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Abstract

Stable networks of order r where r is a natural number refer to those networks that are immune to coalitional deviation of size r or less. In this paper, we introduce stability of a finite order and examine its relation with efficient networks under anonymous and component additive value functions and the component-wise egalitarian allocation rule. In particular, we examine shapes of networks or network architectures that would resolve the conflict between stability and efficiency in the sense that if stable networks assume those shapes they would be efficient and if efficient networks assume those shapes, they would be stable with minimal further restrictions on value functions.

JEL Code: **C62**, **C71**.

Keywords: **Stability of order** r, **Efficiency**, **Network architecture**.

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1 Introduction

The last few years have witnessed a rapid growth of papers devoted to social and economic networks. A subset of those papers have focussed on stability and efficiency and the conflict between them. Stable networks refer to those networks which once formed will be immune to further changes. The formation process itself is sometimes left unspecified. Efficient networks are those which maximize a social welfare function that is a sum of the utility or payoff functions of individuals. It is well documented there is a general conflict between stability and efficiency, namely, in many settings stable networks are not necessarily efficient and efficient networks are not necessarily stable.

The notion of stability is itself not uniquely defined but depend on what coordination possibilities are available to players or individuals. The existing models of network formation have focussed on two ends of a coordination spectrum. On one hand they have focussed on individual and pairwise solution concepts such as pairwise stability (see Jackson and Wolinsky (1996)) and strong pairwise stability (see Chakrabarti and Gilles (2007)). On the other hand, they have assumed complete coordination with the result being any arbitrary coalition including the grand coalition can freely form and members can alter the structure of links in the coalition (see Jackson and van den Nouweland (2005)).

In the real world very often, we can get intermediate levels of coordination resulting in formation of coalitions which are relatively small (smaller than the grand coalition) but at the same time bigger than a coalition of 1. Therefore, from both a mathematical point of view as well as an economic one, such coalition formation is of interest and this is the subject of the present paper. Stable networks of order r where r is a natural number refer to those networks that are immune to coalitional deviation of size r or less. In this paper, we introduce stability of a finite order and examine its relation with efficient networks.

The main objective is to identify specific classes of value functions where the conflict between stability of a certain order and efficiency is resolved. The focus of this paper is on the architecture based resolution of the conflict, namely, we investigate shapes of networks such that if efficient networks take those shapes, they are necessarily stable, and if stable networks take those shapes, they are necessarily efficient.

The question of network topologies or shapes or architectures for which there is no conflict between stability and efficiency may be significant. If there are a vast number of such networks and one is looking for networks which are simultaneously stable and efficient, one may first focus on those topologies for which there is no conflict between stability and efficiency under reasonable assumptions.

We shall pursue the transferable utility framework introduced by Jackson and Wolinsky (1996). Under this framework, the network produces a certain value as defined by a value function which in turn is allocated among the players based on an allocation rule which determines the payoffs of the players. An example of such a framework would be, for instance, airline code-sharing, where the passenger embarking on a long distance flight pays an up-front fee which is distributed among airlines participating in the code-sharing network.

We assume that the value function follows certain well-known properties like anonymity and component additivity (see below for formal definitions) and focus on a certain allocation rule that has been popular in the literature, namely, the component-wise egalitarian allocation rule. Anonymity implies that value function only depends on the shape of the network and not the labels of the players. A component is a subnetwork where there is a path between any two players and no path from a player within the component to a player outside the component. Component additivity assumes away externalities across components. The component-wise egalitarian allocation rule divides the value produced by a component equally across the members of a component. These assumptions are undoubtedly strong and may not hold in real world settings but may serve as a starting point before we analyze more complicated networks.

The rest of the paper proceeds as follows. Section 2 introduces the notation and terminology. In Section 3 we impose the assumption of component additivity on value functions and under the component-wise egalitarian allocation rule, we look at that the configuration of efficiency compatible network architectures, by which we mean network shapes which ensure that if efficient networks assume those shapes, they are necessarily stable with no further restrictions on value functions. In Section 4, we impose the additional assumption of anonymity and we look at network architectures under which the conflict between stability of a certain order and efficiency is resolved.

We find a broad class of architectures for which stable networks of a certain order are necessarily efficient subject to existence of both stable and efficient networks which assume those architectures. Section 5 concludes.

2 Modelling Principles

2.1 Networks

In this section we define the formal elements to describe network formation along with some concepts borrowed from graph theory. Let $N = \{1, 2, \ldots, n\}$ be a finite set of players. A *coalition* refers to any subset of the player set and the *size* of a coalition is the number of players in that coalition. Two distinct players $i, j \in N$ with $i \neq j$ are *linked* if i and j are related in some capacity. Usually we think of such links as economically productive relationships between players. These relationships are *undirected* in the sense that the two players forming a relationship are equals within that relationship. We do not rule out that these relationships have spillover effects on the productive relations between other players.

Formally, an (undirected) link between i and j is defined as the set $\{i, j\}$. Throughout we use the shorthand notation ij to denote the link $\{i, j\}$. It should be clear that ij is completely equivalent to ji.

In total there are $\frac{1}{2}n(n-1)$ potential links on the player set N. The collection of these potential links on N is denoted by

$$g_N = \{ij \mid i, j \in N \text{ and } i \neq j\}\,. \tag{1}$$

A *network* g is now defined as any collection of links $g \subset g_N$. The collection of all networks on N is denoted by $\mathbb{G}^N = \{g \mid g \subset g_N\}$. The collection $\mathbb{G}^N$ consists of $2^{\frac{1}{2}n(n-1)}$ networks. The network g_N consisting of all links is called the *complete network* on N and the network $g_0 = \varnothing$ consisting of no links is the *empty network* on N.

Let $\pi \colon N \to N$ be a permutation on N. For every network $g \in \mathbb{G}^N$ the corresponding permutation is denoted by $g^\pi = \{\pi(i)\pi(j) \mid ij \in g\} \in \mathbb{G}^N$. Two networks $g, h \in \mathbb{G}^N$ have the same *topology* or *architecture* if there exists a permutation

$\pi\colon N \to N$ such that $h = g^\pi$. This is denoted as $g \sim h$. For $g \in \mathbb{G}^N$ the corresponding network topology is denoted by $\overline{g} = \{h \in \mathbb{G}^N \mid h \sim g\}$. Clearly a network topology is a mathematical *equivalence class* with regard to the binary relationship $\sim$. It is obvious that the collection of all networks $\mathbb{G}^N$ is partitioned into network topologies.

For every network $g \in \mathbb{G}^N$ and every player $i \in N$ we denote i's *neighborhood* in g by $N_i(g) = \{j \in N \mid j \neq i \text{ and } ij \in g\}$. Player i therefore is participating in the links in her *link set* $L_i(g) = \{ij \in g \mid j \in N_i(g)\} \subset g$. We also define $N(g) = \cup_{i \in N} N_i(g)$ and let $n(g) = \#N(g)$ with the convention that if $N(g) = \varnothing$, we let $n(g) = 1$.[1] Also, $n_i(g) = \#N_i(g)$. $n(g)$ will be referred to as *size* of the network g.

A *path* in g connecting i and j is a set of distinct players $\{i_1, i_2, \ldots, i_p\} \subset N(g)$ with $p \geqslant 2$ such that $i_1 = i$, $i_p = j$, and $\{i_1 i_2, i_2 i_3, \ldots, i_{p-1} i_p\} \subset g$. We say i and j are *connected* to each other if a path exists between them and they are *disconnected* otherwise. The network $g' \subset g$ is a *component* of g if for all $i \in N(g')$ and $j \in N(g')$, $i \neq j$, there exists a path in g' connecting i and j and for any $i \in N(g')$ and $j \in N(g)$, $ij \in g$ implies $ij \in g'$. In other words, a component is simply a maximally connected subnetwork of g. We denote the set of network components of the network g by $C(g)$. For any component $h \in C(g)$, the cardinality of $N(h)$ is the *size* of h.

