The Uzawa-Lucas Growth Model with Natural Resources

Neustroev, Dmitry

2013

The Uzawa-Lucas Growth Model with Natural Resources

Dmitry Neustroev

Institute of Economics and Industrial Engineering

Siberian Branch, Russian Academy of Sciences

Novosibirsk, Russia

January 14, 2014

Abstract

This article offers the modification of the Uzawa-Lucas growth model. The model also includes natural resources as a factor of production. The necessary and sufficient conditions for this model are considered. Growth rates of the main macroeconomic indicators along balanced growth path are obtained. The analysis of influence of natural resources on economic growth along balanced growth path is considered.

Keywords: Uzawa-Lucas model, human capital, natural resources.

1 Introduction

The question of the influence of human capital on the development of the economic system is quite important today. There is considerable attention devoted to this question in economic literature nowadays. There is a variety of economic models now where human capital is the factor of economic growth, for example the two-sector endogenous Uzawa-Lucas model of economic growth. Nevertheless, we observe that in the real economy the presence of significant amount of accessible and demanded natural resources also plays an important and mixed role in economic development. This role's analysis is especially important for the countries where the economy depends highly on the results of extractive industries. Therefore, the object of this article is to combine the influences of human capital and natural resources within one model. For this consideration the Uzawa-Lucas growth model was chosen and it was modified by including natural resources as the factor of economic growth.

Characteristic feature of the two-sector Uzawa-Lucas growth model is the presence of the human capital and its influence on economic growth. Hirofumi Uzawa in his article (Uzawa 1965) analyzed the model of economic growth with Harrod-neutral technological progress. The aggregate production function of this model is written as

$$F\left[K(t),\, A_U(t)L_P(t)\right],$$

where $K(t)$ is the fixed capital stock, $A_U(t)$ is the labor efficiency, and $L_P(t)$ is the amount of labor employed in production sector. Uzawa assumed that labor efficiency depends on education, health, public goods, etc. In his model the influence of these factors was described by the educational sector. It was written as

$$\dot{A_U}(t)\,/A_U(t) = \phi\,\left[L_E(t)/L(t)\right],$$

where $L_E(t)$ is the labor employed in educational sector, $\dot{A_U}(t)$ is the improvement of the labor productivity at time t and $\frac{\dot{A_U}(t)}{A_U(t)}$ is the growth rate of the labor productivity. Using Pontryagin's Maximum Principle Uzawa analyzes dynamics of his model. While Uzawa didn't mention human capital directly in his article (though this conception is felt intuitively), later, Robert Lucas introduced in his research (Lucas 1988) slightly different model where human capital is included explicitly. Technological progress in his model is mixed and is not strictly Harrod-neutral as it was in Uzawa's model. Production function now is given by

$$Y(t) = AK(t)^{\beta}[b_L(t)h(t)L(t)]^{1-\beta}h_a(t)^{\xi}, \tag{1}$$

where A is a technological level which is assumed to be constant, $b_L(t)$ is the fraction of worker's non-leisure time devoted to production, $h(t)$ is the level of per worker's human capital and $h_a(t)^{\xi}$ is the external effect of human capital. The second, educational sector, which is responsible for accumulation of the human capital is written as

$$\dot{h(t)} = h(t)^{\zeta}G\left(1 - b_L(t)\right), \tag{2}$$

where $1-b_L(t)$ is the fraction of worker's non-leisure time devoted to human capital accumulation; function $G(\centerdot)$ is decreasing in $b_L(t)$ and ζ reflects the degree of influence of existing human capital on its accumulation. In the further analysis Lucas assumed that $\zeta = 1$. Lucas as well as Uzawa considered dynamics of his model using optimal control theory.

Since this model was introduced, it has been used many times by researchers for the analysis and empirical estimation of human capital influence on the economic system development. In Rebelo (1991) author tries to analyze the influence of tax policy on long-term rates of economic growth. He generalizes the described above model by including fixed capital in the educational sector (2). Therefore, the Uzawa-Lucas growth model is becoming the special case of the Rebelo model. At the same time he doesn't take into account the external effect of $h_a(t)^{\xi}$ in production function (1). As the result he concluded that relationship between income tax and rates of economic growth is inverse. Taxation is also considered in Gomez (2003) and Gorostiaga et al. (2011) by using the Uzawa-Lucas growth model. In Benhabib and Perli (1994) authors consider the dynamics of the Uzawa-Lucas growth model and its balanced path in more details. They analyze the range of model parameters which may lead to positive rates of economic growth. The analysis of the balanced path, transitional dynamics and model parameters we can see also in Bethmann (2008), Gomez (2006), Jones and Scrimgeour (2008), Mulligan and Sala-i-Martin (1993), Xie (1994). The two-sector Uzawa-Lucas growth model is an essential part of the monographs which include theoretical concepts of economic growth. We see the analysis of the model in Barro and Sala-i-Martin (2004) and Acemoglu (2009). Also we can find it in Aghion and Howitt (1998) and Savvides and Stengos (2009). It is necessary to mention monograph Mattana (2004) which is completely devoted to the analysis of this model. Taking into account that in this article it will be considered a modification of this model by including natural resources it is worth to mention a few works where natural resources are the factors of economic growth model. For example in Cavalcanti et al. (2011) authors consider oil production, oil rent and oil reserves as a proxy of natural resources and in each case estimate production function. Also we can see other works, for example Benchekroun and Withagen (2008), Färnstrand Damsgaard (2010), Gaitan and Roe (2005), Groth and Schou (2002) where the influence of natural resources on economic system development within macroeconomic models is considered. There are also studies by Bravo-Ortega and De Gregorio (2002) and Valente (2007) where the influence of human capital and natural resources on economic development is considered. In this work we present modified Uzawa-Lucas growth model with natural resources included.

