpha}{3(1+\alpha)}\sigma$

Finally, it is straightforward, albeit tedious to show that no other equilibria exist, which establishes uniqueness.

Proof of Proposition 2

On the equilibrium path, the transitions from one state to another follow a Markov chain with transition matrix

$$P^* = \begin{bmatrix} (1-\alpha)(1-p^*_{|ST}) & (1-\alpha)p^*_{|ST} & \alpha(1-p^*_{|FT}) & \alpha p^*_{|FT} \\ 1-\alpha & 0 & \alpha & 0 \\ (1-\alpha)\frac{1}{2} & (1-\alpha)\frac{1}{2} & \alpha(1-q^*_{|FT}) & \alpha q^*_{|FT} \\ 1-\alpha & 0 & \alpha & 0 \end{bmatrix} \tag{23}$$

As in Colliard and Foucault (2012), the stationary probability distribution $\varphi^* = (\varphi^{LO*}_{ST}, \varphi^{MO*}_{ST}, \varphi^{LO*}_{FT}, \varphi^{MO*}_{FT})$ is then simply given by the left eigenvector of P^* associated with the unit modulus.

Proof of Corollary 1

The result follows immediately from the expected limit order profits computed in the proof of Proposition 1, where $V_0^{LO*} = \frac{1}{3}(2L-\sigma)$ for $\sigma \in \underline{\Sigma}$ and $V_0^{LO*} = \frac{2}{5}L$ for $\sigma \in \overline{\Sigma}$.

Proof of Corollary 2

The trading rates for each type of equilibrium are obtained by substituting the respective execution probabilities from the proof of Proposition 1 into the formula for the stationary probability distribution derived in Proposition 2. We collect them in the following table.

Equilibrium	TR^*
SLFR	$\frac{4+\alpha(1-\alpha)}{(4+\alpha)(5-\alpha)}$
ULFR	$\frac{4+3\alpha(1-\alpha)}{20-\alpha+\alpha^2}$
SHFR	$\frac{4-\alpha(1-\alpha)}{12+\alpha-\alpha^2}$
UHFR	$\frac{1}{3}$

Because $TR_0^* = 1/3$ for $\sigma \in \underline{\Sigma}$ and $TR_0^* = 1/5$ for $\sigma \in \overline{\Sigma}$, the result follows immediately.

Proof of Corollary 3

The result follows directly from STs' execution probabilities computed in the Proof of Proposition 1, where $p_0^* = 1/2$ for $\sigma \in \underline{\Sigma}$ and $p_0^* = 1/4$ for $\sigma \in \overline{\Sigma}$.

Proof of Corollary 4

From the computations involved in the Proof of Proposition 1 it follows that $p^*_{|-\sigma} = q^*_{|FT,-\sigma} = 1/2$ for $\sigma \in \underline{\Sigma}$. It is then easy to deduce that the probability of being picked off for each type of equilibrium is given by

Equilibrium	π^*_{ST}	π^*_{FT}
SLFR	0	0
ULFR	0	0
SHFR	$\frac{1}{2-\alpha}$	$\frac{\alpha}{2}$
UHFR	$\frac{1}{2}$	$\frac{\alpha}{2}$

Because $\pi^*_0 = \frac{1}{2}$ for $\sigma \in \underline{\Sigma}$ and $\pi^*_0 = 0$ for $\sigma \in \overline{\Sigma}$, the result follows.

Proof of Corollary 5

The $\omega^{\varepsilon *}_{ST,k}$ are easily computed by multiplying the probability of observing a limit order by a type$-k$ trader, φ^{LO*}_k, with the respective conditional execution probability. Then, some rather involved computations reveal that

Equilibrium	$E(\tau^*_{ST})$	$E(\tau^*_{FT})$
SLFR	$\frac{3+\alpha}{5-\alpha}L$	$\frac{4+\alpha(7+\alpha)}{(5-\alpha)(4+\alpha)}L$
ULFR	$\frac{4+\alpha(55-\alpha(16-5\alpha))}{(7+3\alpha)(4+\alpha(3+\alpha))}L$	$\frac{1+5\alpha}{7+3\alpha}L$
SHFR	$\frac{2+\alpha}{6-\alpha}L - \frac{16-6\alpha(5-\alpha)}{(4+\alpha(1-\alpha))(6-\alpha)}\sigma$	$\frac{8(1-\alpha)(2+\alpha)^2+4\alpha(5-\alpha)(4+\alpha^2)}{(2+\alpha)(6-\alpha)(8+4\alpha(3-\alpha))}L - \frac{2(5-\alpha)(8(1-\alpha)(2+\alpha)+\alpha(28-8\alpha))}{(2+\alpha)(6-\alpha)(8+4\alpha(3-\alpha))}\sigma$
UHFR	$\frac{1}{3}L - \frac{4-6\alpha}{3(1+\alpha)}\sigma$	$\frac{1}{3}L - \frac{4}{3(1+\alpha)}\sigma$

The result follows.

