# Coalitional Fairness: The Case of Exact Feasibility with Asymmetric Information

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# COALITIONAL FAIRNESS: THE CASE OF EXACT FEASIBILITY WITH ASYMMETRIC INFORMATION

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Abstract. Consider a pure exchange economy with asymmetric information. The space of agents is a mixed measure space and the commodity space is an ordered Banach space whose positive cone has an interior point. The concept of coalitional fairness introduced in [9] is examined in the framework of asymmetric information. It is shown that the private core is contained in the set of privately coalitionally fair allocations under some assumptions. This result provides an extension of Theorem 2 in [9] to an asymmetric information economy with infinitely many commodities.

## 1. Introduction

It is well known that one of the crucial facts in the classical Debreu-Scarf's replica theorem is that each allocation in the core assigns the same consumption to all agents of the same type, that is, no agent prefers his net trade to that of another agent of the same type. Since the comparison was restricted among identical agents, Schmeildler and Vind [11] introduced the concept of *fair net trades* in an exchange economy with finitely many agents, where an agent was able to compare his net trade with that of an another agent with different type. A net trade is fair if the net trade of each agent is at least good for him as the net trade of any other agent would be. Thus, each agent evaluates the other agent's position on the same terms that he judges his own. To define it formally, let $x = (x_1, \cdots, x_n)$ be an allocation of commodities among agents in an exchange economy with $n$ agents. The *net trade* of agent $i$ is $x_i - a_i$, where $x_i$ is the commodity bundle received by $i$ at $x$ and $a_i$ is the initial endowment of agent $i$. The net trade $y = (y_1, \cdots, y_n)$, defined by $y_i = x_i - a_i$, is said to be *fair* if for all agents $i$ and $j$, $y_i \succeq_i y_j$, where $\succeq_i$ denotes the preference relation of agent $i$. In other words, if a net trade is fair then the market does not discriminate among the agents.

An analogous idea of discrimination was considered in Jaskold-Gabszewicz [9] in terms of coalitions and it was termed as the *coalitional fairness*. The allocation $x$ is called *coalitionally unfair* if there exists two disjoint coalitions $S_1$ and $S_2$ such that $\sum_{i\in S_1} y_i < \sum_{i\in S_2} y_i$. In this case, the agents in $S_1$ could have benefited by achieving the net trade of $S_2$. Formally, there exists another allocation $z = (z_1, \cdots, z_n)$ such that $z_i \succ x_i$ for all $i \in S_1$ and $\sum_{i\in S_1}(z_i - a_i) = \sum_{i\in S_2} y_i$. So, $S_1$ is treated under $x$ in a discriminatory way by the market. The allocation $x$ is called *coalitionally fair* [1] if there does not exist any two such disjoint coalitions. In

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[1] See Shitovitz [13] for a similar concept.

a pure exchange mixed economy with finitely many commodities ([12]), Jaskold-Gabszewicz [9] showed that the core is contained in the set of coalitionally fair allocations if coalitions are restricted to those non-null measurable sets which are either atomless or containing all atoms. The result may fail if a coalition is just a non-null measurable set, refer to Proposition 2 in [9].

In the past two decades, an economy involving uncertainty and asymmetric information is one of the most important research areas in the theoretical economics. Thus, it is interesting to know how far one can extend the results in [9] to this framework. Due to different information and communication opportunities among agents, several alternative core concepts had been introduced in [14, 15]. One of them is the notion of the private core, which was based on the fact that agents have no access to the communication system, that is, each member of the coalition uses only his own private information whenever a coalition blocks an allocation, refer to [15]. It is worth to point out that under standard assumptions, the private core is non-empty, Bayesian incentive compatible and rewards the information superiority of agents (see [10, 15]). Thus, it is essential to see whether a relation between the core and the set of coalitionally fair allocations similar to that in [9] can be established in a framework where allocations are privately measurable. The first attempt was made to this problem by Graziano and Pesce[2] in [8]. In fact, they showed that in an asymmetric information economy with a mixed measure space of agents and a finite dimensional commodity space, the private core is a subset of the set of coalitionally fair allocations if coalitions are restricted to those non-null measurable sets which are either atomless or containing all atoms. In their result, the allocations were restricted to a certain class of functions (refer to the assumption (A.6) in [8]) and the feasibility was taken as free disposal. However, when feasibility is defined with free disposal, the private core allocations may not be incentive compatible and contracts may not be enforceable, refer to [1]. Thus, to avoid this problem, it is desirable to consider a framework without free disposal.

The main purpose of this paper is to examine whether a result similar to that in [8] can be obtained without free disposal assumption and the assumption (A.6) in an asymmetric information economy with a mixed measure space of agents and an ordered Banach space having the non-empty positive interior as the commodity space. The rest of the paper is organized as follows. In Section 2, a general description of the model is provided. Section 3 deals with some technical lemmas which are useful in the proofs of the main results. In section 4, the main results are presented.