The set of players that are not connected in the network g are collected in the set of (fully) disconnected players in g denoted by

$$N_0(g) = N \setminus N(g) = \{i \in N \mid N_i(g) = \varnothing\}.$$

Such players are also known as singletons. Furthermore, we define

$$\Gamma(g) = \{N(h) \mid h \in C(g)\} \cup \{\,\{i\} \mid i \in N_0(g)\} \tag{2}$$

as the partitioning of the player set N based on the component structure of the network g. For any $g \in \mathbb{G}^N$, $S \in \Gamma(g)$, let $g(S) = g \cap g_S$ where $g_S = \{ij \mid i, j \in S \subset N \text{ and } i \neq j\}$. Namely, $g(S)$ denotes the subgraph of g on the player set S.

Sometimes, depending on the context, we refer to a singleton as a null component, in which case a component as defined above is called a non-null component.

[1] We emphasize here that if $N(g) \neq \varnothing$, we have that $n(g) \geqslant 2$. Namely, in those cases the network has to consist of at least one link.

We denote by $C^r(g)$ be the set of all non-null components of size less than or equal to r. A connected network is one where each pair of players are connected to each other. In other words, a connected network has one non-null component and no singletons. A network that is not connected is disconnected.

We shall emphasize certain network topologies. A *star* network is a connected network that has all players directly linked to a central player and no two other players directly linked to each other. A *circle* is a connected network in which every player is directly linked to two other players.

A *critical player* refers to a player who can by deleting a well chosen subset of links in her neighborhood link set can disconnect at least two other players. A network that has no critical players is referred to as a *bi-connected network*.[2] A circle, for instance, is a bi-connected network.

2.2 Value and Allocation

The *value function* given by $v\colon \mathbb{G}^N \to \mathbb{R}$ such that $v(\varnothing) = 0$ expresses the collective network benefits stemming from a certain network.[3] A network value function v assigns a total benefit $v(g) \in \mathbb{R}$ to the network $g \in \mathbb{G}^N$. The space of all network value functions v such that $v(\varnothing) = 0$ is denoted by $\mathbb{V}^N$. It is clear that $\mathbb{V}^N$ is a $\left(2^{\frac{1}{2}n(n-1)} - 1\right)$-dimensional Euclidean vector space. The allocated payoff to an individual player is determined by an *allocation rule* $Y\colon \mathbb{G}^N \times \mathbb{V}^N \to \mathbb{R}^N$ which determines how the collective value is distributed over the individual players. $Y_i(g, v)$ is the payoff to player i from the network g under the value function v.

Let $v \in \mathbb{V}^N$ be some network value function. We consider two fundamental properties of such a network value function:

- The network value function v is component additive if $v(g) = \sum_{h \in C(g)} v(h)$. Component additivity immediately implies that disconnected players $i \in N_0(g)$ generate no value.

[2] This is a standard graph theoretic term. Gilles et. al. (2006) refer to these as *well-connected networks*.

[3] The notation and terminology has been borrowed from Jackson and Wolinsky (1996).

- The network value function v is *anonymous* if $v(g^\pi) = v(g)$ for all permutations π and networks g. Anonymity implies that benefits $v(g)$ depend on the topology of the network g only.

Next we define some properties of an allocation rule. Recall that $\pi\colon N \to N$ is a permutation. Let v^π be defined by $v^\pi(g^\pi) = v(g)$.

- An allocation rule Y is *anonymous* if for any permutation π, $Y_{\pi(i)}(g^\pi, v^\pi) = Y_i(g, v)$. Anonymity of the allocation rule simply means payoff of a player depends solely on the position in the network rather than the label of the players.

- An allocation rule Y is *balanced* if $\sum\limits_{i \in N} Y_i(g, v) = v(g)$ for all v and g. Balancedness is a minimal property that will be assumed throughout.

- An allocation rule Y is *component balanced* if $\sum\limits_{i \in N} Y_i(h, v) = v(h)$ for every g and $h \in C(g)$ and every component additive v. It is obvious that component balance implies balance. Component balance along with component additivity implies that fully disconnected players in $N_0(g)$ always have an allocated payoff of zero.

- An allocation rule is *component decomposable* if $Y_i(g, v) = Y_i(h, v)$ for all component additive $v \in \mathbb{V}^N$, $g \in \mathbb{G}^N$, $h \in C(g)$ and $i \in N(h)$. Component decomposability requires that if v is component additive, the way that value is allocated within a component does not depend on the structure of other components.

Let $v \in \mathbb{V}^N$ be component additive. The *component-wise egalitarian allocation rule* is defined by

$$Y_i^{ce}(g, v) = \frac{v(h_i)}{n(h_i)} \tag{3}$$

where $h_i \in C(g)$ such that $i \in N(h_i)$ and $Y_i^{ce}(g,v) = \varnothing$ if there is no $h \in C(g)$ such that $i \in N(h)$. Under this rule, the value generated by a component is split equally among the members of that component. The component-wise egalitarian allocation rule satisfies anonymity, component balance and component decomposability.

Also, we consider the *egalitarian allocation rule* defined by

$$Y_i^e(g,v) = \frac{v(g)}{n} \text{ for all } i \in N. \tag{4}$$

Clearly this allocation rule satisfies anonymity and balance, but not component balance or component decomposability. In our discussion these two allocation rules are featured prominently.

2.3 Stability and Efficiency

In this section we discuss network formation principles from a link-based perspective. Central to this approach is that the formation of a link in principle is considered separately. Each link in the network involves a pair of players. While mutual consent is required for establishing a link, each player can delete a link unilaterally.

Denote by $g + ij$, the network obtained by adding link ij to the existing network g, i.e., $g + ij = g \cup \{i, j\}$. Similarly, $g - ij$ denotes the network that results from deleting link ij from the existing network g, i.e., $g - ij = g \setminus \{ij\}$.

Similarly, denote by $g + h$, the network obtained by adding the link-set h to the existing network g where $h \subset g^N \backslash g$. For $h \subset g$, $g - h$ denotes the network that results by deleting the link-set h from the existing network g.

We introduce three fundamental link formation principles.

- A network $g \in \mathbb{G}^N$ is *link deletion proof* (LDP) if for every player $i \in N$ and every neighbor $j \in N_i(g)$, it holds that $Y_i(g-ij, v) \leqslant Y_i(g)$. Link deletion proofness requires that each individual player has no incentive to sever an existing link with one of his neighbors.

- A network $g \in \mathbb{G}^N$ is *strong link deletion proof* (SLDP) if for every player $i \in N$ and every link-set $h \subset L_i(g)$, it holds that $Y_i(g - h, v) \leqslant Y_i(g, v)$. Strong link deletion proofness requires that each player has no incentive to sever links with one or more of his neighbors. Obviously, SLDP implies LDP.

- A network $g \in \mathbb{G}^N$ is *link addition proof* if for all players $i, j \in N$, it holds that $Y_i(g + ij, v) > Y_i(g, v)$ implies $Y_j(g + ij, v) < Y_j(g, v)$. Link addition proofness states that there are no incentives to form additional links.

These three fundamental stability concepts can be used to define additional stability concepts. A network $g \in \mathbb{G}^N$ is *pairwise stable* if it is link deletion proof and link addition proof. Furthermore, a network $g \in \mathbb{G}^N$ is *strongly pairwise stable* if it is strong link deletion proof and link addition proof.

Next we define certain notions of coalitional stability borrowed from Jackson and van den Nouweland (2005).

Let $S \subset N$ be an arbitrary coalition.

- A network $g' \in \mathbb{G}^N$ is *obtainable from* $g \in \mathbb{G}^N$ *via link deletion* by S if $g' \subset g$ and $ij \in g$ and $ij \notin g'$ implies $\{i, j\} \cap S \neq \varnothing$.

- A network $g' \in \mathbb{G}^N$ is *obtainable from* $g \in \mathbb{G}^N$ *via link addition* by S if $g' \supset g$ and $ij \notin g$ and $ij \in g'$ implies $\{i, j\} \subset S$.