2 The Model

Let's consider the two-sector model of economic growth which is given by

$$Y(t) = A(t)K(t)^{\alpha}S(t)^{\beta}[b(t)H(t)]^{1-\alpha-\beta}, \tag{3}$$

$$\dot{H}(t) = H(t)z\,(1-b(t)) - \delta_H H(t), \tag{4}$$

where $Y(t)$ is the total amount of production of the final goods at time t; $A(t)$ is the technological level which is constant; $K(t)$ is the fixed capital stock; $S(t)$ is the natural resources; $H(t)$ is the total amount of accumulated human capital; δ_H is the level of human capital depreciation; $b(t)$ is the share of human capital devoted to production and z is the parameter which defines effectiveness of human capital accumulation.

It is possible to describe the volume of human capital as a product of labor and per capita accumulated human capital. Using this definition, we can rewrite (3) and (4) as

$$y = Ak^{\alpha}s^{\beta}(bh)^{1-\alpha-\beta}, \tag{5}$$

$$\dot{h} = hz(1-b) - \delta_H h, \tag{6}$$

where y, k, s and h are per capita values of the final product, fixed capital, natural resources and human capital respectively.

Fixed capital and natural resources dynamics can be described in the following way:

$$\dot{k} = i_K - \delta_K k, \tag{7}$$

$$\dot{s} = \eta i_S^{\gamma} - \delta_S s, \tag{8}$$

where i_K is per capita investments in fixed capital; δ_K is the level of fixed capital depreciation; δ_S is the rate of natural resources extraction; i_S is per capita investments in natural resources; γ and η are parameters which describe availability and accessibility of natural resources in economic system.

Instantaneous utility function of representative agent is presented in isoelastic form

$$u(c) = \frac{c^{1-\theta}-1}{1-\theta}, \tag{9}$$

where c is per capita consumption and θ is the coefficient of relative risk aversion which is a constant in our case.

In infinite-horizon economy agents maximize received utility in accordance with

$$\max \int_0^{\infty} \frac{c^{1-\theta}-1}{1-\theta} e^{-\rho t} dt, \tag{10}$$

where ρ is the rate of time preference.

3 Necessary and Sufficient Conditions

To find an optimal path of the economic system development we will use methods of the theory of optimum control for the purpose of utility function maximization. The maximum utility received by individuals corresponds to the most preferable path of economic system development.

For solving optimization problem, we should consider two cases. The first one is the situation when the decision taking process is centralized. This is so called the Social Planner's problem. Let's describe the optimization problem in this case.

Production sector output can be used for consumption and investments in fixed capital and natural resources. The following constraint holds:

$$c + i_K + i_S = y = Ak^{\alpha} s^{\beta} (bh)^{1-\alpha-\beta}. \tag{11}$$

To solve optimization problem using (6), (7), (8), (10) and (11) we can write the present-value Hamiltonian as follows:

$$\begin{aligned} H^S = & \frac{c^{1-\theta} - 1}{1-\theta} + \lambda_1 \left[Ak^{\alpha} s^{\beta} (bh)^{1-\alpha-\beta} - c - i_S - \delta_K k \right] + \\ & + \lambda_2 \left[hz\left(1-b\right) - \delta_H h \right] + \lambda_3 \left(\eta i_S^{\gamma} - \delta_S s \right), \end{aligned} \tag{12}$$

where λ_1, λ_2 and λ_3 are adjoint variables which can be interpreted as shadow prices of the state variables k, h and s; the control variables are c, b and i_S.

Necessary conditions are

$$\begin{aligned} \frac{\partial H^S}{\partial c} = & 0, \ \frac{\partial H^S}{\partial b} = 0, \ \frac{\partial H^S}{\partial i_S} = 0, \\ \dot{\lambda}_1 & = \rho \lambda_1 - \frac{\partial H^S}{\partial k}, \\ \dot{\lambda}_2 & = \rho \lambda_2 - \frac{\partial H^S}{\partial h}, \\ \dot{\lambda}_3 & = \rho \lambda_3 - \frac{\partial H^S}{\partial s}. \end{aligned} \tag{13}$$

Using these conditions we can get the following equalities:

$$c^{-\theta} - \lambda_1 = 0, \tag{14}$$

$$\lambda_1 (1 - \alpha - \beta) \frac{y}{b} - \lambda_2 hz = 0, \tag{15}$$

$$-\lambda_1 + \lambda_3 \gamma \eta i_S^{\gamma - 1} = 0, \tag{16}$$

$$\dot{\lambda}_1 = \rho\lambda_1 - \lambda_1 \left(\alpha \frac{y}{k} - \delta_K\right), \tag{17}$$

$$\dot{\lambda}_2 = \rho\lambda_2 - \lambda_1 (1 - \alpha - \beta) \frac{y}{h} - \lambda_2 \left[z (1 - b) - \delta_H\right], \tag{18}$$

$$\dot{\lambda}_3 = \rho\lambda_3 - \lambda_1 \beta \frac{y}{s} + \lambda_3 \delta_S. \tag{19}$$

The transversality conditions are given by

$$\begin{aligned} &\lim_{t\to\infty} \left[e^{-\rho t}\lambda_1(t)k(t)\right] = 0, \\ &\lim_{t\to\infty} \left[e^{-\rho t}\lambda_2(t)h(t)\right] = 0, \\ &\lim_{t\to\infty} \left[e^{-\rho t}\lambda_3(t)s(t)\right] = 0. \end{aligned} \tag{20}$$

Let's analyze now the second situation when decision taking process is decentralized. Budget constraint can be written now as

$$c + i_K + i_S = w (bh) + rk + qs,$$

where w is the salary r and q are rental prices of fixed capital and natural resources respectively. The present-value Hamiltonian in this case can be expressed as

$$\begin{aligned} H^d =& \frac{c^{1-\theta} - 1}{1 - \theta} + \lambda_1 \left[w (bh) + rk + qs - c - i_S - \delta_K k\right] + \\ &+ \lambda_2 \left[hz (1 - b) - \delta_H h\right] + \lambda_3 \left(\eta i_S^{\gamma} - \delta_S s\right). \end{aligned}$$