## 2. Description of the model

A standard mixed model of a pure exchange economy with asymmetric information is considered. The space of *economic agents* is denoted by a measure space $(T, \mathscr{T}, \mu)$ with a complete, finite, and positive measure $\mu$. Since $\mu(T) < \infty$, the set $T$ can be decomposed into two parts: one is atomelss and the other contains countably many atoms. That is, $T = T_0 \cup T_1$, where $T_0$ is the atomless part and $T_1$ is the countable union of atoms. Let

$$\mathscr{T}_0 = \{S \in \mathscr{T} : S \subseteq T_0\} \text{ and } \mathscr{T}_1 = \{S \in \mathscr{T} : T_1 \subseteq S\}.$$

[2]For the recent developments of different (coalitonal) fairness notions, refer to [6, 8].

Thus, $\mathscr{T}_0$ (resp. $\mathscr{T}_1$) is the subfamily of $\mathscr{T}$ containing no atoms (resp. all atoms). Denote by

$$\mathscr{T}_2 = \mathscr{T}_0 \cup \mathscr{T}_1 = \{S \in \mathscr{T} : S \in \mathscr{T}_0 \text{ or } S \in \mathscr{T}_1\}$$

the subfamily of $\mathscr{T}$ containing either no atoms or all atoms. The *commodity space* is an ordered Banach space $B$ whose positive cone has an interior point. The order on $B$ is denoted by $\leq$, and $B_+ = \{x \in B : x \geq 0\}$ denotes the positive cone of $B$. The symbol $x \gg 0$ is employed to denote that $x$ is an interior point of $B_+$, and put $B_{++} = \{x \in B_+ : x \gg 0\}$. The exogenous uncertainty is described by a measurable space $(\Omega, \mathscr{F})$, where $\Omega$ is a finite set denoting all possible states of nature and the $\sigma$-algebra $\mathscr{F}$ denotes all events. The economy extends over two periods. In the first period, agents arrange contracts that may be contingent on the realized state of nature. Consumption takes place in the second period when agents receive their private information.

Each agent $t \in T$ is associated with a *consumption set* $Y_+$ in every state of nature. The *private information* of agent $t$ is described by a partition $\mathscr{P}_t$ of $\Omega$. If $\omega_*$ is the true state of nature in the second period then agent $t$ observes the unique element of $\mathscr{P}_t$ which contains $\omega_*$. Let $\mathscr{F}_t \subseteq \mathscr{F}$ be the $\sigma$-algebra generated by $\mathscr{P}_t$, denoted by $\mathscr{F}_t = \sigma(\mathscr{P}_t)$. The preferences of agent $t$ is represented by a *state dependent utility function* $U_t : \Omega \times B_+ \to \mathbb{R}$, and $\mathbb{Q}_t$ is a probability measure on $\Omega$, denoting a *prior beliefs* of agent $t$. The *ex ante expected utility* of agent $t$ for a random bundle $x : \Omega \to B_+$ is defined by

$$\mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, x(\cdot))) = \sum_{\omega \in \Omega} U_t(\omega, x(\omega))\mathbb{Q}_t(\omega).$$

An *allocation* is a function $f : T \times \Omega \to B_+$ such that $f(\cdot, \omega)$ is Bochner integrable for all $\omega \in \Omega$, and $f(t, \cdot)$ is $\mathscr{F}_t$-measurable $\mu$-a.e. There is a fixed initial allocation $a$; $a(t, \omega)$ represents the *initial endowment density* of agent $t$ in the state of nature $\omega$. It is assumed that $a(t, \omega) \in B_{++}$ for all $(t, \omega) \in T \times \Omega$. An allocation $f$ is said to be *feasible* if

$$\int_T f(\cdot, \omega) d\mu = \int_T a(\cdot, \omega) d\mu$$

for all $\omega \in \Omega$. Any set $S \in \Sigma$ with $\mu(S) > 0$ is called a *coalition* of $\mathscr{E}$. If $S$ and $S'$ are two coalitions of $\mathscr{E}$ with $S' \subseteq S$ then $S'$ is termed as a *sub-coalition* of $S$. A coalition $S$ *privately blocks an allocation* $f$ in $\mathscr{E}$ if there is an allocation $g$ such that

$$\mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, g(t, \cdot))) > \mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, f(t, \cdot)))$$

