Topological spaces for which every closed and semi-closed preorder respectively admits a continuous multi-utility representation

Bosi, Gianni and Herden, Gerhard

University of Trieste, Italy, University of Essen, Germany

4 February 2014

TOPOLOGICAL SPACES FOR WHICH EVERY CLOSED AND SEMI-CLOSED PREORDER RESPECTIVELY ADMITS A CONTINUOUS MULTI-UTILITY REPRESENTATION

Gianni Bosi [a], Gerhard Herden [b,*]

[a] *Dipartimento di Scienze Economiche, Aziendali, Matematiche e Statistiche, Università di Trieste, Piazzale Europa 1, 34127, Trieste, Italy*

[b] *Fakultät für Mathematik, Mathematikcarrée, Universität Duisburg-Essen, Campus Essen, Thea-Leymann-Straße 9, D-45127 Essen, Germany*

Abstract

On basis of the meanwhile classical continuous multi-utility representation theorem of Levin on locally compact and σ-compact Hausdorff-spaces the question of characterizing all topological spaces (X, t) for which every closed and semi-closed preorder respectively admits a continuous multi-utility representation will be discussed. In this way we are able to provide the fundaments of a purely topological theory that systematically combines topological and order theoretic aspects of the continuous multi-utility representation problem.

Key words: Normal preorder, strongly normal preorder, paracompact space, Lindelöf space, metrizable space

Mathematics subject classification (2010): 54F05, 91B16, 06A05

Proposed running head: Representability of preorders

1 Introduction

A well-known and in some sense best approach (cf. Evren and Ok [5]) of representing a preorder $\precsim$ on a topological space (X, t) is to find a family $\mathcal{F}$

* Corresponding author. Tel.: +49-201-183-2516, Fax: +49-201-183-2426. E-mail address: gerhard.herden@uni-due.de

of increasing continuous real functions f on (X,t) such that

"$x \precsim y \Leftrightarrow f(x) \leq f(y)$" for all $f \in \mathcal{F}$.

Such a representation is called a *continuous multi-utility representation* of $\precsim$.

It has the advantage of fully characterizing the preorder $\precsim$.
A total (complete) preorder on a topological space (X,t) is said to be *continuous* if $\precsim$ satisfies one of the following equivalent conditions.

C1: For every point $x \in X$ both sets $d(x) := \{y \in X \mid y \precsim x\}$ and $i(x) := \{z \in X \mid x \precsim z\}$ are closed subsets of X.

C2: $\precsim$ is a closed subset of $X \times X$ with respect to the product topology $t \times t$ on $X \times X$ that is induced by t.

Of course, the implication "**C2** $\Rightarrow$ **C1**" also holds if $\precsim$ is allowed to be incomplete. In this case, however, the converse implication, in general, fails to be true. This is well-known. Nevertheless, in Theorem 3.2 we shall discuss the problem up to which degree condition **C1** is weaker than condition **C2**.
In the arbitrary case, a preorder $\precsim$ on (X,t) therefore, is said to be *semi-closed* if $\precsim$ satisfies condition **C1**. If $\precsim$ satisfies condition **C2** then $\precsim$ is said to be *closed*.
It is beyond any doubt that closed preorders are of particular importance in mathematical economics (cf., for instance, the literature that has been quoted by Evren and Ok [5], Bosi and Herden [1], Minguzzi [14,15] and many others). Indeed, in some standard textbooks on microeconomics (such as Mas-Colell, Whinston and Green [13, page 46]) the definitions of continuity of an (incomplete) preference relation and of a closed preference relation coincide. In addition closed preorders are of particular interest in the fundamental work of Nachbin on topology and order (cf. [16, Chapter 1]). Because of this observation the most fundamental question in the theory of continuous multi-utility representations of preorders is the question of precisely characterizing (determining) all topological spaces (X,t) that have the property that all their closed preorders respectively admit a continuous multi-utility representation.
In contrast to the importance of closed preorders Theorem 3.2 implies that in mathematical economics the concept of an incomplete semi-closed preorder hardly can be justified.
Although meanwhile many papers on the continuous multi-utility representation of (closed) preorders have been published (cf., for instance, the literature that has been cited in the more recent papers by Bosi and Herden [1], Bosi and Zuanon [2], Evren [4], Evren and Ok [5], Galaabaatar and Karni [6], Minguzzi [14,15] and Pivato [17]) since the pioneering work of Levin [11] (cf. also Evren and Ok [5, Theorem 1]) with respect to the above mentioned fundamen-

tal problem in the theory of continuous multi-utility representations no real progress has been made. Levin's fundamental theorem states that *every closed preorder on a locally and σ-compact Hausdorff space has a continuous multi-utility representation*. Indeed, in combination with its corollaries on separable metric spaces, compact spaces and Euclidean spaces (cf., for instance, Evren and Ok [5, Corollary 1, Corollary 2 and Corollary 3]) this theorem still belongs to the most quoted theorems in Mathematical Utility Theory, in particular, in the theory of appropriate utility representations of incomplete preference relations.

All well-known continuous multi-utility representation theorems only present as well as Levin's theorem *sufficient* conditions for the existence of continuous multi-utility representations. In opinion of the authors this is the great lack of these theorems. This lack pertains to formal mathematics as well as to applications in mathematical economics. Indeed, in order to completely characterizing (determining) topological spaces (X, t) having the property that all their semi-closed and closed preorders respectively admit a continuous multi-utility representation necessary and sufficient conditions have to be presented. In mathematical economics, on the other hand, necessary conditions allow the selection of appropriate topologies. Indeed, necessary conditions imply at least particular difficulties in representing a preorder by a family of continuous increasing real functions. Actually, these conditions even often imply the impossibility of a continuous multi-utility representation of a preorder.

The difficulties of presenting necessary conditions for the existence of a family of continuous increasing real functions that represent a preorder is based upon the fact that corresponding proofs must be constructive. Indeed, proving the necessity of a given condition one, in general, is forced to verify that negating the validity of this condition allows the construction of preorders that do not have a continuous multi-utility representation. Corresponding proofs, therefore, need the intuitive idea of possible conditions that may be necessary for representing a preorder by a family of continuous real functions as well as the ability of constructing preorders that do not have a continuous multi-utility representation if these conditions are not satisfied. Because of these difficulties our approach of approximating the problem of completely characterizing topological spaces (X, t) for which every closed, respectively semi-closed preorder admits a continuous multi-utility representation, therefore, is conservative. This means that Levin's original theorem stands in focus of our approach. In a first attempt we, therefore, want to clarify up to which degree the assumptions of locally and σ-compactness are also necessary for ensuring the existence of continuous multi-utility representations for closed preorders. Indeed, setting $S := \{x \in X \mid \{x\} \in t\}$ and concentrating on metrizable spaces in this way in the third section of this paper, among other results, the following three results will be proved and widely generalized (cf. Theorem 3.5, Theorem 3.7, Theorem 3.8 and corresponding corollaries).

1. *Let* (X,t) *be a metrizable space. Then the following assertions hold:*
(i) $(X \setminus S, t_{|X\setminus S})$ *is compact. In this case every closed preorder* $\precsim$ *on* (X,t) *has a continuous multi-utility representation.*
(ii) $(X \setminus S, t_{|X\setminus S})$ *is not compact. Then in order that every closed preorder* $\precsim$ *on* (X,t) *has a continuous multi-utility representation it is necessary that* (X,t) *is the direct sum of locally compact second countable metric spaces.*

2. *Let* (X,t) *be a second countable space. Then in order that every closed preorder* $\precsim$ *on* (X,t) *has a continuous multi-utility representation it is necessary and sufficient that* $(X \setminus S, t_{|X\setminus S})$ *is a compact and* (X,t) *a second countable metrizable space or that* (X,t) *is a second countable locally compact metrizable space.*

3. *Let* (X,t) *be a connected metrizable space. Then in order that every closed preorder* $\precsim$ *on* (X,t) *has a continuous multi-utility representation it is necessary and sufficient that* (X,t) *is locally compact and second countable.*

In addition, in Theorem 3.2 necessary (and sufficient) conditions for topological spaces that have the property that every semi-closed preorder is closed or admits a continuous multi-utility representation will be presented. Theorem 3.2 (cf. Remark 3.1) states, in particular, that for a first countable space (X,t) the assertions (X,t) to have the property that every semi-closed preorder is closed, (X,t) to have the property that every semi-closed preorder admits a continuous multi-utility representation and (X,t) to contain at most one point $x \in X$ such that $\{x\}$ is not an open subset of X are equivalent. In this paper topological spaces (X,t) for which there exists at most one point $x \in X$ such that $\{x\}$ is not an open subset of X are said to be *almost discrete* . In addition, the negation of the existence of weakly inaccessible cardinal numbers allows us to drop the assumption (X,t) to be first countable in order to nevertheless prove a corresponding very general restrictive result (cf. assertion (ii) of Theorem 3.2). Therefore, it seems that in mathematical economics the concept of a semi-closed incomplete preorder only is of little use.

