Theories and Tests for Bubbles

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1. Introduction

In this dissertation I will explain some of the theory about bubbles, and especially bubbles in the stock market. The intention is to give a presentation of the concept of bubbles in general and how it may affect the Norwegian economy. I will then make an empirical investigation, aiming to reveal whether there are signs of bubbles in the Norwegian stock market or not. Finally, some conclusions will be drawn on the basis of this. Before explaining the contains of this dissertation any further, I believe it is appropriate to suggest an exact definition of what a bubble is for readers that are unfamiliar with this term:

A bubble is the component of an assets price that is expected to pay no dividends

If the buyer is rational the investment in this asset is done purely in the belief that the price will be higher when the asset is sold. The fundamental price (the asset price less the bubble) is calculated as the expected future dividends in infinite time.

The topics of the different chapters are as follows: Chapter two will be a presentation of the theory necessary to derive the concept of rational bubbles in the economy. I will here explain the Overlapping Generations Model. Important concepts will be presented, and then used in chapter three. In chapter three , I will present bubbles that are rational and not completely rational bubbles. Rational bubbles are the easiest to analyse, because we can infer something about which solutions will be beneficial for people in the economy, and thereby limit the possible outcomes. I will derive the theory used to explain rational bubbles and how a stable equilibrium can occur. The theory will be generalised to include bubbles that have a chance of bursting, stochastic bubbles. To give a full perspective of what bubbles are, I will give some examples during the presentation.

In the theoretical chapters, much work has been devoted to make the concept of bubbles more accessible, without compromising accuracy. The main reason for the inaccessibility, was that much literature on bubbles were found to make «leaps» in the theoretical presentation[1]. It has

[1] Probably since readers of this literature are assumed to have required knowledge about the subject or the general theory behind it prior to reading. This is however not assumed in such a dissertation as this.

therefore been necessary to «fill inn », so that there would be no missing parts in the argumentation.

Of non-rational bubbles, there is to my knowledge no precise theory about. I have therefore only discussed this in a more general manner. In the part of this chapter that concern non-rational bubbles, there will also be described some research that investigates irrationality in the market.

After the theory is presented, in chapter four, I will give two examples of ways to test for bubbles which I will perform on Norwegian data in the period 1976 to 1997. The tests that will be applied are Shiller's variance test and West's specification test. I have modified West's test, the most extensive of the tests, to accommodate the short span of the data set.

In chapter five the results will be presented. It was unfortunately impossible to get data from earlier than 1977, so the results can be criticised as suffering from a small sample. Yet, I do not believe that to be a problem in this dissertation since some of the aim of the empirical part is to show how the tests can be performed and analysed, in particular West's test with my modifications. The data consist of time series from thirty companies registered at Oslo Bourse.

A general discussion will be done in chapter six. Conclusions based on the findings in the test and the theory will be presented. The theoretical results discussed in chapter three is used to derive implications for the economy in general and specifically how bubbles affects the Norwegian economy.

1.1 Notation in the theoretical part (prior to chapter 4)

a:	The discount factor ($a = 1/(1+r)$)
A_t:	A factor that reflects productivity within a firm
B_t:	The aggregate value of the bubble at time t
b_t:	The value of the bubble in per capita terms at time t
c_{1t}:	Consumption of the young at time t
c_{2t}:	Consumption of the old at time t
c_t:	Consumption at time t
d_t:	The dividend of an asset at time t
e_t	The error term of the bubbles value at time *t* in the stochastic bubble model
$f(k)$:	The per capita production function at time t.
I_t:	The information set at time t, consisting of all information available to the market at this time
K_t:	The capital stock at time t
k_t:	The per capita capital stock at time t
L:	The initial population - the population at time t: $L(1+n)^t$
M:	The number of bubble assets
m_t:	The number of assets purchased at time t
n:	The population growth rate
O_t:	Bonds issued by a firm
p_t^*:	The fundamental price of an asset at time t (a particular solution to equ. (1))
p_t:	The price of an asset at time t
r:	The short term risk-free rate of return.
s(...):	The per capita saving function.
$\tilde{u}_{jt}$	The stochastic error term of the expected return at time *t* of asset *j*
w_t:	The wage at time t
y:	The exogenous real endowment of the individual
θ:	The individual discount rate
ρ:	The individual discount factor

1.2 Notation in the empirical part (chapter 4 and further)

a:	The discount factor ($a = 1/(1+r)$)
A:	The aggregated data set
b_t:	The bubble component of the market price
δ:	Auto regression parameter for the price process
d_t:	The dividends of an asset at time t
ϕ:	Auto regression parameter for the dividend process
F_t	The information set available at time t (Bond and Thaler (1985) model)
F_t^m	The information set used by the market at time t (Bond and Thaler (1985) model)
$\mathbf{h}_t(\boldsymbol{\theta})$:	The vector of the auto regression equations
I_t:	An information set at time t, consisting of all information available to the market at this time
H_t:	An information set consisting of earlier dividends and a constant, a subset of information set I_t.
k	The number of parameters in a model
m:	Constant in the price process
μ:	Constant in the dividend process
p_t^*:	The fundamental price of an asset at time t
p_t:	The market price of an asset at time t
$\boldsymbol{\theta}$:	The vector of auto regression parameters
R	The constraint equation in the simple case
$\mathbf{R}(\boldsymbol{\theta})$:	The vector of the constraint equations
$\tilde{R}_{it}$:	The return of asset j at time t (Bond and Thaler (1985) model)
RRRS	The restricted residual sum of squares, the sum of squares resulting when there are restrictions imposed on the parameters
$\mathbf{S}$:	The sum of the auto covariance matrices.
URRS	The unrestricted residual sum of squares, the sum of squares resulting when no restrictions are imposed on the parameters
u_t:	The error term of the arbitrage equation
$\mathbf{V}$:	The variance matrix
v_t:	Error term for the dividend process
w_t:	Error term for the price process
X_n	Sub-index n, n=1,2,3,4
z_t:	The error of the expected sum of future dividends caused by misspecification of the information set I_t as H_t

2. Background: The Overlapping Generations model

The Overlapping Generations Model (OLG) was introduced by Allis (1947), Samuelson (1958) and Diamond (1965) (Blanchard & Fischer 1994). In the simplest model, the society consists of two generations, the young and the old, and two sectors, individuals and firms. The young produce and saves capital for the next period when they are old. Consequently the young are the workers and the old the capital owners at any time. The economy is a closed one, with markets that work efficiently. The model was developed by Diamond, building on work by Samuelson. This kind of two-generation economy is therefore called a *Diamond economy*.

2.1.1 Definition of the functions

The individuals have a utility function $u(c_t)$ where c_{1t} is consumption for the young and c_{2t} for the old at time t . Since they live in two periods, they have the total discounted utility during their life of:

$$(2.1) \qquad u(c_{1t}) + \frac{u(c_{2t+1})}{1+\theta} \quad \text{where } \theta \geq 0, \quad u'(\cdot) > 0 \quad \text{and } u''(\cdot) < 0$$

Utility is increasing, but at a decreasing rate. It also assumed that the utility function is separable across time (one function for each period). This ensures that the goods are normal. The individuals earn a wage w_t during the first period, some of which is spent on savings s_t which pays the interest rate of r_{t+1} next period, and the rest on consumption. What is saved in the first period is the capital stock next period. The population is L in period t=0 and grows at a rate $(1+n)$. In this model, the growth of the population and the growth of the economy are equivalent terms.

It is assumed that firms act competitively and that the production function for all the firms together at time t can be characterised as $Y_t = F\left(K_t, L \cdot (1+n)^t\right)$, which is a function assumed to be homogenous of degree one. K_t is the total amount of capital invested last period and therefore the capital stock at the beginning of period t (so the capital this period equals investment last period). $F(\cdot)$ is a net production function, so there is no depreciation to take account of. If capital per worker, $k_t = \dfrac{K_t}{L(1+n)^t}$, then output per worker is:

$f(k_t) = \dfrac{F(K_t, L\cdot(1+n)^t)}{L\cdot(1+n)^t} = F\left(\dfrac{K_t}{L\cdot(1+n)^t}, 1\right)$. The last equality holds because the function is assumed to be homogenous of degree one. This production function is assumed to be strictly concave and to satisfy the Inada conditions:

$$(2.2) \qquad f(0)=0 \qquad f'(0)=\infty \qquad f'(\infty)=0.$$

Since the function is strictly concave, we have that $f''(k_t)<0$. The cost of labour and capital is taken as given by the firms.

2.1.2 Savings and the interest rate

Under these conditions an individual born at time t has to solve the following problem

$$(2.3) \qquad \max_{c_{1t}, c_{2t+1}} \left(u(c_{1t}) + \frac{u(c_{2t+1})}{1+\theta} \right)$$

subject to the budget constraints:

$$(2.4) \qquad c_{1t} + s_t = w_t \quad \text{and} \quad c_{2t+1} = (1+r_{t+1})s_t$$

The first condition is that the wage of the young must equal consumption and savings. The next is that the capital gains and the savings of the old equals consumption. The first order condition for an interior maximum is:

$$(2.5) \qquad u'(c_{1t}) - \frac{(1+r_{t+1})\cdot u'(c_{2t+1})}{1+\theta} = 0$$

Using the budget constraints and differentiating with respect to savings, wage and the interest rate yields

$$u''(c_{1t})dw_t - u''(c_{1t})ds_t - \frac{u'(c_{2t+1})dr_{t+1} + (1+r_{t+1})^2 u''(c_{2t+1})ds_t + (1+r_{t+1})u''(c_{2t+1})s_t dr_{t+1}}{1+\theta} = 0$$

Which implies a saving function $s(w_t,r_{t+1})$ with these properties:

$$\frac{ds_t}{dw_t} = \frac{(1+\theta)u''(c_{1t})}{(1+\theta)u''(c_{1t}) + (1+r_{t+1})^2 u''(c_{2t+1})} > 0 \tag{2.6}$$

and

$$\frac{ds_t}{dr_{t+1}} = -\frac{u''(c_{2t+1})\cdot(1+r_{t+1})s_t + u'(c_{2t+1})}{(1+\theta)u''(c_{1t}) + (1+r_{t+1})^2 u''(c_{2t+1})} \gtrless 0 \tag{2.7}$$

As we can see, saving increases with the wage. We know that the income effect will increase consumption in both periods. There are no inferior goods, since we have assumed that the utility function is separable. An increase in the wage, keeping the interest rate constant, must therefore be spent on both more consumption in period one, and more savings.

The interest rate effect is ambiguous due to the presence of both income and substitution effects. The *income effect is in this case negative* in contrast to the positive effect when wage is increased. The income effect of an increased interest rate increases consumption in both periods as mentioned before, but since the available resources in the first period (the wage) do not change, this has a negative effect on savings. This can be derived as follows: a higher interest rate increases consumption by $dc_{1t}>0$, but the wage does not increase, $dw_t=0$. If we use this and differentiate the budget constraint in the first period in (2.4) we get $ds_t = -dc_{t1} < 0$. The first term in the numerator of (2.7) can therefore be interpreted as the income effect and the second term the substitution effect. The substitution effect is positive on savings since an increase in the interest rate makes consumption in next period cheaper so savings are increased.