$\mu$-a.e. on $S$, and

$$\int_S g(\cdot, \omega) d\mu = \int_S a(\cdot, \omega) d\mu$$

for all $\omega \in \Omega$. The *private core* of $\mathscr{E}$, denoted by $\mathscr{PC}(\mathscr{E})$, is the set of feasible allocations which are not privately blocked by any coalition. The family of partitions of $\Omega$ is denoted by $\mathfrak{P}$. Since $\Omega$ is finite, $\mathfrak{P}$ also has finitely many elements: $\mathscr{P}_1, \cdots, \mathscr{P}_n$. It is assumed that the set $T_i = \{t \in T : \mathscr{P}_t = \mathscr{P}_i\}$ is $\mathscr{T}$-measurable for all $1 \leq i \leq n$. For any $S \in \mathscr{T}$, let

$$\mathfrak{P}_S = \{i : S \cap T_i \neq \emptyset\} \text{ and } \mathfrak{P}(S) = \{i : \mu(S \cap T_i) > 0\}.$$

For any $k \geq 1$, the $(k-1)$-simplex of $\mathbb{R}^k$ is defined as

$$\Delta^k = \left\{ x = (x_1, \cdots, x_k) \in \mathbb{R}^k_+ : \sum_{i=1}^{k} x_i = 1 \right\}.$$

Consider a function $\varphi : (T, \mathscr{T}, \mu) \to \Delta^{|\Omega|}$ defined by $\varphi(t) = \mathbb{Q}_t$ for all $t \in T$. For each $\omega \in \Omega$, define a function $\psi_\omega : T \times B_+ \to \mathbb{R}$ by $\psi_\omega(t, x) = U_t(\omega, x)$. The following assumptions are needed to prove the main results of this paper, first three of which are similar to those in [2, 3, 4, 7].

$(\mathbf{A}_1)$ The function $\varphi$ is measurable, where $\Delta^{|\Omega|}$ is endowed with the Borel structure.

$(\mathbf{A}_2)$ For each $\omega \in \Omega$, the function $\psi_\omega$ is Carathéodory, that is, $\psi_\omega(\cdot, x)$ is measurable for all $x \in B_+$, and $\psi_\omega(t, \cdot)$ is norm-continuous for all $t \in T$.

$(\mathbf{A}_3)$ For each $(t, \omega) \in T \times \Omega$, $U_t(\omega, x + y) > U_t(\omega, x)$ if $x, y \in B_+$ with $y \gg 0$.

$(\mathbf{A}_4)$ For each $(t, \omega) \in T_1 \times \Omega$, $U_t(\omega, \cdot)$ is concave.

For any allocation $f$, define a correspondence $P_f : (T, \mathscr{T}, \mu) \rightrightarrows B^\Omega_+$ such that

$$P_f(t) = \left\{ x \in X_t : \mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, x(\cdot))) > \mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, f(t, \cdot))) \right\},$$

where

$$X_t = \{x : \Omega \to B_+ : x \text{ is } \mathscr{F}_t\text{-measurable}\}.$$

Suppose that $B^\Omega$ is endowed with the point-wise algebraic operations, the point-wise order and the product norm. An element $y \in B^\Omega_+$ can be identified with the function $y : \Omega \to B_+$ and vise-versa. An *integrable selection* of $P_f$ is a Bochner integrable function $f : (T, \mathscr{T}, \mu) \to B^\Omega_+$ such that $f(t) \in P_f(t)$ $\mu$-a.e. The *integration* of $P_f$ over a coalition $S$ in the sense of Aumann is a subset of $B$, defined as

$$\int_S P_f d\mu = \left\{ \int_S f d\mu : f \text{ is an integrable selection of } P_f \right\}$$

Note that, under $(\mathbf{A}_4)$, $\mathrm{cl} \int_S P_f d\mu$ is convex for any coalition $S$.

## 3. Blocking Mechanism

In this section, some technical lemmas are established. These results will be employed to prove the main results in the next section.

**Lemma 3.1.** *Assume* $(\mathbf{A}_1)$-$(\mathbf{A}_3)$. *Let* $f, g$ *be two allocations and*

$$\mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, g(t, \cdot))) > \mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, f(t, \cdot)))$$

*$\mu$-a.e. on a coalition $S$. Then there exist a $\lambda \in (0, 1)$, a $z \in B_{++}$, and an allocation $h$ such that*

$$\mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, h(t, \cdot))) > \mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, f(t, \cdot)))$$

*$\mu$-a.e. on $S$, and*

$$\int_S h(\cdot, \omega) d\mu + z = \int_S ((1 - \lambda) g(\cdot, \omega) + \lambda a(\cdot, \omega)) d\mu$$