These definitions reflect the fact that players can delete their links unilaterally but link addition requires cooperation of both players in question. Of course, coalitions can engage simultaneously in link addition and link deletion which we refer to as deviations.

- A network $g' \in \mathbb{G}^N$ is *obtainable from* $g \in \mathbb{G}^N$ *via deviations* by S if (i) $ij \in g$ and $ij \notin g'$ implies $ij \cap S \neq \varnothing$.(ii) $ij \notin g$ and $ij \in g'$ implies $ij \subset S$.

- A deviation by $S \subset N$ is *profitable* if there exists a network g' that is obtainable from g by S via deviation satisfying two properties:

- $Y_i(g', v) \geqslant Y_i(g, v)$ for all $i \in S$.
- There exists $j \in S$ such that $Y_j(g', v) > Y_j(g, v)$.

Stable networks of order r are those that are immune to profitable deviations by any coalition of size r or less.

- A network g is *link deletion proof (link addition proof) of order* $r \in \mathbb{N}$ (where $1 \leqslant r \leqslant n$) with respect to allocation rule Y and value function v if for any $S \subset N$ with $|S| \leqslant r$, g' that is obtainable from g via link deletion (link addition) by S, and $i \in S$ such that $Y_i(g', v) > Y_i(g, v)$, there exists $j \in S$ such that $Y_j(g', v) < Y_j(g, v)$.

- A network g is *stable of order* $r \in \mathbb{N}$ (where $1 \leqslant r \leqslant n$) with respect to allocation rule Y and value function v if for any $S \subset N$ with $|S| \leqslant r$, g' that is obtainable from g via deviations by S, and $i \in S$ such that $Y_i(g', v) > Y_i(g, v)$, there exists $j \in S$ such that $Y_j(g', v) < Y_j(g, v)$.

Pairwise stability was seminally introduced by Jackson and Wolinsky (1996). Strong pairwise stability as defined above has been introduced by Gilles and Sarangi (2004). A similar concept has been introduced independently by Goyal and Joshi (2006) as "pairwise Nash equilibrium". Pairwise Nash equilibria are equivalent to strong pairwise stability. Strong stability as defined by Jackson and van den Nouweland (2005) is equivalent to stability of order n. Any notion of stability of order r is stronger than the corresponding notion of order $r-1$ where $r \geqslant 2$. Stability of order 1 is equivalent to SLDP. Stability of order 2 is equivalent to "pairwise strong Nash equilibrium" as defined by Bloch and Bellafleme (2004).

It is important to note that stability of any order is stronger than link addition proofness and link deletion proofness of that order combined. In other words, a network may be both link addition proof and link deletion proof of order r but not strong stable of order r. This is because it may be immune to both link formation and link deletion but not both simultaneously.

Finally, we define efficiency.

- A network $g \in \mathbb{G}^N$ is *efficient* with respect to value function v if $v(g) \geqslant v(g')$ for all $g' \subset g_N$.

2.4 Cooperative Games and the r-Core

A *TU cooperative game* is a pair (N, w) where N is the set of players, and $w : 2^N \to \mathbb{R}$ defines the productive value of a coalition, and is called the characteristic function. Since the player set is fixed, we will simply refer to the characteristic function as a cooperative game.

An allocation $x \in \mathbb{R}^N$ is in the *r-core* , where r is a natural number less than or equal to n, if $\sum_{i \in N} x_i = w(N)$ and $\sum_{i \in S} x_i \geqslant w(S)$ for all $S \subset N$ such that $|S| \leqslant r$. The n-core is simply referred to as the *core*. Obviously, any allocation in the r-core also belongs to the $(r-1)$-core.

For the purposes of this paper, we are interested in a special type of cooperative game, which we denote by w^v. For all $S \subset N$,

$$w^v(S) = \max_{g \subset g_S} v(g).$$

Thus the value of a coalition is the maximum value that the coalition can obtain by rearranging its members in a network. The anonymity and component additivity of v implies symmetry and additivity of w^v.

3 Component Additive Value Functions

3.1 Deviation Proof Network Topologies

We will begin by extending the analysis of Gilles et. al. (2006) who show that under component additive (but not necessarily anonymous) value functions, once efficient networks take certain shapes (namely, bi-connected graphs), then under Y^{ce}, all efficient networks are strongly pairwise stable without any further restrictions on value functions. This raises the possibility that appropriate architectures can resolve the conflict between stability and efficiency for higher orders of stability. Namely for stronger notions of stability, there is a wide class of architectures such that if efficient networks assume those architectures, they are stable without further restrictions on value functions. Unfortunately, this is not the case and architecture based resolution of the conflict cannot be extended to higher orders of stability.

We shall assume only component additivity of value functions in this section but not anonymity. We shall also restrict ourselves to the component-wise egalitarian allocation rule. We begin with the definition of *inclusive networks.*

Definition 1 *A network g' which is obtainable from g by deviation by a coalition $S \subset N$ is said to be inclusive of g if for any $i, j \in N(h)$, $h \in C(g)$, there exists $h' \in C(g')$ such that $i, j \in N(h')$.*

In other words, the deviation would not result in breaking up existing components into smaller components though it allows the connecting of existing components to form larger components. So all types link addition is allowed while only those link deletions are allowed which do not destroy all existing path between any two players. In Figure 1, g_2 is inclusive of g_1 while g_3 is not because in g_3, there is no path between 4 and 5 which were previously connected.

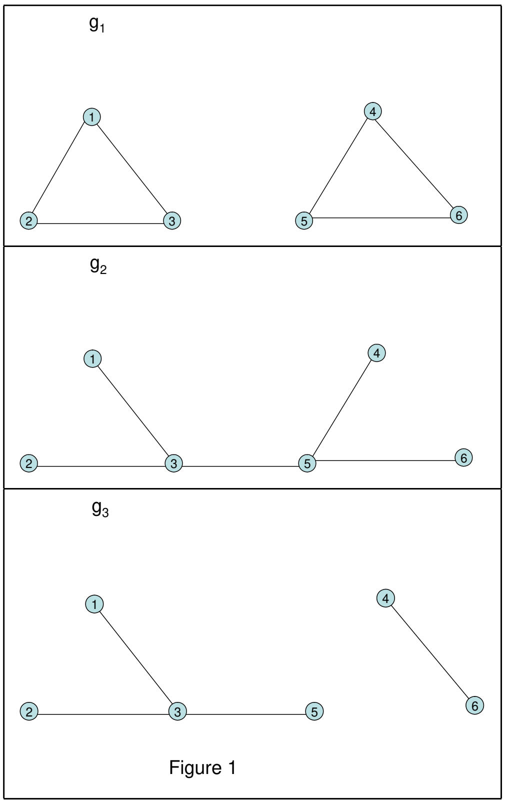

Figure 1

Next we have the following proposition.

Lemma 1 *Let v be component additive and $Y = Y^{ce}$. Given an efficient network g, there does not exist any profitable deviation that yields a network g' that is inclusive of g.*

Proof. Towards a contradiction, suppose there exists a profitable deviation by an arbitrary coalition $S \subset N$ that yields a network g' that is inclusive with regard to g. We begin by showing that there can no player that can be strictly worse off as a result of such a deviation.

Consider any arbitrary player i (which may or may not belong to the deviating coalition S). Let h'_i be a component (possibly a null component if i is a singleton) containing i, namely, $i \in N(h'_i)$ in the network g'. Then there are two possibilities which we delineate as two cases.

Case 1: $N(h'_i) \notin \Gamma(g)$.

Then, given that g' is inclusive of g, there exists a collection of player sets $A_1, A_2, \ldots, A_m \in \Gamma(g)$ such that

$$N(h'_i) = \bigcup_{l=1}^{m} A_l.$$

Two other conditions have to hold. First, i must belong to one of the sets among $A_1, A_2, \ldots, A_m$. Call this set A_i. Second, A_i must contain at least one member of the deviating coalition, namely, $A_i \cap S \neq \varnothing$. Let $j \in A_i \cap S$.

