Improving Fairness and Efficiency in Matching with Distributional Constraints: An Alternative Solution for the Japanese Medical Residency Match

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Improving Fairness and Efficiency in Matching with Distributional Constraints: An Alternative Solution for the Japanese Medical Residency Match*

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Abstract

Regional imbalance of doctors is a serious issue in many countries. In an attempt to average the geographical distribution of doctors, the Japanese government introduced "regional caps" recently, restricting the total number of medical residents matched within each region. Motivated by this policy change, Kamada and Kojima [17] proposed a mechanism called the flexible deferred acceptance mechanism (FDA) that makes every doctor weakly better off than the current system. In this paper, we further study this problem and develop an alternative mechanism that we call the priority-list based deferred acceptance mechanism (PLDA). Both mechanisms enable hospitals in the same region to fill their capacities flexibly until the regional cap is filled. FDA lets hospitals take turns to (tentatively) choose the best remaining doctor, while PLDA lets each *region* directly decide which doctor is (tentatively) matched with which hospital based on its priority list. We show that PLDA performs better than FDA in terms of efficiency and fairness through theoretical and computational analyses.

1 Introduction

The matching theory has been extensively developed for markets in which the agents have *maximum* quotas, the maximum number of agents they can

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be matched with.[1] However, more general distributional constraints may be involved in many real-world markets. This paper considers many-to-one matching problems with distributional constraints. Namely, each agent in the "many" side (hospital, school, etc.) belongs to a region, and each region has a limitation on the number of matches. We refer to the upper limit on region as a *regional cap* whereas that on each agent as a *maximum quota*.

Regional caps are relevant in many matching markets around the world. In particular, the Japanese government recently introduced regional caps to each prefecture in the Japanese Residency Matching Program (JRMP). More specifically, JRMP has started in 2004 in accordance with the reform of the clinical training system in Japan. At the beginning, it employed the standard deferred acceptance (DA) algorithm (Gale and Shapley [10]), where only hospitals' maximum quotas are considered. The mechanism resulted in doctors' excessive concentration in urban areas and the shortage of doctors in rural areas. To avoid the imbalance of the geographical distribution, the Japanese government has adopted regional caps to each prefecture since 2008. Regional caps are also utilized to regulate the geographical distribution in Chinese graduate admission, Ukrainian college admission, Scottish probationary teacher matching, and so on.[2]

An alternative, more direct method to guarantee that a certain number of doctors are allocated to each rural hospital, is to impose a *minimum quota* (Biró *et al.* [5], Monte and Tumennasan [20]). However, if agents on one side of the market can find agents on the other side unacceptable, no individually rational matching may satisfy minimum guotas. That is, to guarantee that all minimum quotas are satisfied every minimum quota is always satisfied, all hospitals must be acceptable for all doctors and vice versa. Clearly, this assumption is too restrictive. Fragiadakis *et al.* [9] develop a method for handling individual minimum quotas by transforming them into regional caps. Thus, if we obtain more fair and efficient mechanisms that can handle regional caps, we can also obtain better mechanisms that can handle individual minimum quotas using their new method.

There exist few mechanisms that can handle regional caps. First, a most straightforward approach, which is currently used in JRMP is to reduce maximum quotas so that the sum of maximum quotas in a region becomes equal to the regional cap, and to apply the standard DA mechanism. We call this mechanism Artificial Cap DA mechanism (ACDA). ACDA satisfies the standard fairness, i.e., even if a doctor wishes to be matched with a hospital, the hospital does not want to replace her with any belonging doctor. However, this mechanism can cause a severe loss of efficiency since a hospital has to reject doctors when its maximum quota is reached, even though the region where the hospital belongs can accept more doctors. To improve

[1] See Roth and Sotomayor [25] for a comprehensive survey of this literature.
[2] Kamada and Kojima [17] explain these applications in detail.

efficiency, Kamada and Kojima [17] propose a mechanism called the Flexible Deferred Acceptance algorithm (FDA), which regards artificial caps as *targets capacities* and allows doctors to be assigned to a hospital beyond its target capacity if its region can still afford to accept her. They prove that, FDA still satisfies the standard fairness while it enhances doctors' welfare.

The contribution of this paper is to propose a new mechanism that can improve both fairness and efficiency compared to FDA. More specifically, we consider a fact that in the presence of regional caps, doctors who are applying to different hospitals in the same region will compete with each other. For example, suppose that three doctors d_1, d_2, and d_3 are applying for three distinct hospitals h_1, h_2, and h_3, respectively. These three hospitals belong to the same region. Also, the region can accept at most one doctor (i.e., the regional cap is 1). Then, these hospitals (or the region) need to agree on which doctor should be accepted. Let us assume each hospital prefers d_3 over d_2, and d_2 over d_1. In FDA, a round-robin ordering among hospitals is used as a tie-breaking rule. Assume the ordering is defined as $h_1 \to h_2 \to h_3 \to h_1 \ldots$. Then, FDA assigns d_1 to h_1 by this tie-breaking rule, and d_2 and d_3 are rejected. It is natural to assume that d_3 (or d_2) has reasonably justifiable envy towards d_1, since every hospital in the region unanimously prefers d_3 (or d_2) over d_1.

In this paper, we assume that there exists an organization or consortium in which all hospitals in the region are involved, and it has reached a consensus on the priority over matches, i.e., 'which doctor is matched with which hospital'. Moreover, the consensus in each region is explicitly expressed as a priority ordering over pairs of a doctor and a hospital in the region, which we call a *priority list*. We further assume that each region's priority list should be consistent with the individual priority orderings of hospitals in the region. For example, a priority list $(d_3, h_1) \succ (d_3, h_2) \succ (d_3, h_3) \succ (d_2, h_1) \succ (d_2, h_2) \succ (d_2, h_3) \succ (d_1, h_1) \succ (d_1, h_2) \succ (d_1, h_3)$ is consistent with the individual priority ordering of each hospital. Here, $(d, h) \succ (d', h')$ indicates that a pair of d and h has a priority over that of d' and h'. According to this priority list, d_3 should be given a priority to be assigned to h_3.

In this paper, we introduce a new concept called *regional fairness*. This concept is stronger than the standard fairness and requires a matching to reflect a priority list for each region as well as a priority ordering for each hospital. Moreover, we also propose a group strategy-proof mechanism called *Priority-List based Deferred Acceptance mechanism* (PLDA) in order to obtain a regionally fair matching. Then, we show that PLDA has an advantage over FDA theoretically and computationally. More specifically, we prove that PLDA always produces a regionally fair matching but FDA does not. In particular, a matching produced by PLDA is the optimal for doctors within the set of all regionally fair matchings.

When regional caps are imposed, a fair and nonwasteful matching does not exist in general.[3] Nonwastefulness means that if doctor d, who is assigned to h in the current matching, wishes to move to hospital h' but rejected by h', then, there must be a good reason, i.e., moving d from h to h' will cause the violation of some maximum quota or regional cap. Since both PLDA and FDA are fair, they cannot be nonwasteful in general. We show that there exists no dominance relationship between these two mechanisms in terms on nonwastefulness. Thus, we evaluate nonwastefulness and doctors' welfare of these mechanisms via computer simulation. The computational results illustrate that PLDA improves doctors' welfare compared to FDA. This is because PLDA can vary the number of doctors assigned to each hospital more flexibly according to its popularity. As a result, it produces a less wasteful matching compared to FDA.

The theory of matching has been extensively developed for markets in which the agents (doctors/hospitals, students/schools, workers/firms) have *maximum* quotas that cannot be exceeded.[4] However, more general distributional constraints may be charged in many real-world markets. This paper considers two-sided many-to-one matching problems with a region structure, where each agent in the 'many' side (hospital, school, etc.) belongs to a region, and each region has a limitation on the number of matches. We refer to the upper limits on each agent and each region as *maximum quotas* and *regional caps*, respectively.

Regional caps are relevant in many concrete matching markets. In particular, the Japanese government recently introduced regional caps to each prefecture in the Japanese Residency Matching Program (JRMP). More specifically, JRMP has started in 2003 in accordance with the reform of the clinical training system in Japan. At the beginning, it employed the standard deferred acceptance (DA) algorithm (Gale and Shapley [10]), where only hospitals' maximum quotas are considered. The mechanism resulted in doctors' excessive concentration in urban areas and the shortage of doctors in rural areas. To avoid the imbalance of the geographical distribution, the Japanese government has adopted regional caps to each prefecture since 2008. Regional caps are utilized to regulate the geographical distribution in Chinese graduate admission, Ukrainian college admission, Scottish probationary teacher matching, and so on (see Kamada and Kojima [17]).

An alternative, more direct method to guarantee that a certain number of doctors are allocated to each rural hospital is to impose a *minimum quota* (Biró *et al.* [5], Monte and Tumennasan [20]). However, to guarantee that all minimum quotas are satisfied, all hospitals must be acceptable for all doctors and vice versa; guaranteeing the existence of an individually rational matching that satisfy all minimum quotas is impossible if agents on one side

[3]The standard definition of a stable matching is equivalent to a fair and nonwasteful matching.

[4]See Roth and Sotomayor [25] for a comprehensive survey of many results in this literature.

of the market can find agents on the other side unacceptable. Clearly, this assumption is too restrictive. Furthermore, Fragiadakis *et al.* [9] develop a method for handling individual minimum quotas by transforming them into regional caps. Thus, if a more fair and efficient mechanism that can handle regional caps becomes available, we can also obtain a better mechanism that can handle individual minimum quotas based on the newly developed mechanism.

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In this paper, we assume that there exists an organization or consortium in which all hospitals in the region are involved, and it has reached a consensus on the priority over matches, i.e., 'which doctor is matched with which hospital'. Moreover, the consensus in each region is explicitly expressed as an ordering over pairs of a doctor and a hospital in the region, which we call a *priority list*. We further assume that each region's priority list should be consistent with the individual preference orderings of hospitals in the region. For example, a priority list $(d_3, h_1) \succ (d_3, h_2) \succ (d_3, h_3) \succ (d_2, h_1) \succ (d_2, h_2) \succ (d_2, h_3) \succ (d_1, h_1) \succ (d_1, h_2) \succ (d_1, h_3)$ is consistent with the individual preference ordering of each hospital. Here, $(d, h) \succ (d', h')$ indicates that a pair of d and h has a priority over that of d' and h'. According to this priority list, d_3 should be given a priority to be assigned to h_3.

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1.1 Related Literature

This section discusses papers related to this study. In the one-to-one matching setting where only maximum quotas are imposed, Mcvitie and Wilson [19] show that a doctor or a hospital that is unmatched at one stable matching is unmatched in every stable matching. This fact is called *rural hospital theorem*. This theorem is extended in more general settings by Gale and Sotomayor [11, 12], Roth [21, 23], Martinez *et al.* [18] and Hatfield and Milgrom [16]. From these results, it seems inevitable that any addition of distributional constraints causes loss of a stable matching.

There exist several works on imposing various distributional constraints in one-to-many matching. Besides Kamada and Kojima [17], regional cap is also studied by Biró et al. [5]. They show that, when regional caps are

[5]The standard definition of a stable matching is equivalent to a fair and nonwasteful matching.

present, finding a stable matching is computationally difficult. As for individual minimum quotas, Biró *et al.* [5] show the nonexistence of a stable matching. Fragiadakis *et al.* [9] develop two DA-based strategy-proof mechanisms that can handle individual minimum quotas, where one mechanism is fair (but wasteful), while the other mechanism is nonwasteful (but not fair). Monte and Tumennasan [20] show that a serial dictatorship mechanism yield an efficient matching in the project assignment problem with quorums, i.e., individual minimum quotas. Furthermore, Westkamp [26] and Braun et al. [6] analyze complex maximum quota constraints in the German university admission systems.

More broadly, this paper is part of a rapidly growing literature on matching market design, which takes into account various constraints in practical markets. In particular, affirmative actions in school choice, where an obtained matching should satisfy some distributional constraints on different types of students, have recently attracted many researchers (see, for example, Abdulkadiroğlu and Sönmez [3], Abdulkadiroğlu [1], Ergin and Sönmez [8], Ehlers et al. [7] and Hafalir *et al.* [13]). The problem structure of affirmative actions in school choice seems to have some similarities with hospital-doctor matching with regional caps. Thus, developing a more fair and efficient mechanism that can handle regional caps might contribute toward developing better mechanisms for affirmative actions.

If agents do not submit their preferences truthfully, a matching mechanism may not bring an intended outcome. Many papers have discussed agents' incentive for truthful reporting. Hatfield and Milgrom [16] introduce 'matching-with-contracts' model, which is a general and abstract many-to-one matching model, including Kelso-Crawford labor market models, ascending package auctions, etc. They propose a condition called law of aggregate demand and prove that, when the workers are substitutes and the condition is satisfied, in a worker-offering matching mechanism, reporting her preference truthfully is a strategy for a worker (i.e., strategy-proof). In the same setting, Hatfield and Kojima [14] show that no coalition of workers can deviate profitably from the matching produced by the mechanism (i.e., group strategy-proof). As for the matching-with-contract model, Hatfield and Kominers [15] prove that the group strategy-proofness is guaranteed even in the framework of many-to-many matching, while Aygün and Sönmez [4] point out that several results in the matching-with-contracts model are invalid in the absence of an implicitly assumed condition called irrelevance of rejected contracts. We establish group strategy-proofness of our mechanism by reformulating our model and mechanism within the matching-with-contract model of doctors and regions, where maximum quotas of hospitals are represented as regions' preferences.

2 Model

Let us consider a residency matching problem with a finite set of doctors and hospitals. A market is a tuple $\left(D, H, R, q_H, q_R, \succ_D, \succ_H, \succ_R^{PL}\right)$. Let us denote $D = \{d_1, \ldots, d_n\}$ as the set of doctors and $H = \{h_1, \ldots, h_m\}$ as the set of hospitals. $R = \{r_1, r_2, \ldots\}$ is the set of regions each of which is simply a nonempty subset of H, i.e., $r \in 2^H \setminus \emptyset$ for any $r \in R$. Here, we assume that every hospital is located in exactly one region. Thus R can be regarded as a partition on H, i.e., $\bigcup_{r \in R} = H$ and $r \cap r' = \emptyset$ for any distinct $r, r' \in R$. The region which hospital h belongs to is denoted by $r\,(h)$.

Let us denote $q_H = \{q_{h_1}, q_{h_2}, \ldots\}$ as the set of *maximum quotas*. Each hospital $h \in H$ is assumed to be matched with at most q_h doctors. We assume maximum quotas are nonnegative integer: $q_h > 0$ for all $h \in H$. On the other hand, $q_R = \{q_{r_1}, q_{r_2}, \ldots\}$ denotes the set of *regional caps*, which is the maximum number of doctors that hospitals in a region can admit. Hence, when a region r is imposed on a regional cap of q_r, the hospitals in the region r can be matched with at most q_r doctors in total. Without loss of generality, we assume $q_r \leq \sum_{h \in r} q_h$ holds for any $r \in R$, that is, the regional caps should be smaller than or at most equal to the sum of maximum quotas of hospitals in the region (if $q_r > \sum_{h \in r} q_h$, this regional cap is nonbinding and we can replace it by $\sum_{h \in r} q_h$).

Each doctor d has a strict preference relation $\succ_d$ over H. We say $h \succ_d h'$ if d prefers h to h' and $h \succeq_d h'$ if $h \succ_d h'$ or $h = h'$. The vector of all such relations are denoted as $\succ_D = (\succ_d)_{d \in D}$. Let $\mathcal{P}$ denote the set of possible preference relations over H, and $\mathcal{P}^{|D|}$ denote the set of all preference vectors for all doctors. Each hospital h has a strict preference relation $\succ_h$ over the set of subsets of D, i.e., 2^D. For any $D', D'' \subseteq D$, we write $D' \succ_h D''$ if h prefers D' to D'' and $D' \succeq_h D''$ if $D' \succ_h D''$ or $D' = D''$.

Doctor d is said to be *acceptable* to h if $d \succ_h \emptyset$.[6]

$$\succ_d \colon h \succ h'$$

means that hospital h is the most preferred, h' is the second most preferred, and only h and h' are acceptable to d.