Proof of Corollary 6

The following table collects MT^*_{ST} and MT^*_{FT} for all equilibria.

Equilibrium	MT^*_{ST}	MT^*_{FT}
SLFR	$\frac{4-5\alpha+\alpha^2}{4+5\alpha-\alpha^2}$	$\frac{2-\alpha}{\alpha}$
ULFR	$\frac{4-\alpha}{4+3\alpha+\alpha^2}$	$\frac{8-4\alpha}{4-\alpha+\alpha^2}$
SHFR	$\frac{4-2\alpha}{4+\alpha-\alpha^2}$	$\frac{20-4\alpha}{8+12\alpha-4\alpha^2}$
UHFR	1	1

In order to show that MT^*_{FT} (MT^*_{ST}) is increasing (decreasing) in σ, we require that an equilibrium transition due to an increase in σ yields an increase (decrease) in the ratio. Under Assumption 1, increases in σ given a fixed level of (α, L) may lead to the following transitions: UHFR→SHFR, SHFR→SLFR, SHFR→ULFR, and UHFR→ULFR. It is easily verified that this is always the case such that the result follows.

Proof of Corollary 7

Expanding equation (11) yields

$$\begin{aligned} PE^* &= \frac{\varphi^{LO*}_{ST}}{\varphi^{LO*}_{ST}+\varphi^{LO*}_{FT}}\left[\frac{1}{2}\left|v+\sigma-B^*_{ST}\right|+\frac{1}{2}\left|v-\sigma-B^*_{ST}\right|\right] \\ &\quad+\frac{\varphi^{LO*}_{FT}}{\varphi^{LO*}_{ST}+\varphi^{LO*}_{FT}}\left[\begin{array}{c}\frac{\alpha}{2}\left|v+\sigma-B^*_{FT}\right|+\frac{\alpha}{2}\left|v-\sigma-B^*_{FT}\right| \\ +\frac{(1-\alpha)}{2}\left|v+\sigma-B^{+\sigma*}_{FT}\right|+\frac{(1-\alpha)}{2}\left|v-\sigma-B^{-\sigma*}_{FT}\right|\end{array}\right] \end{aligned} \tag{24}$$

Then, substituting the respective equilibrium quotes B^*_{ST} and $(B^*_{FT}, B^{+\sigma*}_{FT}, B^{-\sigma*}_{FT})$ leads to the following expressions for the pricing error in the different types of equilibria.

Equilibrium	PE^*	Condition on σ
SLFR	$\frac{(3+\alpha)(4+\alpha)-2\alpha^2(5-\alpha)}{(5-\alpha)(4+\alpha)}L+\frac{4-\alpha(5-\alpha)(1-\alpha)}{4}\sigma$	
ULFR	$\frac{4+35\alpha+36\alpha^2+21\alpha^3}{(7+3\alpha)(4-\alpha+\alpha^2)}L+\frac{4-5\alpha(1-\alpha)}{4-\alpha+\alpha^2}\sigma$	
SHFR	$\frac{(5-\alpha)(1-\alpha)\alpha(2+\alpha)}{(6-\alpha)(4+\alpha(1-\alpha))}L+\frac{24-\alpha(18-\alpha(22-\alpha(9-\alpha)))}{(6-\alpha)(4+\alpha(1-\alpha))}\sigma$	for $\sigma > L\ \frac{2+\alpha}{10-2\alpha}$
	$\frac{(1-\alpha)(2+\alpha)(4+\alpha(5-\alpha))}{(6-\alpha)(4+\alpha(1-\alpha))}L-\frac{16-(5-\alpha)\alpha(6+\alpha(4-\alpha))}{(6-\alpha)(4+\alpha(1-\alpha))}\sigma$	for $L\frac{2+\alpha}{10-2\alpha} \geq \sigma > L\ \frac{4+\alpha^2}{26-\alpha^2}$
	$\frac{16+\alpha(20+\alpha(4-\alpha(23-3\alpha)))}{(2+\alpha)(6-\alpha)(4+\alpha(1-\alpha))}L-\frac{2(16-\alpha(22-\alpha(36-\alpha(11-\alpha))))}{(2+\alpha)(6-\alpha)(4+\alpha(1-\alpha))}\sigma$	for $L\ \frac{4+\alpha^2}{26-\alpha^2} \geq \sigma$
UHFR	$\frac{\alpha(1-\alpha)}{3}L+\frac{2\alpha(1-\alpha)+3(1+\alpha^3)}{3(1+\alpha)}\sigma$	for $\sigma > \frac{1+\alpha}{7+3\alpha}L$
	$\frac{1}{3}L-\frac{4-6\alpha(1-\alpha)}{3(1+\alpha)}\sigma$	for $\frac{1+\alpha}{7+3\alpha}L \geq \sigma$

