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Abstract

In this paper we first develop a theory of almost stochastic dominance for risk-seeking investors to the first three orders. Thereafter, we study the relationship between the preferences of almost stochastic dominance for risk-seekers with that for risk averters.

Keywords: Almost Stochastic Dominance, expected-utility maximization, risk averters, risk seekers.

JEL Classification: C00, D81, G11.

1 Introduction

There are two major types of persons: risk averters and risk seekers. Markowitz (1952) and Tobin (1958) propose the mean-variance (MV) selection rules for risk averters and risk seekers. Stochastic Dominance (SD) is first introduced in mathematics by Mann and Whitney (1947) and Lehmann (1955). Quirk and Saposnik (1962), Hanoch and Levy (1969), and many others develop the theory of SD related to economics and develop the stochastic dominance rules for risk averters. On the other hand, Meyer (1977), Stoyan (1983), Wong (2007), and many others develop the stochastic dominance rules for risk seekers.

The theory of almost stochastic dominance (almost SD) developed by Leshno and Levy (LL, 2002) plays an important role in several fields, particularly in financial research, and has drawn several important applications; see, for example, Levy (2006, 2009), Bali, et al. (2009), and Levy, et al. (2010). Tzeng et al. (2013) show that the almost second-degree almost SD introduced by Leshno and Levy (2002) does not possess the property of expected-utility maximization. They modify the definition of the almost SD to acquire this property. Nonetheless, Guo, et al. (2013a) have constructed some examples to show that the almost SD definition modified by Tzeng et al. (2013) does not possess any hierarchy property while Guo, et al. (2013) establish necessary conditions for Almost Stochastic Dominance criteria of various orders.

2 Definitions, Notations, Motivation, and Background

Random variables, denoted by X and Y, defined on $\Omega = [a, b]$ are considered together with their corresponding distribution functions F and G, their corresponding probability density functions f and g, and means μ_X and μ_Y, respectively. The following notations will be used throughout this paper:

$$H_j^A(x) = \int_a^x H_{j-1}^A(y)\,dy \quad \text{and} \quad H_j^D(x) = \int_x^b H_{j-1}^D(y)\,dy \quad , \tag{2.1}$$

where $h = f$ or g and $H = F$ or G. In addition, we define

$$\begin{aligned} &\big|\big|F_n^A(x) - G_n^A(x)\big|\big| = \int_a^b \big|F_n^A(x) - G_n^A(x)\big|dx \ , \\ &\big|\big|F_n^D(x) - G_n^D(x)\big|\big| = \int_a^b \big|F_n^D(x) - G_n^D(x)\big|dx \ , \\ &S_n^A(F,G) = \{x \in [a,b] : G_n^A(x) < F_n^A(x)\} \ , \quad \text{and} \\ &S_n^D(F,G) = \{x \in [a,b] : F_n^D(x) < G_n^D(x)\} \quad \text{for} \quad n = 1,2,3. \end{aligned} \tag{2.2}$$

We note that the definition of H_i^A can be used to develop the stochastic dominance theory for risk averters (see, for example, Quirk and Saposnik, 1962; Hanoch and Levy, 1969), and thus, we call this type of SD ascending stochastic dominance (ASD) because H_i^A is integrated in ascending order from the leftmost point of downside risk. On the other hand, H_i^D can be used to develop the stochastic dominance theory for risk seekers (see, for example, Hammond, 1974; Li and Wong, 1999), and thus, we call this type of SD descending stochastic dominance (DSD) because H_i^D is integrated in descending order from the rightmost point of upside profit. We first define risk-averse and risk-seeking investors as follows:

Definition 2.1 *For $j = 1,2,3$, U_j^A and U_j^D are sets of utility functions u such that:*

$$\begin{aligned} U_j^A &= \{u : (-1)^i u^{(i)} \leq 0 \ , \ i = 1, \cdots, j\} \ , \\ U_j^D &= \{u : u^{(i)} \geq 0 \ , \ i = 1, \cdots, j\} \ , \end{aligned}$$

where $u^{(i)}$ is the i^{th} derivative of the utility function u.

