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ABSTRACT

This paper proposes a novel Pearson-type quasi maximum likelihood estimator (QMLE) of GARCH(p, q) models. Unlike the existing Gaussian QMLE, Laplacian QMLE, generalized non-Gaussian QMLE, or LAD estimator, our Pearsonian QMLE (PQMLE) captures not just the heavy-tailed but also the skewed innovations. Under strict stationarity and some weak moment conditions, the strong consistency and asymptotical normality of the PQMLE are obtained. With no further efforts, the PQMLE can apply to other conditionally heteroskedastic models. A simulation study is carried out to assess the performance of the PQMLE. Two applications to eight major stock indexes and four exchange rates further highlight the importance of our new method. Heavy-tailed and skewed innovations are often observed together in practice, and the PQMLE now gives us a systematical way to capture these two co-existing features.

Some key words: Asymmetric innovation; Conditionally heteroskedastic model; Exchange rates; GARCH model; Leptokurtic innovation; Non-Gaussian QMLE; Pearson's Type IV distribution; Pearsonian QMLE; Stock indexes.

1. INTRODUCTION

After the seminal work of Engle (1982) and Bollerslev (1986), numerous volatility models have been widely used to capture the feature of conditional heteroscedasticity in economic and financial data sets; see, e.g., Bollerslev, Chou and Kroner (1992), Bera and Higgins (1993) and Francq and Zakoïan (2010). Among them, the most influential model in empirical studies is the GARCH(p, q) model given by

$$y_t = \sigma_t \varepsilon_t, \tag{1}$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2, \tag{2}$$

where $\omega > 0$, $\alpha_i \geq 0 (i = 1, \cdots, p)$, $\beta_j \geq 0 (j = 1, \cdots, q)$, and ε_t is a sequence of i.i.d. random variables. Traditional inference for the GARCH model is based on the Gaussian quasi maximum likelihood estimator (GQMLE), which is proposed by assuming that ε_t follows a standard normal distribution. Berkes, Horváth and Kokoszka (2003) showed that when ε_t has a finite fourth moment with $E\varepsilon_t^2 = 1$ (the identification condition), the GQMLE is consistent and asymptotically normal. However, the GQMLE can not capture the heavy-tailedness and skewness of ε_t, which are two well-observed features of financial data in GARCH model applications; see, e.g., Engle and González-Rivera (1991), Christoffersen, Heston, and Jacobs (2006), and Grigoletto and Lisi (2009). Motivated by this, the MLE, based on a user-chosen heavy-tailed or skewed likelihood function, so far has been largely considered. For instance, ε_t can be the Student's t distribution in Bollerslev (1987), the gamma distribution in Engle and González-Rivera (1991), the generalized error distribution in Nelson (1991), the skewed t distribution in Hansen (1994), the stable distribution in Liu and Brorsen (1995), the noncentral t distribution in Harvey and Siddique (1999), the Pearson's Type IV distribution in Premaratne and Bera (2001), the Gram-Charlier distribution in Leon, Rubio, and Serna (2005) and Cheng et al. (2011), the mixture normal distribution in Bai, Russell and Tiao (2003) and many others. However, the true distribution of ε_t is unknown a priori in practice, and as shown in Newey and Steigerwald (1997), the MLE may lead to inconsistent estimates of models (1)-(2) if the distribution of ε_t is misspecified.

In order to obtain a consistent estimator without knowing the true distribution of ε_t, people prefer to use the non-Gaussian QMLE (NGQMLE), which has efficiency advantage over GQMLE when ε_t is heavy-tailed. Generally, there are two ways to obtain a consistent NGQMLE. First, one can assume a different identification condition rather than $E\varepsilon_t^2 = 1$. For instance, Peng and Yao (2003) proposed a least absolute deviation estimator (LADE) under the identification condition that median$(\varepsilon_t^2) = 1$, and the consistency and asymptotic normality of LADE was proved in Chen and Zhu (2013) under only a finite fractional moment of ε_t. By assuming that ε_t follows a standard Laplace distribution, Berkes and Horváth (2004) considered the Laplacian QMLE (LQMLE) under the identification condition that $E|\varepsilon_t| = 1$, and they showed that the LQMLE is consistent and asymptotically normal when ε_t has a finite second moment; see also Li and Li (2008) and Zhu and Ling (2011) for more discussions in this context. Secondly, one can retain the identification condition $E\varepsilon_t^2 = 1$ for NGQMLE but re-parameterize models (1)-(2). This method has been used for the semi-parametric estimator in Drost and Klaassen (1997), the rank-based estimator in Andrews (2012), and the generalized NGQMLE (GNGQMLE) in Fan, Li and Xiu (2013). By introducing a scale adjustment parameter, the GNGQMLE is consistent and asymptotical normal when ε_t has a finite second moment, while the semi-parametric and rank-based estimators can only estimate the heteroscedastic parameters α_i and β_j under the same re-parameterized GARCH(p, q) model. Morevoer, it is worth noting that when ε_t has an infinite fourth moment, all of LADE, LQMLE and GNGQMLE achieve root-n convergency, while the GQMLE suffers a slower convergence rate as shown in Hall and Yao (2003).

In this paper, we propose a Pearsonian QMLE (PQMLE) of models (1)-(2) by assuming that ε_t follows a Pearson's Type IV distribution. Like the LADE and LQMLE, the PQMLE requires a specified identification condition rather than $E\varepsilon_t^2 = 1$. Under strict stationarity and a finite fractional moment of ε_t, the strong consistency and asymptotic normality of the PQMLE are obtained. Therefore, the PQMLE is applicable to all of the aforementioned non-Gaussian distributions used in the MLE method. Furthermore, we show that the PQMLE can be easily applied to other conditionally heteroskedastic models. A simulation study is carried out to assess the performance of the PQMLE, and two applications to eight major stock indexes and four exchange rates further highlight the importance of our new method. Compared to the existing NGQMLEs, the PQMLE captures not only the heavy-tailed but also the skewed innovations. Heavy-tailed and skewed innovations are often observed together in practice, but none of the existing QMLE methods has been focussed on these co-existing features in the literature. The PQMLE method, which can capture a very large range of the asymmetry and leptokurtosis of ε_t, now gives us a systematical way to achieve this goal.

This paper is organized as follows. Section 2 proposes our PQMLE and studies its asymptotic property. Simulation results are reported in Section 3. Applications are given in Section 4. Concluding remarks are offered in Section 5. The proofs are provided in the Appendix. Throughout the paper, A' is the transpose of matrix A, $|A| = (tr(A'A))^{1/2}$ is the Euclidean norm of a matrix A, $\|A\|_s = (E|A|^s)^{1/s}$ is the L^s-norm $(s \geq 1)$ of a random matrix, $O(1)$ denotes a bounded generic constant, "$\rightarrow_d$" denotes convergence in distribution, and "$\rightarrow_p$" denotes convergence in probability.

2. THE PQMLE AND ASYMPTOTIC THEORY

2·1. *Some basic assumptions*

Let $\theta = (\omega, \alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_q)'$ be the unknown parameters of the model given by (1)-(2) and its true value be θ_0. Denote the parameter space by Θ, where $\Theta \in \mathcal{R}_0^{1+p+q}$ is compact and $\mathcal{R}_0 = [0, \infty)$. Then, we need the following assumptions:

Assumption 1. y_t is strictly stationary.

Assumption 2. For each $\theta \in \Theta$, $\alpha(z)$ and $\beta(z)$ have no common root, $\alpha(1) \neq 0, \alpha_p + \beta_q \neq 0$ and $\sum_{j=1}^{q} \beta_j < 1$, where $\alpha(z) = \sum_{i=1}^{p} \alpha_i z^i$ and $\beta(z) = 1 - \sum_{j=1}^{q} \beta_j z^j$.

Assumption 3. (i) ε_t^2 is a nondegenerate random variable; (ii) $\lim_{s\rightarrow 0} s^{-\mu} P(\varepsilon_t^2 \leq s) = 0$ for some $\mu > 0$; (iii) $E|\varepsilon_t|^{2\kappa} < \infty$ for some $\kappa > 0$.

Assumption 1 is a basic set-up for model (1)-(2), and its necessary and sufficient conditions are given in Bougerol and Picard (1992). Assumption 2 and Assumption 3(i) are the identifiability conditions for model (1)-(2) as shown in Berkes, Horváth and Kokoszka (2003). Assumptions 3(ii)-(iii) from Berkes and Horváth (2004) are the technical conditions for proving our asymptotic theory. Note that only a finite fractional moment of ε_t is required in this case, and so our method applies to very heavy-tailed innovations.

2·2. *The Pearson's Type IV distribution*

We briefly review the Pearson's Type IV distribution in Nagahara (1999) and Heinrich (2004). The Pearson's Type IV (PIV) distribution, as one of the asymmetric and leptokurtic distributions, has the following pdf:

$$f(x;\lambda,a,\nu,m)=K\left[1+\left(\frac{x-\lambda}{a}\right)^2\right]^{-m}\exp\left[-\nu\tan^{-1}\left(\frac{x-\lambda}{a}\right)\right], \tag{3}$$

where $x\in\mathcal{R}$ and (λ,a,ν,m) are real parameters with $m\geq 1/2$ and $a>0$. Here, K is the normalizing constant given by

$$K=\frac{2^{2m-2}\left|\Gamma(m+i\nu/2)\right|^2}{a\pi\Gamma(2m-1)},$$

where $i=\sqrt{-1}$ is the imaginary number and $\Gamma(\cdot)$ is the complex Gamma function. In (3), λ and a are the location and the scale parameters, respectively; the parameter ν is related to the asymmetry of the distribution, and a positive (or negative) ν stands for a negatively (or positively) skewed distribution; the parameter m captures the leptokurtosis of the distribution, and a smaller value of m represents a heavier tail of the distribution. To further illustrate this, Figure 1 plots four different $f(x;0,1,\nu,m)$ densities. From Figure 1, we know that PIV(λ,a,ν,m) distribution with a small (or large) m can have a heavier (or lighter) tail than N(0,1) distribution. Also, it is worth mentioning that if $\varepsilon_t\sim$ PIV(λ,a,ν,m), its j-th moment exists only when $j<r+1$ for $r=2(m-1)$. That is, ε_t has a finite second moment when $m>1.5$, and it has a finite fourth moment when $m>2.5$. Particularly, the skewness and kurtosis of ε_t are as follows:

$$\begin{aligned}\text{skew}(\varepsilon_t)&=\frac{-4\nu}{r-2}\sqrt{\frac{r-1}{r^2+\nu^2}}\quad\text{for } m>2,\\ \text{kurt}(\varepsilon_t)&=\frac{3(r-1)\left[(r+6)(r^2+\nu^2)-8r^2\right]}{(r-2)(r-3)(r^2+\nu^2)}\quad\text{for } m>2.5.\end{aligned}$$

Figure 2 gives a 3-dimensional (3-D) plot of the skewness and kurtosis of ε_t. From this figure, we can see that when $|\nu|$ (or m) increases, the absolute value of the skewness increase (or decrease) for fixed m (or ν); and the same conclusion holds for the kurtosis. Hence, we know that the PIV distribution can capture a very large range of the asymmetry and leptokurtosis of the innovation. For more discussions on the PIV distributions, we refer to Bauwens and Laurent (2005), Yan (2005), and Grigoletto and Lisi (2009).