It is difficult to know which effect is the strongest, but if the elasticity of substitution between the two periods is independent of the interest rate, savings will be too. (2.5) may be rewritten in terms of the elasticity of substitution between the two periods ε_t, as:

$$\frac{1}{(1+r_{t+1})}\cdot\frac{c_{2t+1}}{c_{1t}} = \frac{u'(c_{2t+1})}{(1+\theta)u'(c_{1t})}\cdot\frac{c_{2t+1}}{c_{1t}} = -\varepsilon_t \tag{2.8}$$

By using the budget constraints and rearranging we get $s_t = -c_{1t}\varepsilon_t$. This is independent of the interest rate if the elasticity of substitution is independent of it.

This is basically the only thing we can know for sure of the effect of the interest rate on savings in this model. In a more complicated and perhaps realistic model where individuals earn a wage in all periods, Blanchard and Fischer argue that the interest rate has positive effects on savings. I will however not explain this model any further, since it is the model described previously that will be used in relation to the theories about bubbles.

To see why the influence of the interest rate on savings is important for a theory about bubbles, we must turn to how firms behave. Firms hire labour and capital up to the point where the costs equal the marginal product of the input factor, which implies that:

$$\text{(2.9)} \qquad \begin{aligned} w_t &= f(k_t) - k_t \cdot f'(k_t) \Rightarrow \frac{dw_t}{dk_t} > 0 \\ r_t &= f'(k_t) \qquad\qquad\quad\; \Rightarrow \frac{dr_t}{dk_t} < 0 \end{aligned}$$

The rate of interest equals the marginal product of capital because people are considered risk neutral, there are no implementation costs and all capital can be consumed. According to the earlier assumptions of the production function, the wage increases with the capital stock and the interest rate decreases. The two effects are compared in steady state.[2]

In steady state, the dating of the variables can be ignored, so the effect of the capital stock on savings in that case can be written:

$$\text{(2.10)} \; \frac{ds}{dk} = \frac{ds}{dw} \cdot \frac{dw}{dk} > 0 \text{ and } \frac{ds}{dk} = \frac{ds}{dr} \cdot \frac{dr}{dk} \gtrless 0 .$$

In the more complicated model mentioned earlier, interest rates had positive effects on saving. If this is the case, then the effect of an increased capital stock works different ways through interest rates and wage. For a bubble to arise, the sum of the two effects must be positive and

[2] The term «steady state» and also the term «equilibrium» refers to the state where the arguments in the function are stable over time, so that the function itself is stable (does not change). These terms will be used both as a description of single equations and a system of equations. Which meaning it has will appear from the context.

more than (*1+n*), because savings must increase faster than capital. If $\frac{dk}{ds} < 1+n$, savings in one period will be exceeded by capital in per capita terms the next period. The reason for this is that the growth in the economy reduces the per capita earnings next period. This will be explained more thoroughly in the next chapter.

2.1.3 Diamond equilibrium and dynamic efficiency

If there is no bubble in the economy, net savings (savings less the present capital stock) must equal investment:

$$K_{t+1} - K_t = L(1+n)^t s(w_t, r_{t+1}) - K_t \qquad (2.11)$$

In per capita terms, and eliminating the present capital stock we can write the *Diamond equilibrium* as:

$$(1+n)k_{t+1} = s(w_t, r_{t+1}) \qquad (2.12)$$

This is the condition of equilibrium in the goods market if there are no bubbles. The reason it is called the Diamond equilibrium here, is to stress just that. The term *«Diamond equilibrium»* is used when an economy in which bubbles can possibly arise, does not actually exhibit a bubble.

In the OLG model the competitive equilibrium may not be Pareto efficient. This situation is called *dynamic inefficiency*. A Diamond economy is dynamically inefficient if the capital stock is in excess of that consistent with the golden rule since Pareto improvements are possible. The golden rule is that the marginal product of capital equals growth, $f'(k) = n$. If $f'(k) < n$ the capital stock is in excess of this golden rule level, because the marginal product of capital is a decreasing function its argument. The reason Pareto improvements are possible can be derived as follows.

The capital stock and the production function in period t determine how much can be used to invest in capital and how much can be consumed next period:

$$K_t + F(K_t, L\cdot(1+n)^t) = L\cdot(1+n)^t c_t + K_{t+1} \qquad (2.13)$$

Dividing this by the population at time t, noting that $k_{t+1}(1+\mathrm{n}) = \dfrac{K_{t+1}}{L(1+n)^{t+1}}(1+n) =$

$\dfrac{K_{t+1}}{L(1+n)^{t}}$ and rearranging, yields the equation in per capita terms:

(2.14) $$k_t + f(k_t) = (1+n)k_{t+1} + c_t$$

In steady state, capital accumulation will be:

(2.15) $$k^* + f(k^*) = (1+n)k^* + c^* \qquad \text{or} \qquad f(k^*) - nk^* = c^*$$

Differentiating:

(2.16) $$\frac{dc^*}{dk^*} = f'(k^*) - n$$

If the marginal product of capital is less than the growth, $f'(k) < n$, a given permanent *decrease* in the capital stock will *increase* consumption in *all periods but the last.* That consumption in the last period will decrease can be seen from equation (2.16). If $f'(k) < n$, consumption and the capital stock is negatively related, so a *decrease in the capital stock will increase consumption*. In the last period however, there will be no future period to invest in, so *consumption will decrease*: differentiating equation (2.14) at time T with respect to the permanent change in capital *k*, noting that k_{T+1}=0 yields: $\dfrac{dc_T}{dk^*} = f'(k^*) + 1 > 0$. But since period T is in the infinite future, this decrease in consumption due to a reduction in the capital stock can be ignored.

Consumption increases utility, so a decrease in the capital stock is therefore a *Pareto improvement* as long as $f'(k) < n$. This state can therefore not be *Pareto optimal* and is therefore called *dynamically inefficient*. If the economy continues to reduce its capital stock it will eventually reach the point where $f'(k) = n$, and there will be no gains from reducing it further. If $f'(k) > n$ neither an increase nor a decrease of the capital stock will give a Pareto improvement. A permanent *decrease* in the capital stock will *decrease consumption* in all future periods and is therefore not a Pareto improvement. The opposite, *an increase* in *k*, will decrease the consumption in *the first period* and making the old in this period worse off. Thus, this can also be ruled out as a Pareto improvement. Consequently, since no Pareto improvement is possible, the economy is *dynamically efficient*. The capital is less than the golden rule level, so that $f'(k) \geq n$.

Since dynamic inefficiency arises when the capital is in excess of the golden rule level, such an economy is said to have overaccumulated capital.

3. Theories about bubbles

3.1 Introduction

A bubble is the difference between an asset's fundamental value and its market price. The fundamental value is the amount of discounted future dividends and the price of the asset when it is sold in infinite future. This is the price that we would expect in a economy that consists of rational individuals with infinite horizons. If the price deviates from this level, the deviation is called *a bubble*.

A bubble on an asset may arise when the market values an asset more because it previously has increased in value. The traders believe that since the asset has increased before, it will pay off to hold it for a limited period of time. The previous increase promises a continued increase in the future. This is often called a self fulfilling prophecy, since the increase in itself leads to a higher demand for the asset and hence a further increase in the price. Other reasons for the market to expect a future increase in the bubble are possible too, but this is often used as a plausible explanation.

An important feature of the bubble is that if the participants in the market are rational, *a bubble will normally not arise if the market consists of rational individuals with infinite horizons* [3] *and there are a finite number of individuals in it*. People having infinite horizons will never buy an asset for more than what they consider the fundamental price, and hold it forever. If they do, they will decrease their utility today by more than the utility that they will receive later as future income (the dividends) [4]. Therefore everybody knows that no one will hold such assets infinitely, expecting a price drop in the future. If the price drop is expected, you will make a sure profit by selling the asset as soon as its price is above the fundamental value, thus making a bubble impossible. A similar argument can be made in the case of a negative bubble if this is allowed.

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[3] Provided that the transversallity condition is fulfilled , something that will be explained later.

[4] Further discussion about utility maximisation is contained in section 3.3

But if the market is expanding and people with finite horizons exists in the market too, these people will not care if the asset is priced above the fundamental level. This will be of no matter to them as long as it is possible to make a profit by leaving the market before it falls back to the fundamental level. It may even never fall to a fundamental level if new traders enter the market constantly in infinite future.

There may exist rational or non-rational bubbles, based on whether the individuals in the market are rational or not. If the market is not fully rational, one can not exclude bubbles, even if traders have infinite horizons or there is a limited number of them.

3.2 The fundamental price in the stock market

Among others, Blanchard and Fischer (1994) use an assumption of arbitrage to show the relationship between the fundamental price and expectations of the stock market. Arbitrage implies that the expected price-increase and the dividend relative to the price should equal the risk free interest rate (which is viewed as constant here). Let a be the discount factor and therefore positive but less than one. Then if people are rational, the price p_t of a stock at time t should equal the expected discounted dividend d_{t+1} and the expected discounted value of the asset next period based on the information set I_t. This implies the arbitrage equation below:

(3.1) $$\frac{E\left[p_{t+1}\middle|I_t\right]-p_t}{p_t}+\frac{E\left[d_{t+1}\middle|I_t\right]}{p_t}=r$$

This can be written:

(3.2) $$p_t = aE\left[p_{t+1}+d_{t+1}\middle|I_t\right] \qquad \text{where} \qquad a=\frac{1}{1+r}$$

Expectations are assumed to be *Independent from Irrelevant Alternatives* as proposed by Luce in 1956 or also known as *the Law of Iterated Expectations*. This implies that $E\left[E\left[p_{t+2}\middle|I_{t+1}\right]\middle|I_t\right]=E\left[p_{t+2}\middle|I_t\right]$ holds on average.

Substituting forward one time period in equation (3.2) gives:

$$p_t = aE\left[aE\left[p_{t+2} + d_{t+2} \middle| I_{tt+1}\right] + d_{t+1} \middle| I_t\right] = a^2 E\left[p_{t+2} \middle| I_t\right] + aE\left[d_{t+1} \middle| I_t\right] + a^2 E\left[d_{t+2} \middle| I_t\right]$$

If we repeat this T-1 times, we get:

(3.3) $$p_t = a^T E\left[p_{t+T} \middle| I_t\right] + \sum_{i=1}^{T} a^i E\left[d_{t+i} \middle| I_t\right]$$

If the expectations of the fundamental price increase less than at the rate of interest, then:

(3.4) $$\lim_{T \to \infty} a^T E\left[p_{t+T} \middle| I_t\right] = 0$$

This is called the *transversallity condition*. Since people care little about the price they will get for the asset in the infinite future, the present value of the asset will be zero as T goes to infinity. This is one way of explaining this condition.

The fact that an increase of the expected price at a rate less than the rate of interest implies (3.4) can easily be seen by assuming that price expectations increase at a rate x. We can then write this as: $a^T E\left[p_{t+T} \middle| I_t\right] = \left(\frac{1}{1+r}\right)^{T-t} p_t (1+x)^{T-t} = \left(\frac{1+x}{1+r}\right)^{T-t} p_t$. If x < r this expression becomes zero as T goes to infinity.

If prices increase less than the rate of interest, dividends must be expected to do the same. If not, when the present value of the price goes to zero as time goes to infinity, the present value of the dividend at this time has to be an infinite percentage of the price. This is very unlikely, so the transversallity condition will normally imply that the present value of the dividends also go to zero as time goes to infinity.

