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On weak Condorcet winners: existence and uniqueness

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Abstract

Consider a society where each agent has weak preferences over alternatives. The purpose of the society is the selection of alternatives using the majority principle. Then, an alternative x (a weak Condorcet winner) is selected if half or more agents agree that x is at least as good as any other alternative. We give necessary and sufficient conditions on preference profiles under which (i) the set of Weak Condorcet Winners is non-empty, and (ii) the Weak Condorcet Winner is unique.

Keywords: Condorcet winner, majority principle.

JEL classification: D70.

Resumen

Considere una sociedad donde cada agente tiene preferencias débiles frente a otras alternativas. El propósito de la sociedad es la selección de alternativas utilizando el principio de mayoría. Entonces, una alternativa x (un ganador débil de Condorcet) se selecciona si la mitad o más agentes coinciden en que x es al menos tan buena como cualquier otra alternativa. Damos condiciones necesarias y suficientes de perfiles de preferencia en que (i) el conjunto de los ganadores débiles de Condorcet es no vacío, y (ii) el ganador débil de Condorcet es único.

Palabras clave: Ganador de Condorcet, principio de mayoría.

Clasificación JEL: D70.

1 Introduction

Consider a society formed by a finite set of agents with individual weak preferences over a finite set of alternatives with cardinality bigger than three. Condorcet (1785) proposed the majority principle which states that an alternative y could not be selected if more than half of the individuals preferred x over y. However, it could be the case that there exists no alternative which beats by majority every other alternative or there might not be a unique winner. The existence is an important issue if the society agrees to use a rule which selects a Weak Condorcet Winner among the available alternatives. Rules as the Majority Voting or the Copeland Rule select the Weak Condorcet Winner when it exists. Thus, we propose a condition in preference profiles to ensure the existence of a Weak Condorcet Winner. This is the first step to achieve domain restrictions compatible with rules which select the Weak Condorcet Winner. On the other hand, the uniqueness of a Weak Condorcet Winner will be relevant given the objective of the society. If its purpose is the selection of a single alternative by the majority principle, we would like to identify conditions on preference profiles which ensure the uniqueness of a Weak Condorcet Winner. Society could have a disagreement in the tie-breaking rule and might not reach a solution if more than one Weak Condorcet Winner exists.

Black (1958) was aware that a Weak Condorcet Winner would exist; thus, the majority decision was going to be plausible, when the aggregation of preferences yields transitive social preferences. Sen and Pattanaik (1969) improved this observation and came up with the Value Restriction as a sufficient condition to have a Weak Condorcet Winner in the preference profile. In this paper we propose necessary and sufficient conditions in the preference profile for the existence and the uniqueness of the Weak Condorcet Winner. The Upmost Condition will ensure that the set of Weak Condorcet Winners is non-empty. This condition will look only at those alternatives which are strictly preferred by a given alternative x. The Cycle Condition will guarantee that the Weak Condorcet Winner is unique looking at each triple of alternatives.

The paper is organized as follows. In section 2 we introduce the basic notation and the concept of the majority principle. Section 3 contains the results for the conditions in preferences profiles to ensure the existence and uniqueness of a Weak Condorcet Winner. We conclude with some final remarks.

2 Preliminaries

Let X be the non-empty set of alternatives with cardinality m. Let $N = \{1, ..., n\}$ be the set of agents. A preference profile $R = (R_1, ..., R_n)$ is an n-tuple of complete, reflexive, and transitive binary relations on the set of alternatives X. We should read xR_iy as x is as least as good as y for agent i. Therefore, for all $x, y, z \in X$ and $i \in N$, either xR_iy or yR_ix; xR_ix; and if xR_iy and yR_iz then xR_iz. Let P_i be the asymmetric part of the binary relation R_i and I_i denote the indifference relation of the binary relation R_i. Denote by $\mathcal{R}$ the set of individual preferences and by $\mathcal{R}^N$ the set of all preference profiles. A domain $\hat{D} = D_1 \times ... \times D_n \subseteq \mathcal{R}^N$ is a cartesian product subset of $\mathcal{R}^N$.

Denote, at a given profile R, the cardinality of the set of individuals who strictly prefer x to y by $N(xPy)$. That is, $N(xPy) \equiv |\{i \in N \ \mid xP_iy\}|$. The concerned individuals are the number of agents who are not indifferent between x and y; that is, $n^*(x, y; R) = N(xPy) + N(yPx)$.

