Smooth economic analysis for general spaces of commodities

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Abstract

This paper provides an extended framework to study general equilibrium theory with commodity spaces possibly of infinite dimensions. Our approach overcomes some difficulties found in the literature since it allows the study of the equilibrium when consumption sets may have an empty interior. It also overcomes the need for separable utilities or utilities that satisfy quadratic concavity. The results are based on restricting the mathematical notions of open neighborhoods, continuity, and derivatives at a point, to only those directions that lie within the positive cone. We prove in this setting "directional" equivalents of the Sard-Smale and Preimage Theorems. With these tools, we define the social equilibrium set and show that it is a directional Banach manifold. Together with a suitable definition of projection map, this framework allows a natural equivalent to infinite dimensions of the "catastrophic" approach to general economic equilibrium.

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1 Introduction

A stumbling block in the modelling of competitive markets with commodity and price spaces of infinite dimensions, arises from having positive cones with an empty interior. This issue precludes the use of tools of differential analysis, ranging from the definition of a derivative, to the use of more sophisticated results needed to understand determinacy of equilibria and, more generally the structure of the equilibrium set.

For example, many models of competitive markets require a consumption space in the positive cone of an ℓ_p or L_p space, for $1 \leq p \leq \infty$.[1] Unfortunately, the only spaces among L_p and ℓ_p space whose positive cone have a nonempty interior are L_∞ and ℓ_∞. To complicate things, prices are elements of the positive cone of the dual space of the commodity space.[2] Recall that the dual space of ℓ_p (L_p, respectively), $1 \leq p < \infty$, is the space ℓ_q (L_q, respectively) where $1/p + 1/q = 1$. In other words, even if the commodity space had a positive cone with a nonempty interior, the positive cone of the dual space -that is, the price space- will have an empty interior, and vice versa. The dual spaces of L_∞ and ℓ_∞ are subtler, but this problem still holds.[3]

There are many examples in different directions, but to name a few consider the following:

- A pioneering work in the modelling of financial markets in an Arrow-Debreu setting with uncertainty is due to Duffie and Huang (1985). In this paper, they consider a wide class of information structures,

[1] Recall the following definitions. Let p be a real number $1 \leq p < \infty$. The space ℓ_p consists of all sequence of scalars $\{x_1, x_2, \dots\}$ for which $\sum_{i=1}^{\infty} \|x_i\|^p < \infty$. The norm of an element $x = \{x_i\}$ in ℓ_p is defined as $\|x\|_p = \left(\sum_{i=1}^{\infty} \|x_i\|^p\right)^{1/p}$. The space ℓ_∞ consists of bounded sequences. The norm of an element $x = \{x_i\}$ in ℓ_∞ is defined as $\|x\|_\infty = \sup_i \|x_i\|$. The L_p spaces are defined analogously. For $p \geq 1$, the space $L[a, b]$ consists of those real-valued functions x on the interval $[a, b]$ for which $\|x(t)\|$ is Lebesgue integrable. The norm on this space is defined as $\|x\|_p = \left(\int_a^b \|x(t)\|^p\right)^{1/p}$. The space $L_\infty[a, b]$ consists of all Lebesgue measurable functions on $[a, b]$ which are bounded, except possible on a set of measure zero. The norm is defined by $\|x\|_\infty = \text{ess sup}\|x(t)\|$.

[2] If X is a normed linear vector space. The space of all bounded linear functionals on X is called the *normed dual* of X and is denoted by X^*. The norm of an element $f \in X^*$ is $\|f\| = \sup_{\|x\|=1} \|f(x)\|$.

[3] The dual space of $L\infty$ can be identified with bounded signed finitely additive measures that are absolutely continuous with respect to the measure. There are relatively consistent extensions of Zermelo-Fraenkel set theory in which the dual of ℓ_∞ is ℓ_1.

including those generated by continuous-time state-variable stochastic processes. The natural consumption space in this setting is the space L_2 since it restricts the analysis to consumption claims with finite variance. Notice that an important characteristic of this framework is that prices are also elements of L_2 (since the dual space of L_2 is itself). This means that neither the consumption set nor the space of prices will have an empty interior. While Kehoe et al. (1989) study the equilibrium set of this model they need to allow for negative prices and negative consumption. Our work will overcome this difficulty.

- The infinite horizon model is represented through consumption streams in the set ℓ_∞, that is, at every moment of time the consumption of each agent is bounded. Kehoe and Levine (1985) show that equilibria are locally unique but do not study the structure of the equilibrium set. Balasko (1997a,b,c) improves this results by showing that the equilibrium set is a manifold, but requires utilities to be separable and restricts the analysis to "truncated economies". Our work will encompass both approaches without the need for separable utilities.

- Bewley (1972) uses the space L_∞ to model infinite variations in any of the characteristics describing commodities. These characteristics could be physical properties, location, the time of delivery, or the state of the world (in the probabilistic sense) at the time of delivery. Bewley imposes the Mackey topology to consumer's preferences in order to ensure that prices are elements of L_1. While he shows existence of equilibria, the structure of the equilibrium set is unknown as far as the authors are aware.

- Chichilnisky and Zhou (1998) study an "approximative" model of competitive markets with infinite dimensional commodity spaces. Since continuous functions are dense in all L_p spaces, that is all functions in L_p can be approximated arbitrarily close by a continuous function, they let the consumption space to be $C(K)$. By supposing separable utilities, they also show that prices are elements of $C(K)$. In this setting Covarrubias (2010, 2011) shows that the price equilibrium set is a manifold and provides a topological characterisation of this set. In this work, we show an equivalent result without requiring utilities to be separable. Similarly, Accinelli (2013) shows that the social equilibrium set is a manifold and also provides a topological characterisation.