2 Notation and preliminaries

As usual t_{nat} denotes the natural topology on the real line. $\mid M \mid$ is for every set M the cardinality of M.
Let, henceforth, $\precsim$ be a preorder, i.e. a reflexive and transitive binary relation on some fixed given set X. Then for every point $x \in X$ besides the sets $d(x)$ and $i(x)$ also the sets $l(x) := \{y \in X \mid y \prec x\}$ and $r(x) := \{z \in X \mid x \prec z\}$ will be considered.
A subset D of X is said to be *decreasing* if $d(x) \subset D$ for all $x \in D$. By duality the concept of an *increasing* subset I of X is defined.
In addition, for every subset T of X we set $d(T) := \{y \in X \mid \exists x \in T\,(y \precsim x)\}$ and $i(T) := \{z \in X \mid \exists x \in T\,(x \precsim z)\}$, i.e. $d(T)$ is the smallest decreasing and $i(T)$ the smallest increasing subset of X that contains T.
$\Delta_X = \{(x, x) \mid x \in X\}$ is the *diagonal* of X.
Let t be a topology on X. As usual we denote for every subset S of X by $\overline{S}$ its topological closure. For every subset T of X we denote, furthermore, by $D(T)$ the smallest closed decreasing subset of X that contains T. Analogously, we denote by $I(T)$ the smallest closed increasing subset of X that contains T.
In addition, the preorder $\precsim$ on (X, t) is said to be

(i) *d-i-closed* if for every closed subset A of X both sets $d(A)$ and $i(A)$ are closed subsets of X. (The reader may recall that (X, t) is a *Fréchet-space* if and only if $\{x\}$ is a closed subset of X for every point $x \in X$. Therefore, in a Fréchet-space (X, t) every *d-i*-closed preorder is semi-closed.).

(ii) *D-I-closed* if for any two closed subsets A and B of X such that *not*$(y \precsim x)$ for all $x \in A$ and all $y \in B$ the sets $D(A)$ and $I(B)$ are disjoint. (Since the relation *not*$(y \precsim x)$ for all $x \in A$ and all $y \in B$ means that $d(A) \cap i(B) = \emptyset$ it follows that a *d-i*-closed preorder $\precsim$ on (X, t) also is *D-I*-closed. If (X, t) is a regular space then also the converse holds (cf. Proposition 3.1)).

(iii) *normal* if for any two disjoint closed decreasing, respectively increasing subsets A and B of X there exist disjoint open decreasing, respectively increasing subsets U and V of X such that $A \subset U$ and $B \subset V$.

(iv) *strongly normal* if for any two closed subsets A and B of X such that *not* $(y \precsim x)$ for all $x \in A$ and all $y \in B$ there exist disjoint open decreasing, respectively increasing subsets U and V of X such that $A \subset U$ and $B \subset V$.

Finally, $\precsim$ is said to satisfy *cmp* (*continuous multi-utility representation property*) if a family $\mathcal{F}$ of continuous real functions f on (X, t) can be chosen in such a way that $x \precsim y$ if and only if $f(x) \leq f(y)$ for every $f \in \mathcal{F}$.

3 The results

Throughout this section (X, t) is an arbitrarily chosen topological space. Considering the equivalence relation $x \sim y \Leftrightarrow \overline{\{x\}} = \overline{\{y\}}$ and instead of (X, t) the quotient space $(X_{|\sim}, t_{|\sim})$ in the remainder of this paper we may assume without loss of generality that (X, t) is a Fréchet space. This means, in particular, that the identity relation " $=$ " is a closed preorder on (X, t). Indeed, the reader will not have any difficulties in proving that a preorder $\precsim$ on (X, t) is d-i-closed (D-I-closed, semi-closed, closed) if and only if the induced preorder $\precsim_{|\sim}$ that is defined on $(X_{|\sim}, t_{|\sim})$ is d-i-closed (D-I-closed, semi-closed, closed). Now we want to prove the validity of the following results that at least approximate our aims that have been presented in the introduction. We start by proving the following three lemmas.

Lemma 3.1 *Let (X, t) be a normal space. Then every d-i-closed as well as every D-I-closed preorder $\precsim$ on (X, t) is strongly normal.*

Proof. Since $d(C) \subset D(C)$ and $i(C) \subset I(C)$ for every closed subset C of X it suffices to verify that every D-I-closed preorder $\precsim$ on (X, t) is strongly normal. Let, therefore, A and B be two closed subsets of X such that *not* $(y \precsim x)$ for all $x \in A$ and all $y \in B$. Then the assumption $D(A)$ and $I(B)$ to be disjoint subsets of X implies with help of the normality of (X, t) that there exists some open subset O of X such that the inclusions $D(A) \subset O \subset \overline{O} \subset X \setminus I(B)$ hold. Since $X \setminus I(B)$ is decreasing it, thus, follows that $d(\overline{O}) \subset X \setminus I(B)$. This inclusion allows us to conclude that *not* $(y \precsim x)$ for all $x \in \overline{O}$ and all $y \in i(B)$. Hence our assumption $\precsim$ to be D-I-closed implies that $D(\overline{O}) \cap I(B) = \emptyset$. Therefore, we set $V := X \setminus D(\overline{O})$ in order to conclude that V is an open decreasing subset of X that contains $I(B)$. We proceed by considering the inclusions $V = X \setminus D(\overline{O}) \subset X \setminus \overline{O} \subset X \setminus O \subset X \setminus D(A)$. These inclusions imply, in particular, that $\overline{V} \subset X \setminus O \subset X \setminus D(A)$. Since $X \setminus D(A)$ is increasing, we, thus, may conclude that $D(A) \cap i(\overline{V}) = \emptyset$, which means that *not* $(y \precsim x)$ for all $x \in D(A)$ and all $y \in \overline{V}$. In the same way as above it, therefore, follows that $D(A) \cap I(\overline{V}) = \emptyset$. Hence, we set $U := X \setminus I(\overline{V})$. Then U is an open decreasing subset of X that contains $D(A)$. In addition, the inclusion $V \subset I(\overline{V})$ implies that $U \cap V = \emptyset$. Since $A \subset D(A)$ and $B \subset I(B)$ this equality completes the proof of the lemma. $\square$

The next lemma already has been proved in Herden [9, Theorem 2.2].

Lemma 3.2 *Let $(X, \preceq, t)$ be a preordered topological space. Then the following assertions are equivalent:*

(i) $\precsim$ *is strongly normal.*

(ii) *For any two closed subsets A and B of X such that not* $(y \precsim x)$ *for all* $x \in A$ *and all* $y \in B$ *there exists a continuous and increasing function* $f_{AB} : (X, \precsim, t) \longrightarrow ([0,1], \leq, t_{nat})$ *such that* $f_{AB}(A) = \{0\}$ *and* $f_{AB}(B) = \{1\}$.

(iii) *For every closed subset C of X and every bounded, continuous and increasing function* $f_C : (C, \precsim_{|C}, t_{|C}) \longrightarrow (\mathbb{R}, \leq, t_{nat})$ *there exists a bounded continuous and increasing function* $f : (X, \precsim, t) \longrightarrow (\mathbb{R}, \leq, t_{nat})$ *such that* $f_{|C} = f_C$.

Finally, the following lemma is at least implicitly well known. Moreover, its proof is straightforward. Therefore, the proof is omitted for the sake of brevity.

Lemma 3.3 *Let $(X, \precsim, t)$ be a preordered topological space. Then the following assertions are equivalent:*

(i) $\precsim$ *satisfies cmp.*

(ii) *For any two points $x \in X$ and $y \in X$ such that not* $(y \precsim x)$ *there exists a continuous and increasing real function f_{xy} on (X, t) such that* $f_{xy}(x) < f_{xy}(y)$.

With help of these three lemmas we now are fully prepared for proving the next theorem (cf. Theorem 4.3 and Theorem 4.5 in Bosi and Herden [1]).

Theorem 3.1 *The following assertions are equivalent:*

(i) *Every d-i-closed preorder $\precsim$ on (X, t) satisfies cmp.*
(ii) *Every D-I-closed preorder $\precsim$ on (X, t) satisfies cmp.*
(iii) *(X, t) is a normal space.*

Proof: (ii) $\Rightarrow$ (i). With help of the corresponding definitions it follows that every *d-i*-closed preorder $\precsim$ on (X, t) is *D-I*-closed (cf. the corresponding part in the proof of Lemma 3.1). Hence, nothing has to be shown.