Using necessary conditions (13) we can get the next equalities:

$$c^{-\theta} - \lambda_1 = 0, \tag{21}$$

$$\lambda_1 wh - \lambda_2 hz = 0, \tag{22}$$

$$-\lambda_1 + \lambda_3 \gamma \eta i_S^{\gamma-1} = 0, \tag{23}$$

$$\dot{\lambda_1} = \rho\lambda_1 - \lambda_1 (r - \delta_K), \tag{24}$$

$$\dot{\lambda_2} = \rho\lambda_2 - \lambda_1 wb - \lambda_2 [z(1-b) - \delta_H], \tag{25}$$

$$\dot{\lambda_3} = \rho\lambda_3 - \lambda_1 q + \lambda_3 \delta_S. \tag{26}$$

The transversality conditions are the same as in the centralized case (20).

Agents profit maximization implies that

$$\begin{aligned} r = y_k = \alpha\frac{y}{k}, \\ q = y_s = \beta\frac{y}{s}, \\ w = y_{bh} = (1-\alpha-\beta)\frac{y}{bh}. \end{aligned} \tag{27}$$

If we substitute (27) into (21)–(26) we will get the same equalities as in centralized case (14)–(19).

Sufficient conditions imply that Hamiltonian (12) should be concave function of state and control variables. This is Mangasarian Sufficient Condition (Mangasarian 1966). Hamiltonian is concave function if and only if its Hessian is negative semidefinite. Taking into account that we have three control and three state variables the Hessian is 6×6 matrix. It is quite complicated to find out the sign of few principal minors because we don't know exactly the values of the model parameters. That's why we can use Arrow Sufficiency Theorem (Arrow and Kurz 1970) which states that if we substitute the values of control variables derived from necessary conditions into Hamiltonian it will be enough to check concavity of this maximized Hamiltonian only for the state variables.

From (14)–(16) we may define the values of control variables:

$$c = \lambda_1^{-\frac{1}{\theta}}, \tag{28}$$

$$b = \frac{1}{h}\left(\frac{\lambda_1\left(1-\alpha-\beta\right)Ak^\alpha s^\beta}{\lambda_2 z}\right)^{\frac{1}{\alpha+\beta}}, \tag{29}$$

$$i_S = \left(\frac{\lambda_1}{\lambda_3\gamma\eta}\right)^{\frac{1}{\gamma-1}}. \tag{30}$$

If we substitute these values into (12) we will receive the following expression for maximized Hamiltonian:

$$\begin{aligned} H_{\max} = \frac{\lambda_1^{\left(1-\frac{1}{\theta}\right)}-1}{1-\theta} + \lambda_1\left[Ak^\alpha s^\beta\left(\frac{\lambda_1\left(1-\alpha-\beta\right)Ak^\alpha s^\beta}{\lambda_2 z}\right)^{\frac{1-\alpha-\beta}{\alpha+\beta}} - \lambda_1^{-\frac{1}{\theta}} - \left(\frac{\lambda_1}{\lambda_3\gamma\eta}\right)^{\frac{1}{\gamma-1}} - \delta_K k\right] + \\ + \lambda_2\left(hz\left[1-\frac{1}{h}\left(\frac{\lambda_1\left(1-\alpha-\beta\right)Ak^\alpha s^\beta}{\lambda_2 z}\right)^{\frac{1}{\alpha+\beta}}\right] - \delta_H h\right) + \lambda_3\left[\eta\left(\frac{\lambda_1}{\lambda_3\gamma\eta}\right)^{\frac{\gamma}{\gamma-1}} - \delta_S s\right]. \end{aligned} \tag{31}$$

According to Arrow Sufficiency Theorem it is necessary for maximized Hamiltonian (31) to be concave in k, h and s. The Hessian is written as

$$Hess = \begin{pmatrix} H_{kk} & H_{kh} & H_{ks} \\ H_{hk} & H_{hh} & H_{hs} \\ H_{sk} & H_{sh} & H_{ss} \end{pmatrix}. \tag{32}$$

It is necessary for the odd principal minors to be ≤ 0 and for the even ones to be ≥ 0. The values of the principal minors are the following:

$$H_{kk} = \frac{-\alpha\beta\lambda_2 z\left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)}{k^2(\alpha+\beta)^2\left(\frac{\lambda_2 z}{\lambda_1(1-\alpha-\beta)Ak^\alpha s^\beta}\right)^{\frac{1}{\alpha+\beta}}} < 0 \ (because\, of\; \alpha+\beta<1),$$

$$H_{hh} = 0,$$

$$H_{ss} = \frac{-\alpha\beta\lambda_2 z\left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)}{s^2(\alpha+\beta)^2\left(\frac{\lambda_2 z}{\lambda_1(1-\alpha-\beta)Ak^\alpha s^\beta}\right)^{\frac{1}{\alpha+\beta}}} < 0 \ (because\, of\; \alpha+\beta<1),$$

$$\begin{vmatrix} H_{kk} & H_{kh} \\ H_{hk} & H_{hh} \end{vmatrix} = 0, \quad \begin{vmatrix} H_{kk} & H_{ks} \\ H_{sk} & H_{ss} \end{vmatrix} = 0, \quad \begin{vmatrix} H_{hh} & H_{hs} \\ H_{sh} & H_{ss} \end{vmatrix} = 0,$$

$$\begin{vmatrix} H_{kk} & H_{kh} & H_{ks} \\ H_{hk} & H_{hh} & H_{hs} \\ H_{sk} & H_{sh} & H_{ss} \end{vmatrix} = 0.$$

We see that the Hessian (32) is negative semidefinite. As the result, we can say that the conditions (14)-(19) are not only necessary but also sufficient.