If for all $y, x \in X$, $y \neq x$, strictly more than half of the concerned agents prefer alternative x over y, then such alternative x is called a Strong Condorcet Winner.

Definition 1 *A **Strong Condorcet Winner** at profile R is an alternative $x \in X$ such that $N(xPy) > N(yPx)$ for all $y \in X \backslash \{x\}$.*

If it exists, a Strong Condorcet Winner is unique and it does not tie with any alternative in pairwise majority comparison. However, there are profiles R where some alternatives defeat or tie the rest of the alternatives by pairwise majority comparison; we refer to such alternatives as the Weak Condorcet Winners.

Definition 2 *A* ***Weak Condorcet Winner*** *at profile R is an alternative $x \in X$ such that $N(xPy) \geq N(yPx)$ for all $y \in X \backslash \{x\}$.*

For some profiles R, we can have two type of situations when we are looking for a Weak Condorcet Winner:

a) Society is not be able to find any Weak Condorcet Winner and face the so called Condorcet Paradox. Consider three agents $N = \{1, 2, 3\}$ and three alternatives $X = \{a, b, c\}$. Let the profile R be such that aP_1bP_1c, bP_2cP_2a, and cP_3aP_3b. Then, by majority a beats b, b beats c, and c beats a. Thus, there exists no alternative which defeats all the rest by majority pairwise comparison.

b) Although a Weak Condorcet Winner exists for every profile $R \in \hat{D} \subseteq \mathcal{R}^N$, it might not be unique. If the objective is the selection of a single alternative by the majority principle, having more than one Weak Condorcet Winner might cause that the society does not reach an agreement.

We will characterize the profiles R where the set of Weak Condorcet Winners is non-empty and we will differentiate between profiles where this set is a singleton or has cardinality strictly bigger than one.

3 Results

There is not a complete characterization of profiles for which Weak Condorcet Winners exists. Sen and Pattanaik (1969) proposed the Value Restriction property on a profile R to ensure the existence of Weak Condorcet Winners.

Definition 3 *A profile R satisfies* ***Value Restriction*** *if for every triple $\{x, y, z\} \subseteq X$ there is some alternative, say x, such that all individuals agree that x is not the worst, or agree x is not the best, or agree x is not medium. That is, one of the following three conditions hold:*

1. for all $i \in N$, either xP_iy or xP_iz; or

2. for all $i \in N$, either yP_ix or zP_ix ; or

3. for all $i \in N$, either $(xP_iy$ and $xP_iz)$ or $(yP_ix$ and $zP_ix)$.

Theorem 1 *(Sen and Pattanaik, 1969) A sufficient condition for a profile R to have a Weak Condorcet Winner is that R satisfies Value Restriction.*

Theorem 1 gives a sufficient condition for the existence of Weak Condorcet Winners at R. However, in Example 1 we observe that there might be cases in which not all triples satisfy Value Restriction and still have a Weak Condorcet Winner.

Example 1 *Let $N = \{1, 2, 3\}$ and $X = \{x, y, z, w\}$. The profile R*

P_1	P_2	P_3
w	z	y
x	x	x
y	w	z
z	y	w

does not satisfy Value Restriction since the triple $\{y, z, w\}$ has the property that no alternative is considered neither the best, nor the worst, nor the medium. Nevertheless, x is the Weak Condorcet Winner. Furthermore, as x does not tie with any other alternative, it is the Strong Condorcet Winner.

3.1 Existence of Weak Condorcet Winners

In this subsection, we will give a necessary and sufficient condition to be satisfied by a preference profile to have a non-empty set of Weak Condorcet Winners. For this purpose we will focus on those alternatives which are stricly better than a given alternatve x for each agent i at a profile R.

Definition 4 *The **strict upper contour set** of a binary relation R_i at x, denoted by $U(R_i, x)$, is the set of alternatives $y \in X \backslash \{x\}$ that are strictly prefered to x; that is $U(R_i, x) = \{y \in X \mid yP_ix\}$.*

Given an alternative $x \in X$, we can look at the upper contour sets and identify the number of agents which prefer another alternative y over x.

Definition 5 *We say that a profile R satisfies the* ***Upmost Condition****, if there exists at least one alternative $x \in X$ such that for all $y \in X \backslash \{x\}$, $|\{i \in N \mid y \in U(R_i, x)\}| \leq \frac{n^*(x, y; R)}{2}$.*

Denote by x_{uc} a generic alternative which makes that profile R satisfies the Upmost Condition. If profile R satisfies the Upmost Condition at $x_{uc} = x$, at most half of the concerned agents have the rest of the alternatives in their upper contour sets at x.