*for all $\omega \in \Omega$.*

*Proof.* Since $f$ and $g$ are Bochner integrable, there exist a sub-coalition $R$ of $S$ and a separable closed linear subspace $Z$ of $B^\Omega$ such that $f(R) \cup g(R) \subseteq Z$, $\mu(S \setminus R) = 0$ and

$$\mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, g(t, \cdot))) > \mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, f(t, \cdot)))$$

for all $t \in R$. Let $\{c_m : m \geq 1\} \subset (0, 1)$ be a monotonically decreasing sequence converging to 0. Define a function $g_m : R \to Z_+$ by

$$g_m(t, \cdot) = (1 - c_m) g(t, \cdot)$$

for all $t \in R$. Note that $g_{m+1}(t, \cdot) \geq g_m(t, \cdot)$ for all $t \in R$ and $m \geq 1$. Pick an $i \in \mathfrak{P}(R)$ and define $Q^i_f : R \cap T_i \rightrightarrows Z_+$ such that

$$Q^i_f(t) = \left\{ x \in Z_+ : \mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, x(\cdot))) > \mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, f(t, \cdot))) \right\}$$

for all $t \in R \cap T_i$. By Remark 6 in [7], $\mathrm{Gr}_{Q^i_f} \in \mathscr{T} \otimes \mathscr{B}(Z)$, where $\mathrm{Gr}_{Q^i_f}$ denotes the graph of $Q^i_f$ and $\mathscr{B}(Z)$ the Borel $\sigma$-algebra on $Z$. For all $m \geq 1$, let

$$A^i_m = \left\{ t \in R \cap T_i : g_m(t, \cdot) \in Q^i_f(t) \right\}$$

and

$$B^i_m = \mathrm{Gr}_{Q^i_f} \cap \{(t, g_m(t, \cdot)) : t \in R \cap T_i\}.$$

Obviously, $A^i_m$ is the projection of $B^i_m$ on $R \cap T_i$. Note that

$$\{(t, g_m(t, \cdot)) : t \in R \cap T_i\} \in \mathscr{T} \otimes \mathscr{B}(Z)$$

for all $m \geq 1$. Thus, by measurable projection theorem, one has $R^i_m \in \mathscr{T}$ for all $m \geq 1$. Define

$$R^i_m = \bigcap \{A^i_k : k \geq m\}.$$

Applying $(\mathbf{A}_2)$, one obtains

$$R \cap T_i = \bigcup \{R^i_m : m \geq 1\}.$$

Further, it is easy to verify that $\{R^i_m : m \geq 1\}$ is monotonically increasing. For all $\omega \in \Omega$, let

$$a_i(\omega) = \frac{1}{2\mu(R \cap T_i)} \int_{R \cap T_i} a(\cdot, \omega) d\mu$$

and then choose an $b \in B_{++}$ such that $b \leq a_i(\omega)$ for all $\omega \in \Omega$ and $i \in \mathfrak{P}(R)$. Thus, there exists some $m_0 \geq 1$ such that $\mu(R^i_{m_0}) > 0$ and

$$b - \frac{1}{\mu(R^i_{m_0})} \int_{(R \cap T_i) \setminus R^i_{m_0}} g(\cdot, \omega) d\mu \gg 0$$

for all $\omega \in \Omega$ and $i \in \mathfrak{P}(R)$. Define $y^i : R^i_{m_0} \times \Omega \to B_+$ such that

$$y^i(t, \omega) = 2a_i(\omega) - \frac{1}{\mu(R^i_{m_0})} \int_{(R \cap T_i) \setminus R^i_{m_0}} g(\cdot, \omega) d\mu.$$

Obviously, $y^i(t, \cdot)$ is $\sigma(\mathscr{P}_i)$-measurable and $y^i(t, \cdot) \gg b$ for all $t \in R^i_{m_0}$. Consider a function $h^i : (R \cap T_i) \times \Omega \to B_+$ defined by

$$h^i(t, \omega) = \begin{cases} g_{m_0}(t, \omega) + c_{m_0}(y^i(t, \omega) - b), & \text{if } (t, \omega) \in R^i_{m_0} \times \Omega; \\ g(t, \omega) + 2c_{m_0} a_i(\omega), & \text{otherwise.} \end{cases}$$

By $(\mathbf{A}_3)$, $\mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, h^i(t,\cdot))) > \mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, f(t,\cdot)))$ for all $t \in R \cap T_{\mathscr{Q}}$ and

$$\int_{R\cap T_i} h^i(\cdot,\omega)d\mu + c_{m_0} b\mu(R^i_{m_0}) = \int_{R\cap T_i} \left(g_{m_0}(\cdot,\omega) + c_{m_0} a(\cdot,\omega)\right) d\mu$$

for all $\omega \in \Omega$. Thus, $\lambda = c_{m_0}$, $z = c_{m_0} b \sum_{i\in\mathfrak{P}(R)} \mu(R^i_{m_0})$, and the allocation $h : T \times \Omega \to B_+$, defined by

$$h(t,\omega) = \begin{cases} h^i(t,\omega) + \frac{c_{m_0} b\mu(R^i_{m_0})}{\mu(R\cap T_i)}, & \text{if } (t,\omega) \in (R\cap T_i)\times\Omega,\ i \in \mathfrak{P}(R); \\ g(t,\omega), & \text{otherwise,} \end{cases}$$

are desired. □

**Lemma 3.2.** *Let $f$ be an allocation and $z \in B_{++}$. Suppose that $g, h : S \times \Omega \to B_+$ are two functions satisfying*

$$\int_S g d\mu, \int_S h d\mu \in \operatorname{cl} \int_S P_f d\mu.$$

*Under $(\mathbf{A}_3)$-$(\mathbf{A}_4)$, there exists a function $y : S \times \Omega \to B_+$ such that $y(t,\cdot) \in P_f(t)$ $\mu$-a.e. on $S$ and*

$$\int_S (y-a)d\mu = \frac{1}{2}\int_S (g-a)d\mu + \frac{1}{2}\int_S (h-a)d\mu + z.$$