(i) $\Rightarrow$ (iii): Let A and B be two disjoint closed subsets of X. Then we consider the preorder $\precsim$ on (X, t) that is defined by setting

$$\precsim := \Delta_X \cup A \times A \cup B \times B.$$

For every closed subset C of X we then may conclude that

$$d(C) = i(C) = \begin{cases} A \cup B \cup C & \text{if } A \cap C \neq \emptyset \text{ and } B \cap C \neq \emptyset \\ A \cup C & \text{if } A \cap C \neq \emptyset \text{ and } B \cap C = \emptyset \\ B \cup C & \text{if } A \cap C = \emptyset \text{ and } B \cap C \neq \emptyset \\ C & \text{if } A \cap C = \emptyset \text{ and } B \cap C = \emptyset \end{cases}.$$

Hence, $\precsim$ is a *d-i*-closed preorder on (X,t). Let, therefore, $x \in A$ and $y \in B$ be arbitrarily chosen. Then the relation *not*$(y \precsim x)$ is satisfied. Lemma 3.3, thus, implies the existence of some continuous and increasing real function f_{xy} on (X,t) such that $f_{xy}(x) < f_{xy}(y)$. Since $x \sim u$ for every $u \in A$ and $y \sim v$ for every $v \in B$ it follows that (X,t) is a normal space.
(iii) $\Rightarrow$ (ii): Let $\precsim$ be some *D-I*-closed preorder on (X,t). Then Lemma 3.1 implies that $\precsim$ is strongly normal. Hence, we may apply assertion (ii) of Lemma 3.2 in order to conclude with help of assertion (ii) of Lemma 3.3 that $\precsim$ satisfies cmp. □

We now come to the already announced

Proposition 3.1 *Let (X,t) be a regular space. Then every D-I-closed preorder $\precsim$ on (X,t) is d-i-closed.*

Proof: Let A be a closed subset of X. Then we must show that $d(A)$ as well as $i(A)$ are closed subsets of X. Of course, it suffices to verify that $d(A)$ is closed. A dual argument then also applies for $i(A)$. Let, therefore, some point $x \in X \setminus d(A)$ be arbitrarily chosen. Then *not*$(x \precsim y)$ for all points $y \in A$. Hence, $D(A) \cap I(\{x\}) = \emptyset$. The regularity of (X,t) now implies the existence of disjoint open subsets U and V of X such that $D(A) \subset U$ and $x \in V$. Since $d(A) \subset D(A)$ we are done. □

The following proposition completes Theorem 3.1.

Proposition 3.2 *The following assertions are equivalent:*

(i) *Every normal preorder $\precsim$ on (X,t) is strongly normal.*
(ii) *Every normal preorder $\precsim$ on (X,t) is D-I-closed.*

Proof:
(i) $\Rightarrow$ (ii): Let A and B be two closed subsets of X such that *not* $(y \precsim x)$ for all $x \in A$ and all $y \in B$. Then assertion (i) allows us to apply assertion (ii)

of Lemma 3.2. This means, in particular, that the sets $D(A)$ and $I(B)$ are disjoint.
(ii) $\Rightarrow$ (i): Let $\precsim$ be normal and let A and B be closed subsets of X such that *not* $(y \precsim x)$ for all $x \in A$ and all $y \in B$. Then assertion (ii) implies that $D(A)$ and $I(B)$ are disjoint closed decreasing, respectively increasing subsets of X. Since $\precsim$ is normal it, thus, follows that there exist disjoint open decreasing, respectively increasing subsets U and V of X such that $D(A) \subset U$ and $I(B) \subset V$. The inclusions $A \subset D(A)$ and $B \subset I(B)$ now imply the validity of assertion (i). $\square$

In order to now discuss the problem of characterizing all Hausdorff spaces (X, t) having the property that every semi-closed preorder $\precsim$ on (X, t) is closed and the problem of characterizing all Hausdorff spaces (X, t) having the property that every semi-closed preorder $\precsim$ on (X, t) satisfies cmp let, for the moment, (X, t) be a fixed given Hausdorff space. Then the reader may recall from the introduction that $S := \{x \in X \mid \{x\} \in t\}$. In addition, the following notation will be used.

N1: By T we abbreviate the set of all points $x \in X$ that have the property that every neighborhood of x contains some point $y \in X \setminus S$.

N2: For every point $x \in X$ we set

$$c(x) := \begin{cases} 0 & \text{if } x \in S \\ \min\{\mid Z \mid\mid Z \subset X \setminus \{x\}\} \text{ and } x \in \overline{Z}\} & \text{if } x \in X \setminus S \end{cases}.$$

For every point $x \in X \setminus S$ it follows that $c(x) \geq \aleph_0$ and that $c(x)$ is a regular cardinal number. Let us abbreviate this observation by (*).

N3: By $ZFC + \neg WI$ we abbreviate the extension of Zermelo-Fraenkel set theory + Axiom of Choice that negates the existence of weakly inaccessible cardinal numbers, i.e. the existence of *uncountable* regular cardinal numbers κ having the property that for every cardinal number γ that is strictly smaller than κ also its successor is strictly smaller than κ. It is well known that the consistency of ZFC implies the consistency of $ZFC + \neg WI$.

With help of this notation we are fully prepared for stating the following theorem.

Theorem 3.2 *The following assertions hold in ZFC and $ZFC + \neg WI$ respectively:*
(i) *In ZFC it can be proved that in order that every semi-closed preorder $\precsim$ on (X, t) is closed it is necessary and sufficient that $c(x) \neq c(y)$ for any two*

different points $x \in X \setminus S$ and $y \in X \setminus S$.

(ii) *In $ZFC + \neg WI$ the following assertions are equivalent:*

SC: *Every semi-closed preorder $\precsim$ on (X, t) is closed.*

SM: *Every semi-closed preorder $\precsim$ on (X, t) satisfies cmp.*

ST: *(X, t) satisfies the following conditions:*

ST1: *$c(x) \neq c(y)$ for any two different points $x \in X \setminus \{x\}$ and $y \in X \setminus \{x\}$.*

ST2: *$X = \overline{S}$.*

ST3: *$t_{|X \setminus S}$ is the discrete topology on $X \setminus S$.*

Proof: (i): *Necessity:* Let us assume, in contrast, that there exist at least two different points $x \in X \setminus S$ and $y \in X \setminus S$ such that $c(x) = c(y)$. Then we choose subsets Z of $X \setminus \{x\}$ and Z' of $X \setminus \{y\}$ in such a way that $| Z |= c(x) = c(y) =| Z' |$ and $x \in \overline{Z}$ and $y \in \overline{Z'}$. Since (X, t) is assumed to be a Hausdorff space we may assume without loss of generality that $Z \cap Z' = \emptyset$. Hence, we may consider some bijective function $\phi : Z \longrightarrow Z'$ in order to choose the (pre)order $\preceq$ on X that is defined by setting

$$\preceq := \Delta_X \cup \{(z, \phi(z))) \mid z \in Z\}.$$

Since for every $z \in Z$ the singletons $\{z\}$ and $\{\phi(z)\}$ are closed subsets of X it follows that $\preceq$ is a semi-closed (pre)order on (X, t). The assumption of assertion (i), thus, implies that $\preceq$ is a closed (pre)order on (X, t). Since (X, t) is a Hausdorff space it, thus, follows that $(x, y) \in \precsim$. But since $(x, y) \notin \precsim$ this conclusion is incompatible with the definition of $\preceq$. This contradiction proves the necessity part of assertion (i).

Sufficiency: Let us assume, in contrast, that there exists some semi-closed preorder $\precsim$ on (X, t) that is not closed. Then there exists some cardinal number κ and a set $\{(x_\alpha, y_\alpha) \mid \alpha < \kappa\}$ of pairs $(x_\alpha, y_\alpha) \in \precsim$ that is not closed with respect to the product topology $t \times t$ on $X \times X$. This means that we may assume without loss of generality that there exists some pair $(x, y) \in X \times (X \setminus \{x\})$ such that $(x, y) \in \overline{\{(x_\alpha, y_\alpha) \mid \alpha < \kappa\}}$ but $(x, y) \notin \precsim$. Of course, we may assume, in addition, that there exists no cardinal number $\lambda < \kappa$ such that $(x, y) \in \overline{\{(x_\alpha, y_\alpha) \mid \alpha < \lambda\}}$. It, thus, follows that $c(x) = c(y) = \kappa$. Since $x \neq y$ this conclusion contradicts the assumption that $c(x) \neq c(y)$ for any two different points $x \in X \setminus S$ and $y \in X \setminus S$ and, therefore, finishes the proof of the sufficiency part of assertion (i).