Moreover, we say that preference relation $\succ_h$ is *responsive with maximum quota* q_h (Roth [22]) if

1. For any $D' \subseteq D$ with $|D'| \leq q_h$, $d \in D \setminus D'$ and $d' \in D$, $(D' \cup d) \setminus d' \succeq_h D'$ if and only if $d \succeq_h d'$,

2. For any $D' \subseteq D$ with $|D'| \leq q_h$ and $d' \in D$, $D' \succeq_h D' \setminus d'$ if and only if $d' \succeq_h \emptyset$,

[6] We write d as singleton set $\{d\}$ when there is no confusion. As well, h is acceptable to d if $h \succ_d \emptyset$. Since only rankings of acceptable partners matter for our analysis, we often write only acceptable partners to denote preferences. For example,

3. $\emptyset \succ_h D'$ for any $D' \subseteq D$ with $|D'| > q_h$.

In short, preference relation $\succ_h$ is responsive with a maximum quota if and only if the ranking of a doctor (or keeping a position vacant) does not depend on her colleagues, and any set of doctors exceeding its maximum quota is unacceptable. We assume that preferences of each hospital h are responsive with maximum quota q_h throughout this paper. Kamada and Kojima [17] also gives this assumption to all hospitals' preference relations.

From the assumption, for any hospital h, preference relation $\succ_h$ can be characterized by a preference ordering over $D \cup \{\emptyset\}$ and q_h. Hence, given a set of maximum quotas q_H, we often denote preferences of each hospital h by an ordering of doctors (including $\emptyset$). Also, when there is no confusion, we sometimes express the ordering only with acceptable doctors. For example, when q_h is set to 2 and $D = \{d, d', d''\}$,

$$\succ_h: d \succ d' \;(\succ \emptyset \succ d'')$$

means that h has the following preference relation:

$$\begin{aligned}&\{d, d'\} \succ_h \{d\} \succ_h \{d'\}\\&(\succ_h \emptyset \succ_h \{d, d', d''\} \succ_h \{d, d''\} \succ_h \{d', d''\} \succ \{d''\}).\end{aligned}$$

For the ease of expression, we sometime use a function $rank_h(d)$, which represents the rank of acceptable doctor d according to hospital h's preferences over doctors. To be more precise, if d is ranked i-th from the top in h's preferences over doctors, then $rank_h(d) = i$.

As discussed in Section 1, doctors who are applying to different hospitals in the same region will compete with each other. We presume that there exists an organization or consortium in which all hospitals in the region are involved, and it has reached a consensus on the priorities over matches, i.e., "which doctor is matched with which hospital".

To be more precise, on the basis of the consensus, we assume each region r has a strict *priority list* $\succ_r^{PL}$ over pairs of a doctor and a hospital in $(D \times r) \cup \{(\emptyset, \emptyset)\}$. The vector of the priority lists is written as $\succ_R^{PL} = \left(\succ_r^{PL}\right)_{r \in R}$. The priority list of each region should reflect the preferences of the hospitals concerned. Thus, each region r's priority list $\succ_r^{PL}$ satisfies the followings.

1. For any $d, d' \in D$ and any $h \in r$, $(d, h) \succ_r^{PL} (d', h)$ holds if and only if $d \succ_h d'$,

2. For any $d \in D$ and any $h \in r$, $(\emptyset, \emptyset) \succ_r^{PL} (d, h)$ holds if and only if $\emptyset \succ_h d$.

Similarly to agents' preferences, we often write only pairs that has a priority over $(\emptyset, \emptyset)$ in order to denote preferences.

One simple and natural way to construct a priority list is to use the ranks of doctors for each hospital and a tie-breaking ordering over hospitals within the region. Here, for each $r \in R$, we specify an order of hospitals in region r. Let us assume $r = \{h_1, h_2, \ldots, h_k\}$ and the tie-breaking ordering is defined as $h_1 \to h_2 \to \ldots \to h_k$. Given this tie-breaking ordering, we define each binary relation in a priority list $\succ_r^{PL}$ as follows: for each $r \in R$, any $d, d' \in D$, and any $h_i, h_j \in r$, $(d, h_i) \succ_r^{PL} (d', h_j)$ holds if one of the following conditions holds:

- $rank_{h_i}(d) < rank_{h_j}(d')$, or
- $rank_{h_i}(d) = rank_{h_j}(d')$ and $i < j$.

In other words, for two pairs of (d, h_i) and (d', h_j), region r gives a higher ranking to (d, h_i) over (d', h_j) if the ranking of d for h is strictly higher than that of d' for h_j. If the rankings are the same, the region uses the tie-breaking ordering among hospitals.

Let us present a simple example.

Example 1 (Priority list) *Let us consider an example with two doctors d_1, d_2, three hospitals h_1, h_2, h_3, and two regions $r_1 = \{h_1, h_2\}$, $r_2 = \{h_3\}$. The preferences of hospitals are given by*

$$\begin{aligned} &\succ_{h_1}: \ d_1, \\ &\succ_{h_2}: \ d_1 \succ d_2, \\ &\succ_{h_3}: \ d_1 \succ d_2. \end{aligned}$$

For r_1, the tie-breaking ordering is defined as $h_1 \to h_2$. Thus, priority lists are given as follows:

$$\begin{aligned} &\succ_{r_1}^{PL}: (d_1, h_1), (d_1, h_2), (d_2, h_2), (\emptyset, \emptyset), (d_1, h_2) \\ &\succ_{r_2}^{PL}: (d_1, h_3), (d_2, h_3), (\emptyset, \emptyset). \end{aligned}$$

Clearly, these priority lists are consistent with hospitals' preferences.

It must be emphasized that the only restriction imposed on each priority list is that it must respect hospitals' preferences. A region can flexibly construct its priority list according to any agreement made by hospitals in the region. For instance, if the maximum quotas of hospitals are different, e.g., h_1's capacity is twice as large as h_2, then, it might be reasonable to modify the priority list so that h_1 can take twice as many doctors as h_2. Assume there are four doctors d_1, d_2, d_3, d_4 and $d_1 \succ_h d_2 \succ_h d_3 \succ_h d_4 \succ_h \emptyset$ for each hospital h. Then, a modified priority list is given as follows: $\succ_r^{PL}$: $(d_1, h_1), (d_2, h_1), (d_1, h_2), (d_3, h_1), (d_4, h_1), (d_2, h_2), \ldots$.

Next, we shall define matching and mechanism. A matching is a mapping $\mu : D \cup H \to 2^{D \cup H}$ that satisfies: (i) $\mu(d) \in H \cup \{\emptyset\}$ for all $d \in D$, (ii)

$\mu(h) \subseteq D$ for all $h \in H$, and (iii) for any $d \in D$ and any $h \in H$, we have $\mu(d) = h$ if and only if $d \in \mu(h)$. Let $\mu(r) := \bigcup_{h \in r} \mu(h)$. We say that matching μ is *feasible* if (i) $|\mu(h)| \leq q_h$ and $|\mu(r)| \leq q_r$ for any $h \in H$ and $r \in R$ and (ii) $\mu(a) \succ_a \emptyset$ for any agent $a \in D \cup H$.[7] The set of feasible matchings is denoted by $\mathcal{M}$.

A mechanism $\chi : \mathcal{P}^{|D|} \to \mathcal{M}$ is a function that takes as an input any possible preference profile of the doctors and gives a feasible matching as an output. If the doctors submit preference profile $\succ_D \in \mathcal{P}^{|D|}$, then $\chi(\succ_D)$ is the produced matching, and we write $\chi_d(\succ_D)$ for the hospital matched with doctor d by χ under $\succ_D$.

Let us discuss several properties that are important both in theory and practice. These properties are commonly used in existing matching literature. Although each of them is intuitively appealing, it will turn out that they cannot coexist with regional caps.

The first property is *fairness* that requires no agent has envy toward another agent that is justified by a particular criterion. Before defining the property, let us introduce a notion of *justifiable envy.*

Definition 1 *Given a matching μ, doctor d has* ***justifiable envy*** *toward doctor d' where $\mu(d') = h$, if $h \succ_d \mu(d)$ and $d \succ_h d'$ hold.*

Assume doctor d wanted to be matched with hospital h but was rejected. Then, d certainly has envy towards doctor d' who is matched to h. This definition means such envy is *justifiable* only when d has a higher ranking than d' according to h's preferences. If such envy is justifiable, once d suggests that h should exchange d' to d, h will agree with the suggestion. Also, the obtained matching after putting this exchange into practice (with other matches unchanged), the new matching is still feasible.

Definition 2 *A feasible matching $\mu \in \mathcal{M}$ is* ***fair*** *if there is no doctor who has justifiable envy toward another doctor. A mechanism χ is* ***fair*** *if the mechanism always produces a fair matching.*

The second property is *nonwastefulness* that requires no doctor claims an empty seat of a hospital.

Definition 3 *Given a matching μ, doctor d* ***claims an empty seat*** *of hospital h, whose region is r, if $h \succ_d \mu(d)$, $|\mu(h)| < q_h$, $|\mu(r) \cup \{d\}| \leq q_r$.*

This definition describes a situation where hospital h can afford to accept another doctor additionally (without rejecting any doctor) and doctor

[7] Condition (ii) is called *individual rationality* in general. Although this condition is usually contained in the conditions of stability, we contain it in those of feasibility for the ease of analysis.

d wishes to move to h from her current matched hospital. However, the regional cap q_r may prevent d from moving to h, even if the maximum quota q_h is not filled yet. Therefore, d is said to claim an empty seat of h if the new matching where only d moves to h is still feasible. Notice that the condition $|\mu(r) \cup \{d\}| \leq q_r$ can be divided into two cases: if $d \notin \mu(r)$, it means $|\mu(r)| < q_r$, and if $d \in \mu(r)$, it means $|\mu(r)| \leq q_r$.

Now, we are ready to define nonwastefulness.

Definition 4 *A feasible matching $\mu \in \mathcal{M}$ is **nonwasteful** if no doctor claims an empty seat of any hospital. A mechanism χ is **nonwasteful** if the mechanism always produces a nonwasteful matching.*

In existing matching literature, fairness and nonwastefulness are usually merged into a single condition called *no blocking pair* condition, and a feasible matching is called *stable* if it is individually rational and has no blocking pair [10]. In our model, any feasible matching is guaranteed to satisfy the individual rationality condition under the assumption that all hospitals are acceptable to all doctors.

Now we define a generalized notion of the standard stability to accommodate the regional caps. This notion is first introduced as 'strong stability' in Kamada and Kojima [17].

Definition 5 *A feasible matching μ is **stable** if it is nonwasteful and fair. A mechanism χ is **stable** if it always produces a stable matching.*

Finally, let us introduce two notions of *strategy-proofness* and *group strategy-proofness*. These properties consider an incentive for doctors to misreport their preferences intentionally. Strategyproofness requires that truthful report is a dominant strategy for any doctor under a mechanism. Let $\succ_{-d} := (\succ_{d'})_{d' \in D \setminus \{d\}}$ for any $d \in D$.

Definition 6 *A mechanism χ is **strategy-proof** if $\chi_d(\succ_D) \succeq_d \chi_d(\succ'_d, \succ_{-d})$ for all $\succ_D \in \mathcal{P}^{|D|}$, $d \in D$ and $\succ'_d \in \mathcal{P}$.*

Namely, a strategy-proof mechanism eliminates each doctor's profitable misreport.

Next, a group strategy-proof mechanism ensures even group manipulation unsuccessful. Let $\succ_{D'} := (\succ_{d'})_{d' \in D'}$ and $\succ_{-D'} := (\succ_{d'})_{d' \in D \setminus D'}$ for any $D' \subseteq D$.

Definition 7 *A mechanism χ is **group strategy-proof** if there does not exist a preference preference $\succ_D \in \mathcal{P}^{|D|}$, a group of doctors $D' \subseteq D$ and a preference profile $\succ'_{D'}$ such that $\chi_d\left(\succ'_{D'}, \succ_{-D'}\right) \succ_d \chi_d(\succ_D)$ for all $d \in D'$.*

That is, there is no coalition of doctors who can jointly misreport their preference and make every member of the set strictly better off. It is clear that a group strategy-proof mechanism is strategy-proof but not vice versa.

3 Impossibility results and existing remedies

We first show that a stable matching does not always exist. We borrow an example cited in [17].

Example 2 (Kamada and Kojima [17]) *Let* $D = \{d_1, d_2\}$*,* $H = \{h_1, h_2\}$ *and* $R = \{r\}$*, where* $r = \{h_1, h_2\}$*. The maximum quotas and the regional caps are given as* $q_{h_1} = q_{h_2} = q_r = 1$*. Doctors' and hospitals' preferences are given as follows:*

$$\begin{aligned}
&\succ_{h_1}: d_1 \succ d_2,\\
&\succ_{h_2}: d_2 \succ d_1,\\
&\succ_{d_1}: h_2 \succ h_1,\\
&\succ_{d_2}: h_1 \succ h_2.
\end{aligned}$$

Now we prove the nonexistence of a stable matching in this setting. By way of contradiction, let μ *be a fair and nonwasteful matching. Since* $|\mu(r)| = 0$ *contradicts nonwastefulness,* $|\mu(r)| = 1$ *must hold. If* $\mu(h_1) = \{d_2\}$*, then* d_1 *has justifiable envy toward* d_2 *since* $\mu(d_2) = \emptyset$ *and* $d_1 \succ_{h_1} d_2$*, which is a contradiction. Similarly,* $\mu(h_2) = \{d_1\}$ *leads to a contradiction.*

However, if $\mu(h_1) = \{d_1\}$*, then* d_1 *claims an empty seat of* h_2 *since* $|\mu(h_2)| = 0 < q_{h_2}$ *and* $h_2 \succ_{d_1} h_1$*. Similarly,* $\mu(h_2) = \{d_2\}$ *leads to a contradiction.*

Since fairness and nonwasteful are incompatible, let us introduce a weaker notion of nonwastefulness.

Definition 8 *Given a matching* μ*, doctor* d ***strongly claims an empty seat*** *of hospital* h*, whose region is* r*, if* $h \succ_d \mu(d)$*,* $|\mu(h)| < q_h$*,* $|\mu(r)| < q_r$*.*

According to this definition, doctor d can strongly claim an empty seat only when not only hospital h but also region r can accept another doctor.

Definition 9 *A feasible matching* $\mu \in \mathcal{M}$ *is* ***weakly nonwasteful*** *if no doctor strongly claims an empty seat of any hospital. A mechanism* χ *is* ***weakly nonwasteful*** *if the mechanism always produces a weakly nonwasteful matching.*

Now we briefly describe the flexible DA (FDA) mechanism [17], which is fair and weakly non-wasteful.[8] In FDA, for each region r, a round-robin ordering among hospitals is defined.

Stage $k \geq 1$

[8]FDA can utilize target capacities. For simplicity, we assume all target capacities are zero. We discuss issues related to target capacities in Appendix A.

Step 1 Each doctor offers to the most preferred hospital, from which she has not been rejected before stage k. If a doctor has already offered to all the acceptable hospitals, the doctor offers to $\emptyset$ and is tentatively accepted by $\emptyset$. Reset μ as an empty matching.

Step 2 For each r, choose hospital h based on the round-robin ordering, and iterate the following procedure until all doctors applying to hospitals in r are either tentatively accepted or rejected:

1. Choose doctor d who is applying to h and is not tentatively accepted or rejected yet, and has the highest ranking according to h's preference relation over the set of singletons of doctors. If there exists no such doctor, then go to the procedure for the next hospital.
2. If $d \succ_h \emptyset$, $|\mu(h)| < q_h$ and $|\mu(r)| < q_r$, d is tentatively accepted by h. Then go to the procedure for the next hospital.
3. Otherwise, d is rejected by h. Then go to the procedure for the next hospital.

Step 3 If all the doctors are tentatively accepted in Step 2, then make μ as a final matching and terminate the mechanism. Otherwise, go to stage $k+1$.