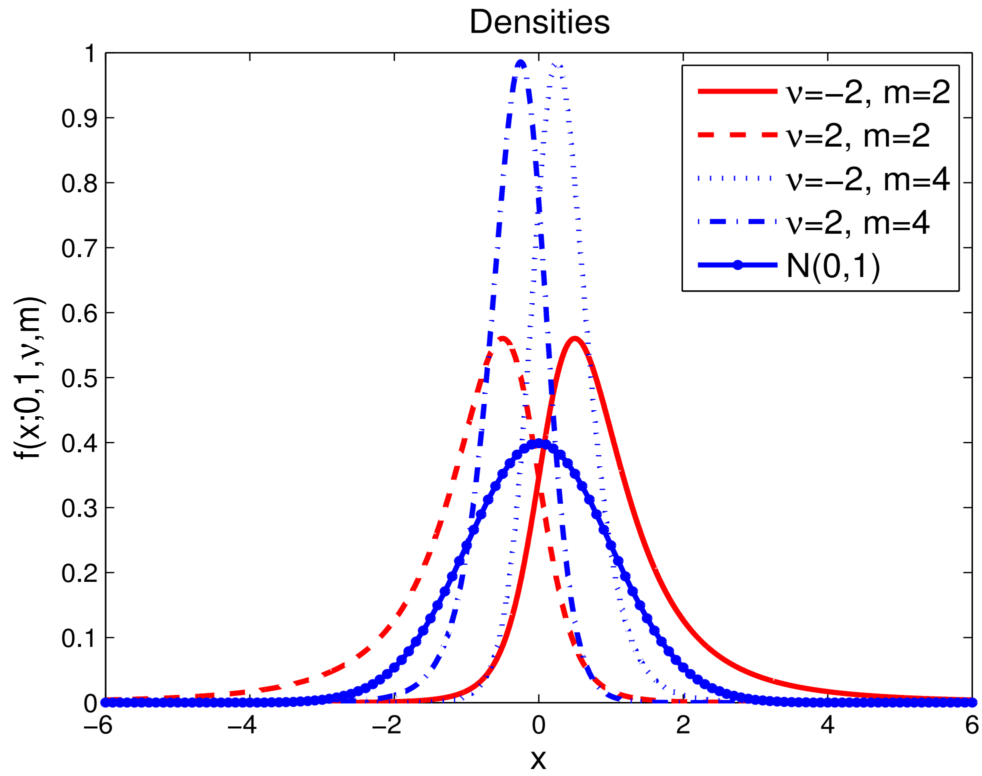

Fig. 1. The plot of four different densities $f(x; 0, 1, \nu, m)$ for the Pearson's Type IV distribution (the solid star line is the density of N(0,1) distribution).

Next, we are interested in the case when ε_t in model (1)-(2) follows the PIV distribution. Figures 3-4 plot one realization for each pair of (ν, m) from the following GARCH(1,1) model:

$$y_t = \varepsilon_t \sigma_t \ \text{ and } \ \sigma_t^2 = 0.01 + 0.01 y_{t-1}^2 + 0.9\sigma_{t-1}^2, \tag{4}$$

where $\varepsilon_t \sim \text{PIV}(0, 1, \nu, m)$ with $(\nu, m) = (\pm 2, 2)$, $(0, 2)$, $(\pm 2, 4)$, and $(0, 4)$. From Figures 3-4, we find that no matter how heavy-tailed ε_t is, y_t has a higher probability to be positive (or negative) when $\nu < 0$(or > 0), and this asymmetric phenomena disappears when $\nu = 0$. Moreover, when m becomes smaller, the absolute value of y_t tends to be larger, especially for its extreme values. All of these findings indicate that the GARCH model with $\text{PIV}(0, 1, \nu, m)$ innovations can capture a very large range of the asymmetry and leptokurtosis of the data set.

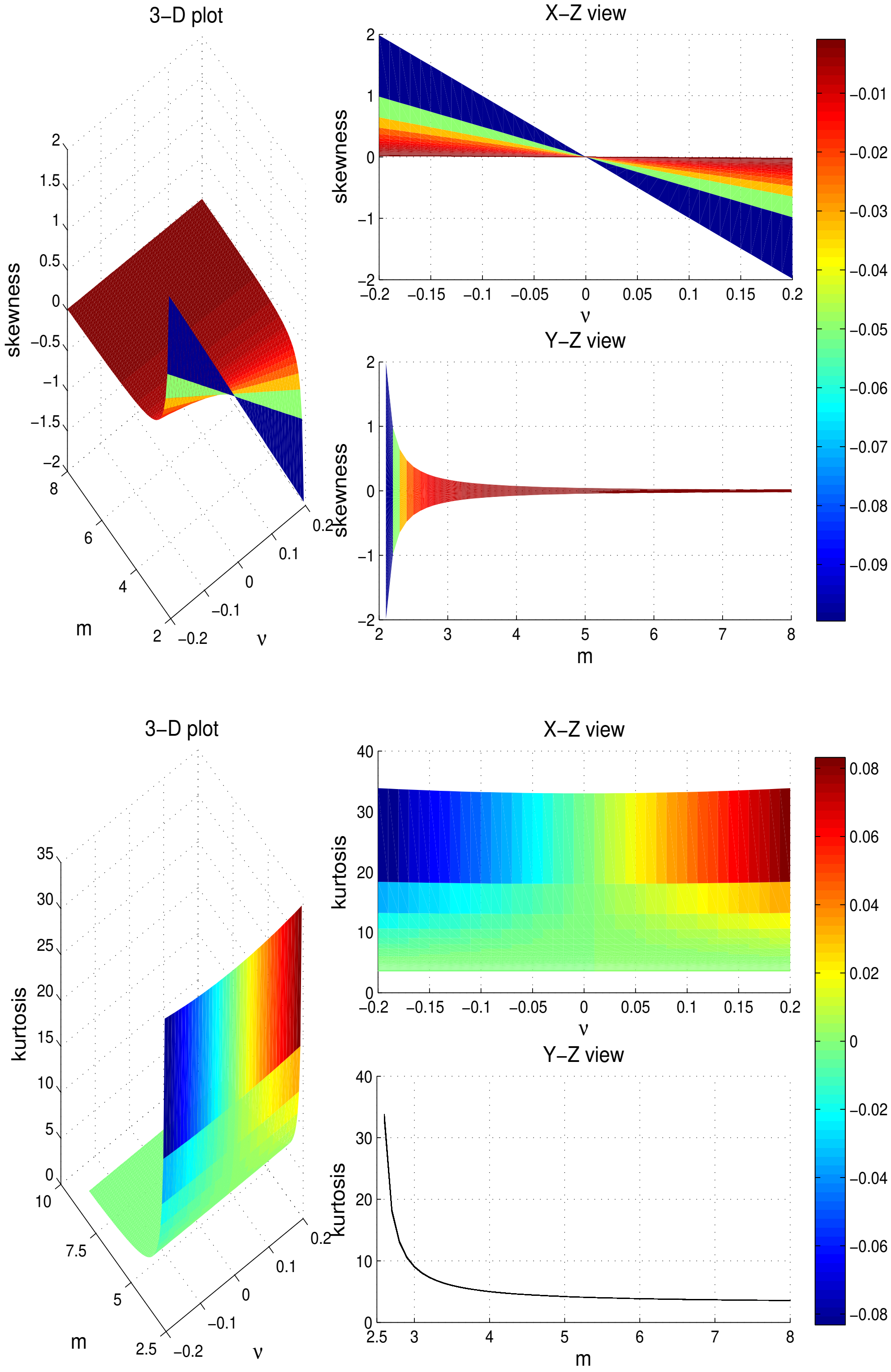

Fig. 2. (top panel) the 3-D plot of the skewness of ε_t, where $\varepsilon_t \sim \mathrm{PIV}(0, 1, \nu, m)$ with $\nu \in [-0.2, 0.2]$ and $m \in (2, 8)$; (bottom panel) the 3-D plot of the kurtosis of ε_t, where $\varepsilon_t \sim \mathrm{PIV}(0, 1, \nu, m)$ with $\nu \in [-0.2, 0.2]$ and $m \in (2.5, 8)$.

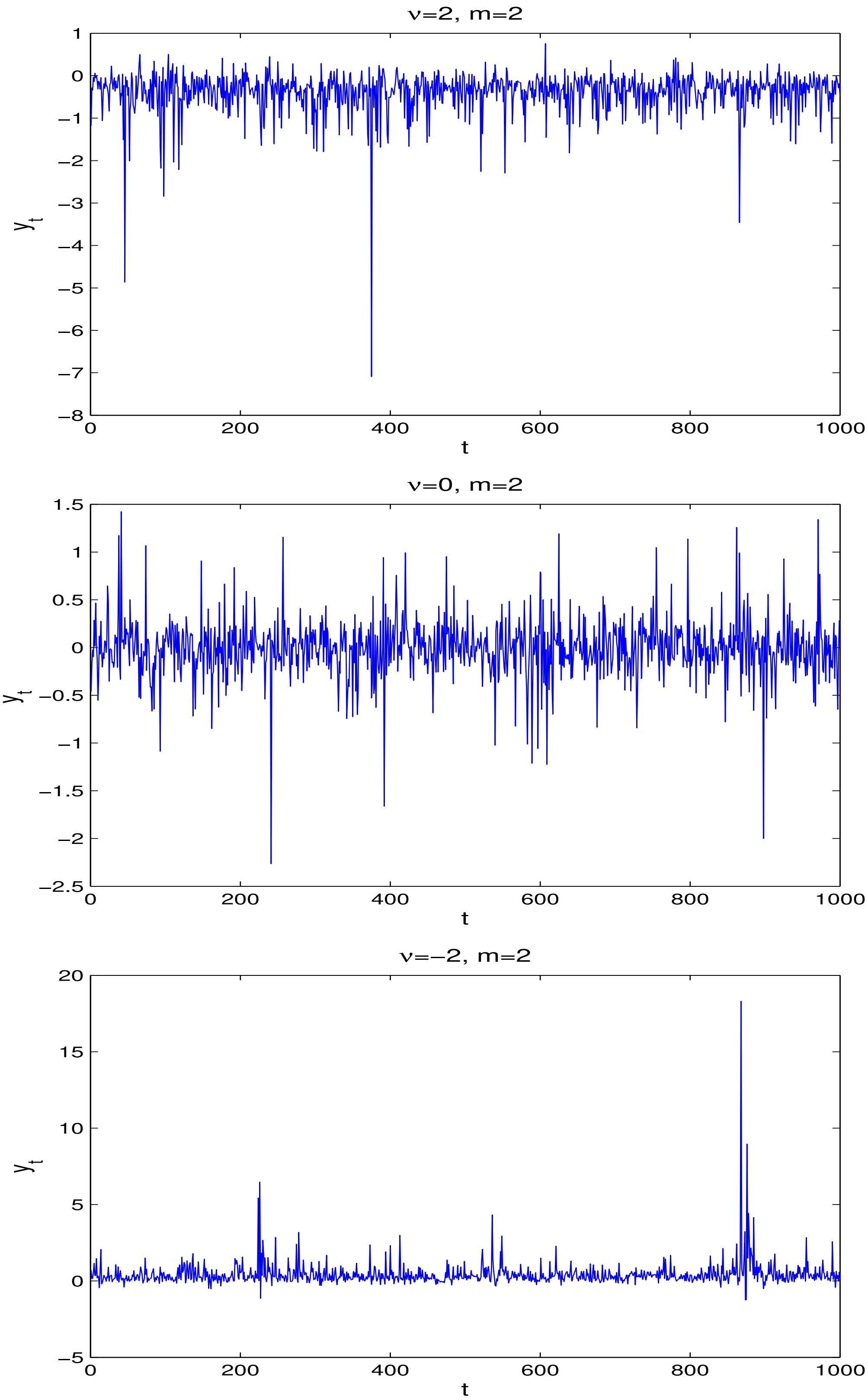

Fig. 3. One realization $\{y_t\}_{t=1}^{1000}$ from model (4), when $\varepsilon_t \sim \mathrm{PIV}(0, 1, \nu, m)$.