(ii): We now assume the validity of $ZFC + \neg WI$. Since a preorder $\precsim$ on (X, t) that admits a continuous multi-utility representation is closed the proof of the validity of the implication "**SM** $\Rightarrow$ **SC**" does not need any additional reflection. We, thus, only have to prove that the implications "**SC** $\Rightarrow$ **ST**" and "**ST** $\Rightarrow$ **SM**" hold. In order to show the validity of the implication "**SC** $\Rightarrow$ **ST**" let every semi-closed preorder $\precsim$ on (X, t) be closed.

Because of assertion (i) it suffices to verify the validity of the conditions **ST2** and **ST3**. The proof of these conditions is divided into two steps.

In the first step we arbitrarily choose some point $x \in D$ in order to then show that $\mid U \mid \geq \min\{c(z) \mid z \in X \text{ and } c(x) < c(z)\}$ for every neighborhood U of x. Because of observation (*) the desired inequality follows for all neighborhoods U of x if we are able to prove that every neighborhood U of x contains at least one point y such that $c(x) < c(y)$. Let us assume, in contrast, that $c(y) \leq c(x)$ for every neighborhood U of x and every point $y \in U \setminus \{x\}$. Then assertion (i) implies that $c(y) < c(x)$ for every neighborhood U of x and every point $y \in U \setminus \{x\}$. Hence, the regularity of $c(x)$ implies with help of assertion (i) that $\sup\limits_{c(y)<c(x)} c(y) = c(x)$. This equation allows us to conclude that for every cardinal number γ that is strictly smaller than $c(x)$ there exists some cardinal number λ that is strictly greater than γ and strictly smaller than $c(x)$. The regularity of $c(x)$, therefore, implies that $c(x)$ is a weakly inaccessible cardinal number which contradicts our assumption that there exist no weakly inaccessible cardinal numbers and, thus, finishes the proof of the first step.
In the second step we, finally, show that D is empty. Then both conditions **ST2** and **ST3** have been proved.. Let us assume, in contrast, that D is not empty. Then there exists some point $x \in D$ such that $c(x) = \min\limits_{w \in T} c(w)$. In addition, the proof of the first step implies the existence of some point $y \in X$ such that $c(y) = \min\{c(v) \mid v \in X \text{ and } c(x) < c(v)\}$. Now the first step allows us to recursively construct a collection $\{Z_\alpha\}_{\alpha < c(y)}$ of pairwise disjoint subsets Z_α of $X \setminus \{x\}$ for which the equation $\mid Z_\alpha \mid = c(x)$ and the inclusion $x \in \overline{Z_\alpha}$ hold. Since the sets Z_α are pairwise disjoint it follows that $\mid \bigcup\limits_{\alpha < c(y)} Z_\alpha \mid = c(y)$. We, thus, set $Z := \bigcup\limits_{\alpha < c(y)} Z_\alpha$ and consider, in addition, some subset Z' of $X \setminus \{y\}$ in such a way that $\mid Z' \mid = c(y)$ and $y \in \overline{Z'}$. As in the proof of assertion (i) we proceed by choosing some bijective map $\phi : Z \longrightarrow Z'$ in order to then considering as in the proof of assertion (i) the semi-closed preorder

$$\preceq := \Delta_X \cup \{(z, \phi(z))) \mid z \in Z\}$$

on (X, t). Now we may conclude as in the proof of assertion (i). This means that the assumption $\precsim$ to be a closed preorder on (X, t) implies that $(x, y) \in \precsim$ in contrast to the definition of $\precsim$ which which finishes the proof of the implication "**SC** $\Rightarrow$ **ST**".

In order to now finally prove the validity of the implication "**ST** $\Rightarrow$ **SM**" we first apply Lemma 3.3 and assertion (i) in order to conclude that it suffices to verify that every closed preorder $\precsim$ on (X, t) is strongly normally preordered. Let, therefore, $\precsim$ be some closed preorder on (X, t). Then we arbitrarily choose closed subsets A and B of X such that $not(y \precsim x)$ for any two points $x \in A$ and $y \in B$. We must show that there exist disjoint open decreasing respectively, increasing subsets U and V of X such that $A \subset U$ and $B \subset V$. U and V will be constructed inductively.

$n = 0$: Indeed, the validity of the conditions **ST2** and **ST3** implies the existence of disjoint open subsets O_0 and P_0 of X such that $A \subset O_0$ and $B \subset P_0$. Unfortunately, it cannot be excluded that there exist points $x \in O_0$ and $y \in P_0$ such that $y \precsim x$. By again applying the condition **ST2** and **ST3** in this case our assumption $\precsim$ to be closed allows us to apply a routine (transfinite) induction procedure in order to step by step eliminate these crucial points in such a way that the finally resulting open subsets O'_0 of O_0 and P'_0 of P_0 have the properties that $A \subset O'_0$ and $B \subset P'_0$ and, furthermore, that $d(\overline{d(\overline{O'_0})}) \cap i(\overline{i(\overline{P'_0})}) = \emptyset$.

$0 \Rightarrow 1$: Now we assume that already open subsets O'_0 and P'_0 of X have been constructed in such a way that $A \subset O'_0$, $B \subset P'_0$ and $d(\overline{d(\overline{O'_0})}) \cap i(\overline{i(\overline{P'_0})}) = \emptyset$. Then the same argument that has been applied for $n = 0$ allows us to construct open subsets O'_1 and P'_1 of X such that $\overline{d(\overline{O'_0})} \subset O'_1$ and $\overline{i(\overline{P'_0})} \subset P'_1$ and $d(\overline{d(\overline{O'_1})}) \cap i(\overline{i(\overline{P'_1})}) = \emptyset$.

$0 < n \Rightarrow n+1$: In this situation we assume that open subsets O'_n and P'_n of X have been constructed in such a way that the inclusions $\overline{d(\overline{O'_{n-1}})} \subset O'_n$ and $\overline{i(\overline{P'_{n-1}})} \subset P'_n$ and the equation $d(\overline{d(\overline{O'_n})}) \cap i(\overline{i(\overline{P'_n})}) = \emptyset$ are satisfied. Then the same argument that has been applied for $n = 0$ allows us to construct open subsets O'_{n+1} and P'_{n+1} of X such that $\overline{d(\overline{O'_n})} \subset O'_{n+1}$ and $\overline{i(\overline{P'_n})} \subset P'_{n+1}$ and $d(\overline{d(\overline{O'_{n+1}})}) \cap i(\overline{i(\overline{P'_{n+1}})}) = \emptyset$.

Continuing in this way we, finally, set $U := \bigcup_{n=0}^{\infty} O'_n$ and $P := \bigcup_{n=0}^{\infty} P'_n$. The inductive construction of the open subsets O'_n and P'_n of X implies that U and V are disjoint open decreasing respectively, increasing subsets of X such that $A \subset U$ and $B \subset V$.This conclusion finishes the proof of the implication "**ST** $\Rightarrow$ **SM**" and, thus, completes the proof of the theorem. □

Remark 3.1 The reader may notice that the proof of the implication "**ST** $\Rightarrow$ **SM**" does not make any use of $\neg WI$. This implication, therefore, also holds in ZFC. In addition, the proof of the implication "**ST** $\Rightarrow$ **SM**" allows us to conclude that for a topological space (X, t) that satisfies the conditions **ST2** and **ST3** every closed preorder on (X, t) is strongly normal. In Proposition 3.5 a somewhat weaker result will be proved for Hausdorff-spaces (X, t) having the property that $(X \setminus S, t_{X \setminus S})$ is compact.

Assertion (i) of Theorem 3.2 implies that a Hausdorff space (X, t) that contains at most one point x such that $\{x\} \in t$ and has the property that every semi-closed preorder $\precsim$ on (X, t) already is closed must be *rigid*. This means that the only one homeomorphism $\phi : (X, t) \longrightarrow (X, t)$ is the identity map on X.

Assertion (ii) of Theorem 3.2 implies that the assertion that there exists no connected Hausdorff space (X, t) for which every semi-closed preorder $\precsim$ on (X, t) already is closed is consistent with ZFC.

Let κ be an infinite regular cardinal number. Then a topological space (X, t) is said to be *κ-countable* if every point $x \in X$ either is contained in S or has a basis of neighborhoods the cardinality of which is κ. If $\kappa = \aleph_0$ then (X, t) is first countable. Now the reader may still recall from the introduction that a topological space (X, t) is said to be *almost discrete* if $\mid X \setminus S \mid \leq 1$. Let (X, t) be a κ-countable Hausdorff space. Then assertion (i) of Theorem 3.2 immediately implies that in ZFC the validity of the equivalence of the following assertions holds, which underlines, in particular, that semi-closed preorders seem to be of very little value in mathematical economics. For the sake of brevity we may omit the trivial details of the corresponding proof.

(i) *Every semi-closed preorder $\precsim$ on (X, t) is closed.*

(ii) *Every semi-closed preorder $\precsim$ on (X, t) satisfies cmp.*

(iii) *t is almost discrete.*

Finally, we want to complete our considerations on semi-closed, closed, *d-i*-closed and *D-I*-closed preorders by proving the validity of the following proposition.