If the dividends are expected to increase at a rate less than the rate of interest, then $\lim_{T \to \infty} \sum_{i=0}^{T} a^i E\left[d_{t+i} \middle| I_t\right]$ will converge. With this particular solution, the fundamental price of the asset can be written:

(3.5) $$p_t^* = \sum_{i=0}^{\infty} a^i E\left[d_{t+i} \middle| I_t\right]$$

This solution can be viewed as the price of the asset that people with infinite horizons would be willing to pay. Since they are planning to hold the asset forever, it is reasonable to assume that the price the asset can be sold for in the infinite future will be of little importance for the individual. This can explain the transversallity condition.

However, this interpretation builds on the assumption that the rate of interest represents peoples individual discount rate, since it is their individual valuation of the price in the infinite future that counts. This is not always a reasonable assumption. Because of this, a similar model where the individual discount rate is present may be adequate to explain the transversallity condition.

3.3 A Model with an explicit utility function

Flood & Garber (1994) present a utility maximising model for asset pricing. This model links the utility maximising behaviour of a representative agent to the asset pricing theory used to explain bubbles.

The problem of the agent is to maximise discounted utility (the utility U discounted by the discount factor ρ) over an infinite horizon with respect to consumption, c_{t+i}, in each period and subject to the budget constraints in each period. These constraints ensure that in each period $t+i$, consumption and the amount used to buy m_{t+i} assets is the same as the available resources. These are an exogenous real endowment, y, the current value of the assets saved from last period, $p_{t+i}m_{t+i-1}$, and dividends of the assets held in the previous period which are paid out in this period, $d_{t+i}m_{t+i-1}$. The number of assets held in the previous period is m_{t+i-1} , so the total value and dividends of the assets held is $(p_{t+i}+d_{t+i})m_{t+i-1}$ in period $t+i$. The problem can therefore be written:

$$\underset{\{c_{t+i}\}_{i=0}^{\infty}}{Max} \; E\left[\sum_{i=0}^{\infty}\rho^{i}U(c_{t+i})\middle|I_t\right] \tag{3.6}$$

Subject to the budget constraints in each period:

$$c_{t+i}+p_{t+i}m_{t+i}=y+(p_{t+i}+d_{t+i})m_{t+i-1} \tag{3.7}$$

The first order conditions are:

(3.2b) $$E\left[U'(c_{t+i})p_{t+i}\middle|I_t\right] = \rho E\left[U'(c_{t+i+1})\cdot(p_{t+i+1}+d_{t+i+1})\middle|I_t\right]$$ $$\text{for } i = 0,1,\cdots,\infty$$

Where $U'(c_{t+i})$ is the marginal utility of consumption in period $t+i$.

By substituting T-1 times using the law of iterated expectations as previously mentioned, we get:

(3.3b) $$U'(c_t)p_t = \rho^{\mathrm{T}} E\left[U'(c_{t+T})p_{t+T}\middle|I_t\right] + \sum_{i=1}^{T}\rho^i E\left[U'(c_{t+i})d_{t+i}\middle|I_t\right]$$

Which can be rewritten.

(3.3b)´ $$p_t = E\left[\rho^{\mathrm{T}}\frac{U'(c_{t+T})}{U'(c_t)}p_{t+T}\middle|I_t\right] + \sum_{i=1}^{T} E\left[\rho^i\frac{U'(c_{t+i})}{U'(c_t)}d_{t+i}\middle|I_t\right]$$

We now have an expression for the price that depends on the marginal rate of substitution between time t and the future periods up to time $t+T$, namely $\rho^i\dfrac{U'(c_{t+i})}{U'(c_t)}$ for i=1,...,T. It is reasonable to assume that the individual will care little for the price he gets by selling the asset in the infinite future. As T goes to infinity it is therefore reasonable to assume that the first term on the right hand side of (3.3b)´ goes to zero. Thus, even if the price is expected to increase more than the rate of interest, we can assume that the increase is too small to offset the low marginal rate of substitution between today and the infinite future. This probably explains the transversallity condition better than in the last section, since the price increase in this case does not have to be less than the exogenous interest rate. If the individual cares little about the price he gets for the asset in infinite future, he should value the stock as the future stream of dividends only. If the transversallity condition holds for the individual, it should hold for the entire market also.

As in the case of the pure asset pricing model, we can now write the fundamental price in terms of the marginal rate of substitution as:

$$p_t = \sum_{i=1}^{\infty} E\left[\rho^i \frac{U'(c_{t+i})}{U'(c_t)} d_{t+i} \middle| I_t\right] \qquad (3.5b)$$

3.4 Rational bubbles

Even though the theory requires that there is a finite number of investors and that the market participants have infinite horizons, which may seem unlikely in any market, the fundamental price is often viewed as an efficient market price. That the price of a security is equal the future stream of dividends is therefore called the Efficient Market Hypothesis (EMH) (Lee et al., 1991). Thus, in an efficiently working asset market, the price should be an estimate of future dividends only. If the asset market does not work efficiently, according to this terminology, there is a bubble.

In the case of a rational bubble, it is possible to derive a dynamic model which reveals specific paths of the bubble over time. The bubble-asset that people are buying cannot increase too much, because this would in the end drive out all investment. Rational individuals would not allow that, since it would leave them worse off eventually. This knowledge makes it possible to rule out the paths towards an ever expanding bubble, making a dynamic equilibrium analysis possible.

3.4.1 If the transversallity condition does not hold

If bubbles are allowed, condition (3.4) may not hold. The sum of the dividends is still expected to converge, and the discounted resale price in infinite future can also be expected to be zero. However, people may wish to hold the asset for a limited period of time, so that saving without investing is possible. That is, there might be a demand for an asset that have a higher price than the one corresponding to the stream of dividends in the infinite future. The reason people want to hold such an asset is to save. While buying an asset previously in this chapter has been described as a investment which pays some kind of return each period, buying an asset can now also be pure saving. It is possible that the return corresponding to the market equilibrium is higher than the rate of return corresponding to an equilibrium in the goods market (the Diamond Equilibrium mentioned in section 2.1.3)[5]. The investments are reduced so that the returns on investment is brought in to line with the market rate of return (due to decreasing marginal rate of return on capital), which is equivalent to an increase of the bubble. Thus some of the savings are not invested, but consumed. This will be explained more thoroughly in the next sections. Since the price now will be a bubble component in addition to a fundamental price, we can write a solution to (3.2) as the sum of the discounted dividends and a bubble component, b_t:

$$p_t = p_t^* + b_t \qquad \text{or} \qquad p_t = \sum_{i=0}^{\infty} a^i E\left[d_{t+i} \middle| I_t\right] + b_t \tag{3.8}$$

Where p_t^* is the fundamental price given by (3.5) and b_t is called a bubble.

Let us assume that all participants in the economy are risk neutral. Then all assets have to pay the same rate of return so that the price increase and the dividends equal the market rate of return. The market rate of return must in turn equal the interest rate.

If the price of one asset increases above the initial fundamental level, then the price increase (the bubble) must pay the same rate of return as the market. Thus, the bubble must increase at the rate of interest, but the fundamental of an asset price will increase at a rate less than the required return since the asset also pays dividends. Therefore rational individuals must expect that the bubble component will increase at the rate of interest. If the bubble is expected to increase less, no one would buy the bubble asset. Investors would be better off making an alternative investment that paid the rate of interest. If the bubble is expected to increase more, more people would invest in the bubble asset, forcing the price up and the expected return down. Since we use the same discount factor for all investments, it will be expected that the *bubble component increase at the rate of interest*. This can be derived as follows:

Since both p_t^* and p_t are solutions to (3.2), they both have to satisfy this equation. Therefore $p_t = aE\left[p_{t+1} + d_{t+1} \middle| I_t\right]$ and $p_t^* = aE\left[p_{t+1}^* + d_{t+1} \middle| I_t\right]$. Subtracting the second equation from the

[5] This can be explained as a situation where some individuals in the economy are willing to borrow in order to increase their present consumption.

first yields $p_t - p_t^* = aE\left[p_{t+1}\middle|I_t\right] - aE\left[p_{t+1}^*\middle|I_t\right]$ (the dividends disappear). From (3.8) we have $b_t = p_t - p_t^*$, so $b_t = aE\left[p_{t+1}\middle|I_t\right] - aE\left[p_{t+1}^*\middle|I_t\right]$. If we take expectations of (3.8) in period t+1, and discount it to the present value, we get $aE\left[b_{t+1}\middle|I_t\right] = aE\left[p_{t+1}\middle|I_t\right] - aE\left[p_{t+1}^*\middle|I_t\right]$. Therefore:

(3.9) $$b_t = aE\left[b_{t+1}\middle|I_t\right]$$

This equation points out that a bubble can exist as a solution to (3.2) only when it is expected to grow at the rate of interest. Equivalently we can say that the expected future bubble at any time must have a present value equal to today's bubble. On important implication here, is that the bubble must be expected to increase at a faster rate than asset prices as long as there is paid dividend of the assets.

The risk neutrality condition *can* be relaxed by assuming that the discount factor is risk adjusted. In that case, the bubble will be expected to increase at a rate equal to the rate of return on assets viewed as equally risky by the market. It will be assumed risk neutrality in the further discussion though.

3.5 Dynamics

A dynamic model based on the OLG model, is presented by Blanchard & Fischer (1994). Suppose an economy where people can save by either *investing* or *holding intrinsically useless papers*. That is, it has no value other than an expected price increase in the future, so it does not and will never pay any dividends. The latter alternative is the bubble asset, which we assume cannot be negative (thus negative bubbles are not allowed in this model). The bubble asset has thus a fundamental value of zero. Let us denote the price of the bubble asset by p_t. The part of savings that goes to investment in period t becomes the capital stock in period t+1, k_{t+1}. This capital stock gives a return in period t+1 of $f'(k_{t+1})$. The people of this economy are considered risk neutral so a possible difference in return caused by different assets having various risks, can be ignored. This enables us to assume that the interest rate is equal to the marginal product of capital, $f'(k) = r(k)$, since arbitrage in this case implies that investment can give neither more nor less return than savings. It is assumed that the capital stock cannot become negative (this is a closed economy), neither can the price of an asset.

As previously shown, a bubble can exist only if it is expected to grow at the rate of interest, which in turn equals the marginal product of capital. Therefore [6]:

$$(3.10) \qquad 1 + f'(k_{t+1}) = \frac{p_{t+1}}{p_t}$$

Let M be the number of bubble assets (a fixed number), so that $B_t = Mp_t$ is the aggregate value of the bubble. The population (economy) starts at size L at t=0, and grows at a rate of n. Then the per capita value of the bubble is $b_t = \frac{B_t}{L(1+n)^t} = \frac{Mp_t}{L(1+n)^t}$, and we can write

$\frac{b_{t+1}}{b_t}(1+n) = \frac{B_{t+1}}{B_t} = \frac{p_{t+1}}{p_t}$. Using (3.10) we can derive:

$$(3.11) \qquad b_{t+1} = \frac{b_t\left[1 + f'(k_{t+1})\right]}{1+n}$$

The condition for the bubble to grow in per capita terms is therefore that the marginal product of capital (the interest rate) exceeds the growth rate of the economy. When this is the case, the Diamond economy is also *dynamically efficient*.