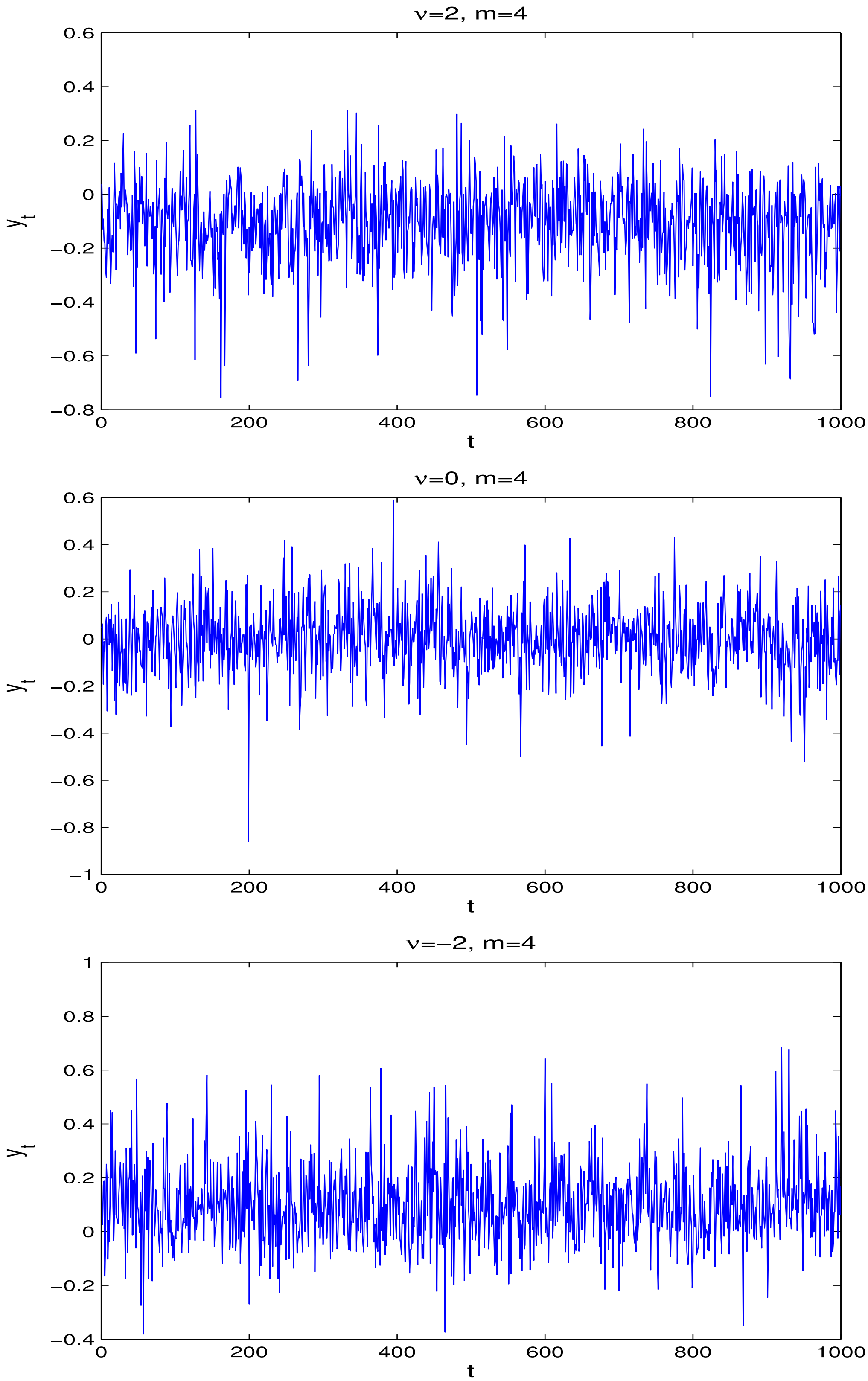

Fig. 4. One realization $\{y_t\}_{t=1}^{1000}$ from model (4), when $\varepsilon_t \sim \mathrm{PIV}(0, 1, \nu, m)$.

2·3. *The PQMLE*

Given the observations $\{y_n, \cdots, y_1\}$ and the initial values $Y_0 =: \{y_i; i \leq 0\}$, we first rewrite the parametric models (1)-(2) as

$$\varepsilon_t(\theta) = y_t/\sqrt{h_t(\theta)} \text{ and}$$
$$h_t(\theta) = c_0(\theta) + \sum_{i=1}^{\infty} c_i(\theta) y_{t-i}^2,$$

where all expressions for $c_i(\theta)(i \geq 0)$ are given in Berkes and Horváth (2004, pages 635-636). Clearly, $\varepsilon_t(\theta_0) = \varepsilon_t$ and $h_t(\theta_0) = \sigma_t^2$. In practice, since the values of Y_0 are unobservable, we can replace them by zeros, and then use $\tilde{h}_t(\theta)$ instead of $h_t(\theta)$ to calculate our estimator, where

$$\tilde{h}_t(\theta) = c_0(\theta) + \sum_{i=1}^{t-1} c_i(\theta) y_{t-i}^2 \text{ for } t = 2, \cdots, n, \tag{5}$$

and $\tilde{h}_1(\theta) = c_0(\theta)$. For given (ν, m), when ε_t follows the PIV$(0, 1, \nu, m)$ distribution, the log-likelihood function (ignoring some constants) can be written as

$$\tilde{L}_n(\theta) = -\sum_{t=1}^{n} \left\{ \log \sqrt{\tilde{h}_t(\theta)} + m \log \left[1 + \frac{y_t^2}{\tilde{h}_t(\theta)} \right] + \nu \tan^{-1} \left(\frac{y_t}{\sqrt{\tilde{h}_t(\theta)}} \right) \right\}, \tag{6}$$

where $m \geq 1/2$. We look for the maximizer of $\tilde{L}_n(\theta)$ on Θ, that is,

$$\tilde{\theta}_n = \arg\max_{\theta \in \Theta} \tilde{L}_n(\theta). \tag{7}$$

Because we do not assume that ε_t follows the PIV$(0, 1, \nu, m)$ distribution, $\tilde{\theta}_n$ is called the Pearsonian quasi-maximum likelihood estimator (PQMLE) of θ_0. Note that equation (6) depends on the distribution parameters (ν, m), and so we should specify them before the calculation of $\tilde{L}_n(\theta)$. Particularly, when $\nu = 0$, the log-likelihood function $\tilde{L}_n(\theta)$ is symmetric. The detailed procedure to select (ν, m) is discussed in Remark 3.

Next, let $\bar{f}(x) = f(x; 0, 1, \nu, m)/K$, $g(y, s) = \log \left[s\bar{f}(ys) \right]$ and $w(s) := E\left[g(\varepsilon_t, s) \right]$, where $y \in \mathcal{R}$ and $s > 0$. Then, it is straightforward to see that

$$\tilde{L}_n(\theta) = \sum_{t=1}^{n} g\left(y_t, 1/\sqrt{\tilde{h}_t(\theta)} \right).$$

In order to derive the asymptotic property of $\tilde{\theta}_n$, we need two more assumptions below:

Assumption 4. The innovation ε_t satisfies that

$$E\left[\frac{2m\varepsilon_t^2 + \nu\varepsilon_t}{1+\varepsilon_t^2}\right] = 1.$$

Assumption 5. $w(s)$ has a unique maximum at $s = 1$.

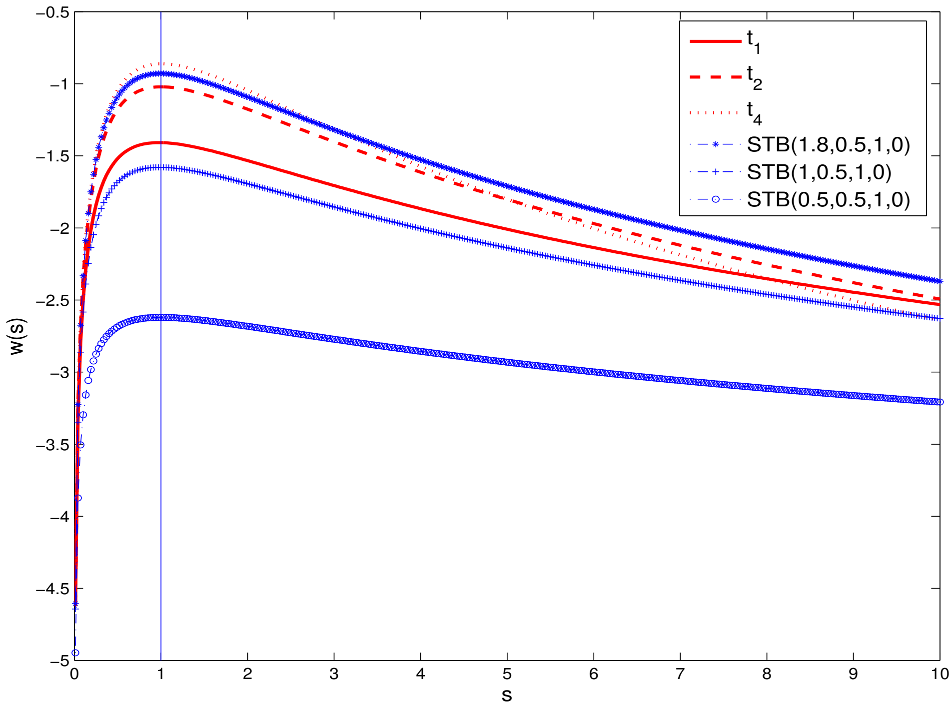

Fig. 5. The plot of $w(s)$ for Student's t and stable (STB) distributions.

Assumption 4 is the identification condition for the PQMLE. Unlike the GQMLE, the condition $E\varepsilon_t^2 = 1$ is not needed, and the conditional variance of y_t in this case is $\sigma_t^2 var(\varepsilon_t)$, provided that $E\varepsilon_t^2 < \infty$. Assumption 5 is a technical condition for proving the strong consistency of the PQMLE. After some simple algebra, we can show that a sufficient condition for Assumption 5 is that (i) $w(s)$ is concave on $\{s : s > 0\}$; and (ii) $E\left[\nu\varepsilon_t/(1+\varepsilon_t^2)\right] \le 0$. Figure 5 plots the function $w(s)$ for Student's t_i $(i = 1, 2, 4)$ distributions and stable (STB) distributions such that Assumption 4 holds, where (ν, m) are set to be $(-1, 1)$ for t_1, $(-1.16, 1.16)$ for t_2, $(-1.3, 1.3)$ for t_4, $(1.11, 1.11)$ for STB$(1.8, 0.5, 1, 0)$, $(0.97, 0.97)$ for STB$(1, 0.5, 1, 0)$, and $(0.76, 0.76)$

for STB$(0.5, 0.5, 1, 0)$. Here, the STB$(\check{\alpha}, \check{\beta}, c, \mu)$ distribution has the following characteristic function:

$$\psi(t; \check{\alpha}, \check{\beta}, c, \mu) = \exp\left[it\mu - |ct|^{\check{\alpha}}(1 - i\check{\beta}\mathrm{sgn}(t)\Phi)\right],$$

where $\check{\alpha} \in (0, 2]$, $\check{\beta} \in [-1, 1]$, $c \in (0, \infty)$, $\mu \in \mathcal{R}$, and

$$\Phi = \begin{cases} \tan(\pi\check{\alpha}/2) & \text{if } \check{\alpha} \neq 1, \\ -(2/\pi)\log|t| & \text{if } \check{\alpha} = 1. \end{cases}$$

Clearly, $w(s)$ in Figure 5 is concave with a unique maximum at $s = 1$ for all six distributions.

Denote the first and second derivatives of $g(y, s)$ with respective to s by $g_1(y, s)$ and $g_2(y, s)$, respectively. We now are ready to give our main results:

THEOREM 1. *Suppose that Assumptions* 1-5 *hold. Then, as* $n \to \infty$, *(i)* $\tilde{\theta}_n \to \theta_0$ *almost surely (a.s.); and (ii)* $\sqrt{n}\left(\tilde{\theta}_n - \theta_0\right) \to_d N(0, 4\tau^2 A^{-1})$, *where*

$$\tau^2 = \frac{Eg_1^2(\varepsilon_t, 1)}{[Eg_2(\varepsilon_t, 1)]^2} \text{ and } A = E\left[\frac{1}{h_t^2(\theta_0)}\frac{\partial h_t(\theta_0)}{\partial\theta}\frac{\partial h_t(\theta_0)}{\partial\theta'}\right].$$

Remark 1. The PQMLE only needs a finite fractional moment of ε_t for its asymptotic normality, which is weaker than the moment condition $E\varepsilon_t^4 < \infty$ for the GQMLE in Berkes, Horváth, and Kokoszka (2003) and Francq and Zakoïan (2004), or the moment condition $E\varepsilon_t^2 < \infty$ for the LQMLE in Berkes and Horváth (2004) and the GNGQMLE in Fan, Li, and Xiu (2013). Note that as shown in Chen and Zhu (2013), the LADE in Peng and Yao (2003) also only needs a finite fractional moment of ε_t for its asymptotic normality.