As mentioned above, this paper overcomes several difficulties related to consumption sets with an empty interior. Mathematically, the difficulties arise from the fact that for any point in the positive cone of a Banach space any neighbourhood will contain points within the ambient space, but outside of the positive cone. Intuitively, this paper restricts the mathematical analysis to only those directions that fall within the positive cone. Once the appropriate mathematical tools have been developed, this paper will follow closely the Negishi approach used in virtually all works previously mentioned. It consists of parameterising a pure exchange economy by the welfare weights of individuals, instead of their initial endowments of commodities, and studying the endogenous social equilibria rather than price equilibria. It can be shown that there is a bijection between the set of social equilibria and the set of price equilibria, making both methods equivalent (Accinelli, 2013). Once the structure of the equilibrium set has been identified, together with a systematic study of the natural projection map, our work becomes a direct generalisation of the Balasko approach to the study of general economic equilibrium.

At this point it is important to mention that Shannon (1999) and Shannon and Zame (2002) have studied the question of determinacy of equilibria in infinite dimensions in much detail, and their results are applicable to all models previously mentioned. Indeed, they show that if utilities satisfy "quadratic concavity", equilibria is then generically determinate. Our results on determinacy are strictly speaking not comparable since our utility functions will not be required to satisfy quadratic concavity but we will however consider a topologically-weaker notion of determinacy. Nevertheless these two papers do not provide any information on the structure of the equilibrium set that we are able to do via proving a directional Preimage Theorem.

The rest of this paper is organized as follows. In the next Section we give a first insight to the market and its main characteristics.Next, in Section 3, we introduce some mathematical tools needed to develop the modeling of the behavior of economic agents in economies whose consumption sets are positive cones with empty interior. In section 4, in light of the mathematical instruments previously considered, we return to the study of the behavior of agents in the market. We will define individual and aggregate excess utility functions and study their properties. In section 5, we study the social equilibrium set, showing that it is a directional Banach manifold. Section 6, is devoted to analyze the determinacy of equilibria while Section 7 presents properties of the natural projection map.

2 The market: a first insght

The aim of this section is to define a pure exchange economy in infinite dimensions. We will define the commodity space, consumption set, utilities, endowments, feasible allocations, and individual and aggregate excess utility functions. The ultimate goal is to define the social equilibrium set which will be the main object of study of this paper.

2.1 Commodity space and consumption space

The **commodity space** will be any Banach lattice B equipped with a norm $\|\cdot\|$, and with a partial order $\geq$. The **consumption set** will be $B_+ = \{x \in B : x \geq 0\}$, the positive cone of B. For this, recall the notion of a positive cone.

Definition 1. *Let B a Banach lattice. The subset $B_+ \subset B$ will be called its **positive cone** if it satisfies:*

1. *For all $x, y \in B_+$ and for all $\alpha, \beta \geq 0$, $\alpha x + \beta y \in B_+$.*
2. *$B_+ \cup (-B_+) = \{0\}$.*

We will use the following notation:

We symbolize by B_{++} the strictly positive cone, i.e,

$$B_{++} = \{x \in B : x \bigvee 0 > 0\},$$

where by $x \bigvee y$ we denote the supremum of the pair of elements $x, y \in B$. The expression $x > y$ means that $x \geq y$ and $x \neq y$.

2.2 Agents and preferences

We consider a pure exchange economy $\mathcal{E}$ where the consumption space is B_+ and a finite number of agents index by $i = 1, \ldots, n$ each of them equiped with a strictly positive endowmnets w_i. We symbolize by $w = (w_1, ..., w_n)$ the vector of the initial endowments, $w \in B^n_{++}$ and utility function $u_i : B_+ \to \mathbb{R}$ strongly monotone and strictly concave. Before to continuous with the characterization of the utility functions we need to introduce some considerations on the topology of the positive cone B_+.

Let us finalize this section with the characterization of feasible allocations. An $n-$tuple $x = (x_1, ..., x_n)$ is called a **feasible allocation**, whenever $x_i \in B^+$ holds for each i and $\sum_{i=1}^n x_i \leq \sum_{i=1}^n w_i$.

3 Directional analysis

This section is devoted to explaining the tools of analysis that will be used throughout this paper. As explained above, the goal is to restrict the notions of continuity, concavity and derivatives at a point, only in the directions contained within the positive cone. With the usual approach, one would have to consider points within the space, but outside the positive cone, which in economic terms would be equivalent to consider negative prices, consumption and endowments.

3.1 Admissible directions

The first concept we define is that of an "admissible direction". The intuition behind this concept is a follows. Suppose w is a point in X, a convex subset of B, and consider a sphere of radius β around w. Then, a direction is β-admissible with respect to w if all the points between w and the edge of the sphere in this direction lie within the convex subset. For example, Figure 3.1 shows two types of convex subsets X, in each case with a specified point w and a sphere or readies β around each. In the left, one can see that h_1 is a β-admissible direction while h_2 is not. Similarly, the right diagram shows two β admissible directions.[4]

Definition 2. *Let X be a convex subset of B. Given a point $w \in X$, we say that $h \in B$ is a $\beta-$**admissible direction of** w, if there exist $\beta > 0$ such that $w + \alpha \frac{h}{\|h\|} \in X$, for all $0 < \alpha \leq \beta$. The set of $\beta-$admissible directions of $w \in X$ will be denoted by*

$$\mathcal{A}_w(\beta) = \left\{ h \in B : w + \alpha \frac{h}{\|h\|} \in X, 0 < \alpha \leq \beta \right\}.$$

[4]It should be remarked that in infinite dimensions the intuition behind many mathematical results fail to be true. Indeed, one of the main motivations behind this paper comes from the mathematical obstacle that many spaces in infinite dimensions have a positive cone with an empty interior, a concept that does not hold in finite dimensions.