The capital stock at time $t+1$ is equal to the total net savings in the economy. Each person has a savings function which depends on the capital stock at time t (through income (wage)) and at time $t+1$ (through the interest rate):

$$(3.12) \qquad s\left(w\left[k_t\right], r\left[k_{t+1}\right]\right) = s\left(k_t, k_{t+1}\right)$$

Net saving at time t is $L(1+n)^t s(k_t,k_{t+1})-K_t$. But part of the savings can be used to buy bubble assets, so investment is not equal to savings. This can be written:

$$(3.13) \qquad K_{t+1} - K_t = L(1+n)^t s\left(k_t, k_{t+1}\right) - K_t - B_t$$

---

[6] Note that since we now take account of the growth of the economy, it is necessary to distinguish between the price of the bubble asset p_t and the price of the bubble asset in per capita terms b_t.

or in per capita terms:

(3.13b) $$k_{t+1} - k_t = \frac{1}{(1+n)}\left[s(k_t, k_{t+1}) - b_t\right] - k_t$$

Since the capital stock is the argument in the production function, equation (3.13b) has to be substituted into (3.11). At the same time b_t is subtracted from both sides.

(3.11b) $$b_{t+1} - b_t = \frac{b_t\left[f'\left(\frac{s(k_t, k_{t+1}) - b_t}{(1+n)}\right) - n\right]}{1+n}$$

We have now two equations, which represent two different effects. I will call these effects the arbitrage effect (equ. (3.11b)) and the investment effect (equ. (3.13b)). Equation (3.11b) is therefore the arbitrage equation, and equation (3.13b) is the investment equation.

3.5.1 The investment effect

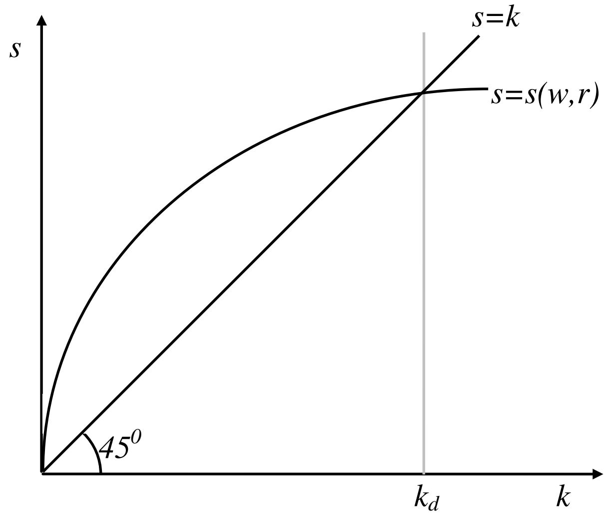
The effect of a increased capital stock on savings in equilibrium will depend on an ambiguous interest rate effect and a positive wage effect. In order for a bubble to arise in equilibrium, savings must increase more than capital at some stage. The effect of capital on savings were derived in chapter 2. As was mentioned then, we cannot know which way an increase in the interest rate affects savings. Blanchard and Fischer argue that interest rates are positively related to savings. If this is the case, the effect of an increased wage must be the far greater.

Since we do not allow for a negative bubble, net savings must be positive. On the assumption that no production is possible without any capital, in equilibrium savings must be zero when there is no capital stock. Therefore in order for savings to increase more than capital and become positive when initially k=0 and s=0, we must have $\frac{ds(w(0), r(0))}{dk} > 1+n$. However, the increase in savings will decrease as the capital stock grows [7] as we can see when we examine the second derivative of (2.6):

$$\frac{d^2 s_t}{dw_t^2} = \frac{(1+\theta)u'''(c_{1t}) \cdot (1+r)^2 u''(c_{2t+1})}{\left[(1+\theta)u''(c_{1t}) + (1+r_{t+1})^2 u''(c_{2t+1})\right]^2} < 0 \text{ if } u'''(c_{1t}) > 0 \qquad (3.14)$$

So the relationship between savings and the capital stock is increasing, but at a decreasing rate. The relationship between saving in equilibrium and the capital stock is illustrated in Figure 3.1.

Figure 3.1: Saving and the capital stock



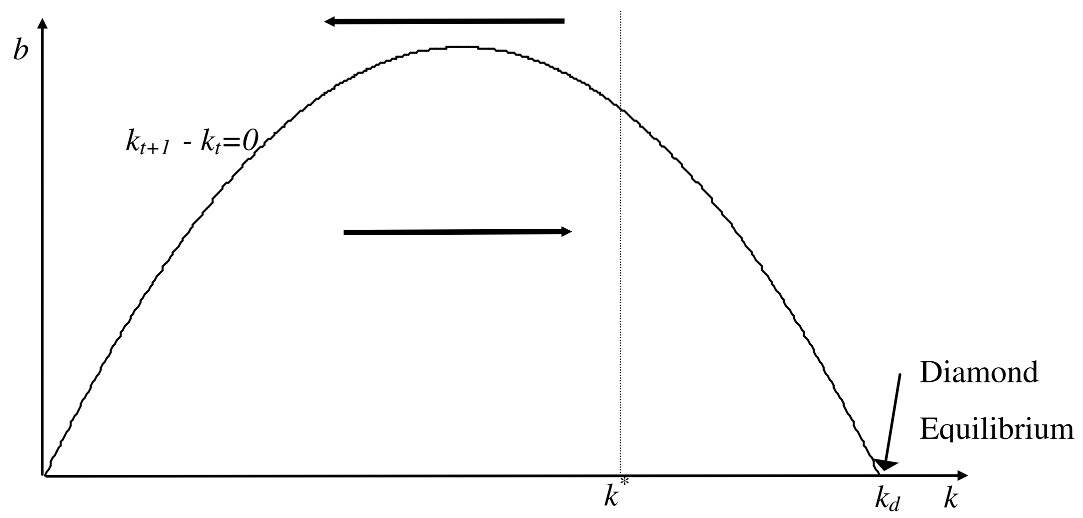
The positive difference between savings and the capital stock is the savings that go to buy bubble assets. The bubble cannot be negative, so a capital stock of more than k_d is impossible. Figure 3.1 can be used to draw a figure of the bubble as a function of the capital stock, as in Figure 3.2. In this figure the combinations of b and k that results in zero capital accumulation are presented. Thus the depicted line gives the combinations of capital and the bubble when $k_{t+1}=k_t$ so that $b = s(k,k) - (1+n)k$ according to equ. (3.11b). As mentioned before, for a bubble to exist, savings must be higher than investment for low levels of capital. Since the

[7] Provided that $u'''(c_{1t}) > 0$, an assumption that is reasonable if we assume that the Inada conditions (2.2) holds for utility as well.

savings function is a decreasing function of capital, at some point the equilibrium level of savings for a certain level of capital will be less for additional investment. Thus, the $k_{t+1} - k_t = 0$ curve is first upward sloping and then falling. At point k_d the bubble is zero. This is the Diamond equilibrium that the economy would reach if there were no bubbles. At this point *savings equals investment*.

At any point above the k_{t+1}-k_t=0 line, the bubble takes up so much of savings that not enough is invested. Production next period becomes less and investment is reduced more. The bubble is too large for the economy to support it, and the capital stock decreases. The opposite is true for any point below the line. The bubble is so small that the part of savings that goes to investment will increase production next period so that investment will increase even more. Savings will increase investment even in the presence of a bubble. The investment effect will therefore increase capital below the k_{t+1}-k_t=0 locus, and reduce it above it.

Figure 3.2: Dynamics caused by saving in the bubble asset



3.5.2 The arbitrage effect

Differentiating the steady state solution of (3.11b) yields:

$$0 = \left(\left[f'(s,b) - n \right] - b \frac{f''(s,b)}{1+n} \right) db + b \frac{f''(s,b)}{1+n} \cdot \frac{ds}{dk} dk$$

Which implies:

$$\frac{db}{dk} = -\frac{bf''(s,b)\cdot\frac{ds}{dk}}{\left[\left(f'(s,b)-n\right)\cdot(1+n)-bf''(s,b)\right]} = -\frac{bf''(s,b)\cdot\frac{ds}{dk}}{\left[-bf''(s,b)\right]} = \frac{ds}{dk}$$

The second equality follows since we are in steady state so that $f'(k)=n$. By the assumptions made earlier, we have that $\frac{db}{dk}=\frac{ds}{dk}>0$. So the steady state curve where $b_{t+1}=b_t$ has a positive slope in Figure 3.3 [8]. If we take the second derivative we get $\frac{d^2b}{dk^2}=\frac{d^2s}{dk^2}<0$ if we use (3.14). Thus the $b_{t+1}=b_t$ locus has a positive slope, but at a decreasing rate as in Figure 3.3. These properties can be explained as follows:

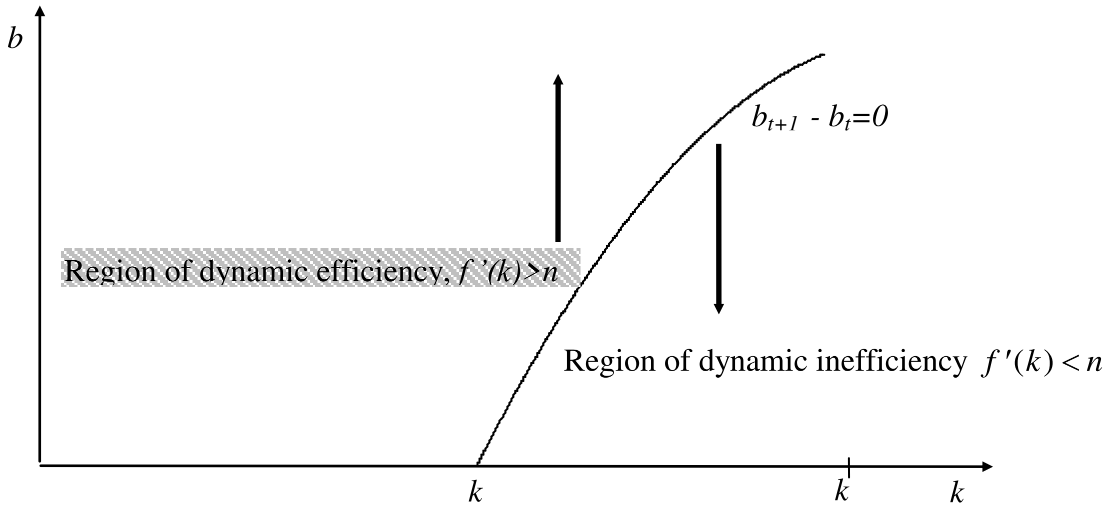
In steady state, when $b_{t+1}=b_t$, the arbitrage equation (3.11b) becomes $f'\left(\frac{s(k,k)-b}{(1+n)}\right)=n$. If we start at the point where the capital stock is such that $f'(k)=n$ and there is no bubble, and then increase the bubble, it takes up more of the savings, leaving less to be invested. The marginal product (the interest rate) will therefore exceed the growth, if the capital stock is not increased. So to keep $f'(k)=n$, k must increase. The steady state locus where $b_{t+1}-b_t=0$ is therefore not vertical but upward sloping as mentioned earlier. The locus is increasing at a *decreasing rate* with the capital k, because more capital is needed to facilitate a given increase in the bubble if the arbitrage equation is to be in steady state ($f'(k)=n$). The forces at the right and the left hand side of the arbitrage steady state path is as follows

To the left the capital stock is so low that the marginal product exceeds the growth rate. The economy is *dynamically efficient*. Arbitrage implies that the *price per unit bubble* must increases at the same rate as the return on investments, $f'(k)$. For this to happen, we know that the bubble in per capita terms must increase since the return on investment is higher than the population growth. So, on the left hand side of the arbitrage steady state path when $f'(k)>n$ the bubble, b_t, must *expand* if the bubble asset is to give the same return as capital relative to the growth.