Kamada and Kojima [17] prove that FDA is fair, weakly nonwasteful and group strategy-proof.[9]

We present a example which illustrates how FDA works.

Example 3 *Let* $D = \{d_1, d_2, d_3, d_4, d_5, d_6\}$, $H = \{h_1, h_2, h_3\}$, $R = \{r_1, r_2\}$, *where* $r_1 = \{h_1, h_2\}$ *and* $r_2 = \{h_3\}$. *The maximum quotas and regional caps is given as* $q_{h_1} = q_{h_2} = q_{h_3} = 2$, $q_{r_1} = 3$ *and* $q_{r_2} = 2$. *The preferences are defined as follows:*

$$\begin{aligned}
\succ_{d_1}, \succ_{d_2}&: \; h_1 \succ h_2 \succ h_3,\\
\succ_{d_3}, \succ_{d_4}&: \; h_2 \succ h_1 \succ h_3,\\
\succ_{d_5}, \succ_{d_6}&: \; h_3 \succ h_1 \succ h_2,\\
\succ_{h_1}&: \; d_1 \succ d_5 \succ d_6 \succ d_4 \succ \emptyset \succ d_2 \succ d_3,\\
\succ_{h_2}&: \; d_2 \succ d_3 \succ d_4 \succ d_1 \succ \emptyset \succ d_6 \succ d_5,\\
\succ_{h_3}&: \; d_1 \succ d_4 \succ d_5 \succ d_6 \succ \emptyset \succ d_2 \succ d_3.
\end{aligned}$$

The round-robin ordering of r_1 *is given as* $h_1 \to h_2 \to h_1 \to \ldots$.

In Stage 1 in FDA, each doctor applies to her most preferable hospital. Thus, d_1 *and* d_2 *apply to* h_1, d_3 *and* d_4 *apply to* h_2, *and* d_5 *and* d_6 *apply to*

[9] They call a matching (and a mechanism) that is fair and weakly nonwasteful as *weakly stable*.

h_3. *In the following orderings, underlines indicates the doctors applying to each hospital.*

$$\begin{aligned}
h_1 &: \ \underline{d_1} \succ d_5 \succ d_6 \succ d_4 \succ \emptyset \succ \underline{d_2} \succ d_3,\\
h_2 &: \ d_2 \succ \underline{d_3} \succ \underline{d_4} \succ d_1 \succ \emptyset \succ d_6 \succ d_5,\\
h_3 &: \ d_1 \succ d_4 \succ \underline{d_5} \succ \underline{d_6} \succ \emptyset \succ d_2 \succ d_3.
\end{aligned}$$

As for r_1, h_1 *is first chosen according to the round-robin ordering. Then* h_1 *tentatively accepts* d_1 *according to* $\succ_{h_1}$. *Next,* h_2 *is chosen and* h_2 *tentatively accepts* d_3 *according to* $\succ_{h_2}$. *Then,* h_1 *is chosen again and* h_1 *rejects* d_2 *since* $\emptyset \succ_h d_2$. *Next,* h_2 *is chosen again,* h_1 *tentatively accepts* d_4.

As for r_2, h_3 *first tentatively accepts* d_5 *according to* $\succ_{h_3}$. *Then,* h_3 *tentatively accepts* d_6.

In the following, surrounding squares represent the tentatively accepted doctors:

$$\begin{aligned}
h_1 &: \ \boxed{d_1} \succ d_5 \succ d_6 \succ d_4 \succ \emptyset \succ \underline{d_2} \succ d_3,\\
h_2 &: \ d_2 \succ \boxed{d_3} \succ \boxed{d_4} \succ d_1 \succ \emptyset \succ d_6 \succ d_5,\\
h_3 &: \ d_1 \succ d_4 \succ \boxed{d_5} \succ \boxed{d_6} \succ \emptyset \succ d_2 \succ d_3.
\end{aligned}$$

In Stage 2, d_2, *who was rejected in Stage 1, applies to her second preferable hospital* h_2, *while the others apply to the same hospitals as previous. Thus,* d_1 *applies to* h_1, d_2, d_3 *and* d_4 *applies to* h_2, *and* d_3 *and* d_4 *apply to* h_3*:*

$$\begin{aligned}
h_1 &: \ \underline{d_1} \succ d_5 \succ d_6 \succ d_4 \succ \emptyset \succ d_2 \succ d_3,\\
h_2 &: \ \underline{d_2} \succ \underline{d_3} \succ d_4 \succ d_1 \succ \emptyset \succ d_6 \succ d_5,\\
h_3 &: \ d_1 \succ d_4 \succ \underline{d_5} \succ \underline{d_6} \succ \emptyset \succ d_2 \succ d_3.
\end{aligned}$$

As for r_1, h_1 *tentatively accepts* d_1, h_2 *tentatively accepts* d_2. *Then,* h_1 *is chosen, but there exists no remaining doctor. Thus,* h_2 *is chosen and* h_2 *tentatively accepts* d_3. *Then,* h_1 *is chosen again, but there exists no remaining doctor. Thus,* h_2 *is chosen again, but the regional cap of* r_1 *is already filled. Hence,* d_4 *is rejected. The tentative matches for* r_2 *are the same.*

In the end, the tentatively accepted doctors are as follows:

$$\begin{aligned}
h_1 &: \ \boxed{d_1} \succ d_5 \succ d_6 \succ d_4 \succ \emptyset \succ \underline{d_2} \succ d_3,\\
h_2 &: \ \boxed{d_2} \succ \boxed{d_3} \succ \underline{d_4} \succ d_1 \succ \emptyset \succ d_6 \succ d_5,\\
h_3 &: \ d_1 \succ d_4 \succ \boxed{d_5} \succ \boxed{d_6} \succ \emptyset \succ d_2 \succ d_3.
\end{aligned}$$

In Stage 3, d_4, *who was rejected in Stage 2, applies to her second preferable hospital* h_1. *Thus,* d_1 *and* d_4 *apply to* h_1, d_2 *and* d_3 *applies to* h_2, *and* d_5 *and* d_6 *apply to* h_3*:*

$$\begin{aligned}
h_1 &: \ \underline{d_1} \succ d_5 \succ d_6 \succ \underline{d_4} \succ \emptyset \succ d_2 \succ d_3,\\
h_2 &: \ \underline{d_2} \succ \underline{d_3} \succ d_4 \succ \underline{d_1} \succ \emptyset \succ d_6 \succ d_5,\\
h_3 &: \ d_1 \succ d_4 \succ \underline{d_5} \succ \underline{d_6} \succ \emptyset \succ d_2 \succ d_3.
\end{aligned}$$

As for r_1, h_1 tentatively accepts d_1, h_2 tentatively accepts d_2, h_1 tentatively accepts d_4. Then, h_2 rejects d_3 due to the regional cap. The tentative matches for r_2 are the same.

The tentatively accepted doctors in this stage are as follows:

$$
\begin{aligned}
h_1 : &\quad \boxed{d_1} \succ d_5 \succ d_6 \succ \boxed{d_4} \succ \emptyset \succ d_2 \succ d_3, \\
h_2 : &\quad \boxed{d_2} \succ \underline{d_3} \succ d_4 \succ d_1 \succ \emptyset \succ d_6 \succ d_5, \\
h_3 : &\quad d_1 \succ d_4 \succ \boxed{d_5} \succ \boxed{d_6} \succ \emptyset \succ d_2 \succ d_3.
\end{aligned}
$$

In Stage 4, d_3 applies to h_1 but is rejected due to the regional cap. In Stage 5, d_3 applies to h_3 but is rejected due to the maximum quota. In Stage 6, d_3 applies to $\emptyset$ and is tentatively accepted. Then all doctors have been tentatively accepted and the mechanism terminates. Finally, the obtained matching becomes:

$$
\begin{pmatrix} h_1 & h_2 & h_3 \\ \{d_1, d_4\} & \{d_2\} & \{d_5, d_6\} \end{pmatrix}.
$$

The resulting matching is fair and weakly nonwasteful, but it is not nonwasteful, e.g., d_4 prefers h_2 to h_1. Furthermore, d_3 cannot be assigned to any hospital. Note that d_3 is the second-ranked doctor for h_2, while d_4, who is accepted to h_1, is the fourth-ranked doctor for h_1. Due to the round-robin ordering for tie-breaking in r_1, d_4 is accepted while d_3 is rejected. d_3 might feel this result is *unfair*. In the next section, we introduce a new fairness concept that eliminates such an unfair outcome.

4 New Properties

The previous section illustrates that fairness and nonwastefulness cannot coexist when regional caps are imposed; a stable matching may not exist. On the other hand, fairness and weak nonwastefulness can coexist and FDA is guaranteed to find such a matching. In this section, we consider a stability concept, which is stronger than fairness and weak nonwastefulness, but certainly weaker than fairness and nonwastefulness. To be more precise, we propose a *stronger* notion of fairness, which we call *regional fairness*. This notion is based on the idea that any competition within a region should be settled in a fair and transparent fashion, i.e., based on a predefined priority list. Furthermore, we propose another notion of nonwastefulness, which we call *regional nonwastefulness*. This notion is based on the same idea of *regional fairness*, which is weaker than nonwastefulness but stronger than weak nonwastefulness.

First, we define regional fairness.

Definition 10 *Given a matching μ, doctor d has* ***regionally justifiable envy*** *toward doctor $d' \neq d$ where $\mu(d') = h'$, if, for some h such that $r(h) = r(h') = r$, $h \succ_d \mu(d)$, $|\mu(h)| < q_h$, and $(d, h) \succ_r^{PL} (d', h')$ hold.*

An intuitive explanation of this definition is as follows. Since d prefers h to her current assigned hospital $\mu(d)$, d must be rejected from h. Also, since h's maximum quota is not filled, d must be rejected due to the regional cap of r. Then, the competition within r must be settled by r's priority list, i.e., for any doctor d', who is accepted to hospital h' within r, (d', h') must be ranked higher than (d, h) in $\succ_r^{PL}$. Otherwise, d can have regionally justifiable envy towards d'.

Let us consider the situation in Example 3. In the matching produced by FDA, d_3 is unmatched while d_4 is assigned to h_1. Here, d_3 has regionally justifiable envy toward d_4 as long as the priority list of r_1 is defined by the rank-based method since $rank_{h_2}(d_3) = 2 < rank_{h_1}(d_4) = 4$.

Then, regional fairness is defined as follows.

Definition 11 *A feasible matching $\mu \in \mathcal{M}$ is **regionally fair** if there is no doctor who has justifiable envy or regionally justifiable envy toward another doctor. A mechanism χ is **regionally fair** if the mechanism always produces a regionally fair matching.*

By definition, if a matching is regionally fair, it is also fair, but not vice versa.

Next we define regional nonwastefulness.

Definition 12 *Given a matching μ, doctor d, who is assigned to h in μ, **regionally claims an empty seat** of hospital h' if $r(h') = r(h) = r$, $h' \succ_d h$, $|\mu(h')| < q_{h'}$, $|\mu(r)| = q_r$, and $(d, h') \succ_r^{PL} (d, h)$ hold.*

An intuitive explanation of this definition is as follows. Since d prefers h' to her currently assigned hospital h, d must be rejected by h'. Also, since the maximum quota of h' is not filled, d must be rejected due to the regional cap of r. However, d is accepted to another hospital h, which is in the same region r. Thus, if (d, h') is ranked higher than (d, h) in $\succ_r^{PL}$, d can request to reassign her to h'. A regional claim for an empty seat can be interpreted as regionally justifiable envy *toward oneself*.

Let us consider the situation in Example 3. In the matching produced by FDA, d_4 is assigned to h_1. Thus, d_4 regionally claims an empty seat of h_2, if the priority list of r_1 is defined by the rank-based method since $rank_{h_2}(d_4) = 3 < rank_{h_1}(d_4) = 4$.

Now we are ready to introduce regional nonwastefulness.

Definition 13 *A feasible matching $\mu \in \mathcal{M}$ is **regionally nonwasteful** if no doctor weakly or regionally claims an empty seat of any hospital. A mechanism χ is **regionally nonwasteful** if the mechanism always produces a regionally nonwasteful matching.*

By definition, a regionally nonwasteful matching (and mechanism) is also weakly nonwasteful, but not vice versa. Also, any nonwasteful matching (and mechanism) is also regionally nonwasteful, but not vice versa.

5 Priority List-Based Deferred Acceptance Mechanism

In this section, we present a new mechanism called the priority list-based deferred acceptance mechanism (PLDA). This mechanism produces a matching that is regionally fair and regionally nonwasteful for any given input (preferences).

PLDA is quite similar to the standard DA but there exists one fundamental difference; in PLDA, each region (instead of each hospital) makes an decision on which doctor (who are applying hospitals in the region) should be accepted based on its priority list.

Similar to the standard DA, PLDA repeats some stages where doctors offers to hospitals and some doctors are tentatively accepted and the other are rejected. More specifically, each stage proceeds as follows. First, each doctor offers to the most preferred hospital that has not rejected herself before. Then, each region r picks up (d, h), which is a pair of an offering doctor and the hospital offered by the doctor in the region, according to r's priority list. If the match of (d, h) violates a constraint for some quota/cap, then d is rejected by h, and otherwise, d is tentatively accepted by h. The mechanism terminates when no doctor is rejected in a stage and outputs the tentatively accepted pairs at the final matching.

The description of PLDA is as follows:

Stage $k \geq 1$

Step 1 Each doctor offers to the most preferred hospital, from which she has not been rejected before stage k. If a doctor has offered to all the acceptable hospitals, the doctor offers to $\emptyset$ and is tentatively accepted by $\emptyset$. Reset μ as an empty matching.

Step 2 For each r, for each i from 1 to $|D| \cdot |r|$, iterate the following procedure:

1. Choose (d, h), which is i-th ranked pair in r's priority list $\succ_r^{PL}$.
2. If d does not offer to h or $(d, h) = (\emptyset, \emptyset)$, then go to the procedure for $i + 1$.
3. If $(d, h) \succ_r^{PL} (\emptyset, \emptyset)$, $|\mu(h)| < q_h$ and $|\mu(r)| < q_r$, d is tentatively accepted by h. Then go to the procedure for $i + 1$.
4. Otherwise, d is rejected by h. Then go to the procedure for $i + 1$.

Step 3 If all the doctors are tentatively accepted in Step 2, then make μ as a final matching and terminate the mechanism. Otherwise, go to stage $k + 1$.

We present a example which illustrates how PLDA works.

Example 4 *We apply PLDA to the case in Example 3.*

Region r_1's priority list, $\succ_{r_1}^{PL}$, is generated from the tie-breaking ordering $h_1 \to h_2$ and the rank-based method. Region r_2's priority list is identical to h_3's priority list. Thus, the priority lists are given as follows:

$$
\begin{aligned}
\succ_{r_1}^{PL}\colon\ & (d_1,h_1),(d_2,h_2),(d_5,h_1),(d_3,h_2),(d_6,h_1),(d_4,h_2),\\
& (d_4,h_1),(d_1,h_2),(\emptyset,\emptyset),(d_2,h_1),(d_6,h_2),(d_3,h_1),(d_5,h_2),\\
\succ_{r_2}^{PL}\colon\ & (d_1,h_3),(d_4,h_3),(d_5,h_3),(d_6,h_3),(\emptyset,\emptyset),(d_2,h_3),(d_3,h_3).
\end{aligned}
$$

In Stage 1, each doctor applies to her most preferable hospital, i.e., d_1 and d_2 apply to h_1, d_3 and d_4 apply to h_2, and d_5 and d_6 apply to h_3. In the following orderings, underlines indicates the pairs (d,h) such that d applies to h.

$$
\begin{aligned}
\succ_{r_1}^{PL}\colon\ & \underline{(d_1,h_1)},(d_2,h_2),(d_5,h_1),\underline{(d_3,h_2)},(d_6,h_1),\underline{(d_4,h_2)},\\
& (d_4,h_1),(d_1,h_2),(\emptyset,\emptyset),\underline{(d_2,h_1)},(d_6,h_2),(d_3,h_1),(d_5,h_2),\\
\succ_{r_2}^{PL}\colon\ & (d_1,h_3),(d_4,h_3),\underline{(d_5,h_3)},\underline{(d_6,h_3)},(\emptyset,\emptyset),(d_2,h_3),(d_3,h_3).
\end{aligned}
$$

For simplicity, we focus only on the underlined pairs. As for region $r_1 = \{h_1,h_2\}$, according to $\succ_{r_1}^{PL}$, (d_1,h_1) is first chosen and tentatively accepted. Then, (d_3,h_2) is chosen and tentatively accepted. Next, (d_4,h_2) is chosen and tentatively accepted. Then, (d_2,h_1) is chosen but $(\emptyset,\emptyset) \succ_r^{PL} (d_2,h_1)$ holds. Hence, d_2 is rejected by h_1.