*Proof.* Pick an $i \in \mathfrak{P}(S)$. Since $\operatorname{cl}\int_{S\cap T_i} P_f d\mu$ is convex,

$$\frac{1}{2}\int_{S\cap T_i} (g+h)d\mu \in \operatorname{cl}\int_{S\cap T_i} P_f d\mu.$$

Choose an open neighbourhood $W$ of 0 such that

$$\frac{z}{|\mathfrak{P}(S)|} - W \subseteq B_{++}.$$

It follows that

$$\left(\frac{1}{2}\int_{S\cap T_i}(g+h)d\mu + W^\Omega\right) \bigcap \int_{S\cap T_i} P_f d\mu \neq \emptyset.$$

So, there exist a $\sigma(\mathscr{P}_i)$-measurable function $w^i : \Omega \to W$ and an integrable selection $x^i$ of $P_f$ such that

$$\frac{1}{2}\int_{S\cap T_i}(g+h)d\mu + w^i = \int_{S\cap T_i} x^i d\mu.$$

Define a function $y^i : (S\cap T_i)\times\Omega \to B_+$ such that for all $(t,\omega) \in (S\cap T_i)\times\Omega$,

$$y^i(t,\omega) = x^i(t,\omega) + \frac{1}{\mu(S\cap T_i)}\left(\frac{z}{|\mathfrak{P}(S)|} - w^i(\omega)\right).$$

By $(\mathbf{A}_3)$, one has

$$\mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, y^i(t,\cdot))) > \mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, f(t,\cdot)))$$

and $y^i(t,\cdot) \in X_t$ $\mu$-a.e. on $S\cap T_i$, and

$$\int_{S\cap T_i} y^i d\mu = \frac{1}{2}\int_{S\cap T_i}(g+h)d\mu + \frac{z}{|\mathfrak{P}(S)|}.$$

Define the allocation $y : S \times \Omega \to B_+$ such that

$$y(t,\omega) = \begin{cases} y^i(t,\omega), & \text{if } (t,\omega) \in (S \cap T_i) \times \Omega, \ i \in \mathfrak{P}(S); \\ h(t,\omega), & \text{otherwise.} \end{cases}$$

Obviously, $y$ is the desired function. $\square$

**Lemma 3.3.** *Assume $f$ be an allocation and that $S \in \mathscr{T}_0$. Suppose also that $g : S \times \Omega \to B_+$ is a function such that*

$$\int_S g d\mu \in \operatorname{cl} \int_S P_f d\mu.$$

*Let $(\mathbf{A}_3)$-$(\mathbf{A}_4)$ be satisfied, $\lambda \in (0,1)$ and $z \in B_{++}$. Then there exist a sub-coalition $R$ of $S$ and a function $h : R \times \Omega \to B_+$ such that $h(t,\cdot) \in P_f(t)$ $\mu$-a.e. on $R$ and*

$$\int_R (h-a) d\mu = \lambda \int_S (g-a) d\mu + z.$$

*Proof.* Pick an $i \in \mathfrak{P}(S)$. Applying an argument similar to that in the proof of Lemma 3.2, one obtains a function $y^i : (S \cap T_i) \times \Omega \to B_+$ such that $y^i(t,\cdot) \in P_f(t)$ $\mu$-a.e. on $S \cap T_i$, and

$$\int_{S \cap T_i} y^i d\mu = \int_{S \cap T_i} g d\mu + \frac{z}{2\lambda |\mathfrak{P}(S)|}.$$

By Lemma 3.3 in [3], one can find a sequence $\{S_n^i : n \geq 1\} \subseteq \Sigma_{S \cap T_i}$ such that

$$\lim_{n \to \infty} \int_{S_n^i} (y^i - a) d\mu = \lambda \int_{S \cap T_i} (y^i - a) d\mu.$$