[8] As long as savings increase with capital in per capita terms. This is reasonable if, as we have assumed earlier, it is the positive *wage effect* of an increased capital stock that is the most important for changes in savings.

To the right, the capital stock is so high that the growth rate exceeds the marginal product, the economy is *dynamically inefficient*. Since the price per unit bubble must increase at a the same rate as the interest rate, the bubble in per capita terms must decrease. So, on the right hand side of the arbitrage steady state path when $f'(k) < n$ the bubble, b_t, must *contract* if the bubble asset is to give the same return as capital relative to the growth. This is illustrated in Figure 3.3

Figure 3.3: Dynamics caused by arbitrage



3.5.3 Dynamic efficiency and diamond equilibrium

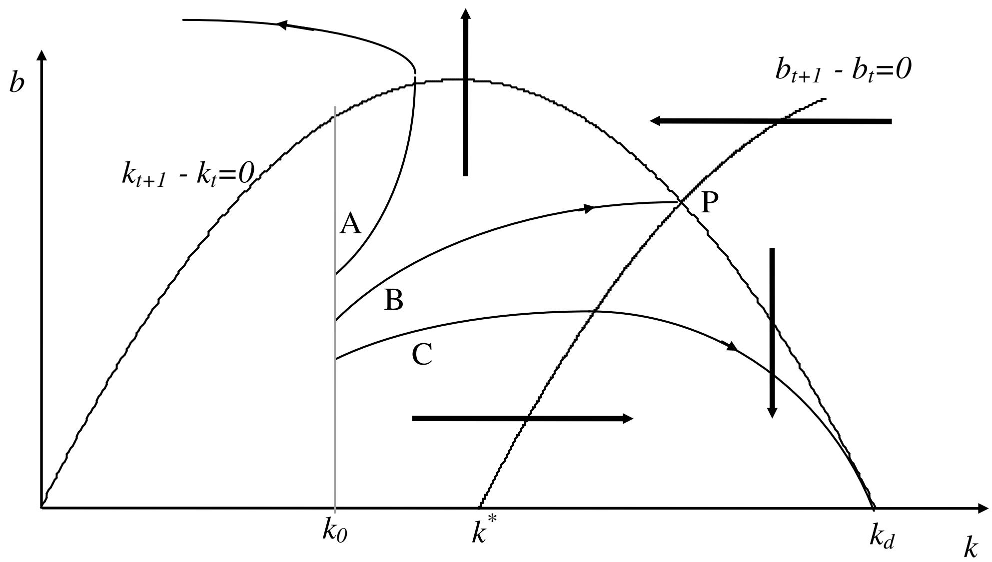
For the bubble to converge to a stable equilibrium, the capital stock k^* necessary for a non-bubble economy to be dynamically efficient, that is $f'(k^*) = n$ and $b=0$, must lie between zero and k_d. If not, the economy can never reach the point where the bubble will decrease (that is where $f'(k) < n$), since there will be no forces under the $k_t = k_{t-1}$ locus that pull in the direction of no bubble. The bubble will increase forever and no equilibrium is possible.

To the right of k^* we will have dynamic inefficiency, since $f'(k) < n$ due to decreasing marginal return on capital. The assumption that $k_d > k^*$ therefore implies that the competitive non bubble economy (the Diamond equilibrium at point k_d) is dynamically inefficient, so that $f'(k_d) < n$. This is a necessary condition for a general equilibrium to exist.

3.5.4 Trajectories

In Figure 3.4, I have drawn possible paths for a bubble starting at three different points, with an initial capital stock at k_0. The bubble starting at point B gives the path leading to the stable equilibrium at P. The arbitrage effect and the investment effect affects the bubble just so much that it will reach a general steady state (both effects are in a steady state).

Figure 3.4: The paths for different initial levels of the bubble



If the initial bubble is too large (e.g. point A), the arbitrage effect becomes very apparent, and the investment effect becomes very small. Most of savings goes to buying bubble assets, which will increase the capital stock too little to account for the strong arbitrage effect. Since the capital stock does not increase as much as if the bubble had started in B, the interest rate decreases and the bubble increases more. Eventually the bubble will cross the k_{t+1}-$k_t=0$ line, and capital will decrease at an increasing rate at the same time as the bubble increase at an accelerating rate. Finally this means that the capital stock becomes negative. Rational behaviour does not allow such a path. People will know that this is not beneficial for them. It is therefore not possible to be on a path above B and reach a general equilibrium, if people are rational.

If the initial bubble is less than B (e.g. point C), it will converge to zero. The initial bubble will increase less relative to investment than if it started at B or equivalently the arbitrage effect will be small compared to the investment effect. This is because the smaller bubble results in higher capital accumulation, so that the interest rate decreases and consequently the bubble increase less. At the point where growth is larger than the rate of interest, the bubble (in per capita terms) will start to decrease if the bubble is to give the same return as investment. The economy will therefore converge towards k_d.

3.5.5 If the bubble asset paid dividends

In the theory described, the bubble asset was intrinsically useless. It was held because people had confidence in that they could sell it later. Does this example apply if the bubble pays dividends? The answer is yes. Since the bubble component of an asset must grow at the rate of interest, and dividends are assumed to increase less than the interest rate, the present value of the asset when it is discounted infinitely, will be the bubble component. The dividend in the future will be negligible, but the bubble component will not.

The same argument rules out that a rational bubble can be negative. A negative bubble will cause the price to become negative since the negative bubble must grow faster than the price. Negative prices are not consistent with rational behaviour and consequently such bubbles in this context are ruled out.

3.5.6 Bubbles, dynamic efficiency, Pareto efficiency and money

As we can see in Figure 3.4, a bubble can exist over time in a general equilibrium at P. A bubble such as this will in fact be dynamically efficient and Pareto efficient, since it prevents the economy from being in the inefficient competitive non-bubble Diamond equilibrium. It can be argued that money is such a bubble. Money may be looked upon as an intrinsically useless asset, which is only valued because people trust that it will be worth something in the future to others.

Money may very well prevent the economy from ending in an inefficient state. In contrast to the Diamond equilibrium where everything the economy saves is invested, money makes it possible to save without investing. It therefore may reduce the capital stock and increases the interest rate and thus make the economy efficient.

However this view may be more appropriate in relation to earlier times, when saving in money was more common and changes in the money supply was less. Today services that money do, such as lowering transactions costs which can be viewed as dividends, are more important. Consumers and firms are therefore more likely to possess money as a fundamentally priced asset. But money is held in large amounts by different investors through out the world (private investors, banks and countries) as an asset. This money does not yield any significant transaction services, but is held to ensure a country's local currency or as pure speculation.

Another, and possibly better example of rational bubbles, is gold. It yields no services/dividends and its use is roughly limited to jewellery and electronic circuits. The finding of new gold is also limited. The limited use of gold of course heightens its value, but most of the known extracted gold resources are hidden in bank vaults, so at least a part of the gold price can be said to be a rational bubble. This fits very well with the stable equilibrium rational bubble. The bubble in «per capita terms» (the gold price adjusted for the economic growth, which is the same as population growth, *n*, according to the OLG model) has been rather constant in the past history, so that the actual gold price has increased at the rate of the economy. Therefore, this may be a stable equilibrium. If everybody, countries and banks included, suddenly found out that saving in gold was pointless, the price might drop to its fundamental level. The price of gold would be equal to what the marginal buyer and seller would be willing to pay for it to use it (since «production» of gold, which is the same as exploring new mines whose sites have not yet been priced for its existence, is very limited). A change in this stable equilibrium is in fact exactly what is happening at present. National banks shifts their holdings from gold to bonds, causing the gold price to drop («The Economist» no. 47, 1997).

3.5.7 Stochastic bubbles

We can now expand the model by introducing the probability that a bubble can burst. In this stochastic model the bubble, b_t, follows a process:

$$(3.15)\quad \begin{aligned} b_{t+1} &= \frac{b_t}{aq} + e_{t+1} && \text{probability q} \\ b_{t+1} &= e_{t+1} && \text{probability 1 - q} \end{aligned}$$

Where e_{t+1} is a stochastic noise term where $E\left[e_{t+1} \middle| I_t\right] = 0$, and a is the discount factor used previously.

This model has interesting features. The noise term allows the bubble to arise after the bubble has burst. If we take the expectation at time t, of this bubble we get:

$$E\left[b_{t+1} \middle| I_t\right] = \frac{1}{a} b_t \qquad \text{or equivalently} \qquad b_t = aE\left[b_{t+1} \middle| I_t\right]$$

So the bubble is expected to increase at the rate of interest as previously described, and is therefore consistent with earlier results. The model also implies that the larger the probability that the bubble will burst, the larger is the expected increase in the value of the bubble, given that it does not burst, becomes. So there is a risk premium.

3.5.8 Stochastic bubbles in general equilibrium

Weil (1987) has shown that such a stochastic bubble can in fact converge to an equilibrium in the same way as deterministic bubbles. However this requires stronger conditions than those used when deterministic bubbles were treated earlier. How Weil derives the model will not be described, but the key equations are:

(3.16) $$(1+n)k_{t+1} = hw(k_t) - b_t$$

(3.17) $$\frac{b_{t+1}}{b_t} = \left\{ \frac{hw(k_t) - b_t}{qhw(k_t) - b_t} \right\} \cdot \frac{1 + r(k_{t+1})}{1+n} \qquad \text{where} \quad h = (1+a)^{-1}$$

Equation (3.16) corresponds to the investment equation (3.13b) in the previous deterministic bubble case. In the same way (3.17) correspond to the arbitrage equation (3.11b). Weil uses an explicit utility maximisation model with a similar specification of the price process as previously mentioned, but without the noise term. He specifies the utility to be logarithmic. This way one can get a closed form expression where q is present, in the difference equations (3.13b) and (3.11b) as seen in (3.16) and (3.11b). The coefficient q is interpreted as the confidence in the bubble. The equations he derives are in fact very similar to these equations. The main difference is that the arbitrage equation is multiplied by a risk premium coefficient. This coefficient is greater than unity, and decreases when the confidence in the bubble bursting, q, increases. The locus for this arbitrage equation will therefore start at the same point where $f'(k) = n$ and $b=0$, but it will increase more than the locus for equation (3.11b) if q<1, due to the risk premium. However the interpretations made in the previous sections are not seriously affected by the introduction of stochastic bubbles instead of deterministic ones. One important and obvious difference is however that this kind of bubble can burst at any moment, and that the longer the bubble runs, the bigger is the chance it will burst. The chance the bubble will last for n periods would be q^n, which will converge to zero. As long as the bubble exists, it can be in a steady state, but the chance it of bursting will increase.