As for region $r_2 = \{h_3\}$, (d_5,h_3) is chosen according to $\succ_{r_2}^{PL}$ and the pair is tentatively accepted. Then, (d_6,h_3) is chosen and tentatively accepted.

In the following, surrounding squares represent the tentatively accepted pairs:

$$
\begin{aligned}
\succ_{r_1}^{PL}\colon\ & \boxed{(d_1,h_1)},(d_2,h_2),(d_5,h_1),\boxed{(d_3,h_2)},(d_6,h_1),\boxed{(d_4,h_2)},\\
& (d_4,h_1),(d_1,h_2),(\emptyset,\emptyset),\underline{(d_2,h_1)},(d_6,h_2),(d_3,h_1),(d_5,h_2),\\
\succ_{r_2}^{PL}\colon\ & (d_1,h_3),(d_4,h_3),\boxed{(d_5,h_3)},\boxed{(d_6,h_3)},(\emptyset,\emptyset),(d_2,h_3),(d_3,h_3).
\end{aligned}
$$

In Stage 2, d_2, who was rejected in Stage 1, applies to her second preferable hospital h_2, while the others apply to the same hospitals as previous. Thus, d_1 applies to h_1, d_2, d_3 and d_4 apply to h_2, and d_5 and d_6 apply to h_3:

$$
\begin{aligned}
\succ_{r_1}^{PL}\colon\ & \underline{(d_1,h_1)},\underline{(d_2,h_2)},(d_5,h_1),\underline{(d_3,h_2)},(d_6,h_1),\underline{(d_4,h_2)},\\
& (d_4,h_1),(d_1,h_2),(\emptyset,\emptyset),(d_2,h_1),(d_6,h_2),(d_3,h_1),(d_5,h_2),\\
\succ_{r_2}^{PL}\colon\ & (d_1,h_3),(d_4,h_3),\underline{(d_5,h_3)},\underline{(d_6,h_3)},(\emptyset,\emptyset),(d_2,h_3),(d_3,h_3).
\end{aligned}
$$

As for region r_1, (d_1,h_1), (d_2,h_2), (d_3,h_2) are tentatively accepted, while (d_4,h_2) cannot be matched since the regional cap is already filled. The tentative acceptance for region r_2 are the same.

In the end, the tentatively accepted pairs are as follows:

$$\begin{aligned}
\succ_{r_1}^{PL}:\ & \boxed{(d_1,h_1)}, \boxed{(d_2,h_2)}, (d_5,h_1), \boxed{(d_3,h_2)}, (d_6,h_1), \underline{(d_4,h_2)},\\
& (d_4,h_1), (d_1,h_2), (\emptyset,\emptyset), (d_2,h_1), (d_6,h_2), (d_3,h_1), (d_5,h_2),\\
\succ_{r_2}^{PL}:\ & (d_1,h_3), (d_4,h_3), \boxed{(d_5,h_3)}, \boxed{(d_6,h_3)}, (\emptyset,\emptyset), (d_2,h_3), (d_3,h_3).
\end{aligned}$$

In Stage 3, d_4, who was rejected in Stage 2, applies to her second preferable hospital h_1. Thus, d_1 and d_4 apply to h_1, d_2 and d_3 apply to h_2, and d_5 and d_6 apply to h_3:

$$\begin{aligned}
\succ_{r_1}^{PL}:\ & \underline{(d_1,h_1)}, \underline{(d_2,h_2)}, (d_5,h_1), \underline{(d_3,h_2)}, (d_6,h_1), (d_4,h_2),\\
& \underline{(d_4,h_1)}, (d_1,h_2), (\emptyset,\emptyset), (d_2,h_1), (d_6,h_2), (d_3,h_1), (d_5,h_2),\\
\succ_{r_2}^{PL}:\ & (d_1,h_3), (d_4,h_3), \underline{(d_5,h_3)}, \underline{(d_6,h_3)}, (\emptyset,\emptyset), (d_2,h_3), (d_3,h_3).
\end{aligned}$$

Then, as for r_1, (d_1,h_1), (d_2,h_2), (d_3,h_2) are tentatively accepted, while (d_4,h_1) cannot be matched since the regional cap is already filled. The tentative acceptance for region r_2 are the same.

The tentatively accepted pairs in Stage 3 are as follows:

$$\begin{aligned}
\succ_{r_1}^{PL}:\ & \boxed{(d_1,h_1)}, \boxed{(d_2,h_2)}, (d_5,h_1), \boxed{(d_3,h_2)}, (d_6,h_1), (d_4,h_2),\\
& \underline{(d_4,h_1)}, (d_1,h_2), (\emptyset,\emptyset), (d_2,h_1), (d_6,h_2), (d_3,h_1), (d_5,h_2),\\
\succ_{r_2}^{PL}:\ & (d_1,h_3), (d_4,h_3), \boxed{(d_5,h_3)}, \boxed{(d_6,h_3)}, (\emptyset,\emptyset), (d_2,h_3), (d_3,h_3).
\end{aligned}$$

In Stage 4, the rejected doctor in stage 3, i.e., d_4, applies to her third preferable hospital h_3:

$$\begin{aligned}
\succ_{r_1}^{PL}:\ & \underline{(d_1,h_1)}, \underline{(d_2,h_2)}, (d_5,h_1), \underline{(d_3,h_2)}, (d_6,h_1), (d_4,h_2),\\
& (d_4,h_1), (d_1,h_2), (\emptyset,\emptyset), (d_2,h_1), (d_6,h_2), (d_3,h_1), (d_5,h_2),\\
\succ_{r_2}^{PL}:\ & (d_1,h_3), \underline{(d_4,h_3)}, \underline{(d_5,h_3)}, \underline{(d_6,h_3)}, (\emptyset,\emptyset), (d_2,h_3), (d_3,h_3).
\end{aligned}$$

Then, the tentative acceptance for region r_1 are the same. As for region r_2, (d_4,h_3) and (d_5,h_3) are tentatively accepted, while (d_6,h_3) cannot be matched since the regional cap, as well as h_3's maximum quota, is already filled.

The tentatively accepted pairs in Stage 4 are as follows (the underlined pair is rejected):

$$\begin{aligned}
\succ_{r_1}^{PL}:\ & \boxed{(d_1,h_1)}, \boxed{(d_2,h_2)}, (d_5,h_1), \boxed{(d_3,h_2)}, (d_6,h_1), (d_4,h_2),\\
& (d_4,h_1), (d_1,h_2), (\emptyset,\emptyset), (d_2,h_1), (d_6,h_2), (d_3,h_1), (d_5,h_2),\\
\succ_{r_2}^{PL}:\ & (d_1,h_3), \boxed{(d_4,h_3)}, \boxed{(d_5,h_3)}, \underline{(d_6,h_3)}, (\emptyset,\emptyset), (d_2,h_3), (d_3,h_3).
\end{aligned}$$

In Stage 5, the rejected doctor in stage 4, i.e., d_6, applies to her second preferable hospital h_1:

$$\begin{aligned}
\succ_{r_1}^{PL}:\ & \underline{(d_1,h_1)}, \underline{(d_2,h_2)}, (d_5,h_1), \underline{(d_3,h_2)}, \underline{(d_6,h_1)}, (d_4,h_2),\\
& (d_4,h_1), (d_1,h_2), (\emptyset,\emptyset), (d_2,h_1), (d_6,h_2), (d_3,h_1), (d_5,h_2),\\
\succ_{r_2}^{PL}:\ & (d_1,h_3), \underline{(d_4,h_3)}, \underline{(d_5,h_3)}, (d_6,h_3), (\emptyset,\emptyset), (d_2,h_3), (d_3,h_3).
\end{aligned}$$

As for region r_1, (d_1, h_1), (d_2, h_2), (d_3, h_2) are tentatively accepted, while (d_6, h_1) cannot be matched since the regional cap is already filled. The tentative acceptance for region r_2 are the same.

The tentatively accepted pairs in Stage 4 are as follows:

$$
\begin{aligned}
\succ^{PL}_{r_1}:\ & \boxed{(d_1,h_1)}, \boxed{(d_2,h_2)}, (d_5,h_1), \boxed{(d_3,h_2)}, \underline{(d_6,h_1)}, (d_4,h_2),\\
& (d_4,h_1), (d_1,h_2), (\emptyset,\emptyset), (d_2,h_1), (d_6,h_2), (d_3,h_1), (d_5,h_2),\\
\succ^{PL}_{r_2}:\ & (d_1,h_3), \boxed{(d_4,h_3)}, \boxed{(d_5,h_3)}, (d_6,h_3), (\emptyset,\emptyset), (d_2,h_3), (d_3,h_3).
\end{aligned}
$$

In Stage 6, the rejected doctor in stage 4, i.e., d_6, applies to her second preferable hospital h_2. However, (d_6, h_2) is rejected due to the regional cap.

In Stage 7, d_6, applies to $\emptyset$ and is tentatively accepted. The tentative acceptance for region r_1 and r_2 are the same. Then all doctors have been tentatively accepted and the mechanism terminates. Finally, the obtained matching becomes:

$$
\begin{pmatrix} h_1 & h_2 & h_3 \\ \{d_1\} & \{d_2, d_3\} & \{d_4, d_5\} \end{pmatrix}.
$$

Note that this result is not stable; d_2 claims an empty seat of h_1 (Definition 3). However, d_2 cannot regionally claim an empty seat of h_1 (Definition 12), since $(d_2, h_2) \succ^{PL}_{r_1} (d_2, h_1)$. This result is regionally nonwasteful and regionally fair.

The following is the main result of this section.

Theorem 1 *PLDA terminates in finite stages. The mechanism is regionally fair, regionally nonwasteful and group strategy-proof.*

The fact that PLDA is regionally fair and regionally nonwasteful is intuitively natural, since in PLDA, each region makes its decision on which doctor should be accepted based on its priority list. Assume doctor d is rejected by hospital h, which belongs to region r. Then, either h or r is full. Also, if r is full, while h is not full, all pairs accepted in hospitals in r should have higher rankings than (d, h) according to r's priority list.

The fact that PLDA is group strategy-proof relies on the fact that the choice of each region satisfies *substitutability*, i.e., if doctor d is rejected from hospital h in a certain stage, then d cannot be accepted even if d manipulates her preference and applies to h in a different stage.

The formal proof of Theorem 1 is described in Appendix C. In the proof, we reformulate our model within the matching-with-contract model presented in Hatfield and Milgrom [16]. In the reformulation, we regard each region (not each hospital) as an 'agent'. In this way, each region is supposed to have a preference over the set of contracts such that the region obeys its own priority list and dislikes any violation of the concerning hospitals' quotas. Thus, the maximum quotas are incorporated into the regions' preferences, while regional caps are interpreted as a new 'maximum quota'. Then, we re-define PLDA in that model and prove that it satisfies group strategy-proofness by the coincidence with the generalized Gale-Shapley algorithm.[10]

In addition to the above theorem, we can also characterize PLDA matching from the viewpoint of the optimality. The result is presented in Section 6.1.

6 Analysis

6.1 The doctor-optimality of the PLDA matching

This section characterizes the matching produced by PLDA in terms of the optimality for doctors. First, let us formally define notions of *doctor-optimality* and *doctor-maximality*. Recall that $\mathcal{M}$ denotes the set of feasible matchings.

Definition 14 *Let $\mathcal{M}'$ be any subset of $\mathcal{M}$. The matching μ is* ***doctor-optimal*** *in $\mathcal{M}'$ if $\mu \succeq_D \mu'$ for any $\mu' \in \mathcal{M}'$. The matching μ is* ***doctor-maximal*** *in $\mathcal{M}'$ if there is no matching $\mu' \in \mathcal{M}'$ such that $\mu' \succeq_D \mu$ and $\mu'(d) \succ_d \mu(d)$ for some $d \in D$.*

Here, $\mu \succeq_D \mu'$ is defined as $\mu(d) \succeq_d \mu'(d)$, $\forall d \in D$. It is clear that if a matching is doctor-optimal, it is doctor-maximal, but not vice versa. Also, a doctor-maximal matching always exists but may not be unique. On the other hand, a doctor-optimal matching does not always exists, but if such a matching exists, it is guaranteed to be unique. In the terminology of lattice theory, the doctor-optimal (or a doctor-maximal) matching corresponds to the maximum (or a maximal) element in a partially ordered set $(\mathcal{M}', \succeq_D)$.

The following theorem holds.

Theorem 2 *The matching produced by PLDA is the doctor-optimal in the set of all regionally fair matchings.*

Thus, if we assume a matching should be regionally fair, then, the matching obtained by PLDA is the best, i.e., it optimizes doctors' welfare. Note that to guarantee the optimality, we do not require regional nonwastefulness, though the matching produced by PLDA is regionally nonwasteful. The matching produced by PLDA is the doctor-optimal not only in the set of all regionally fair and regionally nonwasteful matchings, but also in the set of all regionally fair matchings.

The detailed proof of the above theorem is presented in Appendix C.

[10]To be more precise, Hatfield and Milgrom [16] prove only that the generalized Gale-Shapley algorithm satisfies strategy-proofness. Group strategy-proofness is guaranteed by Hatfield and Kojima [14].

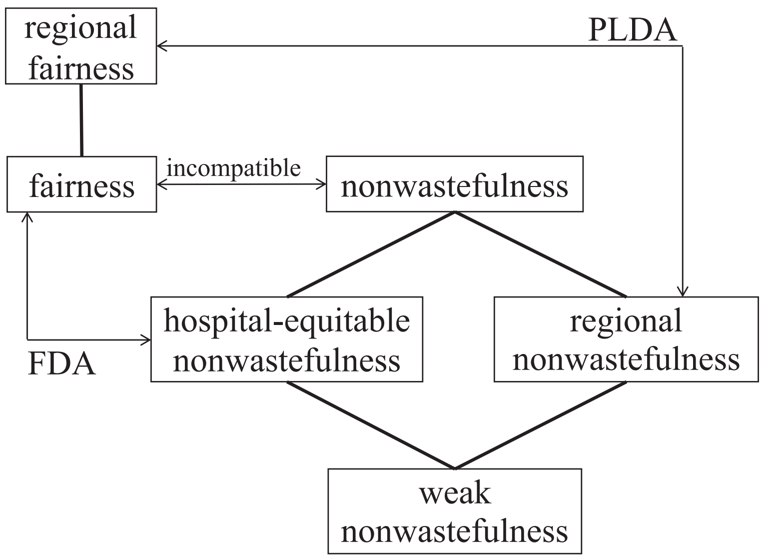

Figure 1: Relation between Properties

6.2 Comparison with FDA

In this subsection, we compare PLDA and FDA. FDA is weakly nonwasteful. Furthermore, it satisfies a stronger property than weak nonwastefulness, which we call hospital-equitably nonwastefulness. [11]

Definition 15 *Given a matching μ, doctor d, who is assigned to h,* ***hospital-equitably claims an empty seat*** *of hospital h' if $r(h') = r(h) = r$, $h' \succ_d h$, $|\mu(h')| < q_{h'}$, $|\mu(r)| = q_r$, and $|\mu(h)| - |\mu(h')| \geq 2$ hold.*

A feasible matching $\mu \in \mathcal{M}$ is ***hospital-equitably nonwasteful*** *if no doctor weakly or hospital-equitably claims an empty seat of any hospital. A mechanism χ is* ***hospital-equitably nonwasteful*** *if the mechanism always produces a hospital-equitably nonwasteful matching.*

This definition means that doctor d, who is assigned to hospital h, can claim an empty seat of another hospital h' in the same region, if moving d from h to h' strictly improves the imbalance of the number of assigned doctors.