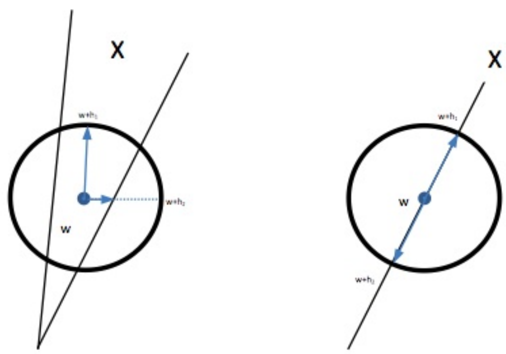

Figure 1:

Remark 1. *Note that since we assume that X is a convex set, if a direction h is β-admissible for w, it will also be admissible for any smaller sphere around w. That is, if h is β-admissible, then it is also α-admissible for any $\alpha \in (0, \beta)$. In particular, $\mathcal{A}_w(\alpha) \subset \mathcal{A}_w(\beta)$, $\forall\, 0 < \alpha < \beta$.*

The next definition is that of an admissible direction, that is, a direction that is admissible for any β. Formally:

Definition 3. *Let X be a convex subset of B. Given a point $w \in X$, we say that $h \in B$ is an* ***admissible direction of*** *w, if $w + \beta\frac{h}{\|h\|} \in X$, for all $0 < \beta$. The subset*

$$\mathcal{A}_w = \left\{ h \in B : w + \beta \frac{h}{\|h\|} \in B_+,\ \forall\, 0 < \beta \right\},$$

is the set of all admissible directions.

From Figure 3.1 it can bee seen on the right diagram that h_1 is for instance an admissible direction, that is, it is a $\beta-$admissible direction for any $\beta > 0$. Also, notice that $\mathcal{A}_w = \bigcup_{\beta > 0} \mathcal{A}_w(\beta)$. **Note that since B_+ is a convex set, then if x and y are two points in B_+ then $h = (x - y) \in \mathcal{A}_y(\alpha)\ \forall\, 0 \leq \alpha \leq 1$.**

3.2 Directional continuity

In what follows we now introduce the notion of directional continuity. As previously, we will restrict the notion of continuity only to those directions within the convex set being analysis, which for our analysis will eventually become the positive cone.

Definition 4. *A map $f : X \to Y$, between a convex subset X of a Banach space, and a topological space Y, is said to be* ***directional-continuous*** *(or *continuous) at $x \in X$, if for any open neighborhood V_y of $y = f(x)$, there exist an $\epsilon-$star neighborhood of x, $U_x^* \subseteq X$, such that, for all $w \in U_x^*$ it follows that $f(w) \in V_y$ or that, equivalently, $\forall\ h \in \mathcal{A}_x$ and $0 < \alpha < \epsilon$ $f(x + \alpha \frac{h}{\|h\|}) \in V_y$.*

Definition 5. *We say that $f : X \to Y$ is a directional-homeomorphism, or *homeomorphism, between a convex subset $X \subset B_+$ and the topological space Y, if f is a star continuous and bijective function whose inverse f^{-1} is continuous.*

3.3 Gateaux-directional derivatives

Let us now to introduce the concept of Gateaux derivative in admissible directions.

Definition 6. *We say that the map $f : X \to Y$ where X is a convex subset of B_+, and Y is a topological space, is* ***Gateaux-directional-differentiable*** *(Gateaux -*differentiable), symbolically G-*differentiable, at $x \in X$ if and only if there exists a map $T \in L(B, Y)$ such that for all $k \in \mathcal{A}_x(t)$ it follows that*

$$f(x + tk) - f(x) = tTk + o(t),\ t \downarrow 0$$

In this case, T is called the ***G-star derivative of*** f ***at*** x *and we define $f'(x) = T$. The G^* differential at x is defined by $d_{G^*} f(x, k) = f'(x)k$*

Higher differential and higher derivatives are defined as habitual, but with the restriction to consider only derivatives in admissible directions.

Having defined to notion of admissible open neighborhoods, continuity and derivatives, Section 2 will define a pure exchange economy in our setting.

3.4 Directional topology

We introduce now the directional-topology in a convex set X of a Banach space, it will be denoted by $^*\tau$. With this objective, considering the concept of admissible direction as defined above, we introduce the notion of directional open ϵ **directional-neighborhood** (or ϵ *neighborhood) of $x \in X$, they will be denoted by $U_x^*(\epsilon)$, where $\epsilon > 0$ and defined by

$$^*U_x(\epsilon) = \{y \in X : \text{there exists} h \in \mathcal{A}_x : y = x + \alpha \frac{h}{\|h\|}, \forall\, 0 \leq \alpha \leq \epsilon\}$$

The family of such sets for all $\epsilon > 0$ and $x \in X$ is a basis[5] for the star-topology. We will denote family of all *neighborhoods of a point x in X by $\mathcal{N}_x^*$ this family is *neighborhood base at x.[6] Again, the intuition behind such a definition is that a neighborhood around a point within a convex set will consist only of those points within the convex set -rather than within the entire ambient space-. As it is straightforward from the definition $U_x^*(\epsilon) = U_x(\epsilon) \cap X$ where $U_(\epsilon)$ is the ball of radius $\epsilon > 0$ centered at x. It follows that the * topology on X is the relative topology generated by the norm of B. Figure 3.4 shows examples of ϵ-*neighborhoods.

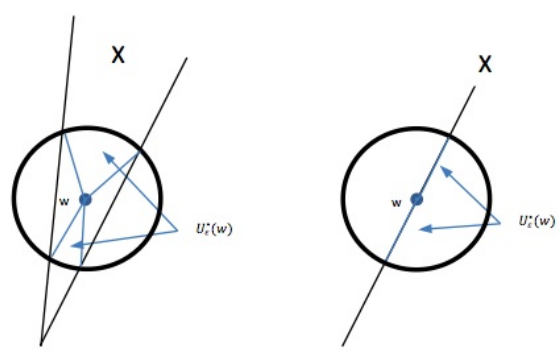

Figure 2:

[5]Recall that a base for a topology τ in a set X is a collection of subsets $\mathcal{B}$ such that for each $x \in X$ and every open set U containing x there is a basic open set $V \in \mathcal{B}$ satisfying $x \in V \subset U$.

[6]A collection $\mathcal{B}$ of neighborhoods of x is a neighborhood base at x if for all nieghborhood U of x there is a $V \in \mathcal{B}$ with $V \subset U$.

Now we can characterize the open sets.