It is easy to show that we have $PE^*_0 = \frac{3}{5}L+\sigma$ for $\sigma \in \overline{\Sigma}$, $PE^*_0 = \sigma$ for $\underline{\sigma} \geq \sigma > \frac{1}{5}L$ and $PE^*_0 = \frac{1}{3}L-\frac{2}{3}\sigma$ for $\frac{1}{5}L \geq \sigma$. It follows that $PE^* < PE^*_0$ for all $\alpha \in (0,1)$ and $\sigma \in \overline{\Sigma}$. Similarly, one may verify that we have $PE^* > PE_0^*$ for all $\alpha \in (0,1)$ when $\sigma \in \widehat{\Sigma} \equiv (\hat{\sigma}, \breve{\sigma})$, where $\hat{\sigma} = L/2$ and $\breve{\sigma} \simeq 0.17785L$. In order to see this, first focus on the case where $\sigma > \frac{1}{5}L$. In this parameter range, it is easy to see that $PE^* > PE_0^*$ for all $\alpha \in (0,1)$ whenever $\sigma < L/2$. Now consider the case where $\frac{1}{5}L \geq \sigma$. While tedious, it can be verified that for this parameter range we have $PE^* > PE_0^*$ in any SHFR. Now consider the UFHR equilibrium and suppose that $L/5 \geq \sigma > \frac{1+\alpha}{7+3\alpha}L$. In this case, $PE^* > PE_0^*$ implies that $\sigma >$. Recall from Proposition 1 that such an equilibrium can only arise for $\sigma_2^*(\alpha) > \sigma$. It follows that $\breve{\sigma} = \sigma_2^*(\tilde{\alpha})$ where $\tilde{\alpha} \simeq 0.17436$ is the unique real positive solution to $\sigma_2^*(\alpha) = \frac{(1+\alpha)(1-\alpha(1-\alpha))}{5+4\alpha-2\alpha^2+3\alpha^3}L$. Finally, for $\frac{1+\alpha}{7+3\alpha}L \geq \sigma$ we have $PE^* < PE_0^*$ ($PE^* > PE_0^*$) in the UHFR equilibrium for $\alpha < 1/3$ ($\alpha > 1/3$).

Proof of Corollary 8

Direct substitution into equation (12) yields the following.

Equilibrium	W_{ST}^*	W_{FT}^*
SLFR	$(2L)\frac{1-\alpha}{5-\alpha}$	$(2L)\frac{8-\alpha(3+\alpha)}{(5-\alpha)(4+\alpha)}$
ULFR	$\frac{28+\alpha(1+\alpha(12-\alpha))}{(7+3\alpha)(20-\alpha+\alpha^2)}(2L)$	$(2L)\frac{3-\alpha}{7+3\alpha}$
SHFR	$\frac{2-\alpha}{6-\alpha}(2L) - \frac{6\alpha(5-\alpha)}{(12+\alpha(1-\alpha))(6-\alpha)}\sigma$	$\frac{48+\alpha(24-\alpha(10+\alpha(3-\alpha)))}{(2+\alpha)(6-\alpha)(12+\alpha(1-\alpha))}(2L) + \frac{60-\alpha(42+\alpha(24-6\alpha))}{(2+\alpha)(6-\alpha)(12+\alpha(1-\alpha))}\sigma$
UHFR	$(2L)\frac{1}{3} - \frac{2\alpha}{3(1+\alpha)}\sigma$	$(2L)\frac{1}{3} + \frac{2(1-\alpha)}{3(1+\alpha)}\sigma$

We have $W_0^* = \frac{2}{3}L$ for $\sigma \in \underline{\Sigma}$ and $W_0^* = \frac{2}{5}L$ for $\sigma \in \overline{\Sigma}$, such that the result follows immediately.