Proposition 1 *Let $n \geq 3$ and $m \geq 3$. A profile R satisfies the Upmost Condition if and only if the set of Weak Condorcet Winners is non-empty.*

Proof. To prove sufficiency, suppose R satisfies the Upmost Condition. Hence, there exists at least one alternative $x \in X$ such that for all $y \in X \backslash \{x\}$, $|\{i \in N \mid y \in U_i(R_i, x)\}| \leq \frac{n^*(x, y; R)}{2}$. By definition, if $y \in U_i(R_i, x)$ then yP_ix. Therefore, the number of agents who strictly prefer y to x is equal or less than half of the concerned agents; that is, $|\{i \in N \mid y \in U_i(R_i, x)\}| = N(yPx) \leq \frac{n^*(x, y; R)}{2}$. Thus, $2N(yPx) \leq n^*(x, y; R)$, then $N(yPx) \leq n^*(x, y; R) - N(yPx)$, which indeed is $N(xPy) \geq N(yPx)$ for all $y \in X \backslash \{x\}$. Hence, x is a Weak Condorcet Winner.

Now, to prove necessity, assume that the set of Weak Condorcet Winners is non-empty at R. Hence, there exists $x \in X$ such that $N(xPy) \geq N(yPx)$ for all $y \in X \backslash \{x\}$. This means that $n^*(x, y; R) - N(yPx) \geq N(yPx)$, which is only possible if $\frac{n^*(x, y; R)}{2} \geq N(yPx)$. As $y \in U(R_i, x)$ if yP_ix then we have that $N(yPx) = |\{i \in N \mid y \in U(R_i, x)\}| \leq \frac{n^*(x, y; R)}{2}$ for all $y \in X \backslash \{x\}$. Thus, R satisfies the Upmost Condition. ■

The Upmost Condition is giving us the existence of Weak Condorcet Winners at profile R. The following example illustrates a profile R satisfying the Upmost Condition.

Example 2 *Consider* $N = \{1,2,3,4\}$*,* $X = \{x,y,z\}$*, and the profile* R *where*

R_1	R_2	R_3	R_4
y	x	z, y	x
x	z	x	z, y
z	y		

Take $x \in X$ *and all the strict upper contour sets* $U(R_i, x)$*. Then,*
$|\{i \in N \mid y \in U_i(R_i, x)\}| = 2 = \dfrac{n^*(x,y;R)}{2} = 2,$ *and*
$|\{i \in N \mid z \in U_i(R_i, x)\}| = 1 < \dfrac{n^*(x,z;R)}{2} = 2.$

Hence, R *satisfies the Upmost Condition with* $x_{uc} = x$*. Alternative* $x \in X$ *beats or ties all the rest of the alternatives by pairwise majority comparison. Hence,* x *belongs to the set of Weak Condorcet Winners.*

Eventhough a profile R satisfies the Upmost Condition, we are not excluding the possibility of having more than one Weak Condorcet Winner.

3.2 Uniqueness of the Weak Condorcet Winner

Having more than one Weak Condorcet Winner can be a problem to define a social choice function or to reach an agreement between individuals. However, it is not always the case in which the objective of a society is to select only one alternative under the majority principle. For example, there might be a pre-selection of alternatives from a pool of candidates previous to the final voting procedure. In either case, it is interesting to know if the Weak Condorcet Winner is unique. We next identify a condition at a profile R to determine if the set of Weak Condorcet Winners is a singleton or not. Like the Value Restriction property, we will focus on properties by each triple of alternatives in X. Hence, we will first define two configurations a triple of alternatives might have.

Definition 6 *Let* R *be a profile. A triple of alternatives* $\{w,y,z\} \subseteq X$ *is a* ***cycle*** *at* R *if:*

$N(wPy) > N(yPw)$,
$N(yPz) > N(zPy)$, *and*
$N(zPw) > N(wPz)$.

Definition 7 *Let R be a profile. A triple of alternatives $\{w, y, z\} \subseteq X$ is a* ***semi-cycle*** *at R if:*
$N(wPy) > N(yPw)$,
$N(yPz) > N(zPy)$, *and*
$N(zPw) = N(wPz)$.