Definition 7. *Let X be a convex set of a Banach space B. We say that a set $U \subset X$ is an* ***open set*** *in X if for each point $w \in U$ there is an ϵ-*neighborhood of w, $V^*_\epsilon(w)$, such that $V^*_\epsilon(w) \in U$.*

Finally, recall that a *topology* is intuitively the choice of open sets. Naturally, we will choose those open sets from Definition 7.

Our main concern is to introduce a topology for the convex subset B_+. The above definitions allow us to do this.

Definition 8. *The topology generated for the family $\mathcal{N}_w, w \in B_+$ with will be called the* ***directional-topology*** *(or d-topology) for B_+ and will be symbolized by $*\tau$.*

4 Agents and preference revisited

We can now finish the characterization of the agents by adding the assumption that the utility functions are G^* differentiable.

4.1 Individual excess utility function

For each $\lambda \in \Delta^n$ there exists a solution for the maximization problem

$$\max_{x \in \Omega_+} \sum_{i=1}^{n} \lambda_i u_i(x) \ \ s.t \ \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} w_i$$

For each (λ, W) where $W = \sum_{i=1}^{n} w_i$ the solution of this problem is denoted by $x(\lambda, W)$ and it is a feasible allocation.[7]

We define the **excess utility function of consumer** i by

$$e_i(\lambda, w) = \lambda_i u_i'(x(\lambda, W)\left[x_i(\lambda, W) - w_i\right].$$

where $u_i'(x)(x_i - w_i) = d_{G^*} u_i(x, x_i - w_i)$.

[7] An allocation $x \in B^n_+$ is feasible if correspond to a redistribution of the initial endowments.

4.2 Aggregate excess utility function

Now, let $\Omega = B \times ... \times B = (B)^n$ be the Cartesian product of n copies of B_+. and let Ω_+ be the positive cone of Ω. That is, $\Omega_+ = (B_+)^n$. Analogously by Ω_{++} we symbolize the cartesian product of n copies of B_{++}. Similarly, let Δ^n denote the $n-1$ simplex. That is,

$$\Delta^n = \left\{ \lambda = (\lambda_1, \ldots, \lambda_n) \in R^n : \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i \geq 0\ \forall i \right\}.$$

Definition 9. *We define the **excess utility function** to be the map* $e : \Delta^n \times \Omega_+ \to \mathbb{R}^{n-1}$ *given by*

$$e(\lambda, w) = (e_1(\lambda, w), ..., e_n(\lambda, w)) .$$

Recall that the excess utility function satisfies the following properties:

Proposition 1. *The excess utility function* $e : \Delta^n \times \Omega_+ \to \mathbb{R}^{n-1}$ *satisfies the following properties:*

(a) $e(\lambda, w)$ *is homogeneous of degree zero, i.e.*

$$e(a\lambda, w) = e(\lambda, w), \forall \lambda \in \Delta^n, \forall a > 0.$$

(b) e *verifies a generalized Walras law [4], i.e.*

$$\lambda e(\lambda, w) = 0, \forall\ \lambda \in \Delta^n.$$

Thus, without loss of generality we can consider that $\lambda_n \neq 0$ and then normalize so that

$$e_n(\lambda, w) = -\frac{\lambda_1}{\lambda_n} e_1(\lambda, w) - .... - \frac{\lambda_{n-1}}{\lambda_n} e_{n-1}(\lambda, w).$$

Then we define the equivalent map $\bar{e} : \Delta^n \times \Omega_+ \to R^{n-1}$, such that the first n coordinates of $e(\lambda, w)$ are equal to the $n-1$ coordinates of $\bar{e}(\lambda, w)$. This map is called the **restricted excess utility function** [4]. In addition we assume that there exists some $k \in B_{++}$ such that $w_i > k$.

5 The social equilibrium set

The main issue in this paper is to study the topological properties characteristics of the social equilibrium set as defined below.

Definition 10. *The* ***social equilibrium set*** *is defined as*

$$\mathcal{SE} = \{(\lambda, w) \in \Delta^n \times \Omega_+ : e(\lambda, w) = 0\}.$$

At this stage, the set $\mathcal{SE}$ only has the structure of a subset of the Cartesian product $\Delta^n \times \Omega_+$. We will prove eventually that it actually has the structure of a Banach manifold. We will do this by showing that there exist a dense and open subset $\Omega^* \subset \Omega_+$ such that 0 is a regular value for $e : \Delta \times \Omega^*$.

This means that for each $w \in \Omega^*$ if we move in an admissible direction h, then locally, $e'(\lambda, w)h$ is a first order linear approximation of $\mathcal{SE}$ in the fixed direction h.

The concept of manifolds arises naturally when one attempts to describe the structure of the solutions set of $f(x) = y$, with $x \in X$ and $y \in Y$, where X and Y are Banach manifolds or open subset of Banach spaces. In economics the set of Walrasian equilibrium $\mathcal{WE}$ is the set of pairs $(x, p) \in B^n \times B'$ where $x = (x_1, ..., x_n)$ is a feasible allocation and p is a set of prices, i.e; an element of the dual space of B (here symbolized by B') such that, x_i maximize the preferences of the $i-$th consumer in her budget set $B(w, p) = \{x \in \Omega : px \leq pw\}$. In [4] is show that there is a a one to one correspondence between this set and the set $\mathcal{SE}$ i.e.: the set of solutions of the equation $e(\lambda, W) = 0$ with $\lambda \in \Delta^n$ and $W \in B_+$ the positive cone of a Banach space (the consumptions set) see [4].

In [9] a detailed analysis of the $\mathcal{WE}$ is performed and is show that, when the consumption set is the positive cone of R^n, this set is a manifold. In [**?**] is shown that in cases where the consumptions set is a Banach space whose positive cone has non-empty interior $\mathcal{SE}$ is a Banach manifold. This means that, under these assumptions, $\mathcal{WE}$ looks locally as a real finite dimensional space, and $\mathcal{SE}$ as a Banach space. The main difficulty is that the set B_+ usually is a positive cone with empty interior, of a Banach space.