Remark 2. The identification condition for the PQMLE in Assumption 4 is different from the identification condition $E\varepsilon_t^2 = 1$ for the GQMLE and the GNGQMLE, the identification condition $E|\varepsilon_t| = 1$ for the LQMLE, or the identification condition median$(\varepsilon_t^2) = 1$ for the LADE. Thus, it is not straightforward to compare the efficiency of the PQMLE with that of other estimators in formal, and the simulation comparison in Section 3 is necessary.

Remark 3. In order to calculate the PQMLE, we need to first select the parameters ν and m. This can be simply done by using the maximum likelihood estimation method; see Premaratne and Bera (2001), Verhoeven and McAleer (2004), and Bhattacharyya, Mirsa, and Kodase (2009). Assume that $\varepsilon_t \sim$ PIV$(0, 1, \nu, m)$. Then, we can estimate (ν, m, θ) jointly by maximizing the full log-likelihood function $\mathrm{LLF}_P(\nu, m, \theta)$, where

$$\mathrm{LLF}_P(\nu, m, \theta) = \tilde{L}_n(\theta) + n \log K. \tag{8}$$

Now, we can choose (ν, m) to be the corresponding estimators from this MLE method. Although the parameters ν and m selected by the MLE method may not be optimal, the practical usefulness of this method will be illustrated by the empirical examples in Section 4.

Remark 4. Note that the value of (ν, m) can be anywhere in $(-\infty, \infty) \times [1/2, \infty]$, and a different value of (ν, m) will imply a different stationarity region of y_t. To see this, Figure 6 plots the strict stationarity region of the GARCH(1,1) model: $y_t = \varepsilon_t \sigma_t$ and $\sigma_t^2 = \omega + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2$, where $\varepsilon_t \sim \mathrm{PIV}(0, 1, \nu, m)$. As a comparison, the region for $Ey_t^2 < \infty$ is also plotted in Figure 6. From this figure, we find that the parameter region for strict stationarity is much larger than that for $Ey_t^2 < \infty$. Moreover, a smaller value of ν or a larger value of m will give a larger strict stationarity region. Particularly, except $\varepsilon_t \sim \mathrm{PIV}(0, 1, 2, 2)$, each strict stationarity region in Figure 6 is much larger than that in Nelson (1990) when $\varepsilon_t \sim N(0, 1)$ or that in Zhu and Ling (2011) when $\varepsilon_t \sim \mathrm{Laplace}(0, 1)$. Therefore, our PQMLE can have a much larger admissible parameter region than the GQMLE, the GNGQMLE or the LQMLE.

2·4. *Extension to conditionally heteroskedastic models*

In this subsection, we study the PMLE for the following conditionally heteroskedastic models:

$$y_t = \sigma_t \varepsilon_t \quad \text{and} \quad \sigma_t = \sigma(y_{t-1}, y_{t-2}, \cdots; \theta_0), \tag{9}$$

where ε_t being independent of $\{y_j; j < t\}$ is a sequence of i.i.d. random variables, the parameter space $\Theta \subset \mathcal{R}^l$ is compact, the true value θ_0 is an interior point in Θ, and $\sigma : \mathcal{R}^\infty \times \Theta \to (0, \infty)$. Many existing models, such as GARCH model in (1)-(2), asymmetric power GARCH model in Ding, Granger, and Engle (1993) and asymmetric log-GARCH model in Geweke (1986), can be embedded into model (9); see e.g., Bollerslev, Chou, and Kroner (1992) and Francq and Zakoïan (2010) for more discussions in this context.

As (5), let $h_t(\theta) = [\sigma(y_{t-1}, y_{t-2}, \cdots; \theta)]^2$ and define $\tilde{h}_t(\theta)$ in the same way as $h_t(\theta)$ by replacing Y_0 by zeros. Then, based on $\{\tilde{h}_t(\theta)\}$, we can define the PMLE for model (9) as in (7). To derive the asymptotic property of the PMLE, three more technical assumptions are needed.

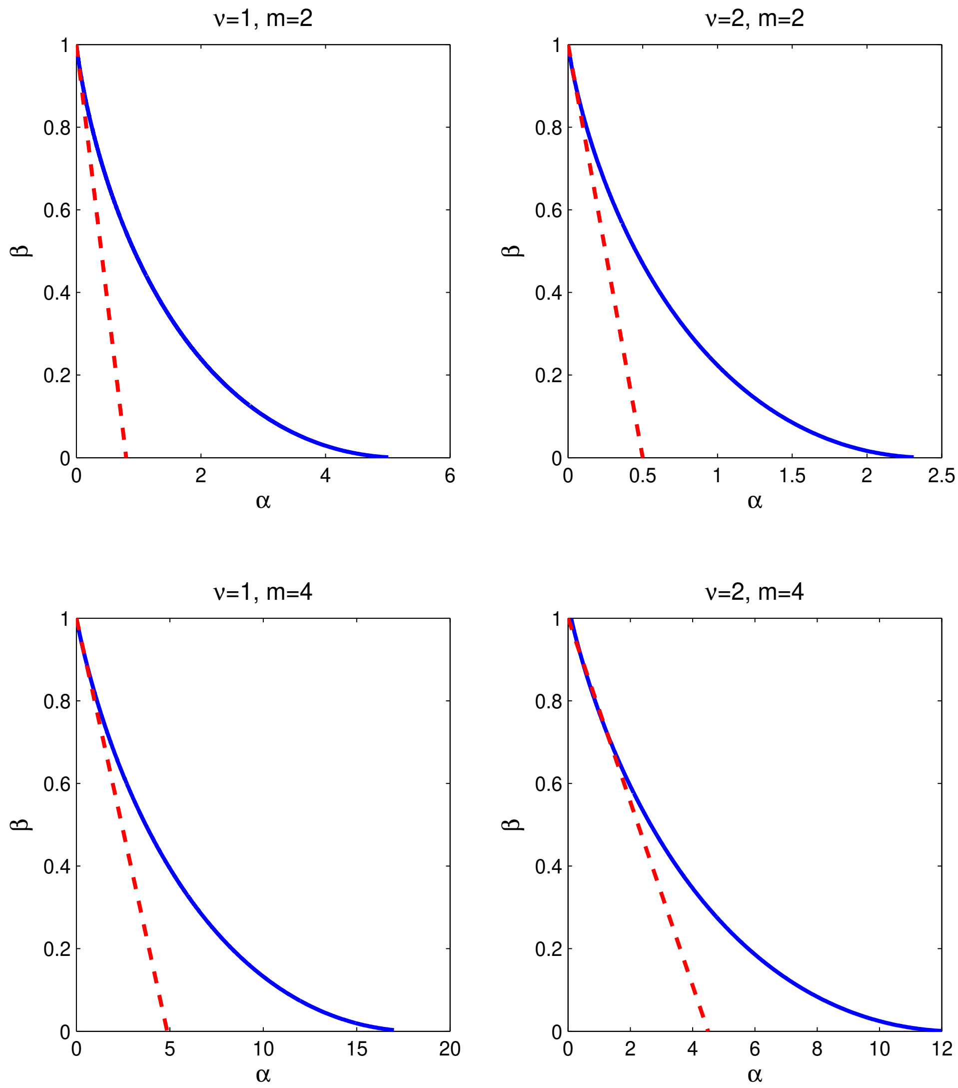

Fig. 6. The regions bounded by the solid and dashed curves are for the strict stationarity (i.e., $E[log(\alpha\varepsilon_t^2 + \beta)] < 0$) and for $Ey_t^2 < \infty$ (i.e., $E\varepsilon_t^2\alpha + \beta < 1$), respectively, where $E\varepsilon_t^2 = (r^2 + \nu^2)/(r^2(r-1))$ with $r = 2(m-1)$.

Assumption 6. (i) $h_t(\theta) \geq \underline{w}$ (a.s.) for some $\underline{w} > 0$ and all $\theta \in \Theta$. Moreover, $h_t(\theta) = h_t(\theta_0)$ (a.s.) if and only if $\theta = \theta_0$; (ii) if $x'(\partial h_t(\theta)/\partial\theta_i)_{i=1\cdots l} = 0$ (a.s.) for any $x \in \mathcal{R}^l$, then $x = 0$.

Assumption 7.

$$(i)\ E\left[\sup_{\theta\in\Theta}\left\|\frac{1}{h_t(\theta)}\frac{\partial h_t(\theta)}{\partial\theta}\right\|\right]^2 < \infty;\ \ (ii)\ E\left[\sup_{\theta\in\Theta}\left\|\frac{1}{h_t(\theta)}\frac{\partial^2 h_t(\theta)}{\partial\theta\partial\theta'}\right\|\right] < \infty.$$

Assumption 8.

$$(i)\ \sup_{\theta\in\Theta}\left\|\frac{1}{\tilde{h}_t(\theta)}\frac{\partial\tilde{h}_t(\theta)}{\partial\theta}-\frac{1}{h_t(\theta)}\frac{\partial h_t(\theta)}{\partial\theta}\right\|\leq O(\rho^t)R_t,$$
$$(ii)\ \sup_{\theta\in\Theta}\left\|\frac{1}{\tilde{h}_t(\theta)}\frac{\partial^2\tilde{h}_t(\theta)}{\partial\theta\partial\theta'}-\frac{1}{h_t(\theta)}\frac{\partial^2 h_t(\theta)}{\partial\theta\partial\theta'}\right\|\leq O(\rho^t)R_t$$

for some constant $\rho\in(0,1)$ and positive random variable R_t such that $ER_t^2<\infty$.

Assumption 6 imposes some basic requirements on the function $h_t(\theta)$, and they are satisfied by most of the conditionally heteroskedastic models; see, e.g., Francq and Zakoïan (2004, 2013). Assumptions 7-8 give some moment conditions, which have been verified for GARCH models in Ling (2007), asymmetric power GARCH models in Hamadeh and Zakoïan (2011) and asymmetric log-GARCH models in Francq, Wintenberger, and Zakoïan (2013). The following corollary gives the strong consistency and asymptotic normality the PQMLE for model (9), and its proof is omitted because it follows the same ones as for Theorems 1.1-1.2 in Berkes and Horváth (2004).

COROLLARY 1. *Assume that y_t follows model (9). If Assumptions* 1, 2*(iii) and* 3-8 *hold, then the conclusions in Theorem 1 hold.*

3. SIMULATION STUDY

In this section, we compare the performance of the PQMLE with those of the GQMLE, the LQMLE, the LADE and the GNGQMLE in finite samples. We generate 1000 replications of sample size $n=1000$ from the following model:

$$y_t=\sigma_t\varepsilon_t \text{ and } \sigma_t^2=\omega_0+\alpha_0 y_{t-1}^2+\beta_0\sigma_{t-1}^2, \tag{10}$$

where we choose $(\omega_0,\alpha_0,\beta_0)=(0.25,0.15,0.3)$ as in Fan, Li, and Xiu (2013), and ε_t is chosen to be the PIV distributions, the STB distributions, and the Student's t distributions, respectively.