Since j is a member of the deviating coalition, it must be the case that

$$Y_j^{ce}(g', v) \geqslant Y_j^{ce}(g, v). \tag{5}$$

But, both i and j belong to A_i and $N(h'_i)$. Hence,

$$Y_i^{ce}(g, v) = Y_j^{ce}(g, v); \tag{6}$$

$$Y_i^{ce}(g', v) = Y_j^{ce}(g', v). \tag{7}$$

Hence,

$$Y_i^{ce}(g', v) \geqslant Y_i^{ce}(g, v).$$

Case 2: $N(h'_i) \in \Gamma(g)$.

Then, there exists $h \in C(g) \cup N_0(g) : N(h) = N(h'_i)$. If $S \cap N(h'_i) = \varnothing$, then $h'_i = h$ and payoffs of i remain unchanged from the definition of Y^{ce}. Next let $S \cap N(h'_i) \neq \varnothing$ and let $j \in S \cap N(h'_i)$.

But, both i and j belong to $N(h)$ and $N(h'_i)$. Hence,

$$Y_i^{ce}(g, v) = Y_j^{ce}(g, v); \tag{8}$$

$$Y_i^{ce}(g', v) = Y_j^{ce}(g', v). \tag{9}$$

Since j is a member of the deviating coalition, it must be the case that

$$Y_j^{ce}(g', v) \geqslant Y_j^{ce}(g, v). \tag{10}$$

From (8) to (10), we get

$$Y_i^{ce}(g', v) \geqslant Y_i^{ce}(g, v).$$

Now, i is a completely arbitrary player. Hence, it follows that in deviations leading to inclusive networks, there is no player in the player set N that is worse off. But there is at least one player in the player set S who is strictly better off from the definition of a profitable deviation. This contradicts that g is efficient. ■

Proposition 1 immediately leads to the following corollary.

Corollary 1 *Under any component additive v and allocation rule Y^{ce}, an efficient network is link addition proof of any arbitrary order.*

Proof. Link addition would only lead to inclusive networks. Hence, by Lemma 1, the result follows. ■

Hence, starting from an efficient network any profitable deviation must lead to a non-inclusive network. Next, we identify another type of deviation that is not profitable as well.

Definition 2 *A player is said to isolate herself if she deletes all links in her neighborhood.*

It is fairly simple of show that starting from an efficient network, isolation cannot be a profitable deviation.

Lemma 2 *Let v be component additive and $Y = Y^{ce}$. Given any efficient network, there cannot be any profitable deviation involving a player isolating herself.*

Proof. The empty network always yields a value of zero. So any component of an efficient network must yield a non-negative value. Because otherwise value can be increased by deleting all links in that component contradicting efficiency. Hence, each player gets a non-negative payoff. Isolation would always lead to a zero payoff. So isolation cannot be a profitable deviation. ■

By *allowable deviations* corresponding to a certain notion of stability, we mean the processes of link addition/link deletion by a coalition of appropriate size such that these processes do not yield profitable deviations for networks satisfying that notion of stability. For instance, when we are talking of strong pairwise stability, allowable deviations are link formation by coalitions of two players and link deletion by coalitions of one. With regard to strong stability, allowable deviation would include any conceivable deviation. Next, we can define *deviation proof* topologies.

Definition 3 *A network architecture or topology is deviation proof with regard to some allowable deviation if it is not possible to form non-inclusive networks without isolation using such deviation.*

Below in Table 1, we list deviation proof architectures for important types of allowable deviations. Next, we have the following proposition. It is an extension of Corollary 3.2 of Gilles et al. (2006).

Lemma 3 *Let v be component additive and $Y = Y^{ce}$. Then, if an efficient network has a deviation proof architecture, it is stable.*

Proof. The only two things a coalition can do by deviating is either form an inclusive network or isolate themselves. By Lemma 1 and Lemma 2, neither is profitable. ■

Next, let us consider architectures such that if under a component additive value function, they turn out to be efficient, they must be stable as well. Let us call those architectures as *efficiency compatible.* From Lemma 3, deviation proof architectures are efficiency compatible. We show below that if a network is not deviation proof, we can find value functions for which it is not efficiency compatible.

Nature of Stability	**Nature of allowable deviation**	**All possible deviation proof architectures**
SLDP	Link deletion by coalitions of size 1	Bi-connected networks
Strong pairwise stability	Link deletion by coalitions of size 1, link formation by coalitions of size 2	Bi-connected networks
Link addition proofness of order $r \geqslant 2$	Link addition by arbitrary coalitions of size 2 or more	All possible networks
Link deletion proofness of order $r \geqslant 2$	Link deletion by coalitions of size 2 or more	Networks where maximum size of each non-null component does not exceed 2
Stability of order $r \geqslant 2$	Simultaneous link addition and link deletion by coalitions of size 2 or more	Empty network

Table 1

Lemma 4 *Let v be component additive and $Y = Y^{ce}$. Given a network that is not deviation proof, we can find value functions for which it is not efficiency compatible.*

Proof. Consider a network g that has an architecture that is not deviation proof under the allowable deviation in question. We shall construct a value function under which g may be efficient but not stable. Hence, we show that it is not efficiency compatible.

From Table 1, the allowable deviations must include link deletion because there does not exist a network that is not deviation proof under link addition alone.

First, suppose there exists a component $\widehat{h} \in C(g)$ such that $n(\widehat{h}) \geqslant 3$. Then there must exist a coalition $S \subset N(\widehat{h})$ where $|S| \geqslant 1$ that can form a smaller non-null component h' by deleting links, namely $n(h') < n(\widehat{h})$ such that $S \subset N(h')$. This is obvious if allowable deviations include link deletion by coalitions of size 2 or more. If link deletion by coalitions of only size 1 are allowed, then, by Table 1, the network is not a bi-connected graph. Then take any critical player. By definition it can form a smaller component by deleting an appropriately chosen set of links in its neighborhood link-set.

Call the resulting network g'. Assign a value 1 to $\widehat{h}$ and 0 to all other components, if any in g. Further, assign a value 1 to h' and a value 0 to all components other that h' that are formed by link deletion by S. Finally, assign values 0 to all other networks. Then, g is an efficient network. But for all $j \in S$,

$$Y_j(g', v) = \frac{1}{n(h')} > \frac{1}{n(\widehat{h})} = Y_j(\widehat{h}, v) = Y_j(g, v).$$

Therefore, g is not stable.

Next consider a network g where the size of the largest component does not exceed 2 and g is not deviation proof with regard to the allowable deviation in question. We know from Table 1 that there must be at least one non-null component because the empty network is deviation proof for all possible allowable deviations. Further the only allowable deviation for which this is not deviation proof is simultaneous link formation and link deletion by coalitions of size 2 or more. Suppose, $n = 3$. Then g consists of a component of size 2 and a singleton, say $g = \{12\}$. Then assign a value 1 to the any components 12 and 23 and zero to all other components. Then 2 and 3 can induce a profitable deviation by 2 deleting its link with 1 and forming a link with 3. Hence, g cannot be stable even though it is efficient. The same logic can be used to construct value functions under which any network with at least one singleton and one non-null component is efficient but not stable.

Finally consider a network g where all components have size 2. There must necessarily be an even number of players. Consider an anonymous value function and assign a value 1 to components of size 2 or less and a value $\frac{7}{4}$ to a star component of size 3. Let all other components earn zero value. The network is efficient but not stable since a coalition of 2 can form a star and earn a higher payoff of $\frac{7}{12} > \frac{1}{2}$. ■

Lemma 4 is making a somewhat obvious point. A deviation proof architecture is efficiency compatible. A non-deviation proof architecture is not guaranteed to be efficiency compatible. The importance of the result lies in its negative implications. Deviation proof networks are few. So, the kind of analysis pioneered by Gilles et. al. (2006) cannot be extended to higher orders of stability.