5.1 Directional submersions

Definition 11. *Let $f : M \to N$ be a k times derivable functions in the Gateaux sense, where M is a convex subset of a Banach space B and N a topological space. Let $\mathcal{A}_x$ the set of admissible directions for $x \in M$. We say*

f is a **star-submersion** *(or *submersion) at x if and only if $f'(x)$ is onto, and the null subset $N^* = N^*(f'(x))$ splits $T^*M^*_x$,*[8] *where*

$$T^*M^*_x = \left\{ v \in B : v = x + \alpha \frac{h}{\|h\|}, \forall\, h \in \mathcal{A}_x, \text{ and } 0 < \alpha \right\},$$

and $N^*(f'(x)) = \{h \in \mathcal{A}_x : f'(x)h = 0\}$.

A couple of remarks are in order. First, notice that clearly $N^*(f'(x))$ is a subset of the kernel of $f'(x)$. Second, the splitting condition plays an important role because under this assumption there exist a continuous projection $P : M^* \to T^*M^*$.

5.2 Regular points and regular values

Below we define the notion of a regular point, singular point, regular value and singular value.

Definition 12. *Assume $f : X \to N$ is a mapping between Banach spaces.*

*(a) A point $w \in X$ is called a **regular point** of f if and only if f is a star-submersion at x. Otherwise is called a **singular point**.*

*(b) A point $y \in N$ is called a **regular value** if and only if the set $f^{-1}(y)$ is empty or consists only of regular points. Otherwise is called a **singular value**, i.e.; $f^{-1}(y)$ contains at least one singular value.*

5.3 0 is a regular value

In what follows, we show that there exist an open and dense subset $\Omega^* \subseteq \Omega_+$ in the *topology such that 0 is a regular value for $\bar{e} : \Delta^n \times \Omega_+ \to \mathbb{R}^{n-1}$. By extension, we will say that in this case 0 is a regular value of the excess utility function.

Theorem 1. *There exists an open and dense subset $\Omega^* \subset \Omega_+$ in the product topological space $(\Omega_+, (*\tau)^n)$, such that 0 is a regular value for the excess utility function restricted to the subset $int(\Delta^n) \times \Omega^* \subset \Delta^n \times \Omega_+$. That is, 0 is a regular value of $e : int(\Delta^n) \times \Omega^* \to R^{n-1}$.*

We leave the proof to the appendix for now.

[8] We say that the set the subset N split $T^*M^*_x$ if there is another closed set if N can be complemented, in the sense that $T^*M^*_x = N \oplus N^\perp$ i.e.; if there exists a continuous projection on $T^*M^*_x$ whose rank is N.

5.4 Directional Banach manifolds

In order to characterize the social equilibrium set, we begin by introducing the definition of a directional-Banach manifold or (*Banach-manifold). Roughly speaking, a directional Banach manifold is a topological space homeomorphic to a subset of a Banach space with empty interior considered with the topology of the norm. Using the concept of directional-homeomorphism, i.e; and homeomorphism, between an neighborhood in a topological space and a $\epsilon-$directional neighborhood in a the positive cone of a Banach space, we introduce the concept of directional- Banach manifold. The formal definitions is the following:

Definition 13. (Directional-Banach manifold or *B-manifold) *A directional Banach manifold M^* is a topological space with the following additional properties:*

1. Local directional coordinate systems. *For every point $u \in M^*$ there is a neighborhood $U(u)$ and a *homeomorphism ϕ_u^* which maps $U(u)$ onto an $\epsilon-$*neighborhood $V^*(u)$ in B_+*

   If $v \in U(u)$, then $x = \phi_u(v)$ is called the star-coordinate of v in the local directional coordinate or local star coordinate system for ϕ_u

2. Coordinate transformations. *If $w \in U(u) \cap U(v)$, then w has local *coordinates $x = \phi_u^*(w)$ and $y = \phi_v^*(w)$.*

3. *The directional manifold is said C^k manifold if the mappings $\phi_u^* o(\phi_v^*)^{-1} : V_v^* \to V_u^*$ and $\phi_v^* o(\phi^*)_u^{-1} : V^*(u) \to V_v^*$ are C^k mappings (i,e.; there exists and are *continuous, the G-*derivatives up to order k) for all $u, v \in M^*$.*

From now on, to simplify the notation, if there is no risk of confusion we will avoid using the asterisk in the parameterizations.

Definition 14. *A star Banach manifold M^* is a C^m* ****Banach manifold****, if the mappings $\phi_u o\psi_u^{-1}$ and $\psi_u o\phi_u^{-1}$ admit G^*-derivatives up to order m for all $u, v \in M^*$*

Since for al $p \in M^*$ there exists a *homeomorphism $\phi : U(p) \to V^*(a)$, where $\phi(p) = a$ then a *Banach manifold, is locally, in the sense of the *topology, homomorphic to B_+.

Notice that if M_1^* and M_2^* are star-Banach-manifolds, with star local coordinates ϕ and ψ then the cartesian product $M_1^* \times M_2^*$ is also a star-Banach-manifold with the local star coordinate $\phi \times \psi : (v, w) = (\phi(u), \phi(w))$.

Let $\Omega_+ = (B_+)^n$ be the Cartesian product of n copies of B_+. The concept of star Banach manifold can be naturally extended to a topological space M^* locally *homeomorphic to Ω^*.

5.5 The tangent set

Let U an open subset of $p \in M^*$ and $\phi : U \to U_a^* : \; \forall\, u \in U_p$ there $h \in \mathcal{A}_a \epsilon$ and $0 \leq \alpha \leq \epsilon : \phi(u) = a + \alpha h$.