Proposition 3.3 *Let (X, t) contain at least one point that has a countable and infinite basis of neighborhoods. Then the following assertions hold:*

(i) *In order that every closed preorder $\precsim$ on (X, t) is d-i-closed it is necessary that (X, t) is sequentially compact.*
(ii) *In order that every closed preorder $\precsim$ on (X, t) is D-I-closed it is necessary that (X, t) is sequentially compact.*

Proof: We shall prove both assertions of the proposition in one step. Therefore, we assume, in contrast, that there exists some countable infinite subset C of X that does not have a limit point in order to then use our assumption on (X, t) in order to choose a sequence $(x_n)_{n \in \mathbb{N}}$ of points $x_n \in X$ that converges to some point $x \in X$. Let $D := \{x_n \mid n \in \mathbb{N}\}$. Of course, we may assume that $x \notin D$. We, thus, proceed by dividing C into two disjoint infinite subsets A and B and D into two disjoint infinite subsets H and K the union of which is C and D respectively. Now we consider bijective functions $\phi : H \longrightarrow A$ and $\psi : B \longrightarrow K$. With help of these functions we may define a (pre)order $\precsim$ on (X, t) by setting

$$\precsim := \Delta_X \cup \{(h, \phi(h)) \mid h \in H\} \cup \{(b, \psi(b)) \mid b \in B\}.$$

Since both sets A and B are closed subsets of X it follows that $\precsim$, actually, is a closed (pre)order on (X, t). But since x neither is contained in $d(A)$ nor in $i(B)$ we may conclude that $\precsim$ is not d-i-closed. In addition, the relation $x \in \overline{d(A)} \cap \overline{i(B)}$ implies that $\precsim$ is not D-I-closed. These contradictions prove the proposition. $\square$

The following theorem is well known in general topology (cf., for instance, Grotemeyer [7, Satz 93]).

Theorem 3.3 *Let (X, t) be paracompact. Then the following assertions are equivalent:*

(i) (X, t) *is compact..*
(ii) (X, t) *is sequentially compact.*

As an application of Theorem 3.3 we want to apply Proposition 3.3 in order to prove the following proposition.

Proposition 3.4 *Let (X, t) be a paracompact space that contains at least one point that has a countable and infinite basis of neighborhoods. Then the following assertions are equivalent:*

(i) *Every closed preorder $\precsim$ on (X, t) is d-i-closed..*
(ii) *Every closed preorder $\precsim$ on (X, t) is D-I-closed..*
(iii) (X, t) *is a compact Hausdorff space.*

Proof: (i) $\Rightarrow$ (ii): The validity of this implication is trivial (cf. the corresponding remark in the proof of Lemma 3.1).

(ii) $\Rightarrow$ (iii): Since a paracompact space is normal our assumption (X, t) to be a Fréchet space implies that (X, t) is a Hausdorff space. Hence, the validity of the implication "(ii) $\Rightarrow$ (iii)" is a consequence of Proposition 3.3 and Theorem 3.3.

(iii) $\Rightarrow$ (i): The validity of this implication is well known. The reader may consult, for instance, in Nachbin [16] the proof of Proposition 4 in Chapter 3. $\square$

In the remainder of this paper we now solely concentrate on the problem of characterizing all topological spaces (X, t) that have the property that all their closed preorders $\precsim$ satisfy cmp. In order to at least approach this problem the following theorem seems to be of interest (cf. Proposition 3.4).

Theorem 3.4 *The following assertions are equivalent:*

(i) *Every closed preorder $\precsim$ on (X, t) satisfies cmp..*
(ii) *Every closed preorder $\precsim$ on (X, t) is normal.*

Proof: (i) $\Rightarrow$ (ii): Let $\precsim$ a closed preorder on (X, t) and let A and B be two disjoint closed decreasing, respectively increasing subsets of X. Then we construct by transfinite induction a closed preorder $\precsim^e$ on (X, t) that extends $\precsim$ and has the additional properties that both sets $A \times A$ and $B \times B$ are contained in $\precsim^e$ and that *not*$(y \precsim^e x)$ for all $x \in A$ and all $y \in B$. Because of assertion (ii) it then follows that $\precsim^e$ satisfies cmp. This means that there exists a continuous and increasing function $f_{AB} : (X, \precsim, t) \longrightarrow (\mathbb{R}, \leq, t_{nat})$ such that $f_{AB}(x) < f_{AB}(y)$ for all $x \in A$ and all $y \in B$. Since $f_{AB|A}$ as well as $f_{AB|B}$ are constant functions, therefore, the validity of assertion (ii) follows.
α is a limit ordinal. The reader may recall that a limit ordinal is an ordinal number that has no (direct) predecessor. This means, in particular, that "0" may be considered being a limit ordinal. Therefore, we set

$$\precsim_\alpha = \begin{cases} \precsim & \text{if } \alpha = 0 \\ \bigcup\limits_{\beta < \alpha} \precsim_\beta & \text{if } \alpha > 0 \end{cases}.$$

α is not a limit ordinal. In this case there exists a limit ordinal γ and a natural number $n \geq 1$ such that $\alpha = \gamma + n$. Then we distinguish between the following three cases.
Case 1: $\gamma = 0$ and $n = 1$. In this case we set $\precsim_\alpha := \precsim \cup A \times A \cup B \times B$.
Case 2: $\gamma \geq 0$ and $n \geq 3$ is an odd natural number. Now we set

$$\precsim_\alpha := \precsim_{\gamma+n-1} \cup \{(p, q) \in X \times X \mid \text{ there exists some net } ((p_i, q_i)) \text{ of pairs } (p_i, q_i) \in X \times X \text{ such that } (p, q) = \lim_{i \in I} (p_i, q_i)\}.$$

Case 3: $\gamma \geq 0$ and $n \geq 2$ is an even natural number. In this situation we set

$$\precsim_\alpha := \precsim_{\gamma+n-1} \cup \{(a, v) \in A \times X \mid \text{ there exists some } a' \in A \text{ such that } (a', v) \in \precsim_{\gamma+n-1}\} \cup \{(w, b) \in X \times B \mid \text{there exists some } b' \in B \text{ such that } (w, b') \in \precsim_{\gamma+n-1}\}.$$

Let κ be the first ordinal number such that the induction process stops. Then we set $\precsim^e=\precsim_\kappa$. The inductive construction of $\precsim^e=\precsim_\kappa$ implies for all ordinal numbers $1 \leq \alpha \leq \kappa$ that both sets $A \times A$ and $B \times B$ are contained in $\precsim_\alpha$. Since A and B are closed subsets of X it follows, furthermore, that for all ordinal numbers $0 \leq \beta < \alpha \leq \kappa$ the inclusion $\precsim_\alpha \setminus \precsim_\beta \subset A \times X \cup X \times B$ holds. In addition, the transitivity of $\precsim$ allows us to conclude with help of our assumption A to be decreasing and B to be increasing that $\precsim_\alpha$ is transitive for all limit ordinals α and all non-limit ordinals $\alpha = \gamma + n$ for which $n \geq 2$ is an even natural number. Moreover, the construction of $\precsim^e=\precsim_\kappa$ implies for all ordinal numbers $1 \leq \alpha \leq \kappa$ such that $\alpha = \gamma + n$ for some odd natural number n that $\precsim_\alpha$ is a closed subset of $X \times X$. Summarizing these considerations it follows that $\precsim^e=\precsim_\kappa$ is a closed preorder on (X, t) that extends $\precsim$ in such a way that $A \times A$ as well as $B \times B$ is contained in $\precsim^e=\precsim_\kappa$ and $not(y \precsim^e x)$ for all $x \in A$ and all $y \in B$. Our corresponding considerations at the beginning of the proof of the implication "(i) $\Rightarrow$ (ii)", thus, imply the validity of assertion (ii).
(ii) $\Rightarrow$ (i): Let $\precsim$ be a closed preorder on (X, t) and let two points $x \in X$ and $y \in X$ such that $not(y \precsim x)$ be arbitrarily chosen. Since $\precsim$ is closed it follows, in particular, that $\precsim$ is semi-closed. Hence the sets $d(x)$ and $i(y)$ are disjoint closed decreasing, respectively increasing subsets of X. The normality of $\precsim$ implies with help of the well known Separation Theorem of Nachbin [16, Theorem 1] that there exists a continuous and increasing function $f_{xy} : (X, \precsim, t) \longrightarrow ([0, 1], \leq, t_{nat})$ such that $f_{xy}(x) = 0$ and $f_{xy}(y) = 1$. Lemma 3.3, therefore, implies the validity of assertion (i). $\square$

Theorem 3.4 provides a first step towards the complete solution of the problem of characterizing all topological spaces (X, t) for which every closed preorder $\precsim$ on (X, t) satisfies cmp. It proves that this problem is equivalent to the problem of characterizing all topological spaces that have the property that every closed preorder $\precsim$ on (X, t) is normal.
We now come to the main results of this paper. Indeed, the following Theorem 3.5 may be interpreted as being the converse of Levin's theorem. Since a locally and σ-compact Hausdorff-space is paracompact the intimate connection of Theorem 3.5 to Levin's theorem is obvious. Its only lack, therefore, is the additional assumption (X, t) to be first countable. But at present the authors do not see any possibility of how to really avoid this assumption in a satisfactory way.
In order to prove Theorem 3.5 we still must verify the validity of the following proposition.