In this case, we will say that w *leads* the semi-cycle, as it is the only alternative in the triple $\{w, y, z\}$ which is not beaten by pairwise majority comparison with the rest.

A set of alternatives X with cardinality m has ${}_mC_3 = \dfrac{m!}{3!(m-3)!}$ possible triples of alternatives. These triples might be cycles, semi-cycles, or neither of them. Notice that an alternative $y \in X$ is in more than one possible triple; for example, if $m = 4$, then each alternative $y \in X$ will be in 3 of the ${}_4C_3 = 4$ possible triples of alternatives. Looking at each triple, we can see if an alternative is a candidate to be a Weak Condorcet Winner and say something about the cardinality of the set.

Definition 8 *Let R be a profile. An alternative $y \in X$ is* ***beaten inside a cycle*** *at R if either y belongs to a cycle or y belongs to a semi-cycle and it is not the leader.*

Definition 9 *Let R be a profile. An alternative $y \in X$ is* ***beaten outside a cycle*** *at R if y is not beaten inside a cycle and $|\{i \in N \mid y \in U_i(R_i, a)\}| < \dfrac{n^*(y, a; R)}{2}$ for some $a \in X$.*

It is easy to see that there might be alternatives which are not beaten neither inside nor outside the cycle. If there is a unique Weak Condorcet Winner, there will only be one alternative which is not beaten in any cycle.

Definition 10 *We say that a profile R satisfies the* ***Cycle Condition*** *if there exists an alternative $x \in X$, such that for all $y \in X \backslash \{x\}$, y is beaten inside a cycle or y is beaten outside a cycle.*

Denote by x_{cc} a generic alternative that ensures that profile R satisfies the Cycle Condition.

Proposition 2 *Let $n \geq 3$ and $m \geq 3$. There exists a unique Weak Condorcet Winner $x \in X$ at R if and only if the Upmost Condition and the Cycle Condition hold with $x = x_{uc} = x_{cc}$.*

Proof. To prove necessity, suppose $x \in X$ is the unique Weak Condorcet Winner at R. By Proposition 1, the Upmost Condition holds. Moreover, as x is the unique Weak Condorcet Winner, $x = x_{uc}$ and looking at any alternative $y \in X \backslash \{x\}$, we can distinguish two cases:

Case i) $y \in X \backslash \{x\}$ belongs to a cycle or a semi-cycle without being the leader. By definition, we know that $N(wPy) > N(yPw)$ with an alternative w of the cycle or semi-cycle. As y looses by pairwise comparison within that specific triple, y is beaten inside a cycle.

Case ii) $y \in X \backslash \{x\}$ does not belong to any cycle or it is the leader of a semi-cycle; hence, y is not beaten inside a cycle. As x is the unique Weak Condorcet Winner, the rest of the alternatives $y \in X \backslash \{x\}$ should loose by pairwise majority comparison with at least one alternative in X; in other words, $N(aPy) > N(yPa)$ for some $a \in X$. Using the concerned individuals, we can rewrite it as $n^*(y, a; R) - N(yPa) > N(yPa)$; thus, $N(yPa) < \dfrac{n^*(y, a; R)}{2}$. By definition, if yP_ia, then $y \in U_i(R_i, a)$. We conclude that $N(yPa) \equiv |\{i \in N \mid y \in U_i(R_i, a)\}| < \dfrac{n^*(y, a; R)}{2}$. Hence, y is beaten outside the cycle.

Thus, all $y \in X \backslash \{x\}$ are beaten either inside or outside a cycle. Hence, R satisfies the Cycle Condition with $x = x_{cc}$.

To prove sufficiency suppose that R satisfies the Upmost Condition and the Cycle Condition with $x = x_{uc} = x_{cc}$. By Proposition 1, we know that if R satisfies the Upmost Condition, then the set of Weak Condorcet Winners is non-empty. In particular, there is at least one alternative $x_{uc} \in X$ which is a Weak Condorcet Winner. Hence x_{uc} beats or ties with the rest of the alternatives $y \in X \backslash \{x_{uc}\}$. Since R satisfies the Cycle Condition, there exists an alternative $x_{cc} \in X$, such that for the rest of the alternatives $y \in X \backslash \{x_{cc}\}$ one of the following two cases holds:

Case i) y is beaten inside a cycle. Hence, either (1) y belongs to a cycle $\{y, z, w\} \subseteq X$ so $N(yPz) > N(zPy)$, $N(zPw) > N(wPz)$, and $N(wPy) > N(yPw)$ or (2) y belongs to a semi-cycle $\{y, z, w\} \subseteq X$ and does not lead it; that is $N(wPy) > N(yPw)$, $N(yPz) > N(zPy)$, and $N(zPw) = N(wPz)$. Hence y is beaten in pairwise comparison by some alternative within the triple.