Let $M^* \subset B_+$ be a star B-manifold, given $p \in M^*$ the tangent set to M^* at p $T_p^* M^*$ is a subset of B that can be described as follows:

Let ϕ be a chart map for M^* such that $a = \phi^*(p)$ and let h be an $\epsilon-$admissible direction for a. Consider $\gamma : (0, \epsilon) \to M^*$ defined by $\gamma(\alpha) = \phi^{-1}(a + \alpha h)$, $\forall\, 0 \leq \alpha \leq \epsilon$. It follows that $\gamma'(0) = (\phi^{-1})'(a) h$. Let $p = \phi(a)$, we define the subset of B

$$T_a^* M^* = \left\{ v \in B : \exists\, h \in \mathcal{A}_a : v = \left((\phi^{-1})'(a) \right) h \right\}.$$

The tangent set $T_a^* M^*$ is homeomorphic to a $\mathcal{A}_a$.

Notice that, if all direction is admissible for ϕ at p then, $\phi'(p)\mathcal{A}_a = T_a^* M^* = \phi'(p) B$, where $\phi'(p) B$ is the image of B under $\phi'(p)$. Then, we recover the classical definition of the tangent space. In such case the image, is a topological vector space and corresponds to the tangent space of a Banach manifold with cart space X_ϕ.

6 Determinacy of equilibria

Let us starting introducing some topological concepts.

6.1 Another topological space

Let $(B_+, *\tau))$ the topological space of B_* with the *topology and let $\Omega_+ = \Pi_{i=1}^n B_i$ be the Cartesian product of $n-$ copies of B_+, i.e:, $B_i = B_+$ $i = 1, 2, ..., n$ and let $(*\tau)^n$ the product *topology on Ω_+ generate by the n-projections $P : \Omega_+ \to B_+$ where $*\tau$ is the weakest topology on Ω_+ that makes P star-continuous. That is a subbase for the product *topology $(*\tau)^n$ consists of all sets of the form $P^{-1}(V_j^*) = \Pi_{i=1}^n V_i^*$ where $*V_i = B_+$ for all

$i \neq j$ and $*V_j$ is an *open set in B_j. We symbolize this topological space by $(\Omega_+, (*\tau)^n)$.

Proposition 2. *Let M^* a star manifold in Ω_+ and $f : M^* \to \Omega_+$. Then $f'(p) : T_pM^* \to (B)^n$.*

Proof. To see this claim, consider $f(p) = a$ for each $a \in (B_+)^n$ and then take a star-parametrization ϕ. Let $a \in (B_+)^n$ such that $a = \phi(p) \in (B_+)^n$ Let $v \in T_pM^*$ then there exists $h \in \mathcal{A}_a \subset B^n$ such that $v = [(\phi^{-1})'(a)]h$ so

$$f'(p)v = f'((\phi^{-1})(a))h = [f'((\phi^{-1})(a))][(\phi^{-1})'(a)]h \in B^n.$$

□

Proposition 3. *For each parameter $w \in \Omega^*$ the map $\bar{e}(\cdot, w) : int(\Delta^n) \to R^{n-1}$ is a Fredholm map.*

Proof. $\bar{e}'(\cdot, w)$ for each $w \in \Omega^*$ a finite dimensional operator, and so a Fredholm operator. □

Proposition 4. *$\bar{e} : int(\Delta^n) \times \Omega^* \to \mathbb{R}^{n-1}$ is proper.*

Proof. This assertion say that the restricted excess utility function is a proper map. The convergence of $w_n \to w$ in Ω_+ as $n \to \infty$ and $e(\lambda_n, w_n) = 0$ for all n implies the existence of a convergent subsequence $\lambda_{n'} \to \lambda$ as $n \to \infty$ with $\lambda \in int(\Delta^n)$ This follows from the continuity of the excess utility function, from the assumptions that consumers have strictly monotone preferences, and from the additional assumption that $w_i > k \in B_{++}$ for all $i \in \{1, 2, ..., n\}$ (for comments in this assumptions see [4]). □

6.2 Directional Sard-Smale Theorem

Then we have the following generalized version of the theorem of the Sard-Smale.

Theorem 2. *There exists a dense an *open subset Ω_0 of Ω^* such that 0 is a regular value for $\bar{e}(\cdot, w)$ each parameter $w \in \Omega_0$.*

The proof is left to the appendix. Also, the following corollary holds:

Corollary 3. *The solutions of $\bar{e}(\lambda, w) = 0$ is a 0-dimensional *Banach manifold, or the solution is empty.*

Proof. Fix $w_0 \in \Omega^*$. Since $e'_\lambda(\lambda, w_0) : R^{n-1} \to R^{n-1}$ is surjective Fredholm Operator of index zero, then the solutions of $e(\lambda, w_0) = 0$ consists of a $0-$ dimensional Banach manifold. □

7 The natural projection

7.1 The tangent space

The tangent set $T^*_{u_0}\mathcal{SE}$ at the point $u_0 = (\lambda_0, w_0)$ precisely consists of all point $v = (k, h) \in R^{n-1} \times \mathcal{A}_{w_0}$: $e'(u_0)v = 0$, i, e:

$$e'_\lambda(\lambda_0, w_0)k + e'_w(\lambda_0, w_0)h = 0. \tag{1}$$

7.2 The natural projection

Definition 15. *We define the non-linear **natural projection map** to be the map $\pi : \mathcal{SE} \to \Omega^*$, defined by $\pi(\lambda, w) = w$.*

7.3 Properties of the natural projection

Proposition 5. *For a fixed u let $D = T^*_u\mathcal{SE}$ the tangent set, in u. We define the projection $Q : D \to \Omega^*$ through $Q(\lambda, w) = w$. Let $\pi'(\lambda, w)$ be the *Gateaux-derivative of the operator π evaluated at $u = (\lambda, w)$. Then we have: $Q(\lambda, w) = \pi'(u)$.*

Proof. This follows because: Let $(k, h) \in T^*_{u_0}\mathcal{SE}$ there exist a curve $\gamma(s) = (\lambda(s), w(s)) \in \mathcal{SE}$ such that $k = \lambda'(s)$ and $h = w'(s)$, being: $\lambda_0 = \lambda(0)$ and $w_0 = w(0)$. Inserting this curve in equation, $\pi(\lambda(s), w(s)) = \lambda(s)$, and taking derivatives, we obtain the result. □