In order to implement the PQMLE, we choose $(\nu,m)=(\nu_0/\tau_0,m_0/\tau_0)$ such that Assumption 4 holds, where

$$\tau_0=E\left[\frac{2m_0\varepsilon_t^2+\nu_0\varepsilon_t}{1+\varepsilon_t^2}\right]$$

Table 1. *The bias and RMSE of all estimators for model (10)*

ε_t		Estimators											
PIV$(0,1,2,4)$		PQMLE$_1$			PQMLE$_2$			PQMLE$_3$			PQMLE$_4$		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias	-0.0034	0.0072	0.0071	0.0013	0.0023	-0.0050	-0.0033	0.0065	0.0049	-0.0021	0.0018	-0.0040
	RMSE	0.1110	0.1132	0.2979	0.1050	0.1010	0.2821	0.1111	0.1173	0.2983	0.1051	0.1041	0.2848
		GQMLE			LQMLE			LAD			GNGQMLE		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias	-0.0047	0.0102	0.0094	0.0069	0.0041	-0.0185	0.0075	0.0256	-0.0245	0.0016	0.0038	-0.0061
	RMSE	0.1122	0.1254	0.3029	0.1082	0.1075	0.2892	0.1156	0.1454	0.3077	0.1050	0.1049	0.2822
PIV$(0,1,2,2)$		PQMLE$_1$			PQMLE$_2$			PQMLE$_3$			PQMLE$_4$		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias	0.0059	0.0010	-0.0073	0.0047	0.0001	-0.0056	0.0037	0.0000	-0.0028	0.0041	0.0000	-0.0040
	RMSE	0.0445	0.0328	0.0881	0.0456	0.0334	0.0908	0.0547	0.0396	0.1097	0.0490	0.0358	0.0981
		GQMLE			LQMLE			LAD			GNGQMLE		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias	0.0087	0.0122	-0.0306	0.0080	0.0017	-0.0138	0.0036	0.0011	-0.0039	-0.0009	-0.0045	-0.0109
	RMSE	0.0900	0.0905	0.1728	0.0529	0.0419	0.1084	0.0554	0.0400	0.1137	0.0497	0.0354	0.1010
PIV$(0,1,2,1.6)$		PQMLE$_1$			PQMLE$_2$			PQMLE$_3$			PQMLE$_4$		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias	0.0044	0.0002	-0.0007	0.0042	0.0002	-0.0002	0.0042	0.0006	0.0005	0.0042	0.0003	0.0002
	RMSE	0.0420	0.0227	0.0477	0.0431	0.0235	0.0493	0.0498	0.0278	0.0582	0.0457	0.0252	0.0527
		GQMLE			LQMLE			LAD			GNGQMLE		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias	-0.0069	-0.0016	0.0139	0.0084	0.0024	-0.0076	0.0030	0.0011	-0.0010	-0.0189	-0.0149	-0.0045
	RMSE	0.1405	0.0829	0.2018	0.0605	0.0390	0.0767	0.0455	0.0261	0.0601	0.0523	0.0300	0.0575
PIV$(0,1,2,1.5)$		PQMLE$_1$			PQMLE$_2$			PQMLE$_3$			PQMLE$_4$		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias	0.0083	-0.0006	0.0010	0.0079	-0.0007	0.0016	0.0074	-0.0007	0.0029	0.0076	-0.0008	0.0023
	RMSE	0.0465	0.0205	0.0394	0.0477	0.0215	0.0411	0.0557	0.0256	0.0490	0.0507	0.0231	0.0442
		GQMLE			LQMLE			LAD			GNGQMLE		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias		N.A.			N.A.		0.0073	-0.0012	0.0020		N.A.	
	RMSE		N.A.			N.A.		0.0515	0.0221	0.0485		N.A.	
STA$(1.8,0.5,1,0)$		PQMLE$_1$			PQMLE$_2$			PQMLE$_3$			PQMLE$_4$		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias	0.0014	-0.0014	0.0025	0.0013	-0.0014	0.0018	0.0018	-0.0001	-0.0008	0.0014	-0.0010	0.0008
	RMSE	0.0565	0.0375	0.1108	0.0498	0.0335	0.0978	0.0506	0.0336	0.0981	0.0477	0.0321	0.0933
		GQMLE			LQMLE			LAD			GNGQMLE		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias		N.A.			N.A.		0.0016	0.0001	0.0002		N.A.	
	RMSE		N.A.			N.A.		0.0552	0.0373	0.1070		N.A.	
STA$(1.8,0.9,1,0)$		PQMLE$_1$			PQMLE$_2$			PQMLE$_3$			PQMLE$_4$		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias	0.0007	-0.0004	0.0023	0.0010	-0.0010	0.0020	0.0018	-0.0008	0.0008	0.0015	-0.0011	0.0013
	RMSE	0.0573	0.0366	0.1090	0.0518	0.0323	0.0984	0.0530	0.0328	0.1005	0.0504	0.0309	0.0953
		GQMLE			LQMLE			LAD			GNGQMLE		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias		N.A.			N.A.		0.0024	-0.0001	0.0004		N.A.	
	RMSE		N.A.			N.A.		0.0595	0.0374	0.1120		N.A.	

† The invalid estimation results are labeled as "Not Available (N.A.)".

Table 2. *The bias and RMSE of all estimators for model* (10) *(con't)*

ε_t		Estimators											
STA$(1.5, 0, 1, 0)$		PQMLE$_1$			PQMLE$_2$			PQMLE$_3$			PQMLE$_4$		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias	0.0047	-0.0009	0.0007	0.0039	-0.0013	0.0015	0.0029	-0.0011	0.0023	0.0034	-0.0013	0.0019
	RMSE	0.0439	0.0305	0.0656	0.0389	0.0268	0.0580	0.0397	0.0262	0.0584	0.0376	0.0252	0.0556
		GQMLE			LQMLE			LAD			GNGQMLE		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias		N.A.			N.A.		0.0033	-0.0015	0.0018		N.A.	
	RMSE		N.A.			N.A.		0.0425	0.0298	0.0644		N.A.	
STA$(1.5, 0.5, 1, 0)$		PQMLE$_1$			PQMLE$_2$			PQMLE$_3$			PQMLE$_4$		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias	0.0029	-0.0001	-0.0001	0.0025	-0.0006	0.0006	0.0015	-0.0010	0.0027	0.0020	-0.0009	0.0016
	RMSE	0.0425	0.0276	0.0647	0.0391	0.0251	0.0593	0.0403	0.0259	0.0623	0.0382	0.0244	0.0583
		GQMLE			LQMLE			LAD			GNGQMLE		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias		N.A.			N.A.		0.0028	-0.0002	0.0009		N.A.	
	RMSE		N.A.			N.A.		0.0430	0.0277	0.0650		N.A.	
t_5		PQMLE$_1$			PQMLE$_2$			PQMLE$_3$			PQMLE$_4$		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias	0.0047	0.0073	-0.0120	0.0045	0.0041	-0.0131	0.0049	0.0025	-0.0086	0.0025	0.0016	-0.0100
	RMSE	0.0799	0.0533	0.1742	0.0716	0.0464	0.1566	0.0704	0.0453	0.1555	0.0675	0.0434	0.1496
		GQMLE			LQMLE			LAD			GNGQMLE		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias	0.0102	0.0039	-0.0237	0.0072	0.0028	-0.0154	0.0027	0.0059	-0.0078	0.0055	0.0019	-0.0151
	RMSE	0.0743	0.0519	0.1689	0.0634	0.0404	0.1440	0.0774	0.0538	0.1743	0.0622	0.0392	0.1423
t_4		PQMLE$_1$			PQMLE$_2$			PQMLE$_3$			PQMLE$_4$		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias	0.0063	0.0050	-0.0148	0.0068	0.0032	-0.0143	0.0068	0.0017	-0.0122	0.0071	0.0022	-0.0138
	RMSE	0.0700	0.0473	0.1521	0.0631	0.0415	0.1367	0.0636	0.0414	0.1363	0.0608	0.0393	0.1308
		GQMLE			LQMLE			LAD			GNGQMLE		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias		N.A.		0.0081	0.0022	-0.0160	0.0052	0.0039	-0.0104	0.0060	0.0006	-0.0176
	RMSE		N.A.		0.0591	0.0390	0.1289	0.0715	0.0493	0.1557	0.0564	0.0369	0.1235
t_3		PQMLE$_1$			PQMLE$_2$			PQMLE$_3$			PQMLE$_4$		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias	0.0013	-0.0018	-0.0046	0.0058	-0.0001	-0.0038	0.0014	-0.0026	0.0000	0.0073	0.0005	-0.0021
	RMSE	0.0602	0.0430	0.1181	0.0531	0.0384	0.1042	0.0507	0.0375	0.1032	0.0498	0.0366	0.0984
		GQMLE			LQMLE			LAD			GNGQMLE		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias		N.A.		0.0069	0.0016	-0.0116	0.0025	-0.0013	0.0007	-0.0012	-0.0043	-0.0076
	RMSE		N.A.		0.0521	0.0399	0.1080	0.0565	0.0419	0.1155	0.0472	0.0350	0.0993
t_2		PQMLE$_1$			PQMLE$_2$			PQMLE$_3$			PQMLE$_4$		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias	0.0025	0.0007	0.0005	0.0023	0.0001	0.0010	0.0024	0.0003	0.0009	0.0023	0.0000	0.0011
	RMSE	0.0476	0.0357	0.0807	0.0422	0.0319	0.0723	0.0429	0.0312	0.0740	0.0405	0.0303	0.0699
		GQMLE			LQMLE			LAD			GNGQMLE		
		ω	α	β	ω	α	β	ω	α	β	ω	α	β
	Bias		N.A.			N.A.		0.0012	0.0008	0.0014		N.A.	
	RMSE		N.A.			N.A.		0.0471	0.0358	0.0829		N.A.	

† The invalid estimation results are labeled as "Not Available (N.A.)".

with $(\nu_0, m_0) = (2,2), (2,4), (-2,4)$ and $(0,4)$, and the corresponding PQMLEs are called the PQMLE_1, PQMLE_2, PQMLE_3, and PQMLE_4, respectively, Furthermore, since other four estimation methods require different identification conditions for model (10), the GQMLE ($\bar{\theta}^*_{1n}$), LQMLE ($\bar{\theta}^*_{2n}$), LADE ($\bar{\theta}^*_{3n}$), and GNGQMLE ($\bar{\theta}^*_{4n}$) are estimators of $(\tau_1\omega_0, \tau_1\alpha_0, \beta_0)$ with $\tau_1 = E\varepsilon_t^2, (E|\varepsilon_t|)^2$, median$(\varepsilon_t^2)$ and $E\varepsilon_t^2$ respectively. In order to make our comparison feasible, we let

$$\bar{\theta}_{1n} = \left(\frac{\bar{\omega}^*_{1n}}{E\varepsilon_t^2}, \frac{\bar{\alpha}^*_{1n}}{E\varepsilon_t^2}, \bar{\beta}^*_{1n}\right) \quad \bar{\theta}_{2n} = \left(\frac{\bar{\omega}^*_{2n}}{(E|\varepsilon_t|)^2}, \frac{\bar{\alpha}^*_{2n}}{(E|\varepsilon_t|)^2}, \bar{\beta}^*_{2n}\right)$$

and

$$\bar{\theta}_{3n} = \left(\frac{\bar{\omega}^*_{3n}}{\text{median}(\varepsilon_t^2)}, \frac{\bar{\alpha}^*_{3n}}{\text{median}(\varepsilon_t^2)}, \bar{\beta}^*_{3n}\right) \quad \bar{\theta}_{4n} = \left(\frac{\bar{\omega}^*_{4n}}{E\varepsilon_t^2}, \frac{\bar{\alpha}^*_{4n}}{E\varepsilon_t^2}, \bar{\beta}^*_{4n}\right)$$

be the GQMLE, LQMLE, LADE, and GNGQMLE of $(\omega_0, \alpha_0, \beta_0)$, respectively. The estimated asymptotic standard deviations of all estimators were derived in a similar way. In all calculations, we use the true values of $E\varepsilon_t^2, (E|\varepsilon_t|)^2$ and median(ε_t^2), and the GNGQMLE is constructed in the same way as in Section 7.2 of Fan, Li, and Xiu (2013). Note that the PQMLEs and LADE are applicable for all innovations, but the GQMLE is only applicable when $E\varepsilon_t^4 < \infty$, and the LQMLE and GNGQMLE are only applicable when $E\varepsilon_t^2 < \infty$.