The function $x_n^i : \Omega \to B$, defined by

$$x_n^i(\omega) = \lambda \int_{S \cap T_i} (y^i(\cdot,\omega) - a(\cdot,\omega)) d\mu - \int_{S_n^i} (y^i(\cdot,\omega) - a(\cdot,\omega)) d\mu,$$

is $\sigma(\mathscr{P}_i)$-measurable for all $n \geq 1$ and $\lim_{n\to\infty} \|x_n^i(\omega)\| = 0$ for all $\omega \in \Omega$. Choose an $n_i \geq 1$ such that

$$\frac{z}{2|\mathfrak{P}(S)|} + x_{n_i}^i(\omega) \gg 0$$

for each $\omega \in \Omega$ and then consider the function $h^i : S_{n_i}^i \times \Omega \to B_+$ defined by

$$h^i(t,\omega) = y^i(t,\omega) + \frac{1}{\mu(S_{n_i}^i)} \left( \frac{z}{2|\mathfrak{P}(S)|} + x_{n_i}^i(\omega) \right).$$

By $(\mathbf{A}_3)$, one has

$$\mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, h^i(t,\cdot))) > \mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, f(t,\cdot))),$$

and $h^i(t,\cdot)$ is $\sigma(\mathscr{P}_i)$-measurable $\mu$-a.e. on $S_{n_i}^i$. Put

$$R = \bigcup \left\{ S_{n_i}^i : i \in \mathfrak{P}(S) \right\}.$$

Then the coalition $R$ and the function $h : R \times \Omega \to B_+$, defined by $h(t,\omega) = h^i(t,\omega)$ if $(t,\omega) \in S_{n_i}^i \times \Omega$, are desired. $\square$

## 4. The Main Result

In this section, the main results are presented.

**Definition 4.1.** An allocation $f$ is called *privately* $\mathscr{C}^{(\mathscr{T}_0,\mathscr{T}_1)}$*-fair* if there do not exist two disjoint coalitions $S_1, S_2$ and an allocation $g$ such that $S_1 \in \mathscr{T}_0$, $S_2 \in \mathscr{T}_1$,

$$\mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, g(t,\cdot))) > \mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, f(t,\cdot)))$$

$\mu$-a.e. on $S_1$ and

$$\int_{S_1} (g(\cdot,\omega) - a(\cdot,\omega))d\mu = \int_{S_2} (f(\cdot,\omega) - a(\cdot,\omega))d\mu$$

for each $\omega \in \Omega$.

**Theorem 4.2.** *Assume* $(\mathbf{A}_1)$-$(\mathbf{A}_4)$ *and that* $f \in \mathscr{PC}(\mathscr{E})$. *Then* $f$ *is privately* $\mathscr{C}^{(\mathscr{T}_0,\mathscr{T}_1)}$*-fair.*

*Proof.* On the contrary, suppose that $f$ is not privately $\mathscr{C}^{(\mathscr{T}_0,\mathscr{T}_1)}$-fair. Then there exist two disjoint coalitions $S_1, S_2$ with $S_1 \in \mathscr{T}_0$ and $S_2 \in \mathscr{T}_1$ and an allocation $g$ such that

$$\mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, g(t,\cdot))) > \mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, f(t,\cdot)))$$

$\mu$-a.e. on $S_1$ and

$$\int_{S_1} (g - a)d\mu = \int_{S_2} (f - a)d\mu.$$

By Lemma 3.1, one has a $\lambda \in (0,1)$, a $z \in B_{++}$ and an allocation $h$ such that

$$\mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, h(t,\cdot))) > \mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, f(t,\cdot)))$$

$\mu$-a.e. on $S_1$, and

$$\int_{S_1} (h - a)d\mu + 19z = (1-\lambda)\int_{S_1} (g - a)d\mu.$$

Applying Lemma 3.3, one can find a sub-coalition $R_1$ of $S_1$ and a function $g_1 : R_1 \times \Omega \to B_+$ such that $g_1(t,\cdot) \in P_f(t)$ $\mu$-a.e. on $R_1$ and

$$\int_{R_1} (g_1 - a)d\mu = \lambda\int_{S_1} (g - a)d\mu + z.$$

Combining above two equations, one has

$$\int_{S_1} (h - a)d\mu + \int_{R_1} (g_1 - a)d\mu + 18z = \int_{S_1} (g - a)d\mu.$$

Lemma 3.2 implies that there must exist a function $h_1 : R_1 \times \Omega \to B_+$ such that $h_1(t,\cdot) \in P_f(t)$ $\mu$-a.e. on $R_1$ and

$$\int_{R_1} (h_1 - a)d\mu = \frac{1}{2}\int_{R_1} (h - a)d\mu + \frac{1}{2}\int_{R_1} (g_1 - a)d\mu + z.$$

By Lemma 3.3, one has a sub-coalition $R_2$ of $S_1 \backslash R_1$ and a function $h_2 : R_2 \times \Omega \to B_+$ such that $h_2(t,\cdot) \in P_f(t)$ $\mu$-a.e. on $R_2$ and

$$\int_{R_2} (h_2 - a)d\mu = \frac{1}{2}\int_{S_1 \backslash R_1} (h - a)d\mu + z.$$