Proposition 6. *The operator $\pi : \mathcal{SE} \to \Omega^*$ is proper.*

Proof. To see this let P be a compact set is Ω^*. If $\{(\lambda_n, w_n)\}$ is a sequence in $\pi^{-1}(P)$ then $e(\lambda_n, w_n) = 0$. Since P is compact, there exists a convergent subsequence $\lambda_{n'} \to \lambda$ with $\lambda \in P$ and now for $(H4)$ the result follows. □

Proposition 7. *$e'_\lambda(u)$ is surjective if and only if $Q : \mathcal{SE} \to \Omega^*$ is surjective.*

Proof. To see this note that we have

$$Q(\lambda, w) = 0 \Leftrightarrow w = 0, e'_\lambda(u_0)k = 0.$$

This implies $din\ N(Q) = dim\ N(e'_\lambda(u_0))$. □

Proposition 8.

$$codim\ R(Q) = codim\ R(A).$$

Proof. From the definition of Q it follows that

$$R(Q) = (e'_w(u_0))^{-1}(R(e'\lambda(u_0)).$$

Recall that $e'_\lambda(u_0) : R^{n-1} \to R^{n-1}$, and $e'_{u_0}(u_0) : B^n \to R^{n-1}$, are Fredholm operators. We choose, B_0, R_0 subsets of B^n and R^{n-1} which induce the direct sum decompositions:

$$B^n = N(e'_w(u_0) \oplus B_0, \quad R^{n-1} = R(A) \oplus R_0.$$

Let the operator $e'_{w0}(u_0) : B_0 \to R(e'_w(u_0))$ be the restriction of $e'_w(u_0)$ onto B_0. Then $e'_{w0}(u_0)$ is bijective. This gives:

$$B^n = N(e'w(u_0)) \oplus (e'_{w0}(u_0))^{-1}(R(e'\lambda(u_0)) \oplus ((e'_{w0}(u_0))^{-1}(R(e'(u_0))^{-1}(R_0)$$

So we have that

$$B_0 = (e'_w(u_0))^{-1}(R(e'_{w0}(u_0)) \oplus (e'_{w0}(u_0))^{-1}(R_0) = R(Q) + (e'_{w0}(u_0))^{-1}(R_0).$$

Therefore

$$codim\ R(A) = dim\ R_0 = dim\ (e'_{w0}(u_0))^{-1}(R(e'_w(u_0)) = codim\ R(Q).$$

□

It follows that Q is Fredholm with $index\ Q = index\ e'_\lambda(u)$. Since $Q = \pi'(u)$ we obtain $\pi : \mathcal{SE} \to \Omega^*$ is Fredholm with $index\ \pi'(u) = index\ e'_\lambda(u)$. Since $\mathcal{SE}$ is a *Banach manifold then the set of regular values of π is open and dense in B^n. So there exists an open and dense subset Ω_0 of Ω^* such that each $w_0 \in \Omega_0$ is a regular value of π. Moreover, let $w_0 \in \Omega_+$ since

$$e_\lambda(\lambda_0, w_0) : R^{n-1} \to R^{n-1}$$

is surjective. Therefore 0 is a regular value of $e(\cdot, w_0)$.

Appendix

Proof. (of Theorem 1) Let $w \in \Omega_+$ be the initial endowments of the economy and let $h = (h_1, ...h_n)$ be a vector in B^n such that h_i is admissible. Consider the vector $v = (v_1, ..., v_n) \in R^n$ such that $w_i + v_i h_i \in \Omega_+$ and $v_n h_n = \sum_{i_1}^{n-1} v_i h_i$ and define $\eta = (\eta_1, \eta_2, ..., \eta_n)$ verifying that $\eta_i = v_i h_i; i = 1, 2, ..., n-1$. The vector η will be thought as a parameters for redistributions of initial endowments.

The excess utility function for the economy $\mathcal{E}(\eta) = \{u_i, w(\eta)_i, I\}$ will be:

$$e(\lambda, w(\eta)) = (e_1(\lambda, w_1 + v_1 h_1), ..., e_n(\lambda, w_n + v_n h_n)), \tag{2}$$

where

$$e_i(\lambda, w_i + v_i h_i) = u_i'(x_i^*(\lambda, W))[x_i(\lambda, W) - w_i - v_i h_i].$$

Observe that the function $e_i(\lambda, w_i + v_i h_i)$ $i = 1, 2, ..., n$ depends only on the $n-1$ real variables $v_i, i = 1, ..., n-1$. So we can consider the equivalent excess utility function $\bar{e}(v_1, ..., v_{n-1}) = \bar{e}(v)$, observe that $\bar{e} : R^{(n-1)} \to R^{n-1}$.

The derivative of $\bar{e_i}$ with respect to $v_i, i = 1, ..., n-1$ evaluated at $(\lambda, w(\eta))$ is given by:

$$\tfrac{\partial e_i(\lambda, w_i + v_i h)}{\partial v_i} = \tfrac{\partial \tilde{e_i}(v_i)}{\partial v_i} = -u_i'(x_i(\lambda, W))h_i,$$

$$\tfrac{\partial e_n(\lambda, w_n - \sum_{i=1}^{n-1} v_i h_i)}{\partial v_i} = u_n'(x_n(\lambda, W))h_i.$$

Then it follows that:

$$\frac{\partial e(\lambda, w(\eta))}{\partial v_i} = \frac{\partial \tilde{e}(v)}{\partial v_i} = (0, ..., 0, \frac{\partial \tilde{e_i}(v_i)}{\partial v_i} h_i, 0..., 0, \frac{\partial \tilde{e_n}(w_n - \sum_{i=1}^{n-1} v_i h_i)}{\partial v_i} h_i)$$

$$= (0, ..., 0, -u_i'(x_i(\lambda, W))h_i, 0, ..., 0, u_n'(x_n(\lambda, W))h_i).$$