We call investors the j^{th} order risk averters if their utility functions $u \in U_j^A$ and the j^{th} order risk seekers if their utility functions $u \in U_j^D$. Readers may refer to Menezes, et al. (1980), Post and Levy (2005), Post and Versijp (2007), Fong, et al. (2008), Wong and Ma (2008), and Crainich, et al. (2013) for more properties of the utility functions.

Leshno and Levy (2002) and others develop the almost SD rule. We state the almost SD rule developed by Leshno and Levy (2002) and modified by Tzeng et al. (2012) as follows:[1]

Definition 2.2 *Given two random variables X and Y with F and G as their respective distribution functions, for $0 < \epsilon < 1/2$, X is at least as large as Y in the sense of:*

1. *ϵ-almost FASD or ϵ-AFASD, denoted by $X \succeq_{1A}^{almost(\epsilon)} Y$ if and only if*

$$\int_{S_1^A} \left[F_1^A(x) - G_1^A(x)\right] dx \leq \epsilon \left|\left|F_1^A(x) - G_1^A(x)\right|\right|,$$

2. *ϵ-almost SASD or ϵ-ASASD, denoted by $X \succeq_{2A}^{almost(\epsilon)} Y$ if and only if*

$$\int_{S_2^A} \left[F_2^A(x) - G_2^A(x)\right] dx \leq \epsilon \left|\left|F_2^A(x) - G_2^A(x)\right|\right| \quad and \quad \mu_X \geq \mu_Y \ ,$$

3. *ϵ-almost TASD or ϵ-ATASD, denoted by $X \succeq_{3A}^{almost(\epsilon)} Y$ if and only if*

$$\int_{S_3^A} \left[F_3^A(x) - G_3^A(x)\right] dx \leq \epsilon \left|\left|F_3^A(x) - G_3^A(x)\right|\right| \ and \ \ G_n^A(b) \geq F_n^A(b) \ for \ \ n = 2, 3$$

where $S_n^A(F,G)$ and $\left|\left|F_n^A(x) - G_n^A(x)\right|\right|$ for $n = 1, 2, 3$ are defined in (2.2), ϵ-almost FASD, SASD, and TASD stand for ϵ-almost first-, second-, and third-order ASD, respectively.

In this paper we will develop the theory of almost DSD, the almost SD rule for risk seekers. To do so, we first define the almost SD rule for risk seekers in the following definition:

Definition 2.3 *Given two random variables X and Y with F and G as their respective distribution functions, for $0 < \epsilon < 1/2$, X is almost at least as large as Y and F is almost at least as large as G in the sense of:*

1. *ϵ-almost FDSD or ϵ-AFDSD, denoted by $X \succeq_{1D}^{almost(\epsilon)} Y$ or $F \succeq_{1D}^{almost(\epsilon)} G$, if and only if*

$$\int_{S_1^D} \left[G_1^D(x) - F_1^D(x)\right] dx \leq \epsilon \left|\left|F_1^D(x) - G_1^D(x)\right|\right|,$$

[1] We note that we have modified their notations to distinct them from the notations used for the risk seekers.

2. ϵ-*almost SDSD or* ϵ-*ASDSD, denoted by* $X \succeq_{2D}^{almost(\epsilon)} Y$ *or* $F \succeq_{2D}^{almost(\epsilon)} G$, *if and only if*

$$\int_{S_2^D} \left[G_2^D(x) - F_2^D(x)\right] dx \leq \epsilon \left|\left|F_2^D(x) - G_2^D(x)\right|\right| \quad and \quad \mu_X \geq \mu_Y \ ,$$

3. ϵ-*almost TDSD or* ϵ-*ATDSD, denoted by* $X \succeq_{3D}^{almost(\epsilon)} Y$ *or* $F \succeq_{3D}^{almost(\epsilon)} G$, *if and only if*

$$\int_{S_3^D} \left[G_3^D(x) - F_3^D(x)\right] dx \leq \epsilon \left|\left|F_3^D(x) - G_3^D(x)\right|\right| \ and \ \ G_n^D(a) \leq F_n^D(a) \ for \ \ n = 2, 3$$

where $S_n^D(F, G)$ *and* $\left|\left|F_n^D(x) - G_n^D(x)\right|\right|$ *for* $n = 1, 2, 3$ *are defined in (2.2),* ϵ-*almost FDSD, SDSD, and TDSD stand for almost first-, second-, and third-order DSD, respectively.*