4 Growth Rates Analysis

Let's analyze dynamics of macroeconomic indicators along balanced growth path (BGP). It is supposed that the growth rates of $y,\ k,\ s,\ h,\ c,\ i_K$ and i_S are constants along BGP.

From (7) we can derive expression for the per capita fixed capital growth rate (g_k):

$$g_k = \frac{\dot{k}}{k} = \frac{i_K}{k} - \delta_K.$$

Taking into account that δ_K is constant and the growth rate of per capita fixed capital along BGP is constant also, the ratio $\frac{i_K}{k}$ should be constant and the growth rates of per capita investments in fixed capital and per capita fixed capital are equal.

From (14) we can derive the growth rate of per capita consumption (g_c):

$$g_c = \frac{\dot{c}}{c} = -\frac{1}{\theta}\frac{\dot{\lambda}_1}{\lambda_1}. \tag{33}$$

Using equalities (17) and (33) we can get the next expression for the growth rate of per capita consumption:

$$\frac{\dot{c}}{c} = -\frac{\rho}{\theta} + \frac{\alpha}{\theta} \cdot \frac{y}{k} - \frac{\delta_K}{\theta}.$$

Since $\rho,\ \theta,\ \alpha$ and δ_K are constants and the growth rate of per capita consumption along BGP is constant we can come to conclusion that ratio $\frac{y}{k}$ is constant too and it means that along BGP the growth rate of per capita final output is equal to the growth rate of per capita fixed capital.

From (16) we can derive the growth rate of per capita investments in natural resources (g_{i_S}):

$$g_{i_S} = \frac{\dot{i}_S}{i_S} = \frac{1}{(\gamma - 1)} \left(\frac{\dot{\lambda}_1}{\lambda_1} - \frac{\dot{\lambda}_3}{\lambda_3} \right). \tag{34}$$

From (33) we see that along BGP the growth rate of λ_1 is constant and since the growth rate of per capita investments in natural resources along BGP is constant also we have that growth rate of λ_3 is also constant along BGP.

From (16) and (19) we can obtain

$$i_s^{\gamma-1} = \frac{\left(\rho + \delta_S - \frac{\dot{\lambda}_3}{\lambda_3}\right) s}{\beta y \gamma \eta}. \tag{35}$$

From (35) and taking into account that growth rate of λ_3 is constant along BGP we can obtain the next equality:

$$(\gamma - 1)\frac{\dot{i}_S}{i_S} = \frac{\dot{s}}{s} - \frac{\dot{y}}{y}. \tag{36}$$

From (8) we can get the expression for the growth rate of per capita natural resources (g_s):

$$g_s = \frac{\dot{s}}{s} = \frac{\eta i_S^{\gamma}}{s} - \delta_S. \tag{37}$$

Since the growth rate of per capita natural resources is constant along BGP we have that the growth rate of function $f(i) = i_S^{\gamma}$ is equal to the growth rate of per capita natural resources. Because of such relationship we have the following expression:

$$\frac{\dot{f(i)}}{f(i)} = \frac{\gamma i_S^{\gamma-1} \cdot \dot{i}_S}{i_S^{\gamma}} = \gamma \frac{\dot{i}_S}{i_S} = \frac{\dot{s}}{s}. \tag{38}$$

From (36) and (38) we have that growth rate of per capita investments in natural resources is equal to the growth rate of per capita final production and as a consequence equal to the growth rate of per capita fixed capital:

$$\frac{\dot{y}}{y} = \frac{\dot{k}}{k} = \frac{\dot{i}_S}{i_S}. \tag{39}$$

Using (7) and (11) we can get the expression for the growth rate of per capita fixed capital:

$$\frac{\dot{k}}{k} = \frac{y}{k} - \frac{c}{k} - \frac{i_S}{k} - \delta_K.$$

Using (39) we can come to conclusion that along BGP ratio $\frac{c}{k}$ is constant and the growth rates of per capita consumption and fixed capital are equal.

If we assume that the share of human capital devoted to production along BGP is constant then using equality (5) we can derive the next expression:

$$\frac{\dot{y}}{y} = \alpha\frac{\dot{k}}{k} + \beta\frac{\dot{s}}{s} + (1 - \alpha - \beta)\frac{\dot{h}}{h}.$$

From (38) and (39) we can obtain the relationship between the growth rate of per capita human capital and per capita final production:

$$g_h = \frac{\dot{h}}{h} = \psi \cdot \frac{\dot{y}}{y}, \tag{40}$$

where $\psi = \left(\frac{1-\alpha-\beta\gamma}{1-\alpha-\beta}\right)$.

Since $\gamma < 1$ (if we assume diminishing marginal return on investments in natural resources) we have that growth rate of per capita human capital on BGP is higher than growth rate of per capita final production.

According to (6) the growth rate of per capita human capital is

$$\frac{\dot{h}}{h} = z\,(1 - b) - \delta_H.$$

As a result, we obtain the following relationship between the main macroeconomic indicators along BGP:

$$\frac{\dot{y}}{y}=\frac{\dot{k}}{k}=\frac{\dot{c}}{c}=\frac{\dot{i}_K}{i_K}=\frac{\dot{i}_S}{i_S}=\frac{1}{\gamma}\frac{\dot{s}}{s}=\frac{1}{\psi}\frac{\dot{h}}{h}=\frac{\left[z\left(1-b\right)-\delta_H\right]}{\psi}. \tag{41}$$

To find absolute values of these growth rates we need to find the share of human capital devoted to production along BGP (b).