Let $\bar{e} : R^{n-1} \to R^{n-1}$ be the function defined by the $n-1$ first coordinates of $\tilde{e}$, i.e:

$$\bar{e}(\lambda, w + \eta) = (e_1(\lambda, v_1 h_1), ..., e_{n-1}(\lambda, v_{n-1} h_{n-1}) = (\tilde{e_1}(v_1), ..., \tilde{e}_{n-1}(v_{n-1})).$$

Then:

$$\frac{\partial \bar{e}}{\partial \bar{v}}(\lambda, w(v)) = -\begin{bmatrix} u_1'(x_1^*) & 0 & \dots & 0 \\ 0 & u_2'(x_2^*) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & u_{n-1}'(x_{n-1}^*) \end{bmatrix} \in L\left(X^{(n-1)}, R^{(n-1)}\right). \tag{3}$$

The rank of this matrix is equal to $n-1$, as the rank of a matrix is locally invariant, then for all w there exists an arbitrarily close vector $w(\eta)$: such that the rank of $\frac{\partial \bar{e}}{\partial \bar{v}}(\lambda, w(v))$ is equal to $n-1$ this prove the denseness of Ω^*.[9] Let $\Delta_w = \{\lambda \in int(\Delta^n) : u_i(x(\lambda)) \geq u_i(w_i)\}$ be the set of the *individually rational social weights*. Then for a given $\epsilon > 0$ there exists $\delta > 0$ such that if $|e_i(\lambda, w(\eta) - e_i(\lambda, w)| \leq \|u_i'\|\|h_i\| < \epsilon$ for $h_i : \|h_i\| < \delta$, where $\|u_i'\| = sup_{h_1:\|h_i\|=1}|u_i'(x(\lambda, W)|, \lambda \in \Delta_w$, i.e. the excess utility function of the perturbed economy is in a neighborhood of the excess utility function of the original one.

To show that zero is a regular value for e we need to prove that $Ker(e')$ splits $R^{n-1} \times \Omega$. In our case, as the image of the function e is a subset of R^{n-1}, (i.e.; $e(int[\Delta] \times \Omega_+) \subseteq R^{n-1}$) so the quotient space $(R^{n-1} \times \Omega)/Ker(e')$ has finite dimension, then $codim[Ker(e')] < \infty$ and the splitting property is automatically satisfied[10].

□

Proof. (of Theorem 2) Since $\bar{e} : \Delta^n \times \Omega^* \to R^{n-1}$ is a *submersion there exists and admissible *chart, $(U, \phi) \in \Delta^n \times \Omega^*$ with $\phi : \mathcal{SE} \to \Delta^n \times \Omega^*$ such that $x = \phi(\lambda_0, w_0) \in \Delta^n \times \Omega^*$ with $\phi'(\lambda_0, w_0) = I$. Let $U_\phi = \phi(V^*)$ where $W^*_{\lambda_0, w_0} = N_{\lambda_0} \times V^*_w \subset \Delta^n \times \Omega^*$ is an star-neighborhood of (λ_0, w_0) i,e,

$$\phi(W^*) = \left\{(\lambda, w) \in \Delta^n \times \Omega^* : \lambda \in N_{\lambda_0} \ and \ w \in V^*_{w_0}\right\}$$

where N_λ is a relative neighborhood of $\lambda_0 \in \Delta^n$ and $V^*_{w_0}$ is a star-neighborhood of $w_0 \in \Omega^*$.

$$\bar{e}(\phi^{-1}(\lambda_0 + \beta k, w_0 + \alpha h)) = \bar{e}'(\lambda_0, w_0)v + y$$

where $v = (k, h) \in R^{n-1} \times \mathcal{A}_{w_0}$ and $\bar{e}'(\lambda_0, w_0)v = \bar{e}'_\lambda(\lambda_0, w_0)k + \bar{e}'_w(\lambda_0, w_0)h$. The existence of these derivatives, is shown in [2]. A vector $h \in B^n$ belong to $\mathcal{A}_{w_0}$ if and only if $h = (h_1, h_1, ...h_n),$ *and* $h_i \in \mathcal{A}_{w_{0i}}$ being $w_0 = (w_{10}, w_{20}, ..., w_{n0})$.

So, for all $z \in V_x$ the solution of the equation $e(\lambda, w) = y$ corresponds $\hat{v} = (\hat{k}, \hat{h}) \in R^{n-1} \times \mathcal{A}_{w_0}$, such that

$$e'(\lambda_0, w_0)\hat{v} = e'_\lambda(\lambda_0, w_0)\hat{k} + e'_w(\lambda_0, w_0)\hat{h} = 0.$$

[9] This shows that the linearized form of $e(\lambda, w)$ with respect to λ, $e_\lambda(\lambda, w) : T_\lambda \Delta \to R^{n-1}$ is surjective. Then we can use the surjective implicit function theorem.

[10] Recall that in a locally convex Hausdorff space X, every finite dimensional subspace Y can be complemented, that means that there exists a closed vector subspace Z such that $X = Y \oplus Z$ i.e. Y splits X see [20].

We denote by $\mathcal{A}_{(\lambda_0,w_0)}$ the subset of admissible directions in $\Delta^n \times \Omega^n$ i.e; $\mathcal{A}_{(\lambda_0,w_0)} = R^{n-1} \times \mathcal{A}_{w_0}$ So $\mathcal{SE}$ is *homeomorphic to the $Ker^*(e'(\lambda_0, w_0)) \subset T^*_{(\lambda_0,w_0)}\mathcal{SE}$, by $Ker^*(e'(\lambda_0, w_0))$ we denote the subset $Ker(e'(\lambda_0, w_0)) \cap \mathcal{A}_{(\lambda_0,w_0)}$.

We conclude that $\mathcal{SE}$ is a star Banach manifold. To finish the prove we shall show that there exists a subset Ω_0 open and dense in Ω^* with the $(*\tau)^n$ topology, such that for each $w \in \Omega_0$ $e'_\lambda(\lambda, w) : R^{n-1} \to R^{n-1}$ is surjective, and then 0 is a regular values of $e(\cdot, w)$ This proves the theorem.

$\square$

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