Thus, one concludes that

$$\int_{R_1} (h_1 - a)d\mu + \int_{R_2} (h_2 - a)d\mu + 7z = \frac{1}{2}\int_{S_2} (f - a)d\mu.$$

Let $R_3 = R_1 \cup R_2$ and define $h_3 : R_3 \times \Omega \to B_+$ by $h_3(t) = h_1(t)$ if $t \in R_1$; and $h_3(t) = h_2(t)$ if $t \in R_2$. So,

$$\int_{R_3} (h_3 - a)d\mu + 7z = \frac{1}{2}\int_{S_2} (f - a)d\mu.$$

If $\int_{S_2}(f-a)d\mu = 0$ then $f$ is privately blocked by the coalition $R_3$ via the allocation $y : T \times \Omega \to B_+$, defined by

$$y(t,\omega) = \begin{cases} h_3(t,\omega) + \frac{7z}{\mu(R_3)}, & \text{if } (t,\omega) \in R_3 \times \Omega; \\ g(t,\omega), & \text{otherwise,} \end{cases}$$

which is a contraction with the fact that $f \in \mathscr{PC}(\mathscr{E})$. So, $\int_{S_2}(f-a)d\mu \neq 0$ which means $\mu(T \setminus S_2) > 0$. In this case,

$$\int_{R_3} (h_3 - a)d\mu + \frac{1}{2}\int_{T\setminus S_2} (f - a)d\mu + 7z = 0.$$

Using Lemma 3.3, the above equation can be written as

$$\int_{R_3} (h_3 - a)d\mu + \int_{R_4} (h_4 - a)d\mu + 6z = 0$$

for some sub-coalition $R_4$ of $T \setminus S_2$ and function $h_4 : R_4 \times \Omega \to B_+$ satisfying $h_4(t,\cdot) \in P_f(t)$ $\mu$-a.e. on $R_4$. Again, applying Lemma 3.2 for the coalition $R_3 \cap R_4$ and Lemma 3.3 for coalitions $R_3 \setminus R_4$ and $R_4 \setminus R_3$, one can find three sub-coalitions

$$R_5 = R_3 \cap R_4, R_6 \subseteq R_3 \setminus R_4, R_7 \subseteq R_4 \setminus R_3$$

and three functions $h_i : R_i \times \Omega \to B_+$ for $i = 5, 6, 7$ such that

$$\sum_{i=5}^{7} \int_{R_i} (h_i - a)d\mu = 0$$

and $h_i(t,\cdot) \in P_f(t)$ $\mu$-a.e. on $R_i$ for $i = 5, 6, 7$. Thus, the coalition $R = R_5 \cup R_6 \cup R_7$ privately blocks $f$ via the allocation $y : T \times \Omega \to B_+$, defined by

$$y(t,\omega) = \begin{cases} h_i(t,\omega), & \text{if } (t,\omega) \in R_i \times \Omega,\ i = 5,6,7; \\ g(t,\omega), & \text{otherwise,} \end{cases}$$

which is again a contradiction. □

**Definition 4.3.** An allocation $f$ is called *privately* $\mathscr{C}^{(\mathscr{T}_1,\mathscr{T}_0)}$*-fair* if there do not exist two disjoint coalitions $S_1, S_2$ and an allocation $g$ such that $S_1 \in \mathscr{T}_1$, $S_2 \in \mathscr{T}_0$,

$$\mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, g(t,\cdot))) > \mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, f(t,\cdot)))$$

$\mu$-a.e. on $S_1$ and

$$\int_{S_1} (g(\cdot,\omega) - a(\cdot,\omega))d\mu = \int_{S_2} (f(\cdot,\omega) - a(\cdot,\omega))d\mu$$

for each $\omega \in \Omega$.

**Theorem 4.4.** *Assume* $(\mathbf{A}_1)$-$(\mathbf{A}_4)$ *and that* $f \in \mathscr{PC}(\mathscr{E})$. *Then* $f$ *is privately* $\mathscr{C}^{(\mathscr{T}_1,\mathscr{T}_0)}$*-fair.*

*Proof.* On the contrary, suppose that $f$ is not privately $\mathscr{C}^{(\mathscr{T}_1,\mathscr{T}_0)}$-fair. Thus, there must exist two disjoint coalitions $S_1, S_2$ with $S_1 \in \mathscr{T}_1$ and $S_2 \in \mathscr{T}_0$ and an allocation $g$ such that

$$\mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, g(t,\cdot))) > \mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, f(t,\cdot)))$$

$\mu$-a.e. on $S_1$ and

$$\int_{S_1}(g-a)d\mu = \int_{S_2}(f-a)d\mu.$$

Now, Lemma 3.1 yields a $\lambda \in (0,1)$, a $z \in B_{++}$ and an allocation $h$ such that

$$\mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, h(t,\cdot))) > \mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, f(t,\cdot)))$$