To determine steady state share of human capital devoted to production let's introduce the following variables:

$$\varphi_1=\frac{h}{k},\ \varphi_2=\frac{h}{s},\ \varphi_3=\frac{c}{k},\ \varphi_4=\frac{i_s}{k},\ \varphi_5=\frac{\eta i_s^{\gamma}}{s}.$$

From (15) we can obtain the ratio λ_1/λ_2:

$$\frac{\lambda_1}{\lambda_2}=\frac{hzb}{\left(1-\alpha-\beta\right)y}=\frac{z}{\left(1-\alpha-\beta\right)A}\varphi_1^{\alpha}\varphi_2^{\beta}b^{\alpha+\beta}. \tag{42}$$

From (17) we can express the growth rate of λ_1:

$$\frac{\dot{\lambda}_1}{\lambda_1}=\rho-\alpha\frac{y}{k}+\delta_K=\rho-\alpha Ab^{1-\alpha-\beta}\varphi_1^{1-\alpha}\varphi_2^{-\beta}+\delta_K. \tag{43}$$

Using (18) and (42) we can obtain the growth rate of λ_2:

$$\frac{\dot{\lambda}_2}{\lambda_2}=\rho-z+\delta_H. \tag{44}$$

From (42) using (43) and (44) we can get the growth rate of λ_1/λ_2:

$$\begin{aligned}\frac{\dot{\left(\frac{\lambda_1}{\lambda_2}\right)}}{\left(\frac{\lambda_1}{\lambda_2}\right)}=\left(\frac{\dot{\lambda}_1}{\lambda_1}-\frac{\dot{\lambda}_2}{\lambda_2}\right)=\frac{\frac{z}{(1-\alpha-\beta)A}\varphi_1^{\alpha}\varphi_2^{\beta}b^{\alpha+\beta}\left(\alpha\frac{\dot{\varphi}_1}{\varphi_1}+\beta\frac{\dot{\varphi}_2}{\varphi_2}+(\alpha+\beta)\frac{\dot{b}}{b}\right)}{\frac{z}{(1-\alpha-\beta)A}\varphi_1^{\alpha}\varphi_2^{\beta}b^{\alpha+\beta}}=\\ =\left(\alpha\frac{\dot{\varphi}_1}{\varphi_1}+\beta\frac{\dot{\varphi}_2}{\varphi_2}+(\alpha+\beta)\frac{\dot{b}}{b}\right).\end{aligned} \tag{45}$$

From equalities (6), (7) and (11) we can obtain the growth rate of φ_1:

$$\frac{\dot{\varphi}_1}{\varphi_1} = \left(\frac{\dot{h}}{h} - \frac{\dot{k}}{k}\right) = z\,(1-b) - \delta_H - \frac{y}{k} + \varphi_3 + \varphi_4 + \delta_K. \tag{46}$$

From equalities (6), (8) we can get the growth rate of φ_2:

$$\frac{\dot{\varphi}_2}{\varphi_2} = \left(\frac{\dot{h}}{h} - \frac{\dot{s}}{s}\right) = z\,(1-b) - \delta_H - \varphi_5 + \delta_S. \tag{47}$$

If we substitute (43), (44), (46) and (47) into (45) we can obtain the expression for the growth rate of the share of human capital devoted to production:

$$\frac{\dot{b}}{b} = z \cdot b - \frac{\alpha}{(\alpha+\beta)}\,(\varphi_3 + \varphi_4) + \frac{\beta}{(\alpha+\beta)}\varphi_5 + \chi, \tag{48}$$

where $\chi = \frac{(z-\delta_H)(1-\alpha-\beta)+\delta_K(1-\alpha)-\delta_S\beta}{(\alpha+\beta)}$.

From (7), (11), (14) and (17) we can get the growth rate of φ_3:

$$\frac{\dot{\varphi}_3}{\varphi_3} = \left(\frac{\dot{c}}{c} - \frac{\dot{k}}{k}\right) = \frac{y}{k}\left(\frac{\alpha-\theta}{\theta}\right) - \frac{\rho}{\theta} - \delta_K\left(\frac{1}{\theta} - 1\right) + \varphi_3 + \varphi_4. \tag{49}$$

From (7), (8), (11) and (38) we can obtain the growth rate of φ_4:

$$\frac{\dot{\varphi}_4}{\varphi_4} = \left(\frac{\dot{i}_S}{i_s} - \frac{\dot{k}}{k}\right) = \frac{1}{\gamma}\frac{\dot{s}}{s} - \frac{\dot{k}}{k} = \frac{\varphi_5}{\gamma} - \frac{\delta_S}{\gamma} - \frac{y}{k} + \varphi_3 + \varphi_4 + \delta_K. \tag{50}$$

Using (40) we can derive the following equality:

$$z\,(1-b) - \delta_H - \psi\left(\frac{y}{k} - \varphi_3 - \varphi_4 - \delta_K\right) = 0. \tag{51}$$

From (48), (49), (50) and (51) and taking into account that on BGP $\dot{b}$, $\dot{\varphi}_3$, and $\dot{\varphi}_4$ are zeroes we can obtain the following system of equations:

$$\begin{cases} z \cdot b - \frac{\alpha}{(\alpha+\beta)}\,(\varphi_3+\varphi_4) + \frac{\beta}{(\alpha+\beta)}\varphi_5 + \chi = 0 \\ \frac{y}{k}\left(\frac{\alpha-\theta}{\theta}\right) - \frac{\rho}{\theta} - \delta_K\left(\frac{1}{\theta}-1\right) + (\varphi_3+\varphi_4) = 0 \\ \frac{\varphi_5}{\gamma} - \frac{\delta_S}{\gamma} - \frac{y}{k} + (\varphi_3+\varphi_4) + \delta_K = 0 \\ z\,(1-b) - \delta_H - \psi\left[\frac{y}{k} - (\varphi_3+\varphi_4) - \delta_K\right] = 0 \end{cases}.$$