The conditions for a general equilibrium are different though. The previous requirement that the competitive non-bubble economy must be dynamically inefficient for a general equilibrium to exist is replaced by another stronger requirement. This is that $q > \frac{1+f'(k_d)}{1+n}$. (Remember that $f'(k) = r(k)$, the interest rate). It is in other words not enough that the economy is dynamically inefficient, e.g. that the rate of interest is less than growth at k_d, but the rate of interest must be sufficiently low compared with growth so that the relation between them is less than the chance of the bubble bursting. If the bubble is deterministic, q=1, this condition becomes the same as before; the economy has to be dynamically inefficient in a non bubble competitive economy if a general equilibrium shall exist.

If stock markets are assumed to be efficient, stochastic bubbles are probably more adequate to describe the stock market bubbles. While deterministic bubbles display little variance, stochastic bubbles can fluctuate a lot, as asset prices often do.

3.6 How bubbles can arise in a not fully rational market

In the case of non rational bubbles, or a speculative bubble, the bubble element of the price is no longer restricted to follow the rate of interest. Therefore bubbles that are not fully rational are difficult to approach analytically, since there are few kinds of bubbles that can be ruled out a priori. The bubble *can* follow all the three possible paths described in the previous section, but many other paths can be followed too. As a matter of fact one can say very little about how such bubbles develop. The difference equations (3.11) and (3.13) in the previous section 3.4, will not apply in this context.

The arbitrage equation (3.11b) will not necessarily be correct, since opportunities for arbitrage might exist in a not fully rational market. For example a bubble asset might yield a much higher return than the market rate of return, if the investors who buy this asset and therefore cause the price to increase so much, are not rational. Individuals who are not rational will not assess this asset on the basis of what a full information set tells them that they can expect to get in return in the future (as a price increase). They will rather base their assessment on accidental events and a biased selection of information. An asset can therefore be assumed by the market to have a much greater potential for future return than it really has. It will in fact be possible that a non-rational individual can value a bubble asset as a dividend paying asset. This individual will not be able to reveal that the bubble asset is not expected to pay any dividends, so in a non-rational market it will not be clear what is a bubble and what is not. The arbitrage equation will therefore not apply here.

This argument also makes the investment function (3.13b) invalid. How savings changes when the capital stock changes will be impossible to tell, since the resulting rate of return of saving (for instance what the bubbles yields) will be completely unpredictable. The rate of return of savings will not be linked to the rate of return on capital, since the arbitrage equation does not hold in a market that is at least not fully rational.

If positive bubbles are assumed, the return may therefore be much higher than for investments. In that case, such speculative bubbles should at least attract some of the capital that would else have been used for real investment. Thus, some of the conclusions derived in the previous section still hold for non-rational bubbles. The capital stock will be reduced. However, no equilibrium can be found since such bubbles, as mentioned earlier, are completely unpredictable. The reduction in the capital stock may have the same consequences as described earlier, making the economy dynamically efficient.

Rational bubbles arise purely as a means of saving when the return on capital is too low for further investment. They will then follow a specific path based on the start value of the bubble. There can, however, be all sorts of reasons for non-rational bubbles, and one can not derive their paths. A conclusion of this discussion is therefore that dynamic models for non-rational bubbles are difficult to obtain.

A more interesting approach to this issue would therefore be a general discussion of how bubbles can arise in a market which is not necessarily rational. I will base the first part of this discussion on a model proposed by Shleifer and Summers (1990), then present some theories from psychology and finally present some research on the efficiency and rationality of the market.

3.6.1 A two-traders model

Shleifer and Summers divide the traders into two groups, «rational traders» and «noise traders». The rational traders ensure that there is no riskless arbitrage opportunity in the market. If an asset with a certain risk gives higher returns than another equally risky asset, the rational traders instantly demand more of the high return asset and less of the other. This will equalise the expected return and eliminate the possibility of arbitrage. This works well for assets which depend on known properties and therefore are easy to assess, like derivatives and bonds. It also works for individual assets which can be compared to others. But it does not work for a market as a whole. If the entire market is priced wrongly, the only way to hedge against market risk is to buy options. But this may be expensive since no one knows what will happen in the future. Noise traders drive up the prices above the level consistent with the expected dividends. These traders are at least not fully rational and may be subject to systematic biases.

According to S&S there are two types of risks for the rational traders. If the rational traders try to sell short to avoid the present market risk, expecting a price decrease in the future, they risk that their estimates of the future dividends are wrong. It might be that the prices really reflect the true dividends. This is *the fundamental risk.*

There is also a possibility that the bubble will increase further, before it bursts, giving short-sellers a loss if they are going to realise the options before the market falls. Since no one knows when the bubble will burst, short-selling can be a very risky affair. It is also possible that the bubble will never burst, as in the case of a general equilibrium in the previous section. There may also be more traders that enter the market. This is the *future resale risk*

There is obviously no problem if the rational traders have infinite horizons. That assumption may however be incorrect. The structure of the financial market gives incentives for short horizons. To attract capital to a risky market, the traders need to borrow short, giving lenders per period fees. S&S argues that there is a strong bias towards short horizons.

The two-traders model is a based on the presence of both rational and irrational individuals. Even though the rational traders do not have infinite horizons, it is assumed that they would act as if they had if the noise traders were absent. Assets would be priced at their expected fundamental level. But in the presence of noise traders, the market becomes riskier and an irrational bubble arises. The rational traders become noise traders themselves. It will in fact often be beneficial and rational for the traders in such a situation to act «irrationally». Trend spotting is a known strategy in financial market. If a trader is good at spotting trends in a boom, it is not unreasonable that he makes a profit on average even if the market falls from time to time. Since trend spotting is a risky undertaking, the expected return may therefore have a high risk premium.

3.6.2 Market biases

There are a number of biases that influence the demand of assets. These biases are often well known in psychology. But even though psychology is an important part of market behaviour, research on market psychology ended pretty much in the 1950s when expected-utility theory became more popular (Shiller, 1989)

One bias is the price increase often followed by inclusion in a stock index, because many funds acquire the exact same representation of stocks as some indexes. If many mutual funds require the same asset at the same time, the inclusion of this asset in the index will cause a price increase. This is an example of collective irrationality by the fund managers. There would be money to save if the purchase of these stocks where spread over a longer period, and this would probably affect the earnings minimally. The influence of the advice of a financial guru or ideas taken from popular asset market «theories», may also affect the market significantly. This bias reveals clearly irrational behaviour by the market, since the individual traders must know that many others follow the exact same advice.

A basic property in psychology is that humans rarely chose randomly (proven many times in psychological experiments) (Plous, 1993). People tend to emphasise some pieces of information and ignore others. Examples are *overconfidence*. When one has made a decision, it is normal to be unrealistically confident that it is the right one. You will probably consider speculation to be more profitable and less risky after you have entered the marked than before.

If overconfidence is usual in the market, it can bias the capital invested in the market and consequently prices upwards, since investors are unable to acknowledge the true risk. Investors may also stay in the market too long and lose if there is a recession. This behaviour may be caused by *entrapment*. If you lose money on the stock market, you might want to increase your stake to make up for the loss, ignoring the risks. In this kind of situation you are trapped by previous irreversible and irrelevant events that still influence the present decisions.

It is usual to put too little weight on the full information set (*base rates*) and too much on new or more apparent information. This can be an explanation of the often exaggerated reactions of the stock market to new information. The «bad» stocks are undervalued and the «winners» are overvalued. Investors tend to overreact as described in the next section.

Finally there is a *roulette effect*. If you are told a random number and then asked to make a guess on something, your guess will probably be biased towards the random number. This way, a high price may in itself cause the expectations of future incomes of that asset to be biased upwards. This is an effect that will reinforce a bubble, and maybe give some explanation of the reason for it to arise. If the price of an asset starts to increase, people will get used to it and estimate an even higher future income of that asset.

3.6.3 Overreaction

Perception of information can cause bubbles to arise. It may be that the information set initially is the same for all participants in the market, but that the way the market perceive the information is biased in a certain way. For example it is often assumed that traders react more to some pieces of information than others. New information is often given too much weight relative to less apparent or older «base rate» information (as mentioned on page 39).

Bond and Thaler (1985) test what they call the overreaction hypothesis. They use a model where one information set represents all available information at time t-1 (F_{t-1}) used to predict the return of asset j, $\tilde{R}_{jt}$, at time t. Another other information set , F_{t-1}^{m}, consists of the information used by the market to assess the future return of this asset. If traders are rational, they should on average use the complete information set and one should not expect any difference between the estimates based on either of the two information sets. Therefore, letting $\tilde{u}_{jt}$ be the stochastic error term of the expectations, rational traders implies that:

(3.18) $$E\left(\tilde{R}_{jt} - E_m\left(\tilde{R}_{jt} \middle| F^m_{t-1}\right) \middle| F_{t-1}\right) = E\left(\tilde{u}_{jt} \middle| F_{t-1}\right) = 0$$

If (3.18) is true for any asset, it must also be true for a «winner» and a «loser» portfolio. From the market, stocks that have experience extreme losses are selected into a «loser» portfolio and stocks that have experienced extreme gains are selected into a «winner» portfolio. The portfolios are named j={l,w}, where portfolio l is a «loser» and w is a «winner». If the traders are not rational, but rather over reactive, they will put too much weight on the information causing the «losers» to lose and the «winners» to win. The losers will therefore have a higher expected return in the future than the return expected by the market, and the winners will have a lower expected return, thus the *overreaction hypothesis* states that:

(3.19) $$E\left(\tilde{u}_{lt} \middle| F_{t-1}\right) > 0 \qquad \text{and} \qquad E\left(\tilde{u}_{wt} \middle| F_{t-1}\right) < 0$$

In their test, Bond & Thaler sort stocks registered on the New York Stock Exchange (NYSE) every three years, and then calculate the error terms in (3.19) using the average market return as the expected return if the full information set were used. Using the average market rate should only make it more difficult to reject the null that financial markets does not overreact, since one often would expect that a loser portfolio would give a lower and a winner portfolio a higher return than the market portfolio.

Bond & Thaler reject the null hypothesis, thus lending support to the idea that financial markets overreact and that negative or positive information about assets is overrated by the traders.

Bond & Thaler (1990) have also studied whether security analysts overreact when they estimate forecasts on earnings per share (EPS). If the analysts are rational, the actual EPS will be uncorrelated with the initial forecast. If not, the analysts would learn that their estimates was biased and in the next period make an unbiased estimate (more about this on page 47). Therefore the estimated change in EPS for a stock over time, should not be systematically correlated with the estimation error. The overreaction hypothesis is that the forecasts are too extreme so that actual changes are smaller than the changes in forecasts. At the same time they investigate whether this bias gets stronger if the uncertainty is larger. Bond & Thaler (1990) find evidence in favour of both these hypotheses.

3.6.4 A model with feedback

The models described previously assume that the solution to (3.2), $p_t = aE\left[p_{t+1} + d_{t+1} \middle| I_t\right]$, that contains bubbles is the finite sum (3.8), $p_t^* = b_t + \sum_{i=0}^{\infty} a^i E\left[d_{t+i} \middle| I_t\right]$. This builds on the assumption that the dividend process is independent of the price process. However Timmermann (1994) shows that if the dividends follow a stochastic process in which the price is present and the process satisfies certain conditions, then the sum in (3.8) will go to infinity and no finite solution is possible. This way, a bubble may be excluded if dividends follow such a process. This knowledge can be used to check if bubbles can exist at all. If it is found that the dividend process satisfies the conditions, it would lead to a rejection of the hypothesis of bubbles. Such a test would therefore be an alternative procedure compared to the more common tests used to check if bubbles are present or not. However, both testing directly whether there *are* bubbles or not and testing these conditions which would suggest that bubbles does not exist, can be difficult.