Proposition 3.5 *Let (X,t) be a Hausdorff-space and let $(X \setminus S, t_{|X\setminus S})$ be compact. Then every closed preorder $\precsim$ on (X,t) is normal.*

Proof: Let $\precsim$ be some closed preorder on (X,t). In order to prove that $(X,\precsim,t)$ is a normally preordered space, let A and B be two disjoint closed decreasing respectively, increasing subsets of X. We must show that there exists disjoint open decreasing respectively, increasing subsets U and V of X such that $A \subset U$ and $B \subset V$. In order to prove the existence of U and V respectively we distinguish between the case that at least one of the sets A and B is a subset of S and the case that neither A nor B is a subset of S. Because of the properties of S the first case does not need any reflection. Therefore we now may concentrate on the situation that neither $A \cap (X \setminus S)$ nor $B \cap (X \setminus S)$ is empty. Since $\precsim$ is a closed preorder on (X,t) and since $(X \setminus S, t_{|X\setminus S})$ is a compact Hausdorff space we may use, in particular, a straightforward modification of the proof of Theorem 4 in Chapter 3 on compact ordered spaces in Nachbin [16] in order to conclude that disjoint open subsets O_0 and P_0 of X that contain A and B respectively can be chosen in such a way that $O_0 \cap (X \setminus S)$ is decreasing and $P_0 \cap (X \setminus S)$ is increasing with respect to $\precsim_{|X\setminus S}$. Now we proceed by completely following the spirit of the implication "**ST** $\Rightarrow$ **SM**" of the proof of Theorem 3.2 (cf., in particular the argument that has been used in case that $n = 0$ in the proof of Theorem 3.2). Using the assumption $\precsim$ to be a closed preorder on (X,t) we, thus, verify with help of our general assumptions on (X,t) that we may assume without loss of generality that $d(\overline{d(O_0)}) \cap i(\overline{i(P_0)}) = \emptyset$. Hence, $\overline{d(O_0)}$ and $\overline{i(P_0)}$ are disjoint closed subsets of X such that $not(y \precsim x)$ for all points $x \in \overline{d(O_0)}$ and all points $y \in \overline{i(P_0)}$. Now one easily verifies that this conclusion allows us to continue the proof of Proposition 3.5 by using $\overline{d(O_0)}$ and $\overline{i(P_0)}$ respectively instead of A and B respectively. This means that we now construct open subsets O_1 and P_1 of X that contain $\overline{d(O_0)}$ and $\overline{i(P_0)}$ respectively such that $d(\overline{d(O_1)}) \cap i(\overline{i(P_1)}) = \emptyset$. Continuing inductively in this way we, thus, obtain for every $n \in \mathbb{N}$ open subsets O_n and P_n of X such that $A \subset O_n \subset \overline{d(O_n)} \subset O_{n+1}$, $B \subset P_n \subset \overline{i(P_n)} \subset P_{n+1}$ and $d(\overline{d(O_{n+1})}) \cap i(\overline{i(P_{n+1})}) = \emptyset$. As in the proof of the implication "**ST** $\Rightarrow$ **SM**" of the proof of Theorem 3.2 we, finally, may conclude that $U := \bigcup_{n=0}^{\infty} O_n$ and $V := \bigcup_{n=0}^{\infty} P_n$ are disjoint open decreasing respectively, increasing open subsets of X that contain A and B respectively. This conclusion completes the proof of the proposition. $\square$

In order to successfully continue we still need the following immediate corollary that is an immediate consequence of Proposition 3.5 and the proof of the implication "(ii) $\Rightarrow$ (i)" of Theorem 3.4.

Corollary 3.1 *Let (X, t) be a Hausdorff-space and let $(X \setminus S, t_{|X \setminus S})$ be compact. Then every closed preorder $\precsim$ on (X, t) satisfies cmp.*

Theorem 3.5 *Let (X, t) be a first countable paracompact space. Then $(X \setminus S, t_{|X \setminus S})$ is compact and every closed preorder $\precsim$ on (X, t) satisfies cmp or $(X \setminus S, t_{|X \setminus S})$ is not compact and the assumption that every closed preorder $\precsim$ on (X, t) satisfies cmp implies that (X, t) is the direct sum of locally and σ-compact Hausdorff spaces.*

Proof: Since a paracompact space is normal our assumption on (X, t) to be a Fréchet space implies that (X, t) is a Hausdorff space (cf. the above proof of Proposition 3.4). The proof of Theorem 3.5, therefore, is based upon Theorem 3.3 and the following theorem that is well known in general topology (cf., for instance, Grotemeyer [7, Satz 97]).

Theorem 3.6 *Let (X, t) be a locally compact topological space. Then the following assertions are equivalent:*

(i) (X, t) *is paracompact..*
(ii) (X, t) *is the direct sum of locally and σ-compact topological spaces.*

Because of Corollary 3.1 we may continue by arbitrarily choosing some in the remainder of the proof fixed given first countable paracompact space (X, t) that satisfies the additional conditions that $(X \setminus S, t_{|X \setminus S})$ is not compact and that every closed preorder $\precsim$ on (X, t) satisfies cmp. Hence Theorem 3.6 implies that it suffices to prove that (X, t) is locally compact. Let us assume in contrast that (X, t) is not locally compact. Then there exists some point $z \in X \setminus S$ that does not have a compact neighbourhood. Since $(X \setminus S, t_{|X \setminus S})$ is not compact we now apply Theorem 3.3 in order to conclude that there exists some sequence $(x_n)_{n \in \mathbb{N}}$ of points $x_n \in X \setminus S$ that does not have a limit point. This means that we may choose in particular some closed neighborhood $C(z)$ of z and some open neighborhood $O(z)$ of z such that the following conditions hold.

LP1: $O(z) \subset C(z)$.

LP2: $x_n \in C(z) \setminus O(z)$ for every $n \in \mathbb{N}$.

Let C be the closed subset of X that consists of all points x_n where n runs through $\mathbb{N}$. Then we proceed by choosing for every $n \in \mathbb{N}$ some sequence $(x_{nk})_{k \in \mathbb{N}}$ of points $x_{nk} \in X$ such that $\lim_{k \to \infty} x_{nk} = x_n$ for all $n \in \mathbb{N}$. Condition **LP2** allows us to assume, in addition, that there exists some point $q \in C(z) \setminus (O(z) \cup C)$ that is different from all points x_n and x_{nk} respectively. Since there exists no closed neighborhood $C'(z)$ of z for which every sequence $(z_n)_{n\in\mathbb{N}}$ of points $z_n \in C'(z)$ has a limit point we now may construct inductively families $\{C_n(z)\}_{n\in\mathbb{N}}$ and $\{O_n(z)\}_{n\in\mathbb{N}}$ of closed neighborhoods of z and open neighbourhoods of z respectively in such a way that the following conditions are satisfied.

L1: $C_0(z) \subset O(z)$.

L2: $O_{n+1}(z) \subset C_{n+1}(z) \subset O_n(z)$ for all $n \in \mathbb{N}$.

L3: For every $n \in \mathbb{N}$ there exists some sequence $(y_{nk})_{k\in\mathbb{N}}$ of points $y_{nk} \in C_n(z) \setminus O_n(z)$ that has no limit point.

L4: $\bigcap\limits_{n\in\mathbb{N}} O_n(z) = \{z\}$.