Case ii) y is beaten outside a cycle. Hence, either (1) y belongs to a semi-cycle and leads it or (2) y is not in a cycle or semi-cycle. Then, as y is beaten outside a cycle, there exists an alternative $a \in X \backslash \{y\}$ such that $|\{i \in N \mid y \in U_i(R_i, a)\}| < \dfrac{n^*(y, a; R)}{2}$. We can arrage the terms as $2\,|\{i \in N \mid yP_ia\}| < n^*(y, a; R)$, and $N(yPa) < n^*(y, a; R) - N(yPa)$; that is, $N(aPy) > N(yPa)$ for some $a \in X \backslash \{y\}$. Therefore, y is not a Weak Condorcet Winner, as it has been beaten in pairwise comparison by a.

Thus, if the Cycle Condition holds, none of the alternatives $y \in X \backslash \{x_{cc}\}$ can be a Weak Condorcet Winner at R. Furthermore, as x_{uc} is in the set of Weak Condorcet Winners, it has neither being beaten inside nor outside a triple. Hence $x_{uc} = x_{cc} = x$ is the unique Weak Condorcet Winner at R. ■

Example 3 illustrates a profile R satisfying the Cycle Condition.

Example 3 *Consider* $N = \{1, 2, 3, 4\}$, $X = \{w, x, y, z\}$, *and the profile* R *where*

R_1	R_2	R_3	R_4
y	x, w	z, y	x, z
x	z	x	y, w
w	y	w	
z			

Notice that

$N(zPy) > N(yPz),$
$N(yPw) > N(wPy),$
$N(xPw) > N(wPx),$
$N(xPz) > N(zPx),$ *and*
$N(wPz) = N(zPw) = N(yPx) = N(xPy).$

The possible triples of alternatives are $\{w, x, y\}$, $\{w, x, z\}$, $\{w, y, z\}$, *and* $\{x, y, z\}$. *The Cycle Condition holds with* $x = x_{cc}$ *as* $\{w, y, z\} \subseteq X\backslash\{x\}$ *is a semi-cycle with* z *as the leader. Hence,* w *and* y *are beaten inside the cycle and* z *is beaten outside the cycle as* $|\{i \in N \mid z \in U_i(R_i, x)\}| < \dfrac{n^*(z, x; R)}{2}$. *Futhermore, the Upmost Condition holds with* $x = x_{uc}$ *since* $|\{i \in N \mid y \in U_i(R_i, x)\}| \leq \dfrac{n^*(x, y; R)}{2}$ *for all* $y \in X\backslash\{x\}$. *Hence* x *is the unique Weak Condorcet Winner.*

4 Final Remarks

The majority principle is an attractive property of any social choice election procedure. No matter if an alternative or a set of alternatives must be chosen, it is reasonable to think that those selected alternatives should not be defeated by pairwise majority comparison with any other. Nevertheless, even allowing ties, these Weak Condorcet Winners might not exist. Sen and Pattanaik (1969) gave a sufficient condition which helps to identify profiles where Weak Condorcet Winners exist. In particular, Value Restriction is a very strong condition and has been very important in Social Choice Theory, since domains such as Single-Peaked or Group Separable are derived from this property. The Upmost Condition gives a necessary and sufficient condition to have the existence of Weak Condorcet Winners.

We have given examples where the society has to choose several alternatives. As a corollary of the Proposition 1 and Proposition 2, we can state that given $n \geq 3$ and $m \geq 3$, a profile R satisfying the Upmost Condition but not the Cycle Condition has a set of Weak Condorcet Winners with cardinality strictly bigger than one. In such cases, we can be able to apply voting procedures which selects the set of Weak Condorcet Winners. Borm et. al. (2004) introduced the β and λ social choice correspondences which are consistent with the majority principle. Laffond et. al. (1995) analysed and compared several Condorcet consistent rules as the Copeland Rule, the Slater set, the Banks set, and others.

Results do not change if the preference profile R does not admit indifferences; that is, if R is asymmetric. However, in this scenario, we should always have an even number of agents in order to allow ties by pairwise majority comparison and have Weak Condorcet Winners.

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