Tables 1-2 report the bias and root mean square error (RMSE) of all estimators for model (10). From them, we find that all estimators have very small bias. When $\eta_t \sim \text{PIV}(0,1,2,4)$, PQMLE_2 is the efficient estimator and so it has the smallest RMSE, while the performance of LQMLE or GNGQMLE is better than those of the remaining PQMLEs. When $\eta_t \sim \text{PIV}(0,1,2,2)$, PQMLE_1 is the efficient estimator and so it has the smallest RMSE. In this case, all PQMLEs except PQMLE_3 have smaller RMSEs than other estimators. This advantage of the PQMLEs becomes more significant as m becomes smaller. Note that the PQMLE_3 has the worst performance in all PQMLEs, and this nay be because of the sign of ν which is negative for PQMLE_3. Next, we consider the cases that ε_t follows the STB distribution. In this case, only the PQMLEs and LADE are applicable. When $\varepsilon_t \sim \text{STB}(1.8, 0.5, 1, 0)$, all PQMLEs except PQMLE_1 have smaller RMSEs than the LADE; when $\varepsilon_t \sim \text{STB}(1.8, 0.9, 1, 0)$, the innovation becomes more skewed, and then the efficiency advantage of all PQMLEs (including PQMLE_1) over LADE becomes more significant; moreover, when $\varepsilon_t \sim \text{STB}(1.5, 0, 1, 0)$ or $\text{STB}(1.5, 0.5, 1, 0)$, the innovation become more heavy-tailed, and then the similar conclusions can be drawn as before. Thirdly, we consider the cases that ε_t follows the t distribution. In this case, the innovations are symmetric, and hence the PQMLE$_4$ has the best performance among all PQMLEs, although its performance is worse than those of the LQMLE and GNGQMLE. Meanwhile, the GNGQMLE has the best performance in all estimators due to its adaptive property under symmetry, and the performance of the PQMLEs are always better than that of the LADE. Overall, the simulation study shows that all PQMLEs have a good performance in finite samples, especially for the heavy-tailed and skewed innovations.

4. Application

4·1. *Application to stock indexes*

In this subsection, we apply the PQMLE estimation method to eight major stock indexes in the world. The data sets we considered are the daily CAC40, DAX, DJIA, FTSE, HSI, NASDAQ, Nikkei225, and SP500 indexes from January 3, 2000 to December 27, 2007. As usual, we denote the log-return ($\times 100$) of each data set by $\{y_t\}_{t=1}^n$, and the summary statistics for each y_t is given in Table 3. From this table, we find that each y_t is skewed and has a heavier tail than the $N(0,1)$ distribution. Hence, we use a GARCH(1,1) model with the PQMLE estimation method to fit each return series. As a comparison, we also apply the GQMLE, LQMLE, or GNGQMLE estimation method to obtain the fitted GARCH(1,1) model for each return series. For the PQMLE method, μ and m are chosen as in Remark 3. For the GNGQMLE method, the auxiliary likelihood function is based on the standardized t_3, t_5 or t_7 distribution such that it has variance one, and then the corresponding estimator is denoted by GNGQMLE$_1$, GNGQMLE$_2$ or GNGQMLE$_3$, respectively.

Table 3. *Summary of eight major stock indexes*

y_t	n	mean	standard deviation	skewness	kurtosis
CAC40	2049	-0.0025	1.3968	-0.0930	5.9618
DAX	2031	0.0086	1.5495	-0.0455	5.7503
DJIA	2009	0.0081	1.0951	-0.0907	7.4136
FTSE	2017	-0.0012	1.1297	-0.1749	5.8796
HSI	1982	0.0238	1.3533	-0.3596	6.5512
NASDAQ	2007	-0.0216	1.8461	0.1848	7.2060
Nikkei225	1965	-0.0102	1.3796	-0.1581	4.7171
SP500	2007	0.0000	1.1155	0.0469	5.5460

Table 4. *Summary of all estimations for eight major stock indexes*

y_t		PQMLE	GQMLE	LQMLE	GNGQMLE$_1$	GNGQMLE$_2$	GNGQMLE$_3$
CAC40	ω	0.2301	0.0160	0.0071	0.0102	0.0114	0.0121
		(0.0876)†	(0.0060)	(0.0031)	(0.0019)	(0.0021)	(0.0022)
	α	1.3881	0.0851	0.0487	0.0776	0.0800	0.0812
		(0.2084)	(0.0138)	(0.0075)	(0.0003)	(0.0003)	(0.0004)
	β	0.9103	0.9068	0.9164	0.9185	0.9154	0.9137
		(0.0128)	(0.0143)	(0.0122)	(0.0100)	(0.0108)	(0.0113)
	ν	-0.0308					
	m	9.8482					
	$\hat{\eta}_k$				1.3462	1.0872	1.0372
	τ_2	0.9995	0.9995	0.9995	1.0012	1.0006	1.0003
	LLF	-3205.2	-3213.8	-3282.2	-3268.0	-3227.9	-3215.5
DAX	ω	0.3508	0.0240	0.0081	0.0095	0.0125	0.0142
		(0.1277)	(0.0081)	(0.0038)	(0.0022)	(0.0025)	(0.0027)
	α	1.8795	0.1062	0.0591	0.0925	0.0944	0.0954
		(0.2694)	(0.0161)	(0.0087)	(0.0004)	(0.0005)	(0.0005)
	β	0.8947	0.8845	0.9014	0.9074	0.9035	0.9013
		(0.0143)	(0.0167)	(0.0138)	(0.0107)	(0.0117)	(0.0123)
	ν	-0.0830					
	m	10.989					
	$\hat{\eta}_k$				1.3430	1.0880	1.0389
	τ_2	0.9995	0.9996	0.9996	1.0057	1.0036	1.0025
	LLF	-3358.9	-3366.0	-3425.4	-3420.4	-3382.2	-3370.1
DJIA	ω	0.0698	0.0261	0.0075	0.0112	0.0123	0.0128
		(0.0241)	(0.0115)	(0.0027)	(0.0031)	(0.0034)	(0.0035)
	α	0.4584	0.0847	0.0453	0.0801	0.0834	0.0845
		(0.0719)	(0.0246)	(0.0078)	(0.0005)	(0.0005)	(0.0006)
	β	0.9094	0.8934	0.9120	0.9150	0.9109	0.9094
		(0.0132)	(0.0287)	(0.0140)	(0.0186)	(0.0202)	(0.0211)
	ν	-0.0379					
	m	4.2961					
	$\hat{\eta}_k$				1.2666	1.0331	0.9909
	τ_2	0.9995	0.9995	0.9995	1.0086	1.0078	1.0077
	LLF	-2726.4	-2794.5	-2764.7	-2759.0	-2732.1	-2727.6
FTSE	ω	0.5639	0.0152	0.0091	0.0138	0.0138	0.0138
		(0.1743)	(0.0046)	(0.0029)	(0.0018)	(0.0018)	(0.0019)
	α	4.4984	0.1175	0.0699	0.1112	0.1136	0.1148
		(0.6032)	(0.0158)	(0.0099)	(0.0004)	(0.0004)	(0.0004)
	β	0.8728	0.8721	0.8774	0.8794	0.8774	0.8762
		(0.0157)	(0.0159)	(0.0161)	(0.0125)	(0.0130)	(0.0134)
	ν	-0.0028					
	m	20.676					
	$\hat{\eta}_k$				1.3533	1.0933	1.0430
	τ_2	0.9995	0.9996	0.9996	0.9996	0.9996	0.9995
	LLF	-2722.0	-2725.2	-2801.6	-2789.3	-2748.9	-2735.9

† The standard deviations are in parentheses.

Table 5. *Summary of all estimations for eight major stock indexes (con't)*

y_t		PQMLE	GQMLE	LQMLE	GNGQMLE_1	GNGQMLE_2	GNGQMLE_3
HSI	ω	0.0318	0.0414	0.0055	0.0048	0.0073	0.0087
		(0.0192)[†]	(0.0260)	(0.0036)	(0.0055)	(0.0066)	(0.0071)
	α	0.2192	0.1436	0.0378	0.0497	0.0534	0.0559
		(0.0410)	(0.0446)	(0.0079)	(0.0005)	(0.0007)	(0.0008)
	β	0.9463	0.8517	0.9319	0.9529	0.9477	0.9445
		(0.0098)	(0.0437)	(0.0138)	(0.0253)	(0.0283)	(0.0302)
	ν	-0.0741					
	m	3.5529					
	$\hat{\eta}_k$				1.2321	1.0163	0.9795
	τ_2	0.9995	1.0005	1.0002	1.1053	1.0938	1.0873
	LLF	-3174.6	-3272.3	-3191.4	-3195.3	-3177.3	-3176.7
NASDAQ	ω	0.1702	0.0104	0.0037	0.0043	0.0053	0.0059
		(0.0872)	(0.0047)	(0.0025)	(0.0014)	(0.0016)	(0.0017)
	α	1.3844	0.0650	0.0392	0.0620	0.0628	0.0634
		(0.2184)	(0.0112)	(0.0064)	(0.0002)	(0.0002)	(0.0002)
	β	0.9336	0.9319	0.9364	0.9387	0.9373	0.9363
		(0.0099)	(0.0110)	(0.0099)	(0.0080)	(0.0085)	(0.0089)
	ν	-0.0114					
	m	12.195					
	$\hat{\eta}_k$				1.3511	1.0917	1.0411
	τ_2	0.9995	0.9995	0.9995	1.0019	1.0014	1.0010
	LLF	-3576.9	-3583.7	-3652.9	-3643.0	-3602.6	-3589.5
Nikkei225	ω	0.1529	0.0292	0.0099	0.0106	0.0139	0.0161
		(0.0652)	(0.0120)	(0.0046)	(0.0026)	(0.0032)	(0.0035)
	α	0.6068	0.0940	0.0412	0.0573	0.0640	0.0687
		(0.1019)	(0.0179)	(0.0072)	(0.0003)	(0.0004)	(0.0005)
	β	0.9201	0.8960	0.9251	0.9396	0.9316	0.9261
		(0.0132)	(0.0192)	(0.0129)	(0.0093)	(0.0109)	(0.0120)
	ν	0.0013					
	m	5.6151					
	$\hat{\eta}_k$				1.3111	1.0669	1.0213
	τ_2	0.9995	0.9996	0.9996	1.0151	1.0107	1.0078
	LLF	-3289.5	-3310.9	-3333.3	-3330.3	-3299.9	-3292.5
SP500	ω	0.0579	0.0112	0.0036	0.0044	0.0057	0.0064
		(0.0257)	(0.0044)	(0.0017)	(0.0012)	(0.0014)	(0.0015)
	α	0.5753	0.0712	0.0382	0.0623	0.0664	0.0683
		(0.0914)	(0.0135)	(0.0063)	(0.0001)	(0.0002)	(0.0002)
	β	0.9265	0.9200	0.9323	0.9364	0.9311	0.9286
		(0.0111)	(0.0144)	(0.0106)	(0.0091)	(0.0102)	(0.0109)
	ν	-0.0166					
	m	5.6425					
	$\hat{\eta}_k$				1.3060	1.0637	1.0191
	τ_2	0.9995	0.9995	0.9995	1.0054	1.0031	1.0023
	LLF	-2763.5	-2786.5	-2807.6	-2804.4	-2774.1	-2766.6

[†] The standard deviations are in parentheses.

The detailed estimation results for each return series are given in Tables 4-5, in which the full log-likelihood function of the PQMLE is defined as in (8), and the full log-likelihood functions of the GQMLE (LLF$_G$), LQMLE (LLF$_L$), and GNGQMLE (LLF$_{GNG}$) are defined as follows:

$$\mathrm{LLF}_G = -\sum_{t=1}^{n}\left[\log\sqrt{\tilde{h}_t(\bar{\theta}_{1n})} + \frac{y_t^2}{2\tilde{h}_t(\bar{\theta}_{1n})}\right] + n\log\left(\frac{1}{\sqrt{2\pi}}\right),$$

$$\mathrm{LLF}_L = -\sum_{t=1}^{n}\left[\log\sqrt{\tilde{h}_t(\bar{\theta}_{2n})} + \frac{|y_t|}{\sqrt{\tilde{h}_t(\bar{\theta}_{2n})}}\right] + n\log\left(\frac{1}{2}\right),$$

$$\mathrm{LLF}_{GNG} = -\sum_{t=1}^{n}\left[\log\left(\hat{\eta}_k\sqrt{\tilde{h}_t(\bar{\theta}_{4n})}\right) + \frac{k+1}{2}\log\left(1+\frac{y_t^2}{(k-2)\hat{\eta}_k^2\tilde{h}_t(\bar{\theta}_{4n})}\right)\right]$$
$$+ n\log\left(\frac{\Gamma\{(k+1)/2\}}{\sqrt{(k-2)\pi}\Gamma\{k/2\}}\right) \text{ for } k = 3 \text{ (or 5, 7)},$$

where $\bar{\theta}_{1n}$, $\bar{\theta}_{2n}$ and $\bar{\theta}_{4n}$ are the GQMLE, LQMLE and GNGQMLE, respectively, and

$$\hat{\eta}_k = \arg\max_{\eta}\sum_{t=1}^{n}\left[-\log(\eta) - \frac{k+1}{2}\log\left(1+\frac{y_t^2}{(k-2)\eta^2\tilde{h}_t(\bar{\theta}_{1n})}\right)\right].$$

Here, $\hat{\eta}_k$ measures the discrepancy between the correct likelihood function and the given auxiliary likelihood function. Specifically, when $\hat{\eta}_k > 1$(or < 1), the given auxiliary innovation t_k is heavier (or lighter) tailed than the true innovation. Furthermore, Tables 4-5 also report the estimated values of the identification condition τ_2 for each estimation method, that is, τ_2 is the sample mean of $(2m\varepsilon_t^2 + \nu\varepsilon_t)/(1+\varepsilon_t^2)$, ε_t^2 or $|\varepsilon_t|$ for the PQMLE, GQMLE (and GNGQMLE) or LQMLE estimation method, respectively. Meanwhile, it is worth mentioning that all fitted models are adequate by looking at the the ACF and PACF plots (not depicted here) of the squared and absolute residuals.