Now we define a binary relation $\precsim$ on (X, t) by setting

$$\precsim := \Delta_X \cup \{(x_{nk}, y_{nk}) \mid (n,k) \in \mathbb{N} \times \mathbb{N}\} \cup \{(q, x_n) \mid n \in \mathbb{N}\}.$$

Since there do not exist any three points $u \in X$, $v \in X$ and $w \in X$ such that $u \prec v$ and $v \prec w$ the definition of $\precsim$ allows us conclude that $\precsim$ is a (pre)order on X. Since, in addition, none of the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_{nk})_{k\in\mathbb{N}}$ has a limit point and since $(q, x_n) \in \precsim$ for all $n \in \mathbb{N}$ it follows that $\precsim$ is a closed subset of $X \times X$. Hence, $\precsim$ is a closed preorder on (X, t). Furthermore, the definition of $\precsim$ implies that $not(q \precsim z)$. In order to, therefore, finish the theorem it suffices to show that there exists no continuous and increasing function $f_{zq} : (X, \precsim, t) \longrightarrow (\mathbb{R}, \leq, t_{nat})$ such that $f_{zq}(z) < f_{zq}(q)$. Let us assume, in contrast, that there exists some continuous and increasing function $f_{zq} : (X, \precsim, t) \longrightarrow (\mathbb{R}, \leq, t_{nat})$ such that $f_{zq}(z) < f_{zq}(q)$. Then we arbitrarily choose some fixed real number η such that $f_{zq}(z) < \eta < f_{zq}(q)$. The strict relation $x_{nk} \prec y_{nk}$ for all $n \in \mathbb{N}$ and $k \in \mathbb{N}$ implies that $f_{zq}(x_{nk}) \leq f_{zq}(y_{nk})$ for all $n \in \mathbb{N}$ and all $k \in \mathbb{N}$. But since for all $k \in \mathbb{N}$ the sequences $(y_{nk})_{n\in\mathbb{N}}$ uniformly converge to z there exists some $N \in \mathbb{N}$ such that $f_{zq}(y_{Nk}) < \eta$ for all $k \in \mathbb{N}$. Therefore, the equation $\lim\limits_{k\to\infty} x_{Nk} = x_N$ allows us to conclude with help of the definition of $\precsim$ that $f_{zq}(x_N) = \lim\limits_{k\to\infty} f_{zq}(x_{Nk}) \leq \lim\limits_{k\to\infty} f_{zq}(y_{Nk}) \leq \eta < f_{zq}(q)$. Conversely, the inequality $q \prec x_N$ implies that $f_{zq}(q) \leq f_{zq}(x_N)$. Hence, f_{zq} cannot be increasing. This contradiction completes the proof of the second case and, therefore, of the theorem. □

A thorough analysis of the proof of Theorem 3.5 allows us to state the following most general theorem that can be proved with help of the methods that have been developed in this paper.

Theorem 3.7 *Let (X,t) be a Hausdorff space. Then $(X \setminus S, t_{|X \setminus S})$ is compact and every closed preorder $\precsim$ on (X,t) satisfies cmp or $(X \setminus S, t_{|X \setminus S})$ is not compact and the assumptions that (X,t) is first countable and that every closed preorder $\precsim$ on (X,t) satisfies cmp imply that (X,t) is locally sequentially compact.*

Example 3.1 Of course, there exist complete second countable metric spaces (X, t_d) that are not locally compact but have the property that $(X \setminus S, t_{d|X \setminus S})$ is compact. Indeed, let $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ and $X := \{0\} \cup \mathbb{N}^* \times \mathbb{N}^*$ endowed with the metric $d : X \times X \longrightarrow \mathbb{R}^{\geq 0}$ that is defined by setting

$$d(x,y) = d(y,x) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{n} & \text{if } x = 0 \text{ and } y = (n,k) \text{ for some pair} \\ & (n,k) \in \mathbb{N}^* \times \mathbb{N}^* \\ \frac{1}{n} + \frac{1}{m} & \text{if } x = (n,k) \in \mathbb{N}^* \times \mathbb{N}^* \text{ and} \\ & y = (m,t) \in (\mathbb{N}^* \times \mathbb{N}^*) \setminus \{(n,k)\} \end{cases}$$

for all pairs $(x,y) \in X \times X$. Then (X, t_d) has the desired properties.

In order to complete Example 3.1 we mention that $(X,d) := \left(\{\frac{1}{n} \mid n \in N^*\}, |\cdot|\right)$ is a metric space that is not complete but, nevertheless, has the property that all its closed preorders satisfy cmp.
In addition, we still mention that the rationals $\mathbb{Q}$ endowed with its natural metric $d := |\cdot|$ neither have the property that $(\mathbb{Q} \setminus S, t_{d|\mathbb{Q} \setminus S})$ is compact nor the property that $(\mathbb{Q}, t_d)$ is locally compact. Therefore, $(\mathbb{Q}, t_d)$ is a second countable metric space for which not every closed preorder that is definable on $(\mathbb{Q}, t_d)$ satisfies cmp.

Since metrizable spaces are paracompact the first result that has been mentioned in the introduction is an immediate consequence of Theorem 3.5. In addition, with respect to Levin's theorem the following first corollary of Theorem 3.5 is of particular interest.

Corollary 3.2 *Let (X,t) be a first countable Lindelöf space. Then the following assertions are equivalent:*

(i) *Every closed preorder $\precsim$ on (X,t) satisfies cmp.*
(ii) *$(X \setminus S, t_{|X \setminus S})$ is a compact Hausdorff-space and S is a countable set or (X,t) is a locally and σ-compact Hausdorff space.*

Proof: (i) $\Rightarrow$ (ii). Assertion (i) implies with help of Theorem 3.4 that (X, t) is a normal space. Since every regular Lindelöf space is paracompact Proposition 3.4 and Theorem 3.5, therefore, imply with help of the Lindelöf property of (X, t) that $(X \setminus S, t_{|X \setminus S})$ is a a compact Hausdorff-space and S is a countable set or that (X, t) is the direct sum of countable many locally and σ-compact Hausdorff spaces, which means that (X, t) is a locally and σ-compact Hausdorff space.

(ii) $\Rightarrow$ (i). This implication is a consequence of Proposition 3.4 and Levin's theorem respectively. $\square$

Since second countable spaces are Lindelöf and since second countable paracompact Hausdorff spaces are metrizable and since, in addition, compact metrizable spaces are second countable the second result that has been mentioned in the introduction is an immediate consequence of Corollary 3.2. If (X, t) is connected then S is empty. Hence, the third result that has been mentioned in the introduction follows from the following corollary of Theorem 3.5.

Corollary 3.3 *Let (X, t) be a first countable connected paracompact space. Then the following assertions are equivalent:*

(i) *Every closed preorder $\precsim$ on (X, t) satisfies cmp..*
(ii) *(X, t) is a locally and σ-compact Hausdorff space.*

It is well known that a Hausdorff topological vector space $(X, +, t)$ is finite-dimensional if and only if it is locally compact. Hence, the following result that generalizes Example 1 in Evren and Ok [5] also is a corollary of Theorem 3.5.

Corollary 3.4 *Let $(X, || \cdot ||)$ be a normed linear space. Then the following assertions are equivalent:*

(i) *Every closed preorder $\precsim$ on $(X, || \cdot ||)$ satisfies cmp..*
(ii) *X is finite-dimensional.*

Banach spaces $(X, || \cdot ||)$ that are not finite dimensional, therefore, are complete metric spaces that have the property that not every closed preorder that is definable on $(X, || \cdot ||)$ satisfies cmp. The Banach space of all continuous real functions on some compact non-degenerate real interval, thus, is a complete second countable metric space that has the property that not all closed preorders that are definable on this space satisfy cmp (a further example of this type is Example 1 in Evren and Ok [5]).

In the remainder of this section we want to discuss the problem if the necessary condition the validity of which has been proved in Theorem 3.5 also is sufficient in order to guarantee that every closed preorder $\precsim$ on (X,t) satisfies cmp. In order to be more precise, let (X,t) be the direct sum of locally and σ-compact Hausdorff spaces. Then we want to discuss the question if these assumptions imply that every closed preorder $\precsim$ on (X,t) satisfies cmp? Therefore, we somewhat modify Theorem 3.5 by requiring, in addition, (X,t) to be locally connected. Then the following corollary of Theorem 3.5 holds.

Corollary 3.5 *Let (X,t) be a first countable paracompact and locally connected space. Then $(X \setminus S, t_{|X\setminus S})$ is compact and every closed preorder $\precsim$ on (X,t) satisfies cmp or $(X \setminus S, t_{|X\setminus S})$ is not compact and the assumption that every closed preorder $\precsim$ on (X,t) satisfies cmp implies that (X,t) is the direct sum of locally and σ-compact Hausdorff spaces.*

On basis of this corollary we are ready for proving the following proposition that provides a first answer of the question that has been posed above.

Proposition 3.6 *Let (X,t) be the direct sum of connected locally and σ-compact Hausdorff spaces. Then every closed preorder $\precsim$ on (X,t) admits a continuous multi-utility representation.*

Proof: Let $x \in X$ be arbitrarily chosen. Then we denote, for the moment, by $P(x)$ the component of X that contains x. With help of this notation we are able to prove the following lemma that is essential for the proof of the proposition.