3.2 Communication Networks: An Example

Communication networks were introduced by Myerson (1977) and subsequently elaborated upon, among others, by Owen (1986), Slikker and van den Nouweland (2001) and Jackson and van den Nouweland (2005). Let z be a cooperative game. Hence, $z(S)$ indicates the productive value of an arbitrary coalition $S \subset N$.

Let us introduce the restriction that a coalition is productive only when all members of the coalition can communicate with each other. Communication can take place only along links in a communication network g and each link in this network incurs a cost c. This then allows one to define a value function that assigns to each network g the productive value that the players can obtain when they have the communication lines in g available, minus the cost of the network.

A given cooperative game z and a cost per link c lead to a value

$$v^{z,c} = \sum_{S \in \Gamma(g)} z(S) - \sum_{ij \in g} c.$$

In order to ensure that $v^{z,c}(\varnothing) = 0$, we limit ourselves to zero-normalized characteristic functions, i.e. $z(\{i\}) = 0$ for each $i \in N$. Jackson and van den Nouweland (2005) show that if z is convex and symmetric, then a strongly stable and efficient networks exists. We can do away with both convexity and symmetry but impose the restriction that $c = 0$ and prove that there exists an efficient and strongly pairwise stable network. The reason is obvious. If $c = 0$, value depends only on partitioning of the player set Γ rather than the exact network structure. Therefore there exists a certain partition that yields the highest value. One can always find a bi-connected network that yields the efficient partition by replacing each component of an efficient network by a completely intra-connected component. Such a network will be both efficient and strongly pairwise stable. The same cannot be said of any order of stability equal to or higher than 2.

Lemma 5 *For $v^{z,0}$, there always exists a network that is both efficient and strongly pairwise stable.*

4 Component Size, Stability and Efficiency

4.1 Existence of Stable Networks and Efficiency

In this section, we will explore the relationship between stability and efficiency conditional on the fact that one or the other exists but we shall make the additional rather strong assumption of anonymity. Under these two assumptions, namely, component additivity and anonymity of the value function and under the component-wise egalitarian allocation rule, Jackson and van den Nouweland (2005) have shown that the following facts are equivalent:

(i) A stable network of order n exists;

(ii) The set of stable and efficient networks coincide;

(iii) The core of w^v is non-empty.

We extend the results to stability of any arbitrary order. First, we start with some results that are not directly related to the question under investigation but will prove useful later on. The proof of the proposition below is a straightforward extension of Theorem 1 of Jackson and van den Nouweland (2005), but we show the proof for the sake of completeness.

Proposition 1 *Consider any anonymous and component additive value function v. If Y is an anonymous, component decomposable and component balanced allocation and $g \in \mathbb{G}^N$ is a network that is stable of order r with respect to v and Y, then the following would hold:*

(a) If there is more than one component with size less than or equal to r, then all agents in those components earn identical payoffs.

(b) If there is any singleton, then all agents in components with size less than or equal to r will earn zero payoffs.

(c) If there is any component with size less than or equal to r, any agent in a component of size greater than r will earn payoffs that is greater than or equal to those of all agents in components of size less than or equal to r.

Proof. (a) Consider v and Y that satisfy the above properties. Consider $g \in \mathbb{G}^N$ which is stable of order r. Towards a contradiction, let $|C^r(g)| \geqslant 2$ and let there exist two agents $j, k \in N$, $j \neq k$, $j \in N(h)$, $k \in N(h')$ with $h, h' \in C^r(g)$ such that $Y_j(g, v) \neq Y_k(g, v)$. Without loss of generality, assume $Y_j(g, v) > Y_k(g, v)$.

First, suppose j and k belong to different components, namely, $h \neq h'$. Consider a deviation by $N(h) \cup \{k\} \backslash \{j\}$ so that so that k severs all links under g, $N(h) \backslash \{j\}$ severs all links with j and $N(h) \cup \{k\} \backslash \{j\}$ forms a component h'' that is a duplicate of h with k replacing j. By anonymity and component decomposability, $Y_i(h'', v) = Y_i(g, v)$ for all $i \in N(h) \backslash \{j\}$ and $Y_k(h'', v) = Y_j(g, v) > Y_k(g, v)$. This contradicts stability of g of order r since $|N(h) \cup \{k\} \backslash \{j\}| \leqslant r$.

Next consider the situation where j and k belong to the same component, say, $h' = h$. Since $|C^r(g)| \geqslant 2$ then there exists another component $h''' \neq h$, $h''' \in C^r(g)$. Then there exists $i \in N(h''')$ such that either $Y_i(g, v) \neq Y_k(g, v)$ or $Y_i(g, v) \neq Y_j(g, v)$ or both. In that case, we can replicate the above argument to show a contradiction.

(b) Let $k \in N_0(g)$, namely k is a disconnected node or singleton and $|C^r(g)| \geqslant 1$. By component additivity of v and component balance of Y, any singleton earns zero payoff. Hence $Y_k(g, v) = 0$. First note that for all $i \in N(g)$, $Y_i(g, v) \geqslant 0$ because otherwise i can sever all links and earn zero payoffs and gain contradicting deletion proofness of order 1. Hence, towards a contradiction assume

$$\max_{i \in N(h), h \in C^r(g)} Y_i(g, v) > 0.$$

Let $j \in N(h')$ and $h' \in C^r(g)$ such that $Y_j(g, v) > 0$. Consider a deviation by $N(h') \cup \{k\} \backslash \{j\}$ where $N(h') \backslash \{j\}$ severs all links with j and $N(h') \cup \{k\} \backslash \{j\}$ forms a component h'' that is a duplicate of h' replacing j by k. By component decomposability and anonymity, $Y_i(h'', v) = Y_i(g, v)$ for all $i \in N(h') \backslash \{j\}$ and $Y_k(h'', v) = Y_j(g, v) > Y_k(g, v) = 0$. This contradicts stability of order r since $|N(h') \cup \{k\} \backslash \{j\}| \leqslant r$.

(c) Consider v and Y that satisfy the above properties and let $g \in \mathbb{G}^N$ be stable of order r. Let $|C^r(g)| \geqslant 1$ and $|C(g) \backslash C^r(g)| \geqslant 1$. Let $h \in C^r(g)$ and $h' \in C(g) \backslash C^r(g)$ and towards a contradiction

$$\max_{i \in N(h)} Y_i(g, v) > \min_{i \in N(h')} Y_i(g, v)$$

Let

$$j \in \underset{i \in N(h)}{\arg\max} Y_i(g,v) \text{ and } k \in \underset{i \in N(h')}{\arg\min} Y_i(g,v)$$

Hence obviously, $Y_j(g,v) > Y_k(g,v)$. Now consider a deviation by $N(h) \cup \{k\} \backslash \{j\}$ so that k severs all links under g, $N(h) \backslash \{j\}$ severs all links with j, and $N(h) \cup \{k\} \backslash \{j\}$ forms a component h'' that is a duplicate of h replacing j with k. By component decomposability and anonymity, $Y_i(h'',v) = Y_i(g,v)$ for all $i \in N(h) \backslash \{j\}$ and $Y_k(h'',v) = Y_j(g,v) > Y_k(g,v)$. This contradicts stability of order r since $|N(h) \cup \{k\} \backslash \{j\}| \leqslant r$. ■

Next we introduce the notion of *component-size* which plays an important role in resolving the conflict between stability and efficiency. The component-size of a network is the size of the largest component in a network. Formally,

Definition 4 *The component size of a network g denoted by*

$$\vartheta(g) = \max_{S \in \Gamma(g)} |S|.$$

For stable networks of order r, whose component-size is less than or equal to r, the component wise egalitarian allocation rule coincides with the egalitarian allocation rule. If the network has more than one component, then it follows from Proposition 1 given that the component wise egalitarian allocation rule satisfies anonymity, component balance and component decomposability. If it has a single component, it is a immediate consequence of the component-wise egalitarian allocation rule. Next we have the following proposition which is a straightforward extension of Theorem 2 of Jackson and van den Nouweland (2005).