From Tables 4-5, we find that (i) all the values of τ_2 are close to 1 as expected; (ii) for each return series, the PQMLE always has the best fitting in terms of the maximized LLF among all estimation methods; (iii) the GNGQMLE estimation with a t_5 or t_7 likelihood gives the second best fitted models for the DJIA, HSI, Nikkei225 and SP500 return series in which the value of m are smaller, while the GQMLE estimation gives the second best fitted models for the CAC40, DAX, FISE and NASDAQ return series in which the value of m are larger; (iv) the LQMLE has the worst fitting in all cases except for the DJIA and HSI return series, in which the values of m are the smallest, and so the GQMLE has the worst fitting in these two cases; (v) the GNGQMLE estimation with a t_3 likelihood always has the largest value of $\hat{\eta}_k$ among all GNGQMLE estimations, and hence it implies that the auxiliary t_3 innovation is heavier tailed than the true innovation, while the auxiliary t_5 or t_7 innovation has the similar tail as the true innovation because the values of $\hat{\eta}_k$ in these two cases are close to 1; (vi) the values of m are all larger than 2.5, and it suggests that the innovation for each return series has finite fourth moment. Overall, we know that all estimation methods are applicable, and the PQMLE estimation method taking into account both leptokurtosis and asymmetry of the innovation gives the best fitted models for all return series.

Next, we use the conditional coverage test LR_{CC} in Christoffersen (1998, page 847) to examine whether each of the estimation methods can provide us a good interval forecast for its one-step-ahead prediction. For each return series, the out-of-sample data set we used is a length of n_0 consecutive data starting after the last observation of the in-sample data set. Following Christoffersen (1998), the upper-tail predictive interval (UPI) and lower-tail predictive interval (LPI) for each out-of-sample data y_t at the significance level $\bar{p}$ are defined as

$$\text{UPI}_{t|t-1}(\bar{p}) = \left(F^{-1}(1-\bar{p})\bar{\sigma}_t, \infty\right) \text{ and } \text{LPI}_{t|t-1}(\bar{p}) = \left(-\infty, F^{-1}(\bar{p})\bar{\sigma}_t\right),$$

respectively, where $\bar{\sigma}_t$ is the one-step-ahead prediction of σ_t from each estimation method, and $F(\cdot)$ is the cdf of the PIV$(0,1,\nu,m)$, N$(0,1)$, Laplace$(0,1)$, and standardized t_i (for $i = 3, 5, 7$) distribution for the PQMLE, GQMLE, LQMLE, and GNGQMLE_i estimation methods, respectively. Table 6 reports all the results of LR_{CC} with $\bar{p} = 0.95$, which examine whether the UPI or LPI from each estimation method gives us a good conditional coverage rate (CR). From Table 6, we find that (i) no estimation method gives a good CR for the CAC40 and DAX return series; (ii) the p-value of LR_{CC} based on the LQMLE or GNGQMLE_1 method is always close to zero, and hence the CR constructed from these two methods is not satisfactory; (iii) for the DJIA, HSI or Nikkei225 return series, the CR based on the PQMLE or GNGQMLE_3 method is satisfactory in both directions, while the LPI based on the GQMLE method for the DJIA or HSI return series and the UPI based on the GNGQMLE_2 method for the DJIA return series are not satisfactory; (iv) the PQMLE and GQMLE methods indicate that only the LPI is satisfactory for the FTSE return series, and this can not be indicated by all of the GNGQMLE methods; (v) all PQMLE, GQMLE, GNGQMLE_2 and GNGQMLE_3 methods indicate that only the LPI is satisfactory for the NASDAQ and SP500 return series. Overall, we know that when the return series (e.g., FTSE)

Table 6. *The results of LR_{CC} and out-of-sample CR with $\bar{p} = 0.95$ for eight major stock indexes.*

y_t	n_0		PQMLE	GQMLE	LQMLE	GNGQMLE_1	GNGQMLE_2	GNGQMLE_3
CAC40	1515	UPI	8.0768	8.0768	50.927	49.202	16.416	12.642
			(0.0176)†	(0.0176)	(0.0000)	(0.0000)	(0.0003)	(0.0018)
			[0.9340]†	[0.9340]	[0.9842]	[0.9069]	[0.9261]	[0.9294]
		LPI	7.6045	7.7520	69.471	29.858	9.2859	8.6532
			(0.0223)	(0.0207)	(0.0000)	(0.0000)	(0.0096)	(0.0132)
			[0.9518]	[0.9512]	[0.9888]	[0.9248]	[0.9472]	[0.9485]
DAX	1517	UPI	6.8871	8.7585	48.851	38.780	13.213	12.406
			(0.0320)	(0.0125)	(0.0000)	(0.0000)	(0.0014)	(0.0020)
			[0.9394]	[0.9374]	[98.35]	[0.9117]	[0.9334]	[0.9341]
		LPI	7.6019	7.9221	53.464	35.564	11.840	8.6351
			(0.0223)	(0.0190)	(0.0000)	(0.0000)	(0.0027)	(0.0133)
			[0.9519]	[0.9506]	[98.48]	[0.9196]	[0.9433]	[0.9486]
DJIA	1487	UPI	5.9593	4.3344	30.276	33.411	8.5153	5.9593
			(0.0508)	(0.1145)	(0.0000)	(0.0000)	(0.0142)	(0.0508)
			[93.81]	[0.9401]	[0.9771]	[0.9153]	[0.9354]	[0.9381]
		LPI	2.9401	7.2116	61.514	34.740	3.4362	2.7723
			(0.2299)	(0.0272)	(0.0000)	(0.0000)	(0.1794)	(0.2500)
			[0.9509]	[0.9549]	[0.9872]	[0.9159]	[0.9489]	[0.9523]
FTSE	1493	UPI	8.3814	8.3814	44.653	55.661	17.443	12.717
			(0.0151)	(0.0151)	(0.0000)	(0.0000)	(0.0002)	(0.0017)
			[0.9330]	[0.9330]	[0.9826]	[0.9029]	[0.9257]	[0.9297]
		LPI	5.1764	5.1764	73.956	38.891	13.370	8.5952
			(0.0752)	(0.0752)	(0.0000)	(0.0000)	(0.0012)	(0.0136)
			[0.9451]	[0.9451]	[0.9900]	[0.9149]	[0.9357]	[0.9404]
HSI	1490	UPI	0.1211	3.1785	38.049	20.341	1.6155	0.1405
			(0.9412)	(0.2041)	(0.0000)	(0.0000)	(0.4459)	(0.9322)
			[0.9497]	[0.9443]	[0.9805]	[0.9228]	[0.6067]	[0.9490]
		LPI	1.7182	7.6223	56.443	13.132	0.9994	1.1968
			(0.4235)	(0.0221)	(0.0000)	(0.0014)	(0.6067)	(0.5497)
			[0.9564]	[0.9577]	[0.9859]	[0.9302]	[0.9517]	[0.9544]
NASDAQ	1489	UPI	11.414	11.931	35.262	39.303	18.612	14.280
			(0.0033)	(0.0026)	(0.0000)	(0.0000)	(0.0001)	(0.0008)
			[0.9436]	[0.9429]	[0.9792]	[0.9174]	[0.9362]	[0.9402]
		LPI	4.3780	4.3780	84.023	36.208	3.6488	2.9362
			(0.1120)	(0.1120)	(0.0000)	(0.0000)	(0.1613)	(0.2304)
			[0.9597]	[0.9597]	[0.9919]	[0.9174]	[0.9483]	[0.9510]
Nikkei225	1449	UPI	2.5124	1.9710	51.682	51.522	5.2710	3.1463
			(0.2847)	(0.3733)	(0.0000)	(0.0000)	(0.0717)	(0.2074)
			[0.9413]	[0.9420]	[0.9841]	[0.9041]	[0.9365]	[0.9400]
		LPI	0.9494	1.6835	84.317	25.206	0.2882	0.1576
			(0.6221)	(0.4309)	(0.0000)	(0.0000)	(0.8658)	(0.9242)
			[0.9531]	[0.9565]	[0.9924]	[0.9199]	[0.9476]	[0.9503]
SP500	1489	UPI	9.0196	7.7084	26.079	38.624	12.337	8.4437
			(0.0110)	(0.0212)	(0.0000)	(0.0000)	(0.0021)	(0.0147)
			[0.9369]	[0.9382]	[0.9752]	[0.9140]	[0.9308]	[0.9355]
		LPI	1.3253	0.9940	73.623	32.532	1.5354	0.2525
			(0.5155)	(0.6084)	(0.0000)	(0.0000)	(0.4641)	(0.8814)
			[0.9550]	[0.9523]	[0.9899]	[0.9181]	[0.9449]	[0.9503]

† The p-values of LR_{CC} are in open brackets, and the values of CR are in square brackets.

has a large value of m, the PQMLE method like the GQMLE method is applicable to give us a good prediction in the light tail case, while when the return series (e.g., DJIA and HSI) has a small value of m, the PQMLE method, like the GNGQMLE$_3$ method, can give us a more efficient PI than others in the heavy tail case. Finally, it is worth to highlight that unlike the GNGQMLE methods, the performance of PQMLE neither relies on the selection of the auxiliary likelihood function nor becomes worse in the light tail case. This robustness of the PQMLE in constructing the PI may be because of the ability of the PQMLE to take into account both leptokurtosis and asymmetry.

4·2. *Application to exchange rates*

In this subsection, we apply the PQMLE estimation method to four exchange rates. For each exchange rate series, the period of the data we considered is listed under the second column of Table 7. Since the log-return ($\times 100$) of each exchange rates exhibits some correlations in its conditional mean, it is first fitted by an ARMA(2,2) model with the weighted LAD estimation method in Zhu and Ling (2013). Consequently, we denote the residuals from each fitted ARMA(2,2) model by y_t. Table 7 gives the summary statistics for each y_t, from which we find that each y_t is skewed and has a heavier tail than the $N(0,1)$ distribution. Hence, as in Subsection 4.1, we use a GARCH(1,1) model with the PQMLE, GQMLE, LQMLE, GNGQMLE estimation methods to fit each y_t. All of estimation results are summarized in Table 8, and all

Table 7. *Summary of four exchange rates*

y_t	Time Period	n	mean	standard deviation	skewness	kurtosis
HKD/USD	Jan 24, 1996–Jan 08, 2004	2000	0.0000	0.0268	-4.3767	98.122
JPY/USD	Jan 24, 1996–Oct 27, 2000	1200	-0.0127	0.8063	-0.7232	7.7563
SGD/USD	Jan 23, 1996–Jan 13, 2000	1000	0.0000	0.5140	-1.0774	14.847
TWD/USD	Jan 19, 1996–Jan 10, 2000	1000	0.0116	0.4504	1.4054	28.731

fitted models are adequate by looking at the the ACF and PACF plots (not depicted here) of the squared and absolute residuals. From Table 8, we first find that the TWD/USD return series has a very heavy tail because the value of m is smaller than 1.5, from which we may conclude that the innovation has infinite variance, and hence only the PQMLE method is valid. Secondly, we can see that except the JPY/USD return series, the values of m are all smaller than 2.5. So the GQMLE method is only applicable to the JPY/USD return series, and its performance is worst in all cases. Thirdly, we find that the PQMLE has the best fit among all estimation methods in each case. This advantage of PQMLE over LQMLE or GNGQMLE may be caused by including the asymmetry effect in the likelihood.