Figure 1 summarizes the relation between properties discussed so far. If two properties are connected by a thick line, the property placed above is stronger/stricter than the other. As discussed before, fairness and nonwastefulness is incompatible. PLDA satisfies regional fairness, which is stronger than fairness. Also, it satisfies regionally nonwastefulness, which is stronger than weak nonwastefulness but weaker than nonwastefulness. PLDA can obtain the doctor-optimal matching in the set of regionally fair and regionally

[11] In the main text of Kamada and Kojima [17], a matching (and a mechanism) that is fair and hospital-equitably nonwasteful is called *stable*.

nonwasteful matchings. On the other hand, FDA satisfies fairness, as well as hospital-equitable wastefulness, which is stronger than weak nonwastefulness but weaker than nonwastefulness. FDA is not guaranteed to obtain the doctor-optimal matching that satisfies both fairness and hospital-equitable nonwastefulness. For regional nonwastefulness and hospital-equitable nonwastefulness, there is no inclusive relation, i.e., a matching can be regionally nonwasteful but not hospital-equitably nonwasteful, and vice versa. Thus, in terms of nonwastefulness/efficiency, there is no theoretical dominance relation between PLDA and FDA.

Let us examine when these mechanisms waste some seats. The only possibility that doctor d, who is assigned to hospital h, can claim an empty seat is that she hopes to move to another hospital h', where h and h' belong to the same region r. Such a situation can happen when d first applies to h' and has rejected (due to the competition within r). Then, d applies to h and this time, d wins the competition within r and has accepted. In such a case, the preferences of hospitals h and h' are somewhat *inconsistent*, i.e., d's ranking in h' is relatively low, while her ranking in h is relatively high. The situation can be even worse. Let us assume two doctors d_1 and d_2 are trying to obtain a single seat in hospitals within region r. Also assume the preferences of hospitals in r are *inconsistent*, Then, there is a chance that these two doctors keep on competing with each other to obtain one seat in r, and finally d_1 obtains a seat in her lowest-ranked hospital in r, while d_2 is rejected from r. On the other hand, if hospitals in r unanimously give a higher ranking to d_1, d_1 can obtain a seat in her highest-ranked hospital. Thus, the obtained matching weakly dominates the previous result. From these facts, we can expect that when the preferences (over doctors) of hospitals within the same region becomes more *consistent*, then these mechanisms lose less seats and doctors' welfare can increase. We will confirm this conjecture in Section 6.3.

In an extreme case where all hospitals use the same preference ordering, PLDA (when the priority list is created by the rank-based method) becomes identical to a serial-dictatorship mechanism, in which doctors are chosen in a predefined order, and each doctor is assigned to her most preferred hospital as long as the assignment does not violate the maximum quota or regional cap. A serial dictatorship mechanism is guaranteed to be strategy-proof and Pareto efficient, e.g., see [2]. Thus, in such an extreme case, PLDA is nonwasteful.

On the other hand, even if the preferences of hospitals are identical, FDA can lose some seats since FDA tries to equitably assign doctors among hospitals within a region. Let us assume there exist nine hospitals $h_1, \ldots, h_9$ within region r. Also, let us assume $q_r = 90$ (and each individual maximum quota is large enough). There are 100 doctors $d_1, \ldots, d_{100}$. All hospitals are willing to accept any doctor and unanimously give a higher ranking to d_1 over d_2, d_2 over d_3, and so on. Also, all doctors are willing to be matched with any hospital and unanimously prefer h_1 over h_2, h_2 over h_3, and so on. In FDA, all doctors first apply to h_1. Then, h_1 tentatively accepts doctors from d_1 to d_{90}, and doctors from d_{91} to d_{100} are rejected. Then, in the next stage, doctors from d_{91} to d_{100} apply to h_2. Since FDA tries to equitably assign doctors among hospitals within a region, all doctors applying to h_2 are tentatively accepted. Then, doctors from d_{81} to d_{90} are rejected from h_1 and will apply to h_2. Eventually, doctors from d_1 to d_{50} apply to h_1, while doctors from d_{51} to d_{100} apply to h_2. Then, doctors from d_{46} to d_{50}, as well as doctors from d_{96} to d_{100}, are rejected. The final assignment would be, $d_1, \ldots, d_{10}$ to h_1, $d_{11}, \ldots, d_{20}$ to h_2, $\ldots$, $d_{81}, \ldots, d_{90}$ to h_9, and $d_{91}, \ldots, d_{100}$ are rejected. In this case, doctors from d_{11} to d_{90} claim empty seats.

In PLDA, when the priority list is created by the rank-based method, $d_1, \ldots, d_{90}$ are assigned to h_1 and $d_{91}, \ldots, d_{100}$ are rejected. Thus, no doctor claims an empty seat.

From this observation, we conjecture that PLDA would produce a less wasteful matching compared to FDA, since it can vary the number of doctors assigned to each hospital more flexibly according to its popularity. We will confirm this conjecture in Section 6.3.

6.3 Comparative Experiments

In this section, we use computer simulation to compare the doctors' welfare and efficiency of FDA/PLDA. We consider a market with $|D| = 512$ doctors and $|H| = 64$ hospitals, i.e., $D = \{d_1, \cdots, d_{512}\}$ and $H = \{h_1, \cdots, h_{64}\}$. The number of regions $|R|$ is set to 8 and each region has the same number of hospitals, i.e., $R = \{r_1, \ldots, r_8\}$, $r_1 = \{h_1, \cdots, h_8\}$, $r_2 = \{h_9, \cdots, h_{16}\}, \cdots$, $r_8 = \{h_{57}, \cdots, h_{64}\}$. The maximum quota q_h is set to 24 for each hospital h. We set the regional cap q_r to $\frac{|D|}{|R|} = 64$ for each region r, i.e., the sum of regional caps $(64 \times 8 = 512)$ is equal to the number of doctors. Thus, in this parameter setting, regional caps are more restrictive compared to individual maximum quotas. The choice of these parameters is not critical; by changing these parameters, the qualitative properties of obtained results do not change.

We generate preferences of doctors as follows. For simplicity, we assume that all hospitals are acceptable to each doctor. We first construct the cardinal utility values of each doctor. To introduce some correlation within doctors' preferences so that popular/unpopular hospitals can exist, the cardinal utility values of each doctor are constructed using a linear combination of a common value vector and a private vector. To be more precise, we draw one common vector $u_H = (u_h)_{h \in H}$ uniformly at random from set $[0, 1]^{|H|}$. Then, for each doctor, we draw a private vector u_d from the same set uniformly. Then, the utility value vector over hospitals for doctor d is given as $\alpha u_H + (1 - \alpha)\, u_d$, where $\alpha \in [0, 1]$. Here, α is a parameter that controls the correlation of doctors' preferences. When $\alpha = 0$, doctors' preferences are independent, while when $\alpha = 1$, all doctors have the same preferences. Then, we convert these cardinal utility values into an ordinal preference relation for each doctor.[12]

Similarly, we generate hospitals' preferences by constructing cardinal utility values of each hospital. We also assume all doctors are acceptable for each hospital. We draw one common vector $u_D = (u_d)_{d \in D}$ uniformly at random from set $[0, 1]^{|D|}$. For each hospital, we draw a private vector u_h from the same set uniformly. Then, the utility value vector over the doctors for hospital h is constructed as $\beta u_D + (1 - \beta) u_h$, where $\beta \in [0, 1]$. Here, β is a parameter that controls the correlation of hospitals' preferences.

We assume the tie-breaking rule for generating a rank-based priority list, as well as the round-robin ordering among hospitals, is defined as $h_1 \to h_2 \to \ldots \to h_{64}$. We create 100 problem instances for each parameter setting.

We compare the doctors' welfare and efficiency of FDA/PLDA by using the following metrics: (i) the Cumulative Distribution Function (CDF) of the rank of the hospital each doctor is assigned, (ii) the ratio of doctors who prefer the outcome obtained by each mechanism, and (iii) the number of doctors who claim empty seats.

6.3.1 CDF of Rank

In this section, We first measure doctors' welfare by plotting CDFs of the average ratio of doctors matched with their k-th or higher ranked hospital under the two mechanisms. For example, in Figure 3, under PLDA with $\beta = 0.5$, about 30% of doctors are matched with their first choice, about 45% of doctors are matched with their first or second choice, about 55% of doctors are matched with their first or second or third choice, and so on. Thus, if the rank distribution of one mechanism first-order stochastically dominates another, then a strong argument can be made for the use of the stochastically dominant mechanism.

Figures 2 to 4 analyze how different correlations α affect the rank distributions (for these simulation, we fix $\beta = 0.5$ for all hospitals' preferences). We can see that PLDA first-order stochastically dominates FDA regardless of the choice of α, which represents the correlation in the doctors' preferences (while they are almost equivalent for $\alpha = 0$). However, the magnitude of the dominance becomes large as α increases. As α increases, the competition among doctors becomes more severe. Thus, the welfare of doctors decreases. However, PLDA can vary the number of doctors assigned to each hospital more flexibly according to its popularity. As a result, it obtains better welfare of doctors compared to FDA. These results imply that doctors will prefer PLDA to FDA in the first-order stochastic dominance sense.

[12] With a tiny probability, a doctor has the same cardinal utility value for two different hospitals. Then, the doctor breaks the tie by preferring one with the smaller index.

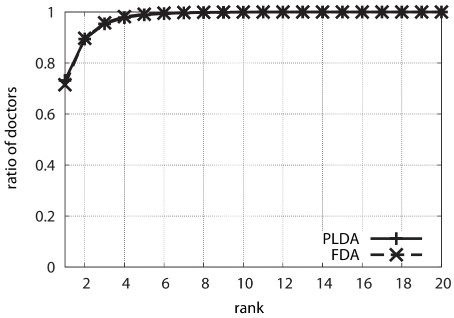

Figure 2: $\alpha = 0$

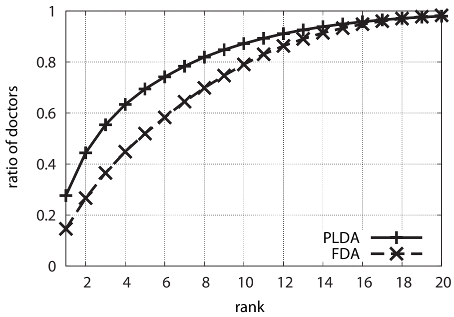

Figure 3: $\alpha = 0.5$

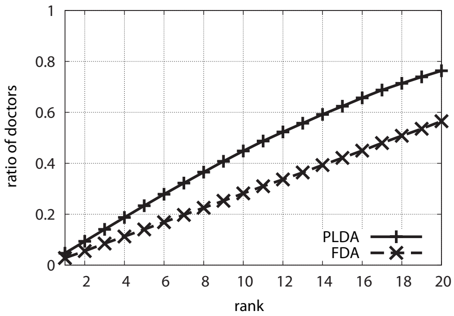

Figure 4: $\alpha = 1.0$

Next, Figures 5, 6, and 7 show the rank functions under $\alpha = 0.5$ when β is set to 0, 0.5, and 1.0, respectively. Here, in contrast to α, by increasing β (which represents the correlation in the hospitals' preferences), doctors' welfare increases slightly. We can confirm our conjecture in Section 6.2, i.e., when the preferences (over doctors) of hospitals within the same region becomes more *consistent*, then the doctors' welfare can increase. Again, we can see that PLDA first-order stochastically dominates FDA. This dominance holds regardless of the choice of β.

6.3.2 Doctors' Preference over Mechanisms

A more rough-grained, but more intuitive way to evaluate the superiority among two mechanisms would be directly asking each doctor which result she prefers. In this subsection, we investigate doctors' preferences over the obtained matchings of two mechanisms. In Figures 8, 9, and 10, the vertical axis presents the share of doctors for $\beta = 0$, 0.5, and 1.0, respectively. The domains at the top (or the bottom) indicate the share of doctors who are better off under FDA (or PLDA). The middle region indicates that doctors are indifferent between FDA and PLDA. The horizontal axis indicates α. We can see that for any combination of α and β, more doctors prefer the outcome of PLDA. Also, when α and β are high, the ratio of doctors who prefer the outcome of PLDA increases. In an extreme case where $\alpha = 1.0$

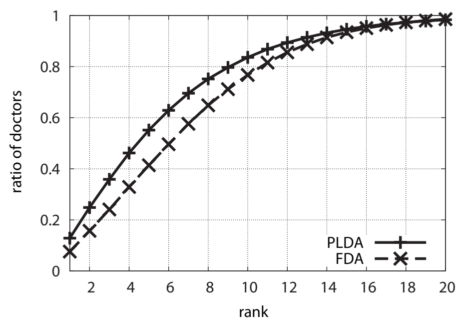

Figure 5: $\beta = 0$

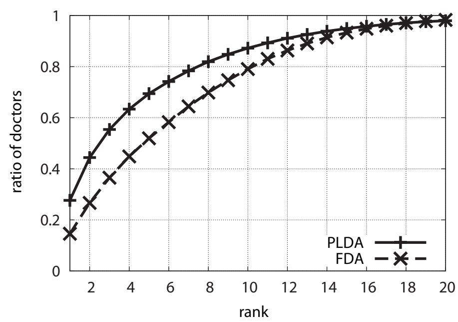

Figure 6: $\beta = 0.5$

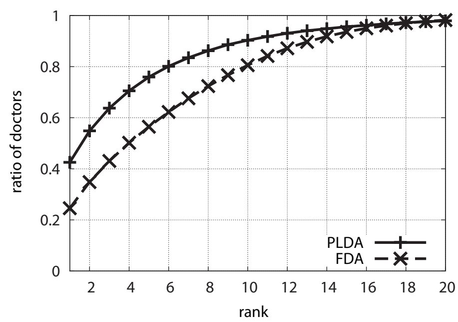

Figure 7: $\beta = 1.0$

and $\beta = 1.0$, a situation almost identical to the case described in Section 6.3 occurs. Thus, PLDA significantly outperforms FDA.

The above results show that the average ratio of doctors who prefer the result of PLDA is larger. However, the average share of votes does not directly reflect the share of winning candidates in an election when the small-constituency system is applied. Thus, instead of calculating the average ratio over different problem instances, we first compare the ratio for each problem instance, and count the number of problem instances where more doctors prefer the result of PLDA (or FDA). We can assume for each problem instance, each doctor votes for her preferred mechanism and the mechanism who obtain more votes wins. The following results show the winning ratio of each mechanism.

Figures 11 to 13 analyze the share of problem instances where one mechanism is preferred by more doctors than the other. The domain at the top (or the bottom) indicates the share of instances where FDA (or PLDA) wins. The middle (thin) region indicates where FDA and PLDA receive the same number of votes. The horizontal axis indicates α. These results show that PLDA is a clear winner if doctors decide which mechanism to use by majority voting.