Proposition 2 *(a) For a component additive and anonymous v and allocation rule Y^{ce}, if there exists an efficient network of component-size less than or equal to r, then all stable networks of order r with component-size less than or equal to r (provided they exist) must be efficient.*
(b) If there exists a network which is stable of order r and has a component-size less than or equal to r, then the r-core of w^v is non-empty.

Proof. (a) Let there exist an efficient network of component-size less than or equal to r and towards a contradiction, consider a stable network g of order r with component-size less than or equal to r which is not efficient. Then there exists g' with component-size less than or equal to r such that $v(g') > v(g)$. We claim that there exists $S' \in \Gamma(g')$ such that

$$\frac{v(g'(S'))}{|S'|} > \frac{v(g)}{n}.$$

The claim can be proved as follows. Suppose not and let

$$\frac{v(g'(S))}{|S|} \leqslant \frac{v(g)}{n} \text{ for all } S \in \Gamma(g').$$

Then

$$\sum_{S\in\Gamma(g')} \frac{v(g'(S))}{|S|} \cdot |S| \leqslant \sum_{S\in\Gamma(g')} \frac{v(g)\cdot|S|}{n} \Rightarrow \sum_{S\in\Gamma(g')} v(g'(S)) \leqslant v(g) \Rightarrow v(g') \leqslant v(g)$$

which is a contradiction. Now,

$$Y_i^{ce}(g,v) = \frac{v(g)}{n} \text{ for all } i.$$

Hence the coalition S' can sever links with the rest of the network, form the network $g'(S')$ and be better off. Since $|S'| \leqslant r$, this contradicts stability of order r of network g.

(b) Let the network g be stable of order r and have a component-size less than or equal to r. Then, we know

$$Y_i^{ce}(g,v) = \frac{v(g)}{n} \text{ for all } i.$$

Towards a contradiction, suppose the r-core is empty. Then, $Y^{ce}(g,v)$ is not a r-core element. Then there exists $S \subset N$ with $|S| \leqslant r$ such that

$$\begin{aligned} w^v(S) &> \sum_{i\in S} \frac{v(g)}{n} \\ &\Rightarrow \frac{w^v(S)}{|S|} > \frac{v(g)}{n}. \end{aligned} \tag{11}$$

By the definition of w^v, it follows that there exists some $T \subset S$ and $g' \subset g_S$ with $T \in \Gamma(g')$ such that

$$\frac{v(g'(T))}{|T|} > \frac{v(g)}{n}. \tag{12}$$

The reason is suppose $g' \in \arg\max_{\widetilde{g} \subset g_S} v(\widetilde{g})$. If for all $T \in \Gamma(g')$, it is the case that

$$\frac{v(g'(T))}{|T|} \leqslant \frac{v(g)}{n}$$

then

$$\begin{aligned}
\sum_{T \in \Gamma(g')} \frac{v(g'(T))}{|T|} \cdot |T| &\leqslant \sum_{T \in \Gamma(g')} \frac{v(g)}{n} \cdot |T| \\
&\Rightarrow v(g') \leqslant \frac{v(g)}{n} \cdot |S| \\
&\Rightarrow \frac{v(g')}{|S|} \leqslant \frac{v(g)}{n} \\
&\Rightarrow \frac{w^v(S)}{|S|} \leqslant \frac{v(g)}{n}
\end{aligned}$$

which contradicts (11). Now $|T| \leqslant r$ combined with (12) contradicts stability of order r of g. ■

The conditions imposed by Proposition 2(a) are rather stringent. There must exist both a stable network of component size less than equal to r and an efficient network with component size less than or equal to r in order to ensure that at least one stable network of order r is efficient as well.

First, we show that if there does not exist an efficient network of component-size less than or equal to r, then stable networks of order r are not necessarily efficient.

Example 1 *In this example, we show that there can be efficient networks of component-size greater than r which are not stable of order r even when the latter exist and have component-size less than r. Let $N = \{1, 2, 3, 4\}$. Consider a component additive and anonymous v and let $v(\{ij, jk, ki\}) = 3.3$, $v(\{ij\}) = -200$, $v(\{ij, jk, ki, jl, kl\}) = 4$. The empty network and all singletons yield zero value. Let all other network topologies yield zero value as well.*
Under Y^{ce}, the empty network is stable of order 2 and has component-size less than 2. The efficient network $\{ij, jk, ki, jl, kl\}$ is not stable of order 2 because the coalition consisting of j and k can sever their link with l and be better off. Neither any stable network (of order 2) is efficient nor is any efficient network stable of of order 2.

Next, we show that if there exists efficient networks of component size less than or equal to r, then stable networks of order r with component size greater than r are not necessarily efficient.

Example 2 *In this example, we show that there can be stable networks of component-size greater than* r *which are not efficient even when there exist efficient networks of component size less than or equal to* r*. Let* $N = \{1, 2, 3, 4, 5, 6\}$*. Let* v *be component additive and anonymous and* $v(\{ij\}) = 0.38$*. Also, let the complete component consisting of 5 players yield a value of* 1*. Let all other network topologies yield a value of zero. The efficient network is given by the network consisting of three components of size* 2*. It yields a total value of* 1.14*. This network is stable of order* 2 *as well under* Y^{ce}*.*
However, there is another network that is stable of order 2*, namely the network consisting of a single complete component of size* 5 *and a singleton. It is inefficient because the total value produced is* 1 *which is less than* 1.14*. Each player gets* 0.20 *under* Y^{ce} *which is greater than* 0.19 *which they get in the efficient network. In all other networks, payoffs are zero. So no pair of players have any incentive to deviate and form a different network.*

The next counter-example shows that existence of stable networks of order r with component-size greater than r does not guarantee that the r-core is non-empty.

Example 3 *In this example, we show that the existence of stable networks of order* r *with component size greater than* r *does not imply that* r*-core of* w^v *is non-empty. Let* $N = \{1, 2, 3, 4, 5\}$*. Let* v *be anonymous and component-additive and let there be two possible network topologies that produce non-zero values. The complete component of size* 4 *produces a value* 1*. The component (obviously complete) of size* 2 *produces a value equal to* 0.44*. Towards a contradiction, suppose a* 2*-core allocation exists. This implies by Lemma 6 (see below), that the allocation giving* 0.2 *to each player is a* 2*-core allocation. But this allocation can be blocked by a coalition of* 2 *which is a contradiction. But the reader can verify that the network that comprises of a complete component of size* 4 *and a singleton is stable of order* 2*.*

The above examples are meant to illustrate that we cannot strengthen the Proposition 2 in any way. Let us now consider the converse question. When are efficient networks stable of order r? First, we show that an efficient network of component-size less than or equal to r is not necessarily stable of order r.

Example 4 *In this example, we show that efficient network of component size less than or equal to* r *are not necessarily stable of order* r*. Let* $N = \{1, 2, \ldots, 8\}$*. Let* v *be component additive and anonymous and a component which is a star network of size* 4 *yields a value* 4*. Also, let any component with shape like that of* g_2 *in Figure 1 yield value* 7*. Singletons yield zero value and so do all other network topologies. The efficient network is given by two star components of size* 4*. But it is not stable of order* 4 *because the centers of stars can delete one link each, form a mutual link and earn a higher payoff of* $7/6$ *which is greater than their payoff in the efficient network, namely* 1*. In fact, in this example there does not exist any stable network of order* 4*.*

Can the problem be resolved by existence of a stable network of order r? Namely, if there exists a stable network of order r, are all efficient networks of component size less than equal to r stable of order r. If $r = n$, Jackson and van den Nouweland (2005) have shown that this is the case and they utilize the core properties of the cooperative game w^v (see above) in order to show that.

However, their result cannot be extended in our context. Even if there exists a stable network of order r with component-size less than or equal to r, and an efficient network with component size less than or equal to r, while it follows from Proposition 2(a) that the stable network must be efficient, the converse is not true. There can be efficient networks with component size less than or equal to r which are not stable of order r. Below, we have a counter-example.