Proof of Corollary 9

Follows from Corollary 8, equation (14) and the fact that $\hat{W}^*(1) < \hat{W}^*(0)$.

Appendix B - Figures

Figure 1: Timing of events in the limit order market

This figure depicts the timing of events from the arrival of a trader at time t until the arrival of the next trader.

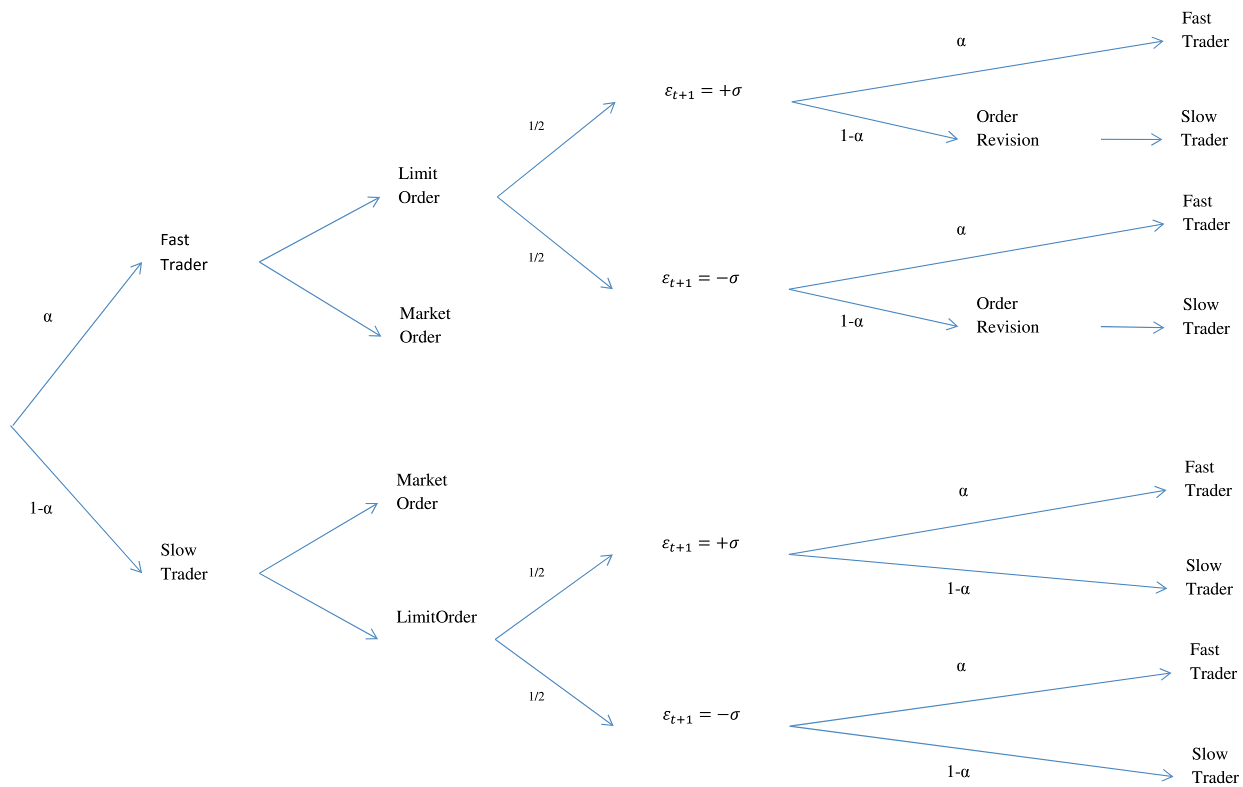

Figure 2: Equilibrium Map

This figure depicts the functions $\sigma^*_{FT}(\alpha)$, $\sigma^*_{ST}(\alpha)$ and $\alpha^*_S(\sigma)$ defined in equations (22a) - (22c) in the (α, σ)-space, where we have set $L = 1$. The shaded grey area indicates the interval $[\underline{\sigma}, \overline{\sigma})$.