Lemma 3.4 *Let $\precsim$ be a closed preorder on (X,t) the indifference classes $[q]$ of which are contained in $P(q)$. Then for any three points $x \in X$, $y \in X$ and $z \in X$ for which the equation $P(x) = P(z)$ and the inequalities $x \prec y \prec z$ hold the equations $P(x) = P(y) = P(z)$ are satisfied.*

Proof: Let us assume, in contrast, that $y \notin P(x) = P(z)$. Then we may conclude with help of the assumptions of Lemma 3.4 that $A := d(y) \cap P(x) = d(y) \cap P(z)$ and $B := i(y) \cap P(x) = i(y) \cap P(z)$ are disjoint closed decreasing respectively, increasing subsets of X. Therefore, we set $P := P(x) \cup [y] = P(z) \cup [y]$ in order to consider the closed preorder $\precsim_{|P}$ on $(P, t_{|P})$. Levin's theorem now implies that $\precsim_{|P}$ has a continuous multi-utility representation. Hence, considering the pairs $(x,y) \in \prec_{|P}$ and $(y,z) \in \prec_{|P}$ we obtain two appropriate increasing functions the sum of which guarantees the existence of some continuous and increasing function $f : (P, \precsim_{|P}, t_{|P}) \longrightarrow (\mathbb{R}, \leq, t_{nat})$ such that $f(u) < f(y) < f(v)$ for all points $u \in P(x) = P(z)$ and $v \in P(x) = P(z)$

such that $u \prec y \prec v$. We proceed by setting $C := \{u \in P(x) = P(z) \mid f(u) \leq f(y)\}$. Then it follows that C and B are disjoint non-empty closed subsets of $P(x) = P(z)$ the union of which is $P(x) = P(z)$. Since $P(x) = P(z)$ is an open subset of X this last conclusion contradicts the connectedness of $P(x) = P(z)$. $\square$

Let $\precsim$ be some fixed given closed preorder on (X, t). In order to now prove Proposition 3.6 we must show that $\precsim$ satisfies cmp. Therefore, we choose in a first step in every indifference class of $\precsim$ some fixed point q in order to then denote the collection of these points by F. Now we replace $\precsim$ by

$$\begin{aligned}\precsim' := \precsim \setminus \{(r, s) \in X \times X \mid \; & \text{there exists some } q \in F \text{ such that } [r] = [s] = [q] \\ & \text{and } r \notin P(q) \text{ or } s \notin P(q)\}.\end{aligned}$$

The definition of $\precsim'$ implies that $\precsim'$ is a closed preorder on (X, t). Furthermore, Lemma 3.3 allows us to conclude that $\precsim$ satisfies cmp if and only if $\precsim'$ satisfies cmp. Hence, it suffices to verify that $\precsim'$ satisfies cmp. Let, therefore, $x \in X$ and $y \in X$ such that $not(y \precsim x)$ be arbitrarily chosen. Then either the equality $P(x) = P(y)$ or the inequality $P(x) \neq P(y)$ is possible. Since the case that $P(x) = P(y)$ is somewhat more complicated than the case that $P(x) \neq P(y)$ and since, in addition, both cases can be settled by analogous arguments, in the remainder of the proof of Proposition 3.6, we merely concentrate on the equation $P(x) = P(y)$. We, thus, set $P := P(x) = P(y)$ in order to then conclude that Levin's theorem guarantees the existence of some continuous increasing function $f : (P, \precsim'_{|P}, t_{|P}) \longrightarrow (\mathbb{R}, \leq, t_{nat})$ such that $f(x) < f(y)$. In order to now finish the proof of the proposition it suffices to show because of Lemma 3.3 that f can be lifted to some continuous and increasing function $h : (X, \precsim', t) \longrightarrow (\mathbb{R}, \leq, t_{nat})$. Let, therefore, some point $z \in X \setminus P$ be arbitrarily chosen. Then we have to distinguish between the following three cases.

Case 1: $z \prec' x$. In this case we set $h(v) := f(x)$ for all $v \in P(z)$.

Case 2: $x \prec' z$. Now we set $h(v) := f(y)$ for all $v \in P(z)$.

Case 3: There exists neither a point $u \in P(z)$ such that $u \prec' x$ nor a point $w \in P(z)$ such that $x \prec' w$. In this situation we set $h(v) := f(x)$ for all $v \in P(z)$.

Setting $h_{|P} = f$ in this way a lifting of f has been defined. In addition, Lemma 3.4 allows us to conclude that $h : (X, \precsim', t) \longrightarrow (\mathbb{R}, \leq, t_{nat})$ is continuous and increasing and satisfies the inequality $h(x) < h(y)$. This conclusion finishes the proof of Proposition 3.6. $\square$

With help of Proposition 3.6 the proof of Theorem 3.5 implies the following theorem which, in addition, at least partly answers the above question and, therefore, provides a good opportunity of finishing this section.

Theorem 3.8 *Let (X,t) be a first countable paracompact and locally connected space. Then the following assertions are equivalent:*

(i) *Every closed preorder $\precsim$ on (X,t) satisfies cmp.*
(ii) *$(X \setminus S, t_{|X\setminus S})$ is compact or (X,t) is the direct sum of connected locally and σ-compact Hausdorff spaces.*

4 Conclusion

Although the problem of characterizing all topological spaces for which every closed preorder admits a continuous multi-utility representation could not completely be solved by the authors the corresponding results of this paper are very restrictive. Indeed, they imply, for example, that spaces like $L^\infty(\mu)$, the space of essentially bounded measurable functions relative to a σ-finite measure μ, always allow the definition of closed preorders that do not have a continuous multi-utility representation. On the other hand topological vector spaces such as $L^\infty(\mu)$ meanwhile often are used as infinite-dimensional commodity spaces in economic theory, so it is desirable to have continuous multi-representation theorems that apply to them. This means that one has to replace the particular linearly ordered abelian group $(\mathbb{R},+,\leq)$ by an arbitrary linearly ordered abelian group $(\mathbb{A},+,\leq)$. Such a more general approach meanwhile has been started by Pivato [17].
In addition, in Herden and Mehta [10] it has been underlined by many examples that the real line often is not the appropriate codomain of a utility function and that, consequently, it is worthwhile to develop a theory of continuous non-real valued utility functions. These considerations also underline that particular value of an approach as presented by Pivato [17].
Nevertheless, the problem of characterizing all topological spaces for which every closed preorder allows a continuous multi-utility representation is still pressing. In this sense Theorem 3.4, Theorem 3.5, Theorem 3.7 and Theorem 3.8 provide first results that at least prepare a complete solution of this problem.

References

[1] Bosi, G., Herden, G., Continuous multi-utility representations of preorders, *Journal of Mathematical Economics* **48** (2012), 212-218.

[2] Bosi, G., Zuanon, M., Continuous multi-utility for extremely continuous preorders, *International Journal of Contemporary Mathematical Sciences* **4** (2009), 439 - 445

[3] Burgess, D.C.J., Fitzpatrick, M., On separation axioms for certain types of ordered topological space, *Mathematical Proceedings of the Cambridge Philosophical Society* **82** (1977), 59-65.

[4] Evren, O., On the existence of expected multi-utility representations, *Economic Theory* **35** (2008), 575-592.

[5] Evren, O., Ok, E.A., On the multi-utility representation of preference relations, *Journal of Mathematical Economics* **47** (2011), 554-563.

[6] Galaabaatar, T., Karni, E., Expected multi-utility representations, *Mathematical Social Sciences* **64** (2012), 242-246.

[7] Grotemeyer, K.P., *Topologie*, Mathematisches Institut der Freien Universität, 1966.

[8] Herden, G., On the existence of utility functions, *Mathematical Social Sciences* **17** (1989a), 297-313.

[9] Herden, G., On a lifting theorem of Nachbin, *Mathematical Social Sciences* **19** (1990), 37-44.

[10] Herden, G., Mehta, G.B., The Debreu Gap Lemma and some generalizations, *Journal of Mathematical Economics* **40** (2004), 747-769.

[11] Levin, V.L., A continuous utility theorem for closed preorders on a σ–compact metrizable space, *Soviet Mathematics Doklady* **28** (1983), 715718.

[12] Levin, V.L., Measurable utility theorem for closed and lexicographic preference relations, *Soviet Mathematics Doklady* **27** (1983), 639643.

[13] Mas-Colell, A., Whinston, M.D., Green, J.R., Microeconomic Theory, Oxford University Press, 1995.

[14] Minguzzi, E., Topological conditions for the representation of preorders by continuous utilities, *Applied General Topology* **13** (2012), 81-89.

[15] Minguzzi, E., Normally Preordered Spaces and Utilities, *Order* **30** (2013),137-150.

[16] Nachbin, L.: *Topology and order*, Van Nostrand, Princeton, 1965.

[17] Pivato, M., Multiutility representations for incomplete difference preorders, *Mathematical Social Sciences* **66** (2013), 196-220.