If we solve this system with respect to $(b,\ \varphi_3+\varphi_4,\ \varphi_5,\ \frac{y}{k})$ we will obtain the next solution:

$$b^* = \frac{(z-\delta_H)(1-\theta)(1-\alpha-\beta)-\rho(1-\alpha-\beta\cdot\gamma)}{z\left[\beta(\gamma-1+\theta)+\theta(\alpha-1)\right]}, \tag{52}$$

$$(\varphi_3+\varphi_4)^* = \frac{\alpha^2(\rho+\delta_H-z-\theta\cdot\delta_K)+\rho\left[\beta(\alpha+\gamma-1)-\alpha\right]+\delta_K\left[\beta(\alpha-1)(1-\gamma-\theta)-\theta(1-2\alpha)\right]}{\alpha\left[\theta(\alpha+\beta-1)-\beta(1-\gamma)\right]}+$$
$$+\frac{\delta_H\left[\theta(1-\alpha-\beta)-\alpha(1-\beta)\right]+z\left[(1-\beta)(\alpha-\theta)+\alpha\cdot\theta\right]}{\alpha\left[\theta(\alpha+\beta-1)-\beta(1-\gamma)\right]},$$

$$\varphi_5{}^* = \frac{\gamma(\delta_H-z+\rho)(1-\alpha-\beta)+\delta_S\left[\theta(\beta+\alpha-1)-\beta(1-\gamma)\right]}{\beta(\gamma-1+\theta)-\theta(1-\alpha)},$$

$$\left(\frac{y}{k}\right)^* = \frac{\theta(1-\alpha-\beta)(\delta_H-\delta_K-z)+\beta(\gamma-1)(\delta_K+\rho)}{\alpha\left[\theta(\alpha-1)-\beta(1-\theta-\gamma)\right]}.$$

If we substitute (52) into (41) we will get the expressions for the growth rates of the main macroeconomic indicators along BGP:

$$\frac{\dot{y}}{y}=\frac{\dot{k}}{k}=\frac{\dot{c}}{c}=\frac{\dot{i}_K}{i_K}=\frac{\dot{i}_S}{i_S}=\frac{z-\delta_H}{\psi}-\frac{(z-\delta_H)(1-\theta)(1-\alpha-\beta)-\rho(1-\alpha-\beta\cdot\gamma)}{\psi\left[\beta(\gamma-1+\theta)+\theta(\alpha-1)\right]},$$

$$\frac{\dot{s}}{s}=\frac{\gamma(z-\delta_H)}{\psi}-\frac{\gamma\left[(z-\delta_H)(1-\theta)(1-\alpha-\beta)-\rho(1-\alpha-\beta\cdot\gamma)\right]}{\psi\left[\beta(\gamma-1+\theta)+\theta(\alpha-1)\right]},$$

$$\frac{\dot{h}}{h}=z-\delta_H-\frac{(z-\delta_H)(1-\theta)(1-\alpha-\beta)-\rho(1-\alpha-\beta\cdot\gamma)}{\beta(\gamma-1+\theta)+\theta(\alpha-1)}.$$

Let's try to analyze the influence of natural resources on economic growth along BGP. We will try to check two hypotheses under the plausible values of parameters. These hypotheses are concerned with the influence of natural resources on the economic development.

Hypothesis 1. The parameter γ increasing causes reduced human capital growth rate as scientific and innovation incentives might be reduced in economics along BGP. (γ defines marginal return on investments in natural resources (equation (11)). As has been already mentioned this parameter maybe understood as availability and accessibility of natural resources. Thus, the more accessible natural resources are, the closer γ to unity is).

Hypothesis 2. The parameter β increasing causes decrease of the economic growth along BGP. (β defines the influence of natural resources dynamics on final production. Under the final production we will mean GDP in further analysis).

In order to check hypotheses let's assign values to other parameters.

$\alpha = 0.3$. This approximate value of the physical capital's elasticity is frequently used in economic literature.

$\delta_H = 0.033.$. Depreciation of human capital is quite an abstract value. Its empirical estimation is complicated. There is a lot of literature where this question is discussed. As usual its estimation varies between 0% and 5% depending on the age, level of education and sector of economic activity. We will take 1/30 corresponding to the 30-year period of full depreciation calculated by straight line depreciation.

$z = 0.2$. This parameter shows an impact of accumulated human capital on its growth (equation (6)) and its estimation is quite difficult. It should be less than one. Let's take it as 0.2.

$\theta = 2$. Estimation of the coefficient of relative risk aversion is also quite difficult. We will take this coefficient equal to 2. Such level we may see in Conniffe and O'Neill (2012).

$\rho = 0.05$. This value more or less corresponds to the interest rates of developed economics, which probably are located closer to BGP.

The range of the parameter β, which is an elasticity of natural resources, will be taken between 0.01 and 0.69. Thus, if $\beta = 0.01$ huge influence of human capital on economic growth will be assumed (because elasticity of human capital will be 0.69) and if $\beta = 0.69$ influence of human capital will be extremely insignificant (elasticity of human capital will be 0.01).

The parameter γ will be varied between 0 and 1.

The parameter ψ is defined through α, β and γ.

In figures 1-4 we can see the results of computation which show the influence of γ and β variation on the growth rates of per capita GDP (figure 1), human capital (figure 2), natural resources (figure 3) and influence on the share of human capital devoted to production (figure 4).