Example 5 *In this example, we show that the mere existence of stable networks of order* r *with component-size less than or equal to* r *does not guarantee that all efficient networks with component-size less than or equal to* r *are stable of order* r*. Let* $N = \{1, 2, \ldots, 8\}$ *and* v *be component additive and anonymous. Further, there are three network topologies that produce a positive value (rest producing zero value). A component which is a star of size* 4 *produces a value* 4*. A component which looks like* h_4 *below produces a value* 4 *as well.*

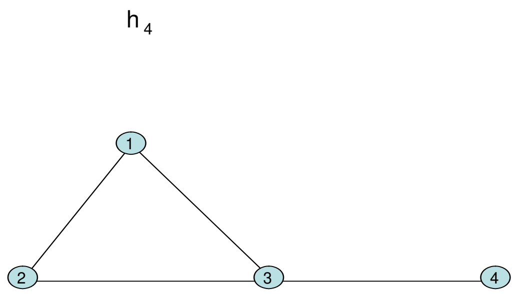

Figure 2

Finally, a component that looks like h_5 *shown below produces a value equal to* 7.

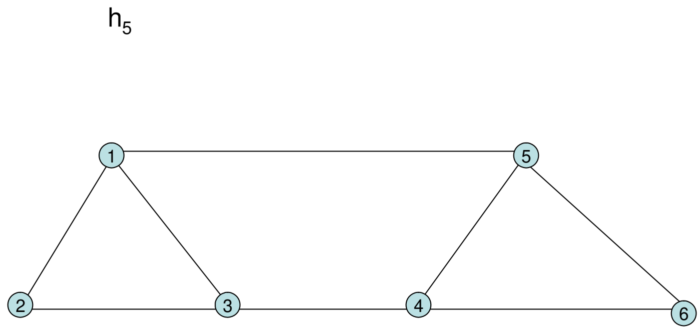

Figure 3

Hence, the efficient networks are networks with two components such as h_4 *(say* g_4*) and networks with two components which are stars of size* 4 *(say* g_6*). A network with one component such as* h_5 *and either two singletons or a component of size* 2 *produces a positive value as well, namely* 7 *but is inefficient. Call a network with one component such as* h_5 *and two singletons* g_5. *All other networks produce value less than* 7.

Now, we claim that g_6 *is stable of order* 4 *but* g_4 *is not. In* g_4 *and* g_6, *each player earns a payoff of* 1. *In a network such as* g_5, *every player belonging to a non-null component earns a payoff of* $7/6$. *The only thing that needs to be checked is that if it is possible to reach the network* g_5 *by a coalition of size* 4 *(or less). The reader can verify that this is indeed the case with* g_4 *but not so with* g_6. *In fact, starting from* g_6, *it takes a coalition of at least 6 players to reach the network* g_5.

We note that this example does not contradict Proposition 2. There exists both one stable network and two efficient network of component size less than 4. *By Proposition 2, the stable network must be efficient which is indeed the case. But Proposition 2* *does not say that the efficient network must be stable as well. Here one of the efficient networks is not stable.*

We will end this subsection with a comment on Jackson and van den Nouweland's (2005) Theorem 2. If efficient networks are not necessarily stable of order r even when there exists stable networks of order r with component-size less than or equal to r, why does the result hold for $r = n$? Below, we prove a result that directly leads to the aforesaid fact. We start with a lemma.

Lemma 6 *Let v be component additive and anonymous. If the r-core of w^v is non-empty, then, the allocation $\widehat{x}$ defined by $\widehat{x}_i = \dfrac{w^v(N)}{n}$ belongs to the r-core.*

Proof. v is anonymous and hence w^v is symmetric. Hence, the r-core is symmetric as well. The r-core is convex by standard arguments. Therefore, taking any core allocation and averaging over all its permutations leads to identical payoffs of $\dfrac{w^v(N)}{n}$.
■

Then, we can prove the following proposition.

Lemma 7 *Let v be component additive and anonymous. If the r-core of w^v is non-empty, then starting from any efficient network with component-size less than or equal to r, there cannot be any profitable deviation that leads to another network with component-size less than or equal to r.*

Proof. Let the network g be efficient relative to v and have a component-size less than or equal to r. Suppose the r-core of w^v is non-empty. Then, from Lemma 6, the allocation $\widehat{x}$ given by $\widehat{x}_i = \dfrac{w^v(N)}{n}$ belongs to the r-core. Then,

$$\sum_{S \in \Gamma(g)} v(g(S)) = v(g) = \sum_{i \in N} \widehat{x}_i.$$

Further, since the component-size of g is less than or equal to r, from the definition of the r-core,

$$\sum_{i \in S} \widehat{x}_i \geqslant w^v(S) \geqslant v(g(S)) \text{ for each } S \in \Gamma(g).$$

Hence, all weak inequalities must hold with equality and therefore for each $S \in \Gamma(g)$,

$$\sum_{i \in S} \widehat{x}_i = v(g(S)).$$

Now, given that under $\widehat{x}$, each player gets equal payoffs, namely, $\widehat{x}_i = \widehat{x}_j$ for each pair i, j, $i \neq j$, it implies

$$Y_i^{ce}(g, v) = \widehat{x}_i \text{ for each } i \in N. \tag{13}$$

Now, consider any arbitrary network g' with component size less than or equal to r. From the definition of the r-core, it follows that

$$\begin{aligned} \sum_{i \in S} \widehat{x}_i &\geqslant w^v(S) \geqslant v(g'(S)) \text{ for each } S \in \Gamma(g') \\ &\Rightarrow \frac{1}{|S|} \sum_{i \in S} \widehat{x}_i \geqslant \frac{1}{|S|} \cdot v(g'(S)) \\ &\Rightarrow \widehat{x}_i \geqslant Y_i^{ce}(g', v) \end{aligned} \tag{14}$$

since $\widehat{x}_i = \widehat{x}_j$ for each pair i, j, $i \neq j$.

From (13) and (14), it follows that

$$Y_i^{ce}(g, v) \geqslant Y_i^{ce}(g', v).$$

Therefore, the lemma follows. ■

We can combine Proposition 2(b) and Lemma 7 to get the following result.

Proposition 3 *Let v be component additive and anonymous. If there exists a network which is stable of order r and has a component-size less than or equal to r, then starting from any efficient network with component-size less than or equal to r, there cannot be any profitable deviation that leads to another network with component-size less than or equal to r.*

Any arbitrary network has component-size less than or equal to n. So starting from a certain network, if there cannot be any profitable deviation leading to a network with component-size less than or equal to n, it simply means that network is strongly stable (or stable of order n). If $r = n$, Proposition 3 translates to the fact that, if there exists a strongly stable network, every efficient network is strongly stable. This is precisely part of Theorem 2 of Jackson and van den Nouweland (2005). Also, Lemma 7 implies that if the core of w^v is non-empty, then every efficient network is strongly stable. Given that there always exists an efficient network, it implies that if the core of w^v is non-empty, then there exists a strongly stable network.

Finally, note that the converse of Proposition 2(b) does not necessarily hold. Namely, the r-core can be non-empty but there need not exist any network that is stable of order r. To see this, reconsider Example 4. The allocation $x_i = 1$ for all $i \in N$ belongs to the 4-core even though no stable networks exists.

We can summarize the results of this section as follows. For a given value function v (which we assume is anonymous and component-additive), let $E(v)$ be the set of efficient networks and $SS^r(v, Y)$ be the set of stable networks of order r under the allocation rule Y. Let

$$\mathbb{G}_r^N = \{g|C(g) = C^r(g))\} = \{g|\vartheta(g) \leqslant r\}.$$

Hence, $\mathbb{G}_r^N \subset \mathbb{G}^N$ denotes the set of networks with component-size less than or equal to r.