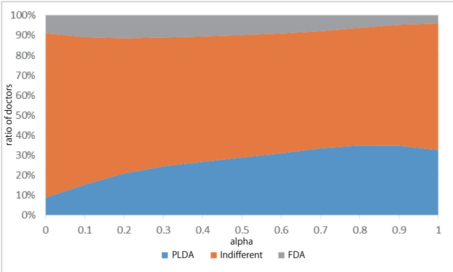

Figure 8: $\beta = 0$

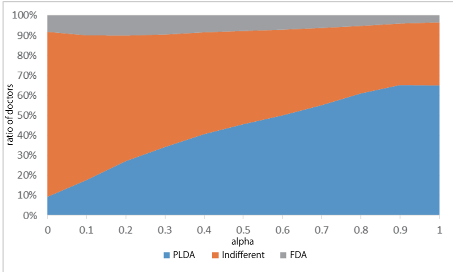

Figure 9: $\beta = 0.5$

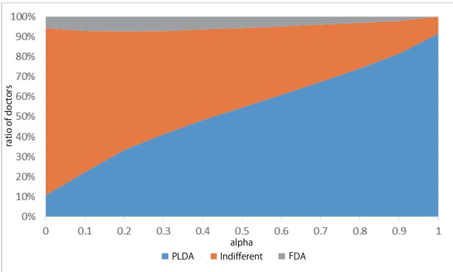

Figure 10: $\beta = 1.0$

6.3.3 Number of claiming doctors

Next, let us examine why doctors' welfare of PLDA is superior to that of FDA. Both mechanisms are fair, thus they cannot be nonwasteful (though they are weakly nonwasteful). Our conjecture is that although both mechanisms inevitably waste some seats, the number of wasted seats in PLDA would be smaller than that of FDA. As a result, the obtained result of PLDA is more *efficient* and doctors prefer the result of PLDA. To confirm this conjecture, we compare the number of doctors who claim empty seats of some hospitals. Figures 14, 15, and 16 indicate the ratio of such doctors under each mechanism for each $\beta = 0$, 0.5, and 1.0, respectively. The horizontal axis indicates α. We can see that for any combination of α and β, the ratio of doctors who claim empty seats is smaller in PLDA. Also, the difference between two mechanisms becomes large when α and β become large. As α becomes large, the competition among doctors, in particular, among doctors who are applying different hospitals in the same region, become more severe. As a result, more seats are tend to be wasted. However, PLDA can vary the number of doctors assigned to each hospital more flexibly according to its popularity. As a result, it wastes less seats than FDA. Furthermore, we can see that by increasing β, the ratio of claiming doctors becomes small. As described before, when the preferences (over doctors) of hospitals within the same region becomes more *consistent*, then less seats are wasted.

In an extreme case where $\beta = 1$, i.e., all hospitals use the same preference

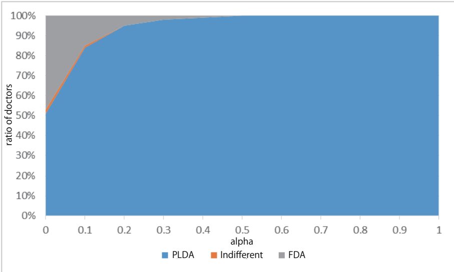

Figure 11: $\beta = 0$

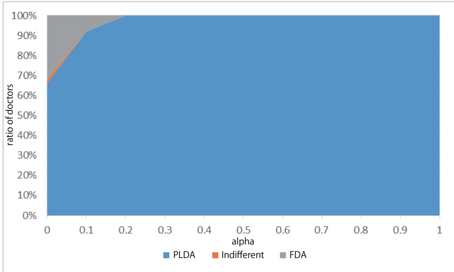

Figure 12: $\beta = 0.5$

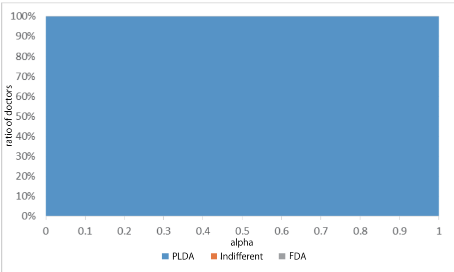

Figure 13: $\beta = 1.0$

ordering, PLDA is nonwasteful. Thus, the number of claiming students is zero.

7 Conclusion

This paper proposed PLDA mechanism that can handle regional caps. We introduced regional fairness, which is a refinement of standard fairness considering the competition among doctors who are applying to different hospitals in the same region. We show PLDA is group strategy-proof and always gives the doctor-optimal regionally fair matching. Comparing PLDA with FDA, we proved that PLDA has properties that FDA cannot satisfy and then confirmed via simulation that PLDA has an advantage over FDA in terms of doctors' welfare and efficiency.

We believe PLDA can be applied/extended in different problem settings. First, by slightly modifying PLDA, we can construct a mechanism that can handle individual minimum quotas, based on the method for transforming individual minimum quotas to regional caps in [9]. Furthermore, we hope to extend PLDA so that it can handle regional minimum quotas (as well as regional caps).

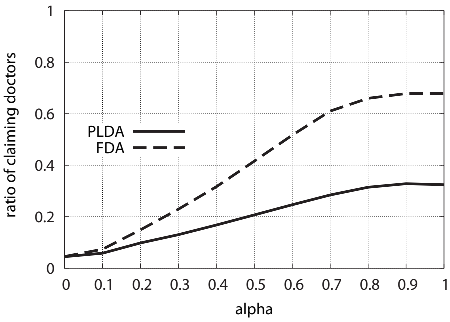

Figure 14: $\beta = 0$

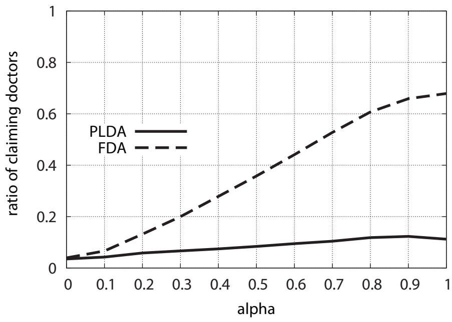

Figure 15: $\beta = 0.5$

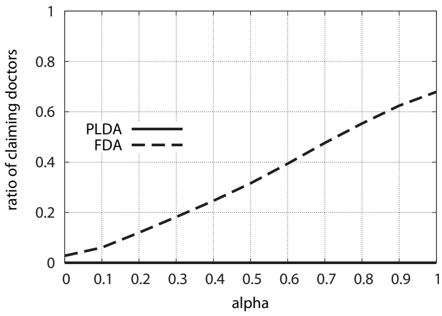

Figure 16: $\beta = 1.0$

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Appendix A Target Capacities

FDA can utilize the *target capacity* of each hospital, i.e., if the target capacity of hospital h is $\bar{q}_h$, then h can accept up to $\bar{q}_h$ doctors without competing with other hospitals in the same region, as long as enough doctors are applying to h. So far, for simplicity, we assume each target capacity is zero in FDA. Setting all target capacities to zero is equivalent to set each $\bar{q}_h$ to $q_r/|r|$ (assuming q_r is a multiple of $|r|$), since by using a round-robin ordering, each hospital can accept at least $q_r/|r|$ doctors.

As shown in Section 6.3, PLDA outperforms FDA in terms of nonwastefulness/efficiency, since it can vary the number of doctors assigned to each hospital more flexibly, while FDA tries to assign doctors among hospitals more equitably. However, a hospital might hope that it can accept a certain number of doctors as long as enough doctors are applying to it.

In this section, we describe how to introduce target capacities in PLDA. We first define target capacity. The set of target capacities is denoted by $\bar{q}_H = \{\bar{q}_{h_1}, \bar{q}_{h_2}, \ldots\}$. We assume $\bar{q}_h \leq q_h$ for any $h \in H$ and $\sum_{h \in r} \bar{q}_h \leq q_r$ for any $r \in R$. The sum of artificial caps must be exactly equal to q_r. On the other hand, target capacities are soft constraints and a hospital can assign more doctors beyond its target capacity. Thus, the sum of target capacities can be less than q_r. A market with target capacities is represented by a tuple $(D, H, R, q_H, \bar{q}_H, q_R, \succ_D, \succ_H, \succ_R^{PL})$.

To apply PLDA to target capacities, we transform the original market with target capacities into another market without target capacities $(D, \tilde{H}, \tilde{R}, \tilde{q}_H, \tilde{q}_{\tilde{R}}, \tilde{\succ}_D, \tilde{\succ}_{\tilde{H}}, \tilde{\succ}_{\tilde{R}}^{PL})$. In the transformed market, the set of doctors is unchanged. Hospital h in the original market with maximum quota q_h and target capacity $\bar{q}_h$ is transformed into two hospitals: *reserved hospital* h^*, which has a maximum quota of $\tilde{q}_{h^*} = \bar{q}_h$, and *standard hospital*, which, with slight abuse of notation, we label h, and that has a maximum quota of $\tilde{q}_h = q_h - \bar{q}_h$. Each of these hospitals uses the original preference ordering of hospital h, i.e., $\tilde{\succ}_{h^*} = \tilde{\succ}_h = \succ_h$. Thus, the set of hospitals is $\tilde{H} = H \cup H^* = \{h_1, \ldots, h_m, h_1^*, \ldots, h_m^*\}$.

For each doctor $d \in D$, a preference over $H \cup H^*$ is created by taking the original preference relation $\succ_d$ and inserting hospital h^* immediately before hospital h. That is, preference $\succ_d$: $h_j \succ_d h_k \succ_d \ldots$ is transformed into $\tilde{\succ}_d :\ h_j^* \tilde{\succ}_d h_j \tilde{\succ}_d h_k^* \tilde{\succ}_d h_k \tilde{\succ}_d \ldots$.

Also, the set of regions $\tilde{R}$ is basically the same as the original market, but each region $\tilde{r}$ includes both reserved hospitals and standard hospitals created from hospitals within r in the original market. The regional cap of $\tilde{r}$ is equal to that of r. Also, region $\tilde{r}$'s priority list over $D \times \tilde{r}$ is created as follows. First, we take the original priority list $\succ_r^{PL}$ and divide its copy for the reserved hospitals into three parts: The first part is the list before $(\emptyset, \emptyset)$, the second one is only $(\emptyset, \emptyset)$, and the third one is the list after $(\emptyset, \emptyset)$. Then, we insert the first part before the original priority list and the third part after $(\emptyset,\emptyset)$ in the original priority list (the second part is not used). That is, priority list

$$\begin{aligned}\succ_r^{PL}\colon\ & (d_1,h_1),\ldots,(d_k,h_l),\\ & (\emptyset,\emptyset),(d_s,h_t),\ldots,(d_x,h_y)\end{aligned}$$

becomes

$$\begin{aligned}\tilde{\succ}_{\tilde{r}}^{PL}:\ & (d_i,h_j^*),\ldots,(d_k,h_l^*),(d_i,h_j),\ldots,(d_k,h_l),\\ & (\emptyset,\emptyset),(d_s,h_t^*),\ldots,(d_x,h_y^*),(d_s,h_t)\ldots,(d_x,h_y).\end{aligned}$$

For instance, assume the setting in Example 1, where $\bar{q}_{h_1}=1$ and $\bar{q}_{h_2}=1$. Then, we can define the priority list as follows:

$$\begin{aligned}\tilde{\succ}_{\tilde{r}_1}^{PL}:\ & (d_1,h_1^*)\,,(d_1,h_2^*)\,,(d_2,h_2^*)\,,\\ & (d_1,h_1)\,,(d_1,h_2)\,,(d_2,h_2)\,,\\ & (\emptyset,\emptyset)\,,(d_2,h_1^*)\,,(d_2,h_1)\,.\end{aligned}$$

Actually, some part of this priority list is never used. For example, since h_1^* always accept d_1 if she applies to h_1^*, d_1 never applies to h_1. Also, the ordering among pairs for different reserved hospitals does not matter. For example, the fact (d_1,h_1^*) appears before (d_1,h_2^*) does not matter. This is because regional conflict never occurs among reserved hospitals, i.e., the regional cap is not binding for them.

Given a corresponding market $(D,\tilde{H},\tilde{R},\tilde{q}_H,\tilde{q}_{\tilde{R}},\tilde{\succ}_D,\tilde{\succ}_{\tilde{H}},\tilde{\succ}_{\tilde{R}}^{PL})$, we can apply PLDA to this market. Let us denote $\tilde{\mu}$ be the obtained matching in the corresponding market. Then, we can obtain a matching μ in the original market by $\mu(h):=\tilde{\mu}(h)\cup\tilde{\mu}(h^*)$ for each $h\in H$. μ is clearly feasible.

The notions of regional fairness and regional nonwastefulness are modified as follows.

Definition 16 *Given a matching μ, doctor d has* ***regionally justifiable envy*** *toward doctor $d'\neq d$, who is assigned to h'* ***under target capacities****, if, for some h such that $r(h)=r(h')=r$, $h\succ_d\mu(d)$, $|\mu(h)|<q_h$, $|\mu(h')|>\bar{q}_{h'}$, and $(d,h)\succ_r^{PL}(d',h')$.*

Compared to the original definition of regional justifiable envy, this condition is more restricted. Doctor d is allowed to have justifiable envy towards another doctor d' who is assigned to h', only when the number of doctors assigned to h' exceeds its target capacity $q_{h'}$. This is because h' is guaranteed to assign up to $q_{h'}$ doctors without any competition in the region.

Definition 17 *A feasible matching $\mu\in\mathcal{M}$ is* ***regionally fair under target capacities*** *if there is no doctor who has justifiable envy or regionally justifiable envy under target capacities toward another doctor.*

On the other hand, the notion of regional nonwasteful under target capacities is not a simple restriction of the original one.

Definition 18 *Given a matching μ, doctor d, who is assigned to h, regionally claims an empty seat of hospital h' under target capacities if $h' \succ_d h$, $|\mu(h')| < q_{h'}$, and either one of the following conditions hold:*

(i) $|\mu(h')| < \bar{q}_{h'}$,

(ii) $|\mu(h')| \geq \bar{q}_{h'}$, $r(h) = r(h') = r$, $|\mu(r)| = q_r$, $|\mu(h)| > \bar{q}_h$, *and* $(d, h') \succ_r^{PL} (d, h)$.

Note that d can claim an empty seat of h' if the target capacity of h' is not filled (regardless of d's position in the priority list). On the other hand, if the target capacity of h' is filled, to claim an empty seat, d must be allocated in a hospital that accepts more doctors than its target capacity, i.e., at least one doctor is allocated in standard hospital h in the transformed market.

Definition 19 *A feasible matching $\mu \in \mathcal{M}$ is **regionally nonwasteful under target capacities** if no doctor strongly claims an empty seat of any hospital or regionally claims an empty seat of any hospital under target capacities.*

The following theorem holds.

Theorem 3 *A matching obtained by PLDA is regionally fair and regionally nonwasteful under target capacities.*

Proof. Let us assume μ is derived from $\tilde{\mu}$ in the transformed market. It is straightforward that no doctor has justifiable envy or strongly claim an empty seat in μ; otherwise, some doctor also has the same justifiable envy or strong claim in $\tilde{\mu}$.

Suppose doctor d has regionally justifiable envy under target capacities toward d', who is assigned to $h' \in r$. Also, there must be another hospital $h \in r$ such that $h \succ_d \mu(d)$, $|\mu(h)| < q_h$, $|\mu(h')| > \bar{q}_{h'}$, and $(d, h) \succ_r^{PL} (d', h')$ hold. If d' is assigned to standard hospital h' in the extended market, d must have regionally justifiable envy toward d' even in $\tilde{\mu}$. Thus, let us assume d' is assigned to reserved hospital h'^*. There exists at least one doctor d'' who is assigned to standard hospital h' in the transformed market, since $|\mu(h')| > \bar{q}_{h'}$. Then, d must have regionally justifiable envy toward d'', since d'' was rejected by h'^* and hence $(d, h) \tilde{\succ}_{\tilde{r}}^{PL} (d', h') \tilde{\succ}_{\tilde{r}}^{PL} (d'', h')$.

Next, suppose doctor d, who is assigned to h, regionally claims an empty seat of $h' \in r$, under target capacities. First, let us assume $|\mu(h')| < \bar{q}_{h'}$. Then, $|\tilde{\mu}(h'^*)| < \tilde{q}_{h'^*}$ holds. Also, $|\tilde{\mu}(\tilde{r})| < \tilde{q}_{\tilde{r}}$ must hold since $h'^* \in \tilde{r}$. Then, d must strongly claim an empty seat of h'^* even in $\tilde{\mu}$.

Next, let us assume $|\mu(h')| \geq \bar{q}_{h'}$. Then, from the definition of regionally claims an empty seat under target capacities, $r(h) = r$, $|\mu(r)| = q_r$, $|\mu(h)| > \bar{q}_h$, and $(d, h') \succ_r^{PL} (d, h)$ hold. These condition implies that there exists at least one doctor d' who is assigned to standard hospital h in $\tilde{\mu}$.