In addition, we specify different types of utility functions as shown in the following definition:

Definition 2.4 *For* $n = 1, 2,$ *and 3, we define*

$$\begin{aligned} U_n^{A*}(\epsilon) &= \left\{u \in U_n^A : (-1)^{n+1} u^{(n)}(x) \leq \inf\{(-1)^{n+1} u^{(n)}(x)\}[1/\epsilon - 1] \ \forall x\right\} \ , \\ U_n^{D*}(\epsilon) &= \left\{u \in U_n^D : u^{(n)}(x) \leq \inf\{u^{(n)}(x)\}[1/\epsilon - 1] \ \forall x\right\} \ . \end{aligned}$$

We call investors the j^{th} order ϵ-risk averters if their utility functions $u \in U_n^{A*}(\epsilon)$ and the j^{th} order ϵ-risk seekers if their utility functions $u \in U_n^{D*}(\epsilon)$.

3 The Theory

Tzeng et al. (2012) modify the almost SD rule developed by Leshno and Levy (2002) so that the almost SD rule for risk averters possesses the property of expected-utility maximization. In this paper we will show that the almost SD rule for risk seekers also possesses the property of expected-utility maximization. Here, we state both results in the following theorem:

Theorem 3.1 *For* $n = 1, 2,$ *and 3,*[2]

[2] We note that one could easily extend our work to $n > 3$. However, though some studies, see, for example, Eeckhoudt and Schlesinger (2006), Eeckhoudt, et al. (2009), and Denuit and Eeckhoudt (2010), study risk to $n > 3$, most academics and practitioners are only interested in studying the case up to $n = 3$. Thus, we stop at $n = 3$.

1. *$X \succeq_{nA}^{almost(\epsilon)} Y$ if and only if $E[u(X)] \geq E[u(Y)]$ for any $u \in U_n^{A*}(\epsilon)$, and*

2. *$X \succeq_{nD}^{almost(\epsilon)} Y$ if and only if $E[u(X)] \geq E[u(Y)]$ for any $u \in U_n^{D*}(\epsilon)$.*

Now, we turn to examine whether there is any relationship between the almost ASD rule and almost DSD rule. We first show in the following theorem that almost ASD and DSD could be a dual problem:

Theorem 3.2 *For any random variables X and Y and for n =1,2 and 3,*

$$X \succeq_{nA}^{almost(\epsilon)} Y \quad \text{if and only if} \quad -Y \succeq_{nD}^{almost(\epsilon)} -X \ .$$

We turn to show that sometimes the preference of assets by using almost ASD could be in the same direction as that by using almost DSD but sometimes they are in the opposite direction. We first show in the following theorem for the first order that they are in the same direction:

Theorem 3.3

For any random variables X and Y,

$$X \succeq_{1A}^{almost(\epsilon)} Y \quad \text{if and only if} \quad X \succeq_{1D}^{almost(\epsilon)} Y \ .$$

Levy and Levy (2002) show that if prospects X and Y have the same finite mean, then sometimes the preference for risk averters and risk seekers could be opposite. Could this property hold for almost SD? We show that this is true as shown in the following theorem:

Theorem 3.4 *If $\mu_X = \mu_Y$, then*

$$X \succeq_{2A}^{almost(\epsilon)} Y \quad \text{if and only if} \quad Y \succeq_{2D}^{almost(\epsilon)} X \ .$$

Chan, et al. (2012) show that it is possible to have non-trivial third order ASD and DSD between prospects X and Y such that their preferences are the same. Is it possible for the almost SD to have a similar property? In this paper we show that this is possible by showing the following theorem:

Theorem 3.5 *If* $\mu_X = \mu_Y$ *and* $F_3^A(b) = G_3^A(b)$, *then* $X \succeq_{3A}^{almost(\epsilon)} Y$ *if and only if* $X \succeq_{3D}^{almost(\epsilon)} Y$.

4 Concluding Remarks

In this paper we first develop a theory of almost stochastic dominance for risk-seeking investors to the first three orders. Thereafter, we study the relationship between the preferences of almost stochastic dominance for risk-seekers with that for risk averters.

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