Next, as in Subsection 4.1, we use the conditional coverage test LR_{CC} to examine whether each of the estimation methods can provide us a good interval forecast for its one-step-ahead prediction. Table 9 reports all the results of LR_{CC} and CR with $\bar{p} = 0.95$. From this table, we first find that for the HKD/USD and TWD/USD return series, only the PQMLE method gives us a satisfactory CR in both directions. Secondly, for the JPY/USD return series, the CRs based on all of PQMLE, GNGQMLE_2 and GNGQMLE_3 methods are satisfactory, while the GQMLE method can only provide us a satisfactory UPI. Thirdly, for the SGD/USD return series, the CRs obtained from all of PQMLE, GNGQMLE_1 and GNGQMLE_2 methods are satisfactory, while the GQMLE or GNGQMLE_3 method is only applicable to provide a satisfactory UPI. Fourth, it is interesting to see that the p-values of LR_{CC} based on the LQMLE method are always close to zeros, and hence the CR constructed from this method is not satisfactory. Fifth, the CRs of the PQMLE are always within one percent from the 95% value, while this is not the case in other methods. Overall, compared with other methods, the performance of PI constructed from the PQMLE method is often satisfactory, and it is not affected by the selection of the auxiliary likelihood function. This advantage of PQMLE becomes more significant when the return series has a smaller value of m.

5. CONCLUDING REMARKS

In this paper, we propose a PQMLE for GARCH models. Under strict stationarity and some weak moment conditions, the strong consistency and asymptotical normality of the PQMLE are obtained. Meanwhile, the PQMLE can apply to other conditionally heteroskedastic models with no further efforts. Unlike the existing QMLE estimators, the PQMLE is the first QMLE in the literature to take into account both leptokurtosis and asymmetry of the innovation, which are two well-known co-existing features in financial and economic data sets. Simulation study demonstrates that the PQMLE can achieve better efficiency than other estimators, especially when ε_t is heavy-tailed and skewed. Two applications to stock indexes and exchange rates further highlight the importance of the PQMLE method. Specifically, the PQMLE method often gives us the best in-sample fit and out-of-sample prediction. This advantage of the PQMLE exists in the both light

Table 8. *Summary of all estimations for four exchange rates*

y_t		PQMLE	GQMLE	LQMLE	GNGQMLE$_1$	GNGQMLE$_2$	GNGQMLE$_3$
HKD/USD	ω	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
		(0.0000)†	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
	α	0.4529	0.2464	0.2718	1.2262	1.0586	0.9734
		(0.0431)	(0.2333)	(0.0544)	(0.0000)	(0.0000)	(0.0000)
	β	0.6526	0.7837	0.6599	0.6795	0.6893	0.6975
		(0.0190)	(0.1265)	(0.0398)	(0.1108)	(0.1126)	(0.1125)
	ν	-0.0248					
	m	1.6612					
	$\hat{\eta}_k$				0.6952	0.6004	0.5977
	τ_2	0.9956	0.9923	0.9956	1.3948	1.3786	1.3966
	LLF	6525.1	5356.8	6389.7	6518.7	6469.4	6421.8
JPY/USD	ω	0.0324	0.0236	0.0068	0.0116	0.0125	0.0131
		(0.1615)	(0.0098)	(0.0027)	(0.0021)	(0.0023)	(0.0025)
	α	0.1615	0.0783	0.0340	0.0601	0.0602	0.0609
		(0.0423)	(0.0255)	(0.0092)	(0.0003)	(0.0003)	(0.0003)
	β	0.9241	0.8853	0.9186	0.9238	0.9218	0.9199
		(0.0177)	(0.0337)	(0.0199)	(0.0174)	(0.0186)	(0.0195)
	ν	-0.0221					
	m	2.8458					
	$\hat{\eta}_k$				1.2109	0.9977	0.9639
	τ_2	0.9993	0.9993	0.9996	1.0054	1.0048	1.0045
	LLF	-1289.8	-1347.1	-1301.4	-1296.8	-1290.6	-1294.1
SGD/USD	ω	0.0013	0.0003	0.0006	0.0017	0.0023	0.0027
		(0.0007)	(0.0008)	(0.0002)	(0.0005)	(0.0006)	(0.0007)
	α	0.2310	0.0693	0.0449	0.2394	0.2474	0.2462
		(0.0418)	(0.0223)	(0.0097)	(0.0002)	(0.0003)	(0.0003)
	β	0.8400	0.9263	0.9132	0.8157	0.8047	0.8013
		(0.0238)	(0.0208)	(0.0163)	(0.0341)	(0.0376)	(0.0397)
	ν	0.0020					
	m	1.9731					
	$\hat{\eta}_k$				1.0717	0.8989	0.8786
	τ_2	1.0001	1.0192	1.0037	1.0492	1.0430	1.0365
	LLF	-322.5	-450.3	-332.5	-323.2	-333.4	-346.9
TWD/USD	ω	0.0000	0.0001	0.0017	0.0002	0.0007	0.0008
		(0.0000)	(0.0005)	(0.0001)	(0.0001)	(0.0003)	(0.0004)
	α	0.4274	1.0917	0.2539	1.1664	1.0186	0.9788
		(0.0615)	(0.6140)	(0.0636)	(0.0003)	(0.0007)	(0.0008)
	β	0.5154	0.6908	0.6843	0.6797	0.6931	0.6958
		(0.0302)	(0.1067)	(0.0482)	(0.0736)	(0.0821)	(0.0848)
	ν	-0.0223					
	m	1.4076					
	$\hat{\eta}_k$				0.6887	0.6003	0.5996
	τ_2	0.9987	0.9975	0.9972	1.0011	0.9949	1.0044
	LLF	294.9	-219.2	247.2	265.2	225.6	193.8

† The standard deviations are in parentheses.

Table 9. *The results of LR_{CC} and out-of-sample CR with $\bar{p} = 0.95$ for four exchange rates series.*

y_t	n_0		PQMLE	GQMLE	LQMLE	GNGQMLE_1	GNGQMLE_2	GNGQMLE_3
HKD/USD	2000	UPI	0.9936	3.0993	20.664	33.411	34.606	34.606
			(0.6085)†	(0.2123)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
			[95.25]†	[0.9575]	[0.9685]	[0.9685]	[0.9750]	[0.9750]
		LPI	3.9696	6.4482	55.627	49.185	63.514	65.891
			(0.1374)	(0.0398)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
			[0.9520]	[0.9585]	[0.9800]	[0.9785]	[0.9835]	[0.9840]
JPY/USD	1200	UPI	0.6719	3.2874	47.840	13.029	0.6719	1.5956
			(0.7146)	(0.1933)	(0.0000)	(0.0015)	(0.7146)	(0.4503)
			[0.9542]	[0.9608]	[0.9867]	[0.9300]	[0.9542]	[0.9575]
		LPI	1.9443	8.9148	47.840	7.2031	1.9443	2.7892
			(0.3783)	(0.0116)	(0.0000)	(0.0273)	(0.3783)	(0.2479)
			[0.9583]	[0.9675]	[0.9867]	[0.9325]	[0.9583]	[0.9600]
SGD/USD	1000	UPI	0.6240	1.1814	24.995	5.7269	0.2007	1.1814
			(0.7320)	(0.5539)	(0.0000)	(0.0571)	(0.9045)	(0.5539)
			[0.9450]	[0.9570]	[0.9800]	[0.9330]	[0.9510]	[0.9570]
		LPI	1.4316	10.788	40.768	0.7966	2.6154	6.9537
			(0.4888)	(0.0045)	(0.0000)	(0.6715)	(0.2704)	(0.0309)
			[0.9560]	[0.9690]	[0.9870]	[0.9470]	[0.9600]	[0.9670]
TWD/USD	1000	UPI	1.1373	23.754	15.057	10.989	21.803	23.754
			(0.5663)	(0.0000)	(0.0000)	(0.0041)	(0.0000)	(0.0000)
			[0.9550]	[0.9780]	[0.9730]	[0.9700]	[0.9770]	[0.9780]
		LPI	1.0623	21.660	13.516	14.773	19.437	23.386
			(0.5879)	(0.0000)	(0.0012)	(0.0006)	(0.0001)	(0.0000)
			[0.9450]	[0.9770]	[0.9720]	[0.9690]	[0.9770]	[0.9790]

† The p-values of LR_{CC} are in open brackets, and the values of CR are in square brackets.

and heavy tail cases, and it becomes more significant when m becomes smaller. Meanwhile, our PQMLE method gives us a simple way to assess the heavy-tailedness and skewness of the innovation by looking at the values of m and ν. Moreover, compared to the GNGQMLE method, the performance of the PQMLE method neither relies on the selection of the auxiliary likelihood function nor becomes worse in the light tail case. All of these findings suggest that the PQMLE estimation method should have a wide application in practice.

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APPENDIX: PROOF OF THEOREM 1

Recall that the first, second and third derivatives of $g(y,s)$ with respective to s are $g_1(y,s)$, $g_2(y,s)$ and $g_3(y,s)$, respectively. By a simple algebra, we can show that

$$\begin{aligned} g_1(y,s) &= \frac{1}{s} - \frac{2my^2s}{1+y^2s^2} - \frac{\nu y}{1+y^2s^2}, \\ g_2(y,s) &= -\frac{1}{s^2} - \frac{2my^2}{1+y^2s^2} + \frac{2y^2s(2my^2s+\nu y)}{[1+y^2s^2]^2}, \\ g_3(y,s) &= \frac{2}{s^3} + \frac{12my^4s+2\nu y^3}{[1+y^2s^2]^2} - \frac{16my^6s^3+8\nu y^5s^2}{[1+y^2s^2]^3}, \end{aligned}$$

where $s > 0$. Next, it is straightforward to see that

$$\begin{aligned} |g_1(y,s)| &\le \frac{1}{s} + \frac{2m}{s} + \frac{|\nu||y|}{2s|y|} = \frac{1+2m+|\nu|/2}{s}, \\ |g_2(y,s)| &\le \frac{1}{s^2} + \frac{2m}{s^2} + \frac{4ms^2y^4}{y^4s^4} + \frac{2s|\nu||y|^3}{[1+y^2s^2]^{3/2}} \\ &\le \frac{1+6m}{s^2} + \frac{2s|\nu||y|^3}{s^3|y|^3} = \frac{1+6m+2|\nu|}{s^2}, \\ |g_3(y,s)| &\le \frac{2}{s^3} + \frac{12m}{s^3} + \frac{2|\nu||y|^3}{[1+y^2s^2]^{3/2}} + \frac{16m}{s^3} + \frac{8|\nu||y|^5s^2}{[1+y^2s^2]^{5/2}} \\ &\le \frac{2+28m}{s^3} + \frac{2|\nu||y|^3}{s^3|y|^3} + \frac{8|\nu||y|^5s^2}{s^5|y|^5} = \frac{2+28m+10|\nu|}{s^3}. \end{aligned}$$

Thirdly, for some $\kappa_0 \in (0,\kappa)$, by Assumption 3(iii) and Jansen's inequality, we have

$$\begin{aligned} E|\log \bar{f}(\varepsilon_t s)| &= E|m\log(1+\varepsilon_t^2 s^2) + \nu \tan^{-1}(\varepsilon_t s)| \\ &\le \frac{m}{\kappa_0} E\log(1+\varepsilon_t^2 s^2)^{\kappa_0} + \frac{\pi}{2}|\nu| \\ &\le O(1)\log[1+E|\varepsilon_t|^{2\kappa_0}s^{2\kappa_0}] + O(1) \\ &\le O(1)(s^{2\kappa_0}+1). \end{aligned}$$

Therefore, under Assumptions 1-5, we have verified all the conditions for Theorems 1.1-1.2 in Berkes and Horváth (2004). Hence, the conclusions in Theorem 1 hold. This completes the proof.

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