If d is assigned to standard hospital h, d must regionally claim an empty seat of h' even in $\tilde{\mu}$, since $(d, h')\tilde{\succ}_{\tilde{r}}^{PL}(d, h)$ holds. If d is assigned to reserved hospital h^*, then $d\tilde{\succ}_{h^*} d'$ holds since d' applies to h^* but is rejected in some stage. Hence, $(d, h)\tilde{\succ}_{\tilde{r}}^{PL} (d', h)$. By transitivity, $(d, h')\tilde{\succ}_{\tilde{r}}^{PL} (d, h)$ and $(d, h)\tilde{\succ}_{\tilde{r}}^{PL} (d', h)$ imply $(d, h')\tilde{\succ}_{\tilde{r}}^{PL} (d', h)$. Then, d must have regionally justifiable envy toward d' even in $\tilde{\mu}$. ■

This theorem shows that PLDA can handle target capacities and still satisfies regional fairness and regional nonwastefulness extended for target capacities.

Appendix B Generalized Priority List

This section proposes a class of mechanisms called generalized PLDA, which included both PLDA and FDA as extreme cases.

In PLDA, for each region r, one priority list is defined. In generalized PLDA, for each region, a round-robin ordering among *multiple* priority lists are defined. Let us assume for region r, there exist k priority lists $PL_1, \ldots, PL_k$. Each PL_j contains pairs in $D \times r$. We assume each pair $(d, h) \in D \times r$ is contained in exactly one priority list. Then, these multiple priority lists are connected by a round-robin ordering, e.g., $PL_1 \rightarrow PL_2 \rightarrow \ldots \rightarrow PL_k \rightarrow PL_1 \rightarrow \ldots$.

By using this round-robin ordering among priority lists, the choice of each region in generalized PLDA is described as follows. First, one priority list PL_j is chosen according to the round-robin ordering. Then, (d, h), which is placed in the top of PL_j and is not tentatively accepted or rejected yet, is chosen. If d is acceptable to h and accepting (d, h) does not violate the maximum quota or regional cap, then tentatively accept (d, h). Otherwise, reject (d, h). Then, move to the next priority list according to the round-robin ordering.

We can consider PLDA is a special case of generalized PLDA, in which only one priority list is used for each region. Furthermore, FDA is one instance of generalized PLDA. To be more precise, assume r contains hospitals $h_1, \ldots, h_k$. Then, there exists k priority lists $PL_1, \ldots, PL_k$ for r. Each PL_j contains pairs related to hospital h_j, and the order in PL_j is identical to the preference ordering of h_j.

Let us show one example of generalized PLDA, which is a hybrid-type mechanism of FDA and PLDA. Assume r is divided into two sub-regions r_1 and r_2. Also, the hospitals in r hope that within each sub-region, obtained matching μ should be fair and efficient as much as possible, while $|\mu(r_1)|$ and $|\mu(r_2)|$ (i.e., the number of doctors assigned to each sub-region) are equitable as much as possible. Then, r can create two priority lists PL_{r_1} and PL_{r_2}, each of which contains pairs related to hospitals within the sub-region, and apply round-robin ordering $PL_{r_1} \rightarrow PL_{r_2} \rightarrow PL_{r_1} \rightarrow \ldots$ to achieve these goals.

In general, the generalized PLDA is not always fair. However, the mechanism can satisfy fairness if all pairs related with each hospital h are included in only one priority list, and the priority list respects the preference ordering of h. We can satisfy this condition in the hybrid mechanism described above (as well as PLDA and FDA, which are represented as special cases of generalized PLDA).

Appendix C The properties of PLDA

We prove several properties of PLDA by reformulating our model within a matching-with-contract model in Hatfield and Milgrom [16]. In the reformulated model, we regard regions instead of hospitals as agents, i.e., we consider a matching between doctors and regions. To be more precise, there is a set of contracts $X = D \times H$, where a contract is a pair of a doctor and a hospital. We sometimes call a subset X' of X an *allocation*. For the ease of expression, x_D and x_H respectively denote the doctor and the hospital concerned with $x \in X$. Also, let us define $X'_d := \{x \mid x \in X', x_D = d\}$, $X'_h := \{x \mid x \in X', x_H = h\}$, and $X'_r := \bigcup_{h \in r} X_h$, for any $X' \subseteq X$

In the model, each doctor can sign at most one contract associated with herself. For each doctor d, her preference $\succ_d$ over $(\{d\} \times H) \cup \{\emptyset\}$ is defined according to her preference in the original, that is, $(d, h) \succ_d (d, h')$ (or $(d, h) \succ_d \emptyset$) in this model if and only if $h \succ_d h'$ (or $h \succ_d \emptyset$) in the original model. According to the preference, each doctor d's choice function Ch_d is given as follows: for any $X' \subseteq X$,

$$Ch_d\left(X'\right) := \begin{cases} \emptyset & \text{if } \{x \in X'_d \mid x \succ_d \emptyset\} = \emptyset, \\ \max_{\succ_d}\{x \in X'_d \mid x \succ_d \emptyset\} & \text{otherwise.} \end{cases}$$

On the other hand, we assume that each region r has $\succ_r^{PL}$ as a preference over contracts in r. Hence, given a set of contracts X', the chosen set by r, $Ch_r(X')$, is a set of q_r (or less) most preferable contracts according to $\succ_r^{PL}$ under the maximum quotas of the hospitals in r. More specifically, Ch_r satisfies the following condition: for any $X' \subseteq X$, $(d, h) \in Ch_r\left(X'\right)$ if and only if

$$\begin{cases} h \in r, \\ d \succ_h \emptyset, \\ \left|\left\{(d', h) \mid (d', h) \in X', (d', h) \succ_r^{PL} (d, h)\right\}\right| < q_h, \\ \sum_{h' \in r} \min\{\left|\left\{(d', h') \mid (d', h') \in X', (d', h') \succ_r^{PL} (d, h)\right\}\right|, q_{h'}\} < q_r. \end{cases}$$

Intuitively, $Ch_r(X')$ chooses a contract one by one in the order of $\succ_r^{PL}$, as long as d is acceptable to h and accepting contract (d, h) does not violate the maximum quota of h (the condition in the middle) and the regional cap of r (the condition in the bottom). The rejection function Re_r is defined by $Re_r(X') := X' - Ch_r(X')$ for any $r \in R$ and any $X' \subseteq X$.

In addition, we define choice functions and rejection functions for R and D as follows:

$$\begin{aligned}
Ch_R(X') &:= \bigcup_{r \in R} Ch_r(X'), \\
Re_R(X') &:= X' - Ch_R(X'), \\
Ch_D(X') &:= \bigcup_{d \in D} Ch_d(X'), \\
Re_D(X') &:= X' - Ch_D(X').
\end{aligned}$$

Definition 20 (Hatfield and Milgrom [16]) *Choice function Ch_r satisfies the* ***substitutes condition*** *if $Re_r(X') \subseteq Re_r(X'')$ whenever $X' \subseteq X'' \subseteq X$.*

In other words, contracts are substitutes if any rejected contract is never chosen even when other contracts are added to alternatives.

Definition 21 (Hatfield and Milgrom [16]) *Choice function Ch_r satisfies the* ***law of aggregate demand*** *if $|Ch_r(X')| \leq |Ch_r(X'')|$ whenever $X' \subseteq X'' \subseteq X$.*

In addition to the above two condition, we introduce a notion of the irrelevance of rejected contracts. Aygün and Sönmez [4] point out that the condition is implicitly assumed in Hatfield and Milgrom [16] but plays an important role in their results.

Definition 22 (Aygün and Sönmez [4]) *Choice function Ch_r satisfies the* ***irrelevance of rejected contracts*** *if, for any $Y \subseteq X$ and any $z \in X \setminus Y$, $Ch_r(Y) = Ch_r(Y \cup \{z\})$ whenever $z \notin Ch_r(Y \cup \{z\})$.*

Proposition 1 *For each region r, the choice function Ch_r defined above satisfies the substitutes condition, the law of aggregate demand and the irrelevance of rejected contracts.*

The proof is presented in Appendix D.1.

A set of contracts $X' \subseteq X$ is said to *feasible* if $|X'_d| \leq 1$ for any $d \in D$, $|X'_h| \leq q_h$ for any $h \in H$ and $\sum_{h \in r} |X'_h| \leq q_r$ for any $r \in R$. Then, we define a notion of stability according to Hatfield and Milgrom [16].

Definition 23 (Hatfield and Milgrom [16]) *A set of contract $X' \subseteq X$ is* ***HM-stable*** *if and only if (i) $X' = Ch_D(X') = Ch_R(X')$, (ii) there is no $(d,h) \notin X'$ such that $(d,h) \in Ch_D(X' \cup (d,h))$ and $(d,h) \in Ch_R(X' \cup (d,h))$.*

Note that condition (i) says not only that any contract $x \in X'$ is acceptable both for x_D and for x_H but also that X' does not violate any maximum quota or any regional cap.

Given a feasible allocation $X' \subseteq X$, we say that matching μ *corresponds* to X' if $\mu(d)$ is h whenever $(d, h) \in X'$ and $\emptyset$ whenever there is no contract associated with d in X'.

Although Hatfield and Milgrom [16] consider the matching between the doctors and the hospitals, we apply their model to the matching between the doctors and the regions. Therefore HM-stability is not directly meaningful in our application. Thus, we translate HM-stability to desirable properties as follows.

Proposition 2 *X' is HM-stable if and only if a matching μ that corresponds to X' is regionally fair and regionally nonwasteful.*

The proof is presented in Appendix D.2.

We say that allocation $X' \subseteq X$ is a **doctor-optimal HM-stable allocation** in the reformulated model if X' is a HM-stable allocation that is preferred to any other HM-stable allocation by each doctor. When every region's choice function satisfies the substitutes condition, the law of aggregate demand and the irrelevance of rejected contracts, the doctor-offering generalized Gale-Shapley algorithm (Hatfield and Milgrom, 2005) finds a doctor-optimal HM-stable allocation in a matching-with-contract model. We will prove that, in the original model, PLDA produces a matching that corresponds to the allocation.

Proposition 3 *There exists the doctor-optimal HM-stable allocation X' in the reformulated model. Moreover, in the original model, PLDA terminates in a finite stages and finally produces a matching that corresponds to X'.*

Proof. As shown before, each region r's choice function Ch_r in the reformulated model satisfies the substitutes condition, the law of aggregate demand and the irrelevance of rejected contracts. Therefore, by Hatfield and Milgrom [16], the doctor-offering generalized Gale-Shapley algorithm (DGGS) produces the doctor-optimal HM-stable allocation in a finite stages. Moreover, each stage of PLDA corresponds to a stage of the DGGS. Indeed, if d applies for h at a stage in PLDA, then (d, h) is offered at the same stage in the DGGS. Also, the set of doctors tentatively accepted/rejected by hospitals at each stage in PLDA corresponds to the set of contracts chosen/rejected at the same stage of the DGGS. ■

By the results of Hatfield and Kojima [14], PLDA is group strategy-proof as it produces a matching that corresponds the doctor-optimal HM-stable allocation. Finally we summarize the properties of PLDA we found in this section.

Conclusion 1 *PLDA is group strategy-proof and produces the doctor-optimal regionally fair and regionally nonwasteful matching.*

From this conclusion, we obtain Theorem 1.

Furthermore, by the above conclusion and the following lemma, we can prove that the doctor-optimal regionally fair and regionally nonwasteful matching is the doctor-optimal even in a broader set, i.e., the set of regionally fair matchings.

Lemma 1 *Any doctor-maximal regionally fair matching is regionally nonwasteful.*

The proof is in Appendix D.3.

Finally, we prove Theorem 2. Assume μ^* is the matching obtained by PLDA. By Propostion 3, μ^* is the doctor-optimal regionally fair and regionally nonwasteful matching. By way of contradition, let us assume there exists amother matching μ', where μ' is regionally fair and $\mu^* \succeq_D \mu'$ does not hold. However, by Lemma 1, μ' is also regionally nonwasteful. This contradicts the fact that μ^* is the doctor-optimal regionally fair and regionally nonwasteful matching.

Appendix D Proofs of propositions and lemmas

Appendix D.1 Proof of Proposition 1

First, we show that Ch_r satisfies the irrelevance of rejected contracts, i.e, for any $X' \subseteq X$ and any $(d,h) \in X \setminus X'$, $(d,h) \notin Ch_r(X' \cup \{(d,h)\})$ implies $Ch_r(X') = Ch_r(X' \cup \{(d,h)\})$. Since, if $h \notin r$ or $\emptyset \succ_h d$, then adding (d,h) does not affect Ch_r at all, without loss of generality, we assume $h \in r$ and $d \succ_h \emptyset$.

By way of contradiction, suppose that $Ch_r(X') \neq Ch_r(X' \cup \{(d,h)\})$. If there exists a contract $(\hat{d},\hat{h}) \in Ch_r(X' \cup \{(d,h)\}) \setminus Ch_r(X')$, $(\hat{d},\hat{h})$ is not included in $Ch_r(X')$. Thus,

$$|\{(d',\hat{h}) \mid (d',\hat{h}) \in X', (d',\hat{h}) \succ_r^{PL} (\hat{d},\hat{h})\}| \geq q_{\hat{h}}, \text{ or} \quad (1)$$

$$\sum_{h' \in r} \min\{|\{(d',h') \mid (d',h') \in X', (d',h') \succ_r^{PL} (\hat{d},\hat{h})\}|, q_{h'}\} \geq q_r. \quad (2)$$

hold.

Let us consider the case where Eq. 1 holds. Then,

$$\begin{aligned} q_{\hat{h}} &\leq |\{(d',\hat{h}) \mid (d',\hat{h}) \in X', (d',\hat{h}) \succ_r^{PL} (\hat{d},\hat{h})\}| \\ &\leq |\{(d',\hat{h}) \mid (d',\hat{h}) \in X' \cup \{(d,h)\}, (d',\hat{h}) \succ_r^{PL} (\hat{d},\hat{h})\}| \end{aligned}$$

This inevitably contradicts that $(\hat{d},\hat{h}) \in Ch_r(X' \cup \{(d,h)\})$. Similarly, for the case where Eq. 2 holds,

$$\begin{aligned} q_r &\leq \textstyle\sum_{h' \in r} \min\{|\{(d',h') \mid (d',h') \in X', (d',h') \succ_r^{PL} (\hat{d},\hat{h})\}|, q_{h'}\} \\ &\leq \textstyle\sum_{h' \in r} \min\{|\{(d',h') \mid (d',h') \in X' \cup \{(d,h)\}, (d',h') \succ_r^{PL} (\hat{d},\hat{h})\}|, q_{h'}\} \end{aligned}$$

leads to a contradiction.

Let us turn to suppose that there is a contract $(\hat{d}, \hat{h}) \in Ch_r(X') \setminus Ch_r(X' \cup \{(d,h)\})$. In this case, $(\hat{d}, \hat{h})$ is not included in $Ch_r(X' \cup \{(d,h)\})$ and thus

$$|\{(d', \hat{h}) \mid (d', \hat{h}) \in X' \cup \{(d,h)\}, (d', \hat{h}) \succ_r^{PL} (\hat{d}, \hat{h})\}| \geq q_{\hat{h}}, \text{ or}$$
$$\sum_{h' \in r} \min\{|\{(d', h') \mid (d', h') \in X' \cup \{(d,h)\}, (d', h') \succ_r^{PL} (\hat{d}, \hat{h})\}|, q_{h'}\} \geq q_r.$$

holds. The contract (d,h) must be ranked higher than $(\hat{d}, \hat{h})$ on $\succ_r^{PL}$, i.e., $(d,h) \succ_r^{PL} (\hat{d}, \hat{h})$, otherwise, the set of contracts ranked higher than $(\hat{d}, \hat{h})$ in $Ch_r(X')$ on $\succ_r^{PL}$ is equivalent to that set in $Ch_r(X' \cup \{(d,h)\})$, violating the assumption that $(\hat{d}, \hat{h}) \notin Ch_r(X' \cup \{(d,h)\})$.

