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6 January 2014

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January 6, 2014

Abstract

The main purpose of this paper is to outline that the definable Debreu map is a local definable diffeomorphism. It implies the equilibrium is locally determined in each connected component partitioning a regular O-minimal equilibrium manifold. It complements the result in Theorem 5 of Arias-R. (2013) and converges to the local determinacy result of definable competitive equilibrium of Blume and Zame (1992).

Key words: O-minimal manifold, cell decomposition, Debreu map, local determinacy

JEL classification: D50, D51

1 Introduction

In this paper we outline that the Debreu map, which is a restriction of the natural projection, is a definable local diffeomorphism for any regular O-minimal equilibrium manifold. We use this result to complement the idea in Arias-R. (2013) regarding to a composition of regular equilibria for any continuous path in the equilibrium manifold. It is directly related with the definable local uniquess of Blume and Zame (1992) in this kind of economies.

2 The note

We follow the same notation in Arias-R. (2013). Let $\mathcal{L} = \{1, ..., L\}$ commodities and $\mathcal{I} = \{1, ..., I\}$ consumers. From Blume and Zame (1992) the preference order $\succsim \subseteq X \times X$ is definable in the space of commodities $X = \mathbb{R}^L$. The space of economies $\Omega = \mathbb{R}^{LI}$ defined by initial endowments $\omega_i = (\omega_i^l)_{i\in\mathcal{I}}^{l\in\mathcal{L}}$ for each consumer and commodity. The space of prices $S = \mathbb{R}_{++}^{L-1}$ allowing the numeraire. Let $f_i(p, p\omega_i)$ be the definable marshallian demand function with $p \in S$. The aggregate demand is $\Sigma_{i=1}^I f_i(p, p\omega_i)$ and aggregate supply is $\Sigma_{i=1}^I \omega_i$.

If $(p, \omega) \in S \times \Omega$ equals aggregate supply and demand, then we say that p is an walrasian equilibrium price for ω. Let $E \subseteq S \times \Omega$ be the regular definable equilibrium manifold. Let $\pi : S \times \Omega \to \Omega$ be the natural projection of $S \times \Omega$ in its Ω-coordinates. This is definable by definition. The Debreu map $\pi\mid_{E\subseteq S\times\Omega}: E \to \Omega$ is also a definable function because its domain is definable. Following Debreu (1970) and Nagata (2004), we could write this function in vectorial form,

$$\pi\mid_{E\subseteq S\times\Omega} = [f_1 + \sum_{i=2} f_i - \sum_{i=2} \omega_i, ..., \omega_I]$$

From Debreu (1970) and Nagata (2004) we could say that a vector of positive prices $p \in S$ is an equilibrium for $\omega \in \Omega$ if $\pi\mid_E (p, \omega) = (\omega_1, ..., \omega_I)$. Balasko (1988) proved that this map is smooth, bijective and proper.

Proposition 1 *The map $\pi\mid_{E\subseteq S\times\Omega}: E \to \Omega$ is definable proper.*

Proof: Balasko (1988) proved that $\pi\mid_{E\subseteq S\times\Omega}$ is proper. Because it is definable, then it is definable proper following Van den Dries (1998). □

Proposition 2 *The map $\pi\mid_{E\subseteq S\times\Omega}: E \to \Omega$ is a definable diffeomorphism.*

Proof: Balasko (1988) proved that $\pi\mid_{E\subseteq S\times\Omega}$ is a diffeomorphism. Because it is definable, then it is a definable diffeomorphism following Van den Dries (1998). □

Example 1 *Consider the definable CES-economy with $\rho = 0$ in example 1 in Arias-R. (2013). From the equilibrium manifold E we do so simple calculations to arrive to the definable Debreu map:*

$$\pi(p, \omega; p_y = 1) = \left[\left(\frac{p_y}{p_x}\right)\sum_{i=1}^{2}\omega_i^y - \omega_2^x; \left(\frac{p_x}{p_y}\right)\sum_{i=1}^{2}\omega_i^x - \omega_2^y; \omega_2^x; \omega_2^y\right]$$

From the Tarsky-Seidemberg theorem in Bochnak, Coste and Roy (1991) we conclude this map is definable. □

By applying the cell decomposition theorem we have $E = \cup E_i$ being E_i definable subset of E. We could restrict the Debreu map in each E_i giving the same definable topological properties to these restrictions.

Theorem 1 *The map $\pi\ |_{E_i \subseteq E}$: $E_i \to \Omega$ is a definable diffeomorphism.*

Proof: Take an economy $\omega^* \in \Omega$ and an equilibrium vector $e^* \in E$ such that $\omega^* = \pi\ |_{E \subseteq S \times \Omega}\ (e^*)$. Use the Cell Decomposition Theorem to divide the manifold E in definable open sets $E = \cup_i E_i$ each one disjunct with $i = 1, 2.., k$. Pick up E_i^* with $e^* \in E_i^*$. The manifold E covers Ω. There exist a definable C^∞-homeomorphism $\pi\ |_{E_i^* \subseteq E}$: $E_i^* \to U_{\omega^*}$ with a definable neighbourhood $U_{\omega^*} \subseteq \Omega$. From this, we deduce the claimed result. $\square$

Example 2 *Let us use the example 1. Consider the components of the Jacobian matrix corresponding to the variations in p_x and p_y*

$$\left[-\frac{1}{p_x^2} \sum_{i=1}^{2} \omega_i^y \right] ; \left[p_x \sum_{i=1}^{2} \omega_i^x \right]$$

The determinant of the Jacobian $\mid J \mid \neq 0$ for every $p \in S$ because $\sum_{i=1}^{2} \omega_i^y \neq 0$ and $\sum_{i=1}^{2} \omega_i^y \neq 0$. Notice that the preferences are strictly convex. We have a determined price vector for each definable CES-economy with $\rho = 0$.

References

[1] Arias-R., Omar Fdo. (2013). *A remark on definable paths in regular O-minimal equilibrium manifolds.* Munich Personal Repec Archive, 51820.

[2] Balasko, Y. (1988). *Foundations of the theory of general equilibrium.* Academic press, Boston.

[3] Blume, L. and Zame, W. (1992). *The algebraic geometry of competitive equilibrium.* Economic theory and international trade; essays in memoriam J. Trout Rader, ed. por W. Neuefeind y R. Reizman, Springer-Verlag, Berlin..

[4] Bochnak, J., Coste, M. and Roy, M. (1991). *Real Algebraic Geometry.* Springer Verlag-Berlin.

[5] Debreu, G. (1970). *Economies with a finite set of equilibria.* Econometrica, Vol. 38, No.3, pp. 387-392.

[6] Nagata, R. (2004). *Theory of regular economies.* World scientific.

[7] Van den Dries, L. (1998). *Tame topology and O-minimal structures.* Cambridge university press.