Then,

$$\begin{aligned} E(v) \cap \mathbb{G}_r^N &\neq \varnothing \Rightarrow SS^r(v, Y^{ce}) \cap \mathbb{G}_r^N \subset E(v); \\ SS^r(v, Y) \cap \mathbb{G}_r^N &\neq \varnothing \Rightarrow Y_i^{ce}(g, v) \geqslant Y_i^{ce}(g', v) \text{ for all } g \in E(v) \cap \mathbb{G}_r^N,\ g' \in \mathbb{G}_r^N. \end{aligned}$$

4.2 Convexity of Value Functions

The next question we shall tackle is what restrictions on value functions guarantee the existence of at least one efficient network that is stable of order r.

From the above section, we have a sufficient condition:

(a) There exists at least one efficient network of component size less than or equal to r, and

(b) There exists of at least one stable network of component-size less than or equal to r.

We call it the *CS condition-* r.

Jackson and van den Nouweland (2005) has given us another sufficient condition, namely, top-covexity. A value function is top-convex if

$$\frac{w^v(N)}{n} \geqslant \frac{w^v(S)}{|S|}$$

for all $S \subset N$.

Top convexity is both a necessary and sufficient condition for a strongly stable network to exist and under this condition, the set of stable networks coincide with the set of efficient networks. Given that there must always exist an efficient network, top-convexity guarantees the existence of an efficient and strongly stable network. It immediately follows that top convexity guarantees the existence of an efficient network which is stable of order r for any arbitrary r.

CS Condition- r is not necessarily weaker than top-convexity. Top-convexity guarantees existence of a stable network but not necessarily one with component-size less than r. In fact, neither condition implies the other. Value functions may satisfy top convexity but not CS condition-r. Also, value functions may also satisfy CS condition-r but not top-convexity. When $r = n$ do the two conditions coincide. Below, we have two examples.

Example 6 *Link monotone value functions: Consider a component additive and anonymous value function which is strictly increasing in the number of links among the players. Specifically, assume that each link increases value by a fixed amount* $\omega > 0$. *Then,*

$$v(g) = \omega \cdot \sum_{ij \in g} 1.$$

Then, the complete network is the unique strongly stable and efficient network. Also,

$$\frac{w^v(S)}{|S|} = \omega \cdot \left(\frac{|S| - 1}{2} \right).$$

Top-convexity is obviously satisfied. But the CS condition-r is not satisfied for any r *except* $r = n$. *This example also implies that CS condition-r is not a necessary condition for a stable network of order r to being efficient.*

Example 7 *Let* $N = \{1, 2, 3, 4, 5, 6\}$. *Consider an anonymous and component-additive value function. Let a component of size* 2 *produce a value* 2. *Let a complete component of size* 5 *produce value* 5.5. *Let all other network topologies produce zero. Then, top-convexity is not satisfied since for any coalition* S *of size* 5

$$\frac{w^v(S)}{|S|} = 1.1 > 1 = \frac{w^v(N)}{n}.$$

*Consequently, there does not exist a strongly stable and efficient network. But CS condition-*2 *is satisfied which ensures existence of an efficient network which is stable of order* 2.

We can define a condition on value function (which is similar to top-convexity but weaker) which is necessary and sufficient for the existence of the r-core.

Definition 5 *A value function is convex of order r if*

$$\frac{w^v(N)}{n} \geqslant \frac{w^v(S)}{|S|}$$

for all $S \subset N$ such that $|S| \leqslant r$.

Then we have a lemma which is a direct extension of Jackson and van den Nouweland's (2005) Theorem 3.

Lemma 8 *The r-core of w^v is non-empty iff the value function is convex of order r.*

Proof. Suppose the r-core of w^v is non-empty. Then, by Lemma 6, $\widehat{x}$ as defined above belongs to the r-core. Hence, for every $S \subset N$ with $|S| \leqslant r$, it follows that

$$\begin{aligned} \sum_{i \in S} \widehat{x}_i &\geqslant w^v(S) \\ &\Rightarrow \frac{w^v(N)}{n} \geqslant \frac{w^v(S)}{|S|}. \end{aligned}$$

This implies that v is convex of order r.

For the converse let v be convex of order r. Then,

$$\frac{w^v(N)}{n} \geqslant \frac{w^v(S)}{|S|}$$

for all S such that $|S| \leqslant r$. This implies

$$\begin{aligned} |S| \cdot \frac{w^v(N)}{n} &\geqslant w^v(S) \\ &\Rightarrow \sum_{i \in S} \widehat{x}_i \geqslant w^v(S) \end{aligned}$$

for all $S \subset N$ such that $|S| \leqslant r$. Hence, $\widehat{x}$ belongs to the r-core and so the r-core cannot possibly be empty. ■

Of course, convexity of order r is a necessary (from Proposition 2(b)) but not sufficient condition for existence of a stable network of order r. In Example 4, for instance, the allocation of 1 to every player belongs to the 4-core where but there does not exist any network that is stable of order 4. To summarize:

(i) CS condition r is a sufficient but not necessary condition for existence of an efficient network that is stable of order r.

(ii) Top convexity is a sufficient but not necessary condition for existence of an efficient network that is stable of order r.

(iii) Neither of the two above conditions implies the other but the two conditions coincide if $r = n$.

(iv) Convexity of order r is necessary but not sufficient for existence of a stable network of order r.

5 Conclusion

In this paper we introduce stability of a finite order and examine its relationship with efficient networks. Stable networks of a finite order are those networks that are immune to changes in terms of alterations in the existing structure of links when only coalition not exceeding a certain size can be formed. We focus in this paper on topologies or architectures of networks which if assumed by stable networks would ensure these are efficient and if assumed by efficient networks would ensure these are stable. Under component additivity of value functions and component-wise egalitarian allocation rule, there are no significant network architectures which if assumed by efficient networks ensures that they are stable of order r (where $r \geqslant 2$) without further restrictions on value functions. This contrasts with the results of Gilles et al. (2006) who find a broad class of such architectures for strong pairwise stability, namely, bi-connected graphs. Once we add anonymity, it holds that if there exists at least one efficient network with component-size less than or equal to r, then all stable networks of order r with component-size less than or equal to r must be efficient. We show using counter-examples that the results cannot be strengthened in any way. From this result, we identify a set of sufficient conditions in terms of stable and efficient networks assuming certain architectures which we call CS condition-r which ensures there exists a network that is stable of order r and efficient. These condition coincides with top-convexity when r is equal to the total number of players but otherwise neither follows from nor implies top convexity.

References

[1] Bloch, F. and P. Bellafleme (2004) "Market Sharing Agreements and Collusive Networks", *International Economic Review*, 45 (2), 387-411.

[2] Chakrabarti, S. and R.P. Gilles (2007) "Network Potentials", *Review of Economic Design*, 11 (1), 13-52.

[3] Gilles, R.P., S. Chakrabarti, S. Sarangi and N. Badasyan (2006) "Critical Agents in Networks", *Mathematical Social Sciences*, 52 (3), 302-310.

[4] Gilles, R.P. and S. Sarangi (2004) "The Role of Trust in Costly Network Formation," Discussion Paper 2003-53, Tilburg University, Center for Economic Research.

[5] Goyal, S. and S. Joshi (2006) "Unequal Connections", *International Journal of Game Theory*, 34 (3), 319-349.

[6] Harary, F. (1995) *Graph Theory*, Addison Wesley, Oxford, U.K.

[7] Jackson, M. and A. Wolinsky (1996) "A Strategic Model of Social and Economic Networks", *Journal of Economic Theory*, 71 (1), 44-74.

[8] Jackson, M. and A. van den Nouweland (2005) "Strongly Stable Networks", *Games and Economic Behavior,* 51 (2), 420-444.

[9] Myerson, R.B., (1977) "Graphs and Cooperation in Games", *Mathematics of Operations Research*, 2(3) 225-229.

[10] Owen, G. (1986) "Value of Graph-Restricted Games", *Siam Journal of Discrete Mathematics*, 7(2), 210-221.

[11] Slikker, M. and A. van den Nouweland (2001) *Social and Economic Networks in Cooperative Game Theory*, Kluwer Academic Publishers, Boston, MA.