$\mu$-a.e. on $S_1$, and

$$\int_{S_1}(h-a)d\mu + 19z = (1-\lambda)\int_{S_1}(g-a)d\mu = (1-\lambda)\int_{S_2}(f-a)d\mu.$$

By Lemma 3.3, one obtains a sub-coalition $R_2$ of $S_2$ and a function $g_2 : R_2 \times \Omega \to B_+$ such that $g_2(t,\cdot) \in P_f(t)$ $\mu$-a.e. on $R_2$ and

$$\int_{R_2}(g_2-a)d\mu = \lambda\int_{S_2}(f-a)d\mu + z.$$

Let $R_1 = S_1 \cup R_2$ and define a function $h_1 : R_1 \times \Omega \to B_+$ by $h_1(t) = h(t)$ if $t \in S_1$; and $h_1(t) = g_2(t)$ if $t \in R_2$. Then

$$\int_{R_1}(h_1-a)d\mu + 18z = \int_{S_2}(f-a)d\mu.$$

Applying lemma 3.2, one has a function $x_1 : R_1 \times \Omega \to B_+$ such that $x_1(t,\cdot) \in P_f(t)$ $\mu$-a.e. on $R_1$ and

$$\int_{R_1}(x_1-a)d\mu = \frac{1}{2}\int_{R_1}(h_1-a)d\mu + \frac{1}{2}\int_{R_1}(f-a)d\mu + z.$$

Thus, one has

$$\int_{R_1}(x_1-a)d\mu + 8z = \frac{1}{2}\int_{R_1\cup S_2}(f-a)d\mu.$$

If $\int_{R_1\cup S_2}(f-a)d\mu = 0$ then $f$ is privately blocked by a coalition $R_1$ via the allocation $y : T \times \Omega \to B_+$ defined by

$$y(t,\omega) = \begin{cases} x_1(t,\omega) + \frac{8z}{\mu(R_1)}, & \text{if } (t,\omega) \in R_1 \times \Omega; \\ g(t,\omega), & \text{otherwise.} \end{cases}$$

This is a contradiction. Now, consider the case when $\int_{R_1\cup S_2}(f-a)d\mu \neq 0$. Since $\mu(T \setminus (R_1 \cup S_2)) \neq 0$ and $T \setminus (R_1 \cup S_2)$ is atomless, by Lemma 3.3, there exist a sub-coalition $R_2$ of $T \setminus (R_1 \cup S_2)$ and a function $h_2 : R_2 \times \Omega \to B_+$ such that $h_2(t,\cdot) \in P_f(t)$ $\mu$-a.e. on $R_2$ and

$$\int_{R_2}(h_2-a)d\mu = \frac{1}{2}\int_{T\setminus(R_1\cup S_2)}(g-a)d\mu + z.$$

Define an allocation $y : T \times \Omega \to B_+$ by

$$y(t,\omega) = \begin{cases} x_1(t,\omega) + \frac{3z}{\mu(R_1)}, & \text{if } (t,\omega) \in R_1 \times \Omega; \\ h_2(t,\omega), & \text{if } (t,\omega) \in R_2 \times \Omega; \\ g(t,\omega), & \text{otherwise.} \end{cases}$$

Note that $f$ is privately blocked by a coalition $R_1 \cup R_2$ via the allocation $y$, which is again a contradiction. $\square$

The following definition and theorem are extensions of those in [9] to an asymmetric information economy.

**Definition 4.5.** An allocation $f$ is said to be *privately* $\mathscr{C}$*-fair* relative to $\mathscr{T}_0$ and $\mathscr{T}_1$ if it is privately $\mathscr{C}^{(\mathscr{T}_0,\mathscr{T}_1)}$-fair and privately $\mathscr{C}^{(\mathscr{T}_1,\mathscr{T}_0)}$-fair. The set of such allocations is denoted by $\mathscr{P}\mathscr{C}^{\{\mathscr{T}_0,\mathscr{T}_1\}}(\mathscr{E})$.

**Theorem 4.6.** *Assume* $(\mathbf{A}_1)$-$(\mathbf{A}_4)$. *Then* $\mathscr{P}\mathscr{C}(\mathscr{E}) \subseteq \mathscr{P}\mathscr{C}^{\{\mathscr{T}_0,\mathscr{T}_1\}}(\mathscr{E})$.

*Proof.* Let $f \in \mathscr{P}\mathscr{C}(\mathscr{E})$. By Theorem 4.2, $f$ is privately $\mathscr{C}^{(\mathscr{T}_0,\mathscr{T}_1)}$-fair. Applying Theorem 4.4, one has $f$ is privately $\mathscr{C}^{(\mathscr{T}_1,\mathscr{T}_0)}$-fair. So, $f \in \mathscr{P}\mathscr{C}^{\{\mathscr{T}_0,\mathscr{T}_1\}}(\mathscr{E})$, and this completes the proof. $\square$

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