I will not present the econometric procedures that Timmermann uses to test if there the price is a variable in the dividend process, nor the rather complex conditions the dividend-price relationship has to satisfy for bubbles to be impossible. I will however present an example from Timmermann of a dividend process that is influenced by the price. This example examines the *leverage effect*. This is a positive effect that stock prices have on the company's dividends. As the price increases, the equity of the firm increases and the debt-to-equity ratio and expected future volatility decrease. This reduces the required return of the firm's shares for this period, which will make the company safer. A safer company will have to pay less to raise capital, which should increase earnings and future dividends. Timmermann assumes a Cobb-Douglas production function to examine this feedback. Capital is as defined previously k_t, and A_t represents productivity.

(3.20) $$f(k_t) = A_t k_t^{\alpha} \qquad 0<\alpha<1$$

To include bonds in the model, it is assumed that there is a fixed cost of production financed by issuing bonds O_t at a rate r^{bonds}. Commodity prices are normalised to 1 so the earnings of the firm are:

$$Earn_t = f(k_t) - O_t r^{bonds} - r(P_t)k_t, \qquad r'(P_t) < 0 \tag{3.21}$$

The negative relationship between *r* and *P* emphasises the leverage effect, as described previously. The company maximises its expected return with respect to capital, so the first order condition for interior maximum becomes:

$$k_t = \left(\frac{r(P_t)}{\alpha A_t}\right)^{-1/(1-\alpha)} \tag{3.22}$$

All the earnings are paid out in form of dividends ($d_t = Earn_t$), so by substituting (3.22) into (3.21) we can find an expression for the dividends:

$$d_t(P_t) = A_t^{1/(1-\alpha)}\left(\alpha^{\alpha/(1-\alpha)} - \alpha^{1/(1-\alpha)}\right)\left(r(P_t)\right)^{-\alpha/(1-\alpha)} - r^b O_t \tag{3.23}$$

$$\Rightarrow d_t{}'(P_t) = \frac{-1}{1-\alpha} A_t^{1/(1-\alpha)}\left(\alpha^{\alpha/(1-\alpha)} - \alpha^{1/(1-\alpha)}\right)\left(r(P_t)\right)^{-1/(1-\alpha)} > 0 \tag{3.24}$$

So if the price increases above the fundamental level, the price will reinforce the dividends and may cause them to increase into infinity [9]. If so, no finite solution is possible and a bubble can be excluded. A bubble will cause the dividends to go to infinity, so the price has to go to infinity too. If this is the case, the price must be infinitely high. This implies that the asset does not sell at all, so there will be no price in the market.

3.7 A look at the Norwegian stock market 1982-1997

In this section, I will discuss in general how the Norwegian stock market has developed in the last two decades. Empirical evidence will be left to the next chapters.

[9] Given that the conditions described by Timmermann are fulfilled.

The period from 1982-1997 in the Norwegian stockmarket will be classified as two exceptional periods of growth and one recession. One growth period is from 1982 to 1985 and one is from 1992 to the present. In Table 3.1 the average real growth rate of the main index of Oslo Stock Exchange is presented, compounded continuously and yearly. Continuous compounding is used since traders often buy and sell many times a day, and therefore have an almost continuous compounding of the value of their portfolio. The average interest rate on 3 month bonds and the increase in investment [10] are also calculated. The real growth rate is obtained by using the Norwegian consumer price index. The average interest rate on 3 month euro bonds are calculated using quarterly compounding because they are calculated using quarterly figures of the return. These are also adjusted using the consumer price index. The changes in real private investment cannot be taken as very precise, since investment changes much from period to period, but these figures at least gives an indication. Stock prices are measured on the last day of the year. The interest rate and the consumer price index are measured in the last quarter of the year and investment is measured yearly. I measure at the end of the year to be consistent in my treatment of the data, since I do this in the empirical part of this dissertation.

Table 3.1: Stock price growth, the interest rate and investment

	Norwegian stock prices, real growth[1]:		**Real interest rate on 3 months euro bonds[2]:**	**Change in real private on-shore investment[3]:**
	Continuously:	***Yearly:***		
82-97:	12,5 %	14,0 %	6,7 %	-0,8 %
82-85:	29,4 %	37,0 %	7,1 %	-1,4 %
85-92:	-2,3 %	-2,3 %	7,9 %	-1,5 %
92-97:	24,4 %	28,3 %	4,0 %	1,0 %

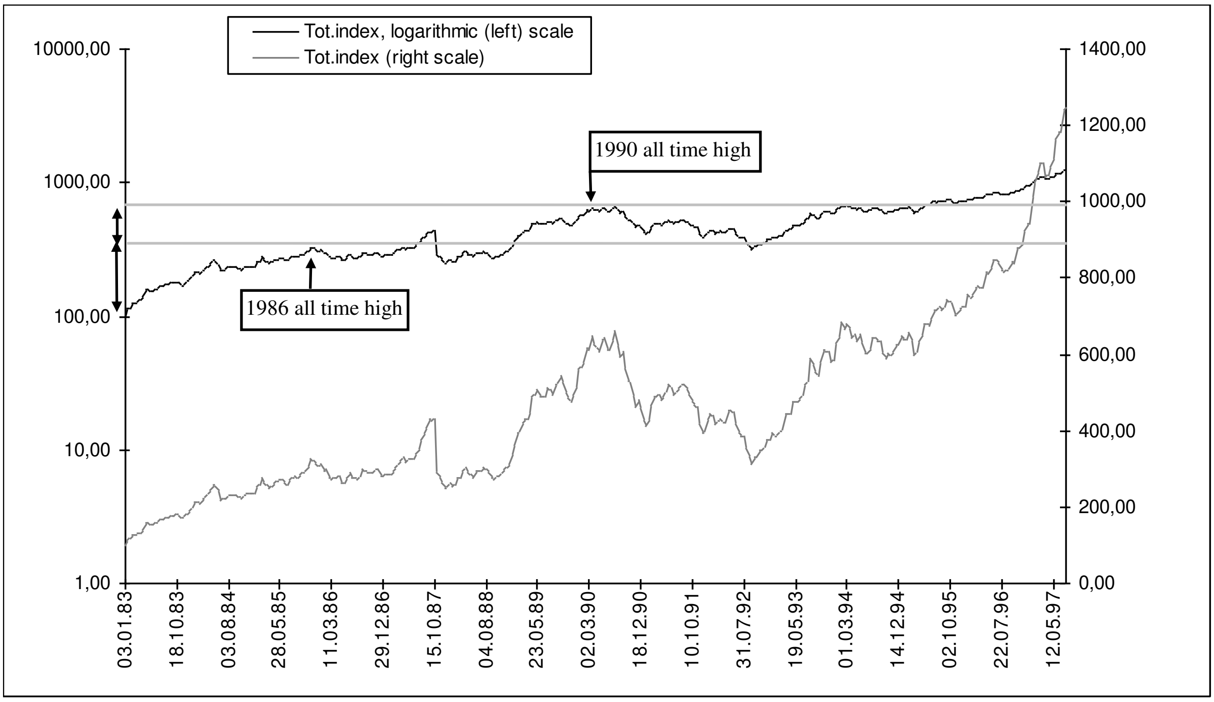
[1]To last quarter of 1996 [2]To July 25. 1997 [3]To second quarter of 1997

It may seem peculiar that I have chosen to use the period 1982-85 and not 1982-87 as one exceptional growth period. First, the sample is taken from the last day in each year as mentioned, so that the total index in 1985 is the latest measurement of that year, and therefore best reflects the price early in 1986. Second, it is often assumed that the problems in western countries during the eighties started with the crash on Wall Street October 1997. In Norway however, problems started as early as 1986 as a result of a vast drop in the oil price. This

[10] Data on the stock index supplied by Oslo Stock Exchange. Data on returns on euro bonds and consumer price index supplied by the Bank of Norway. Data on investment supplied by Statistics Norway.

boosted the economy of many oil-importing countries until 1987, but had unfortunate consequences for Norway. The oil that probably was the reason for some of the exceptional growth prior to 1986, now limited the return in the stockmarket. Although the total index at Oslo Bourse reached an all time high in October 1987 and 1990, the annual real price increase was modest in the years 1986-87 and 1986-1990, and incomparable to the growth that the Norwegian stockmarket experienced in the period 1982 to early in 1986. This is depicted in Figure 3.5, where the actual rise in the total index at Oslo Bourse (the grey curve, right scale) gives the impression that stocks gave high returns in Norway prior to 1987 and 1990 relative to 1986. If we look at the logarithm of the index (the black curve, left scale), which is better suited to spot differences in the increase for different periods, we see that the stock market increased most before 1986. Two help lines are drawn, comparing the 1990 all time high and the 1986 all time high level. The difference is larger between the 1986 all time high and the initial value (100 in 1983), than between 1986 and 1990 if a logarithmic scale is used.

Figure 3.5: The total index of Oslo Bourse, logarithmic and normal scale



As we can see from the table, the growth in the years 1983-85 was very high. The stock market increased at a rate much higher than for example the bond interest rate. But in the recession years in 1985-92 the main stockprice index actually fell both in nominal and in real terms. This happened in a period when the real bond rate was on average eight percent. Then again from 1992 the increase in stock prices has picked up, and has so far almost reached the

82-85 level. The nominal return was much higher in 1982-85, but the low inflation rate during recent years, has made the real return almost as high as in the early eighties period.

It is interesting to look at the changes in the level of investment. In the growth period 82-85, real private investment actually fell. This is surprising since high demand for assets such as stocks should lead to higher investment. Some of the investment could of course go abroad and the level of investment may have been artificially high in the early eighties due to a possible effect of the oil industry on on-shore investments. In addition we do not know which types of investment that the firms registered at the stock market undertake. It could have been the case that other types of investment decreased so much that the statistics did not reveal a possible increase in investment coming from firms listed on Oslo Bourse. But *if* the investment by firms registered at the stock market actually fell, this is not what is expected when stock prices increase. Stock prices represent the price of implemented capital. When the price of stocks increases, one would expect that demand for unimplemented capital, e.g. investment goods, would increase since these are imperfect substitutes. If the supply of investment goods do not decrease, investment should increase.

An explanation of the discrepancy between the increase in the stock price for firms registered at the stockmarket and real investment may be bubbles. If a substantial part of the increase in the stock price resulted from short horizon speculation of a higher future price, this can explain why investment demand did not increase too. It is also a point that firms usually invest more if they expect higher earnings, and therefore possibly higher production in the future. So it seems as if the increased price of stocks is not entirely a result of higher expected dividends, which can be signs of a bubble. However, more thorough investigation into this is necessary to draw any conclusions.