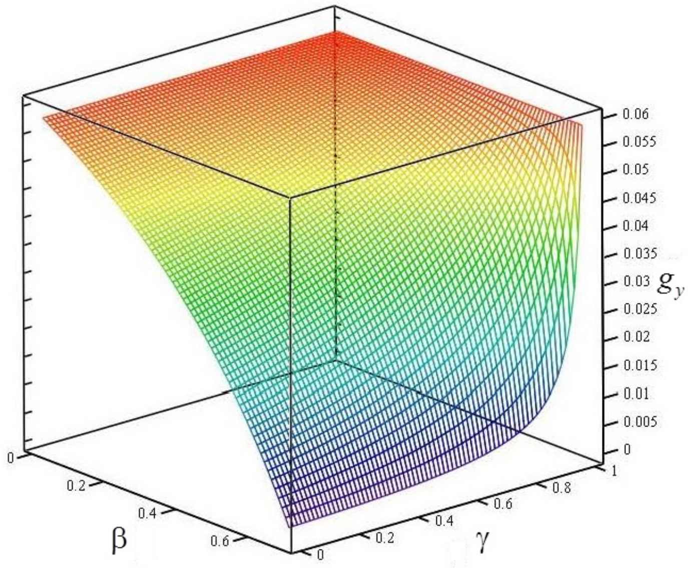

Figure 1. Influence of γ and β on the growth rate of per capita GDP (g_y)

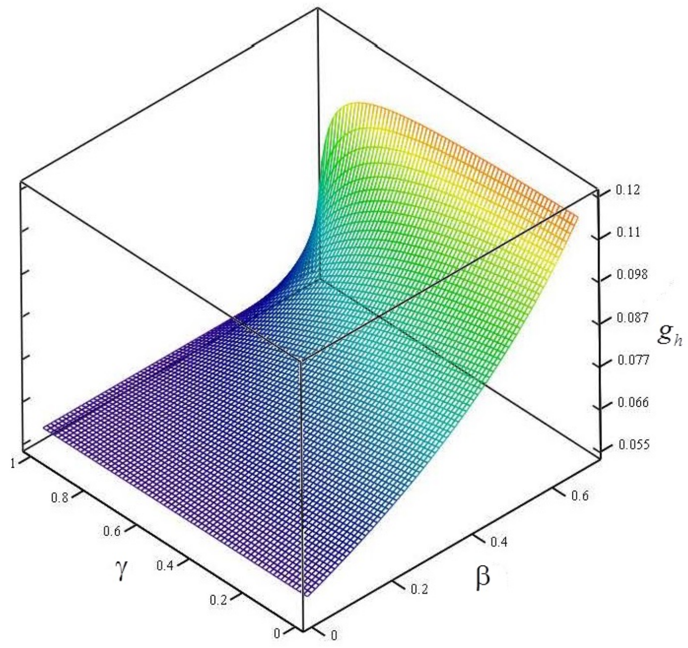

Figure 2. Influence of γ and β on the growth rate of per capita human capital (g_h)

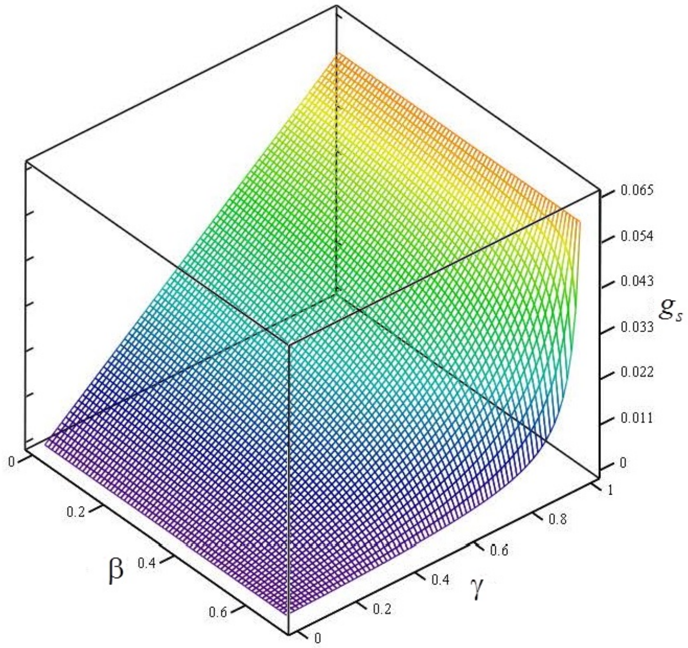

Figure 3. Influence of γ and β on the growth rate of per capita natural resources (g_s)

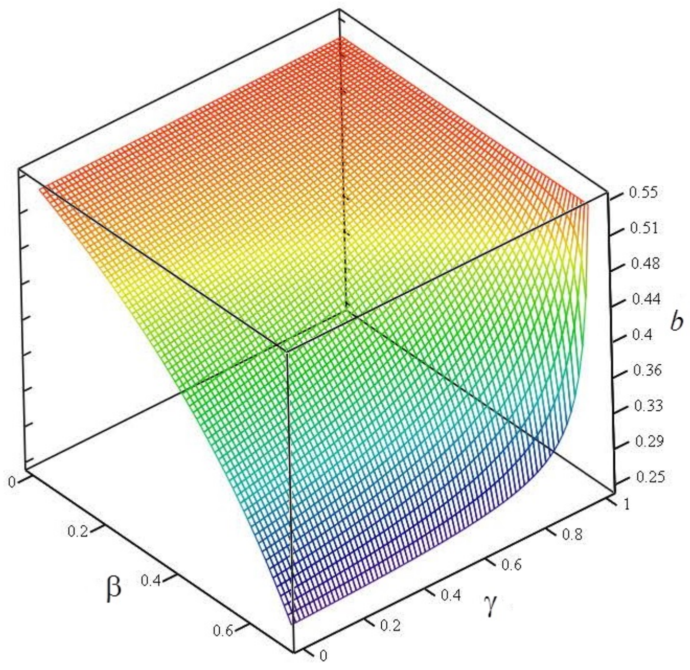

Figure 4. Influence of γ and β on the share of human capital devoted to production (b)

To check the first hypothesis we may look at figure 2. We see that with γ increasing, all other things being equal, the growth rate of per capita human capital decreases. Also, we see that with γ increasing the growth rate of per capita natural resources increases too and it is logical because with γ increasing marginal return on investments in natural resources increases. We see also that the share of human capital devoted to production rises with γ increasing (figure 4) and economic growth increases (figure 1).