Therefore, we have

$$\begin{aligned} q_{\hat{h}} &> |\{(d', \hat{h}) \mid (d', \hat{h}) \in X', (d', \hat{h}) \succ_r^{PL} (\hat{d}, \hat{h})\}| \\ &\geq |\{(d', \hat{h}) \mid (d', \hat{h}) \in X', (d', \hat{h}) \succ_r^{PL} (d, h)\}| \\ &= |\{(d', \hat{h}) \mid (d', \hat{h}) \in X' \cup \{(d,h)\}, (d', \hat{h}) \succ_r^{PL} (d, h)\}| \end{aligned}$$

and

$$\begin{aligned} q_r &> \textstyle\sum_{h' \in r} \min\{|\{(d', h') \mid (d', h') \in X', (d', h') \succ_r^{PL} (\hat{d}, \hat{h})\}|, q_{h'}\} \\ &\geq \textstyle\sum_{h' \in r} \min\{|\{(d', h') \mid (d', h') \in X', (d', h') \succ_r^{PL} (d, h)\}|, q_{h'}\} \\ &= \textstyle\sum_{h' \in r} \min\{|\{(d', h') \mid (d', h') \in X' \cup \{(d,h)\}, (d', h') \succ_r^{PL} (d, h)\}|, q_{h'}\}. \end{aligned}$$

Thus, $(d,h) \in Ch_r(X' \cup \{(d,h)\})$ holds, which leads to a contradiction.

Next, we show that Ch_r satisfies substitute condition. It is sufficient to show that $Re_r(X') \subseteq Re_r(X' \cup \{x\})$ holds for any $X' \subseteq X$ and any $x \in X \setminus X'$. Suppose that $Re_r(X') \subseteq Re_r(X' \cup \{x\})$ does not hold, i.e., $Re_r(X') \setminus Re_r(X' \cup \{x\})$ is non-empty. Then, let us choose any $(d,h) \in Re_r(X') \setminus Re_r(X' \cup \{x\})$. (d,h) is in $Re_r(X')$ and then $(d,h) \in X'$, while x is not in X' from $x \in X \setminus X'$. Clearly, (d,h) is a different contract from x. As a result, $(d,h) \in Ch_r(X' \cup \{x\})$ implies the followings.

$$\begin{aligned} &h \in r, \\ &d \succ_h \emptyset, \\ &\left|\{(d', h) \mid (d', h) \in X' \cup \{x\}, (d', h) \succ_r^{PL} (d, h)\}\right| < q_h, \text{ and} \\ &\textstyle\sum_{h' \in r} \min\{\left|\{(d', h') \mid (d', h') \in X' \cup \{x\}, (d', h') \succ_r^{PL} (d, h)\}\right|, q_h\} < q_r. \end{aligned}$$

It is straightforward that, even if reducing x from $X' \cup \{x\}$, the followings also hold.

$$\begin{aligned} &h \in r, \\ &d \succ_h \emptyset, \\ &\left|\{(d', h) \mid (d', h) \in X', (d', h) \succ_r^{PL} (d, h)\}\right| < q_h, \text{ and} \\ &\textstyle\sum_{h' \in r} \min\{\left|\{(d', h') \mid (d', h') \in X', (d', h') \succ_r^{PL} (d, h)\}\right|, q_h\} < q_r. \end{aligned}$$

Therefore, we have $(d,h) \in Ch_r(X')$. This contradicts that $(d,h) \in Re_r(X')$.

Finally, we show that Ch_r satisfies the law of aggregated demand. It is sufficient to show that $|Ch_r(X')| \leq |Ch_r(X' \cup \{x\})|$ holds for any $X' \subseteq X$ and any $x \in X \setminus X'$.

By way of contradiction, suppose that $|Ch_r(X')| > |Ch_r(X' \cup \{(d,h)\})|$ holds for some set of contracts $X' \subseteq X$ and some contract $(d,h) \in X \setminus X'$. If $(d,h) \notin Ch_r(X' \cup \{(d,h)\})$, then $Ch_r(X') = Ch_r(X' \cup \{(d,h)\})$ holds from the irrelevance of rejected contracts. This implies $|Ch_r(X')| = |Ch_r(X' \cup \{(d,h)\})|$ and hence leads to a contradiction. Thus, we have $(d,h) \in Ch_r(X' \cup \{(d,h)\})$. Then there must be two distinct contracts $(d_1,h_1), (d_2,h_2) \in X'$ such that $(d_1,h_1), (d_2,h_2) \in Ch_r(X') \setminus Ch_r(X' \cup \{(d,h)\})$ $(= Ch_r(X') \cap Re_r(X' \cup \{(d,h)\}))$. Clearly, both h_1 and h_2 belong to r and d_1 and d_2 are acceptable to h_1 and h_2, respectively. With loss of generality, let us assume that $(d_1,h_1) \succ_r^{PL} (d_2,h_2)$. Since $(d_1,h_1) \in Ch_r(X') \cap Re_r(X' \cup \{(d,h)\})$, either of the following two holds:

$$\text{(i)} \sum_{h' \in r} \min\{\left|\{(d',h') \mid (d',h') \in X' \cup \{(d,h)\}, (d',h') \succ_r^{PL} (d_1,h_1)\}\right|, q_{h'}\} \geq q_r, \quad (3)$$

or

$$\text{(ii)} h_1 = h \text{ and } \left|\{(d',h) \mid (d',h) \in X' \cup \{(d,h)\}, (d',h) \succ_r^{PL} (d_1,h_1)\}\right| \geq q_h. \quad (4)$$

Consider the case where Equation (3) holds. Since $(d_1,h_1) \succ_r^{PL} (d_2,h_2)$,

$$\begin{aligned} q_r &\leq \sum_{h' \in r} \min\{|\{(d',h') \mid (d',h') \in X' \cup \{(d,h)\}, (d',h') \succ_r^{PL} (d_1,h_1)\}|, q_{h'}\} \\ &\leq 1 + \sum_{h' \in r} \min\{|\{(d',h') \mid (d',h') \in X', (d',h') \succ_r^{PL} (d_1,h_1)\}|, q_{h'}\} \\ &< 1 + \sum_{h' \in r} \min\{|\{(d',h') \mid (d',h') \in X', (d',h') \succ_r^{PL} (d_2,h_2)\}|, q_{h'}\}. \end{aligned}$$

Thus, we have

$$q_r \leq \sum_{h' \in r} \min\{|\{(d',h') \mid (d',h') \in X', (d',h') \succ_r^{PL} (d_2,h_2)\}|, q_{h'}\}.$$

This implies $(d_2,h_2) \notin Ch_r(X')$, which is a contradiction. On the other hand, consider the case where Equation (4) holds. Then

$$\begin{aligned} q_h &\leq |\{(d',h) \mid (d',h) \in X' \cup \{(d,h)\}, (d',h) \succ_r^{PL} (d_1,h_1)\}| \\ &\leq 1 + |\{(d',h) \mid (d',h) \in X', (d',h) \succ_r^{PL} (d_1,h_1)\}| \\ &< 1 + |\{(d',h) \mid (d',h) \in X', (d',h) \succ_r^{PL} (d_2,h_2)\}|. \end{aligned}$$

Thus, we have

$$q_h \leq |\{(d',h) \mid (d',h) \in X', (d',h) \succ_r^{PL} (d_2,h_2)\}|. \quad (5)$$

If $h_2 = h$, then Equation (5) implies $(d_2,h_2) \notin Ch_r(X')$, which is a contradiction. If $h_2 \neq h$, the following equation holds for any $h' \neq h$:

$$\begin{aligned} &|\{(d',h') \mid (d',h') \in X' \cup \{(d,h)\}, (d',h') \succ_r^{PL} (d_2,h_2)\} \\ &= |\{(d',h') \mid (d',h') \in X', (d',h') \succ_r^{PL} (d_2,h_2)\}. \end{aligned} \quad (6)$$

By Equations (5) and (6), we have

$$\begin{aligned}&\textstyle\sum_{h'\in r}\min\{|\{(d',h') \mid (d',h')\in X', (d',h')\succ_r^{PL}(d_2,h_2)\}|, q_{h'}\}\\ &= \textstyle\sum_{h'\in r}\min\{|\{(d',h') \mid (d',h')\in X'\cup\{(d,h)\}, (d',h')\succ_r^{PL}(d_2,h_2)\}|, q_{h'}\}.\end{aligned} \tag{7}$$

Equations (6) (for h_2) and (7) contradicts $(d_2,h_2)\in Ch_r(X')\cap Re_r(X'\cup\{(d,h)\})$.

Appendix D.2 The proof of Proposition 2

($\Rightarrow$) We prove that, if X' is HM-stable, μ is regionally fair and regionally nonwasteful.

Since $X' = Ch_D(X') = Ch_R(X')$, X' is feasible, and for any contract $(d,h)\in X'$, d and h are acceptable to each other. Hence, μ is also a feasible matching.

Suppose that some doctor d has justifiable envy toward doctor d' under μ. Let $h := \mu(d)$ and $h' := \mu(d')$. Then, there are $(d,h),(d',h')\in X'$ satisfying $h'\succ_d h$ and $d\succ_{h'} d'$. Hence we have $(d,h')\in Ch_d(X'\cup\{(d,h')\})$ and $(d,h')\succ_r^{PL}(d',h')$ where $r\ni h'$. These contradict the HM-stability. Also, if some doctor d has regionally justifiable envy toward doctor d', then there exist region r, hospitals $\hat{h},h'\in r$, and contracts $(d,h),(\hat{d},\hat{h})\in X'$ such that $h'\succ_d h$, $|\mu(h')|<q_{h'}$ and $(d,h')\succ_r^{PL}(\hat{d},\hat{h})$. Then, we have $(d,h')\in Ch_r(X'\cup\{(d,h')\})$ and $(d,h')\in Ch_d(X'\cup\{(d,h')\})$. These contradict the HM-stability of X'. Hence, μ is regionally fair.

Furthermore, suppose that some doctor d, who is assigned to h under μ, strongly claims an empty seat of $h'\in r$. Then, $h'\succ_d h$, $|\mu(h')|<q_{h'}$, and $|\mu(r)|<q_r$ hold. Then, we have $(d,h')\in Ch_r(X'\cup\{(d,h')\})$ and $(d,h')\in Ch_d(X'\cup\{(d,h')\})$. These contradict the HM-stability of X'. Also, let us suppose that some doctor d, who is assigned to $h\in r$ under μ, regionally claims an empty seat of $h'\in r$. Then, $h'\succ_d h$, $|\mu(h')|<q_{h'}$, $|\mu(r)|=q_r$, and $(d,h')\succ_r^{PL}(d,h)$ hold. Then, we have $(d,h')\in Ch_r(X'\cup\{(d,h')\})$ and $(d,h')\in Ch_d(X'\cup\{(d,h')\})$. These contradict the HM-stability of X'. Hence μ is regionally nonwasteful.

($\Leftarrow$) We prove that X' is HM-stable if μ is regionally fair and regionally nonwasteful.

First, it is clear that $X' = Ch_D(X') = Ch_R(X')$ holds since μ is feasible, i.e., under μ, all the quotas are satisfied, each student is matched with an acceptable hospital or nothing, and vice versa.

Second, assume that there exists $(d,h)\notin X'$, where $h\in r$ such that $(d,h)\in Ch_d(X'\cup\{(d,h)\})$ and $(d,h)\in Ch_r(X'\cup\{(d,h)\})$ hold. Hence, $h\succ_d h'$ must hold for any $(d,h')\in X'$ or $X'_d=\emptyset$. In both cases, if there exists $(d',h)\in X'$ such that $d\succ_h d'$, then fairness is violated. Thus, we can concentrate on the case where $d'\succ_h d$ holds for any $(d',h)\in X'$. Since $(d,h)\in Ch_r(X'\cup\{(d,h)\})$, $|X'_h|<q_h$ must hold. If $|X'_r|<q_r$, then d can strongly claim an empty seat of h. Thus, let us assume $|X'_r|=q_r$.

Since $(d,h) \in Ch_r\left(X' \cup \{(d,h)\}\right)$, there exists at least one contract $(\hat{d},\hat{h}) \in Re_r(X' \cup \{(d,h)\}$, which satisfies that $(d,h) \succ_r^{PL} (\hat{d},\hat{h})$. If $\hat{d} \neq d$, (regional) fairness is violated. If $\hat{d} = d$, regional nonwastefulness is violated. Thus, such (d,h) cannot exist. Therefore, X' is HM-stable.

Appendix D.3 The proof of Lemma 1

The proof is derived from the following two lemmas.

Lemma 2 *A doctor-maximal regionally fair matching is weakly nonwasteful.*

Proof. Let μ be a doctor-maximal regionally fair matching. By way of contradiction, suppose that doctor d, who is assigned to hospital h that belongs to region r, strongly claims an empty seat of hospital h' that belongs to region r', i.e., $h' \succ_d h$, $|\mu(h')| < q_{h'}$ and $|\mu(r')| < q_{r'}$. We are going to show that there exists another regionally fair matching μ^* such that $\mu^* \succeq_D \mu$ and $\mu^*(d) \succ_d \mu(d)$. This contradicts the fact that μ is doctor-maximal.

Now, we show how to derive such μ^*. Without loss of generality, we assume (d,h') is the most preferable pair in $\succ_{r'}^{PL}$ that satisfies the above. Then, consider another matching μ^1, which is identical to μ except that d is moved from h to h'. If μ^1 is regionally fair, then let $\mu^* := \mu^1$ and we are done. Note that there is a chance that μ^1 is not regionally fair. However, since μ is regionally fair and we choose (d,h') such that (d,h') is the most preferable pair in $\succ_{r'}^{PL}$, no doctor has justifiable envy or regionally justifiable envy towards doctors assigned to r'. Also, since the assignments of doctors except r and r' are the same, no doctor has justifiable envy or regionally justifiable envy towards these doctors. The only possibility is that another doctor d', who is currently assigned to h'', prefers h to h''. Then, h must be full in μ (otherwise, d' should have regionally justifiable envy in μ), but h is not full in μ^1. Thus, d' can have regionally justifiable envy toward some doctor in r. Without loss of generality, we assume (d',h) is the most preferable pair in $\succ_r^{PL}$ that satisfies the above. Then, consider another matching μ^2, which is identical to μ^1 except that d' is moved from h'' to h. If μ^2 is regionally fair, then let $\mu^* := \mu^2$ and we are done. If it is not, we can further repeat this procedure. In this procedure, doctors are re-assigned to a better hospital. Thus, since the number of possible matchings is finite, this procedure terminates in a finite number of iterations, that is, μ^k is regionally fair for some k. Then let $\mu^* := \mu^k$.

By the definition of μ^*, μ^* is regionally fair and satisfies $\mu^* \succeq_D \mu$ and $\mu^*(d) \succ_d \mu(d)$. This is a contradiction to the maximality of μ. ∎

Lemma 3 *In a doctor-maximal regionally fair matching, no doctor regionally claims an empty seat.*

Proof. Let μ be a doctor-maximal regionally fair matching.

By way of contradiction, suppose that doctor d, who is assigned to h, regionally claims an empty seat of a hospital h', where h and h' are in the same region r. Then, $h' \succ_d h$, $|\mu(h')| < q_{h'}$, and $(d, h') \succ_r^{PL} (d, h)$ hold. Without loss of generality, we assume (d, h') is the most preferable pair in $\succ_r^{PL}$ that satisfies the above. Then, consider another matching μ', which is identical to μ' except that d is moved from h to h'.

Under μ', since the assignments of doctors except d are the same, no doctor has justifiable envy or regionally justifiable envy toward a doctor except d; otherwise, the doctor must have the same justifiable envy under μ. Also, since we choose d such that (d, h') is the most preferable pair in $\succ_r^{PL}$, no doctor has justifiable envy or regionally justifiable envy toward d. Thus, μ' is regionally fair and satisfies $\mu' \succeq_D \mu$ and $\mu'(d) \succ_d \mu(d)$. This is a contradiction to the maximality of μ. ■