To check the second hypothesis let's look at figure 1. We see that results of computation, on the whole, confirm this hypothesis. All other things being equal, the higher β is, the lower the growth rate of economy on BGP is.

We see that if γ is quite high, with β increasing the growth rate of per capita GDP varies narrowly but if γ is not so high, β increasing has significant negative influence on economic growth. There is very low (close to zero) growth rate of per capita GDP if $\beta \approx 0,7$ and γ is close to zero. We can conclude that within the Uzawa-Lucas growth model if the economic system is highly depended on natural resources (β is high) and quick exhausting of natural resources takes place (γ is becoming close to zero) the economy faces stagnation. Also we see that with β increasing the share of human capital devoted to production decreases (figure 4). It occurs because the final production is now less depended on human capital. The most part of the human capital is now concentrated in educational sector and we see that the growth rate of per capita human capital increases (figure 2).

5 Conclusions

In this article the modification of the Uzawa-Lucas growth model was offered. It was obtained that the conditions for decentralized situation for this model are the same as for centralized case. Necessary conditions for this model are also sufficient. One of the results of the analysis is that on BGP the growth rate of per capita human capital should be higher than the growth rate of per capita final production. It was shown that the more natural resources economic system has, the less the growth rate of per capita human capital is, all other things being equal. Also it was shown that the more the economic system from natural resources is depended, the less growth rate of per capita GDP it has.

References

Daron Acemoglu. *Introduction to Modern Economic Growth.* Princeton University Press, Princeton, NJ, 2009.

Philippe Aghion and Peter Howitt. *Endogenous Growth Theory.* MIT Press, Cambridge, Mass., 1998.

Kenneth J. Arrow and Mordecai Kurz. *Public Investment, the Rate of Return, and Optimal Fiscal Policy.* Johns Hopkins Press, Baltimore, 1970.

Robert J. Barro and Xavier Sala-i-Martin. *Economic Growth.* MIT Press, Cambridge, Mass., 2nd edition, 2004.

Hassan Benchekroun and Cees Withagen. Global dynamics in a growth model with an exhaustible resource. Technical report, McGill University, Department of Economics, Jan 2008. Departmental Working Papers.

Jess Benhabib and Roberto Perli. Uniqueness and indeterminacy: On the dynamics of endogenous growth. *Journal of Economic Theory*, 63(1):113–142, 1994.

Dirk Bethmann. The open-loop solution of the uzawa-lucas model of endogenous growth with n agents. *Journal of Macroeconomics*, 30(1):396–414, 2008.

Claudio Bravo-Ortega and Jose De Gregorio. The relative richness of the poor? natural resources, human capital and economic growth. Technical report, Central Bank of Chile, 2002. Working Papers Central Bank of Chile.

Tiago Cavalcanti, Kamiar Mohaddes, and Mehdi Raissi. Growth, development and natural resources: New evidence using a heterogeneous panel analysis. *The Quarterly Review of Economics and Finance*, 51(4):305–318, 2011.

Denis Conniffe and Donal O'Neill. An alternative explanation for the variation in reported estimates of risk aversion. Technical report, 2012.

Erika Färnstrand Damsgaard. Exhaustible resources, technology choice and industrialization of developing countries. Technical report, Research Institute of Industrial Economics, Aug 2010.

Beatriz Gaitan and Terry L. Roe. Natural resource abundance and economic growth in a two country world. Technical report, DEGIT, Dynamics, Economic Growth, and International Trade, Jun 2005. DEGIT Conference Papers.

Manuel Gomez. Optimal fiscal policy in the uzawa-lucas model with externalities. *Economic Theory*, 22(4):917–925, 2003.

Manuel Gomez. Equilibrium efficiency in the uzawa-lucas model with sector-specific externalities. *Economics Bulletin*, 8(3):1–8, 2006.

Arantza Gorostiaga, Jana Hromcova, and Miguel-Angel Lopez-Garca. Optimal taxation in the uzawa-lucas model with externality in human capital. Technical report, Instituto Valenciano de Investigaciones Economicas, S.A. (Ivie), Sep 2011. Working Papers. Serie AD.

Christian Groth and Poul Schou. Can non-renewable resources alleviate the knife-edge character of endogenous growth? *Oxford Economic Papers*, 54(3):386–411, 2002.

Charles Jones and Dean Scrimgeour. A new proof of uzawa's steady-state growth theorem. *The Review of Economics and Statistics*, 90(1):180–182, 2008.

Robert Lucas, Jr. On the mechanics of economic development. *Journal of Monetary Economics*, 22(1):3–42, 1988.

Olvi L. Mangasarian. Sufficient conditions for the optimal control of nonlinear systems. *SIAM Journal on Control*, 4(1):139–152, 1966.

Paolo Mattana. *The Uzawa-Lucas Endogenous Growth Model*. Ashgate, Aldershot, Hants, England; Burlington, VT, 2004.

Casey B. Mulligan and Xavier Sala-i-Martin. Transitional dynamics in two-sector models of endogenous growth. *The Quarterly Journal of Economics*, 108(3):739–773, 1993.

Sergio Rebelo. Long-run policy analysis and long-run growth. *Journal of Political Economy*, 99(3):500–521, 1991.

Andreas Savvides and Thanasis Stengos. *Human Capital and Economic Growth*. Stanford Economics and Finance, Stanford, Calif., 2009.

Hirofumi Uzawa. Optimum technical change in an aggregative model of economic growth. *International Economic Review*, 6(1):18–31, 1965.

Simone Valente. Human capital, resource constraints and intergenerational fairness. Technical report, CER-ETH - Center of Economic Research (CER-ETH) at ETH Zurich, Jun 2007. CER-ETH Economics working paper series.

Danyang Xie. Divergence in economic performance: Transitional dynamics with multiple equilibria. *Journal of Economic Theory*, 63(1):97–112, 1994.