For the period 1992-1997, the stock index also grew very fast. Investment growth is however positive, but low at only one percent, so some of the arguments presented previously may hold here too. However there are some other interesting features of the stock market recently. There has been a vast increase in firms that have earned close to nothing, but their stock market value has nearly exploded. On October 3rd 1997, 30 companies that had earned little or nothing, not all of them represented on Oslo Bourse, had a total estimated value of NOK 17 billion (Dagens Næringsliv, October 4th 1997) based on the current stock prices. Either the companies are expected to earn amazing profits, or the owners are planing to make a profit by selling the stocks at a higher price in the future, thus indicating a bubble.

4. Testing for bubbles

Many tests have been developed to check for the presence of bubbles. A common way to do this, is to test the fundamental price in equation (3.8) in various ways. No matter what method is used though, one has to trace the markets expectations in one way or another to be able to investigate whether there are bubbles or not. This way the fundamental price can be estimated or given certain restrictions. There are however differences in how these expectations of future dividends are traced. It is in the determination of the expected dividends and assumptions that can be drawn from this, the problem of testing bubbles often arises. The methods used to find a reliable information set to estimate future prices or dividends range from using macroeconomic data, look at the companies accounts books, making interviews and using data on prices and dividends. One is rarely concerned with what kind of bubble there exists (e.g. non-rational, rational, stable, ever expanding or converging to zero), but some tests displays a weaker ability to detect rational bubbles than others and some tests is clearly made to reveal non-rational bubbles (as some of the tests described at the end of the previous chapter).

In this part of the dissertation I will perform one test which assumes that past dividends are on average the information set applied by the market to estimate future dividends. This is West's (1987) specification test. I will also do a less comprehensive test, the well known Shiller's (1981) variance test. In this test, the assumption that the market price when there is no bubble is a unbiased estimate of the future dividends will imply something about the variance of the dividends and prices. This test is performed to compare with West's test.

The test usually make use of an assumption, that the error term is uncorrelated with the markets expectations. If there was a systematic correlation between the expectation error and the information set, the expectations would not reflect the best guess one could make. By taking into consideration the connection between the expectation and the information set, people would do better estimates of in the future and eliminate the correlation. We can therefore state that:

Assumption 4.1

The error term of an expectation is uncorrelated with the expectation itself. That is, if $E(x_T)=x_t+\varepsilon_t$*, where* ε_t *is an error term or «white noise» with mean zero, then* $Cov[E(X_T),\varepsilon_t]=0$

4.1 Shiller's variance test.

Shiller (1981) assumed that the market price was the optimal estimate of the fundamental price of an asset. Since the expectation of a sum of stochastic variables should vary less than the sum itself, he argued that this could be used to test for bubbles. If the market price, p_t, is an optimal estimate of the fundamental price, p_t^* , then:

(4.1) $$E\left[p_t^* \middle| I_t\right] = p_t$$

Writing the ex post fundamental price p_t^* as the market price and an error term gives:

(4.1b) $$p_t^* = p_t + \varepsilon_t$$

The error term should be uncorrelated with the market price since this is the optimal estimate of the fundamental value (Assumption 4.1). The variance of this expression then becomes:

(4.2) $$Var\left(p_t^*\right) = Var\left(p_t\right) + Var\left(\varepsilon_t\right)$$

So we have the inequality:

(4.3) $$Var\left(p_t^*\right) > Var\left(p_t\right).$$

The fundamental price is estimated as:

(4.4) $$\hat{p}_t^* = \sum_{i=1}^{T-t} a^i d_{t+i} + a^{T-t} p_T$$

The estimated interest rate used to calculate the discount rate is found by Shiller to be simply the average dividends divided by the average price. The estimated discount factor is therefore:

$$\hat{a} = \frac{1}{1 + E(d)/E(p)} \tag{4.5}$$

Shiller tests the null hypothesis that there are no bubbles and that the inequality of equ. (4.3) holds against the alternative hypothesis that there are bubbles and that (4.3) does not hold. In his test on US stock market indexes for a hundred years sample period (1871-1979), he finds that the inequality goes the other way. Thus the theory of efficient markets where the price is an estimate of the fundamental price and there are no bubbles does not hold.

Critics of this method of testing for bubbles argue that Shiller assumes stationary time series both for prices and dividends after detrending and deflating (Shiller deflates the prices by the producer price index), so that the variance is supposed to be independent of time.

If the time series is not stationary after detrending and deflating, it will be impossible to difference the variables in this test to get stationarity, because the autocorrelation (both within a variable, and between them) would make the inequality ambiguous. To assume stationarity, and to design a test that does not have a option for this problem is a general weakness of the test. One may test for stationarity first and decide afterwards whether the test is applicable, but this only emphasises the weakness of the test since in many cases the test must be discarded. Since stock prices are often initially not stationary even with a trend, which the data from the Norwegian stock market does not refute [11], a test should have an option for differencing.

In addition it is assumed that there is no correlation between the market price and dividends, an assumption which has been discussed in the previous chapter (section 3.6.4). It is also a problem that bubbles can be included in the null hypothesis. If the terminal price in (4.4) contains a bubble, the null hypothesis will do so too. Mankiw, Romer and Shapiro (1985) (Flood & Hodrick, 1990) try to solve these problems with their revised test. I will however not

[11] The Prices in the aggregate index presented in chapter six are found stationary according to the ADF test, when a trend and a fifth order of autocorrelation is assumed. However, due to the small sample, a test for such high orders of autocorrelation have little power. In addition the data is measured yearly, so such high orders of autocorrelation is not very likely. I therefore conclude that there is little evidence of stationarity for this variable in my sample (see appendix for the exact results).

discuss this test any further, since the aim of presenting the test is to compare a simple, well known and early developed test for stock market bubbles with the one presented by West (1987).

4.2 West specification test

West (1987) presents a rather sophisticated way to test for bubbles. To test the null hypothesis that there is no bubbles, one compares two different ways of estimating the discount rate and assumes that the expected dividends is dependent on earlier dividends. I will first present the simplest version of this test, where the expected dividend in next period is assumed to be dependent on the current dividend *only*, so there is no constant in the regression equation. I will then move to a more complicated model where the dividend follow an ordinary AR(q) process (with a constant term and more than one lag). In this presentation it is necessary to use a variance-covariance matrix, which will be presented. Because the data has a short span, I will introduce a way to test several series together. West's tests will then be discussed in a more general matter.

4.2.1 Introduction: A simple test

It is assumed that the expected dividends depend on the current dividends, thus dividends follow a simple AR(1) process with no constant term.

(4.6) $$d_t = \phi d_{t-1} + v_t$$

Where $|\phi| < 1$ with , and v_t is an error term with zero mean.

The discount factor, a, is assumed to be unknown. By iterated substitution we get $d_{t+T} = \phi^T d_t + \sum_{i=0}^{T-1} \phi^i v_{t+T-i}$, so the expected dividend is $E[d_{t+T}] = \phi^T d_t$. We can then find the fundamental value (equation (3.5)) by using the theorem of an infinite geometric series,

(4.7) $$p_t^* = \sum_{i=1}^{\infty} a^i E[d_{t+i} | I_t] = (a\phi) \sum_{i=0}^{\infty} (a\phi)^i d_t + w_t = \delta_1 d_t + w_t$$

(4.8) $$\delta_1 = \frac{a\phi}{1 - a\phi} \qquad \text{or} \qquad a = \frac{\delta_1}{(\delta_1 + 1)\phi}$$

where w_t is a error term with mean zero. So if there is no bubble (4.8) should hold.

(4.6) and (4.7) may be estimated by OLS, yielding point estimates, $\hat{\phi}$ and $\hat{\delta}_1$, for ϕ and δ_1.

A second way of estimating the discount factor, a, is to estimate a third equation, the asset price arbitrage equation (3.2), $p_t = aE\left[p_{t+1} + d_{t+1} \middle| I_t\right]$ by rewriting it as:

(3.2c)
$$\begin{aligned} p_t &= a\left(p_{t+1} + d_{t+1}\right) - a\left[p_{t+1} + d_{t+1} - E\left(p_{t+1} + d_{t+1} \middle| I_t\right)\right] \\ p_t &= a\left(p_{t+1} + d_{t+1}\right) + u_{t+1} \end{aligned}$$

where u_t is an error term with zero mean. This gives a point estimate of the discount factor a, $\hat{a}$. a can be estimated with 2SLS, using d_{t-1} as an instrument. If we insert all three estimates into (4.8), we will expect according to the null hypothesis stating that there is no bubbles, that the equality (4.8) holds. If the expression is significantly different from zero, there is evidence of a bubble in the economy.

4.2.2 The information set

A more realistic approach will be to assume that expected dividends depend on more than one previous dividend. This is done by assuming that on average only a part of the full information set is necessary to predict the future dividends. This reduced information set is information about current and previous dividends. At time t it is denoted H_t. H_t consists of a constant and current and lagged dividends. This is a subset of the full information set I_t. The fundamental solution (3.5) can then be written :

(4.9)
$$\begin{aligned} p_t &= \sum_{i=1}^{\infty} a^i E\left(d_{t+i} \middle| H_t\right) + z_t \\ z_t &= \sum_{i=1}^{\infty} a^i \left[E\left(d_{t+i} \middle| I_t\right) - E\left(d_{t+i} \middle| H_t\right)\right] \end{aligned}$$

Where $H_t = \left\{1, d_{t-i} \middle| \; i \geq 0\right\}$

z_t is serially correlated in general. This is because z_t, depends on the estimated future dividends, which in turn depends on previous dividends. Since z_{t-1}, also depends on previous dividends, the error term will be serially correlated.

The model described is used to construct a regression model as follows:

4.2.3 The model

To be able to use the information set H_t in a regression model, we need a closed-form expression of $\sum_1^{\infty} a^i E\left(d_{t+i} \mid H_t\right)$. This is done by calculating the dividends as an AR(q) relationship. If the dividends are not already stationary, they can obtain this property by differencing. West tests regression models that are both undifferenced and differenced once. It is often assumed that this kind of time series will obtain stationarity if differenced once. This builds on the *assumption that dividends follow a difference-stationary process (DSP)*, or a random walk, contrary to the trend-stationary process (TSP) (Maddala 1992). In the DSP case, the error term has a constant non-zero mean, and is a sum of all earlier errors so that the variance depends on time which is not the case for the TSP. Whereas the DSP can be made stationary by first differencing, the TSP must be de-trended by subtracting the linear time trend from the explained variable. If the two processes are confused, and wrong procedure is followed to induce stationarity, a spurious autocorrelation occurs. Tests can be run to check stationarity. The Dickey-Fuller test or the Augmented Dickey-Fuller test is often used to do this. By using different specification of the process, these tests can be used to check for stationarity when there is assumed a trend and when there is not.

Since dividends are supposed to follow the DSP class, they are differenced once to obtain stationarity. If $\{\varepsilon_t\}$ is a purely random series with mean μ and variance σ^2, then a process $\{X_t\}$ is said to be a random walk (DSP) if

$$X_t = X_{t-1} + \varepsilon_t$$

By repeated substitution we get:

$$X_t = \sum_{i=1}^{t} \varepsilon_i \quad \Rightarrow \qquad E\left(X_t\right) = t\mu \quad \text{and} \qquad Var\left(X_t\right) = t\sigma^2$$

If this is differenced once, stationarity will be obtained:

$$E\left(\Delta X_t\right) = \mu \quad \text{and} \qquad Var\left(\Delta X_t\right) = \sigma^2$$

The variables should therefore be differenced once, if there is a random walk. In this case of yearly measurement, there are no seasonal variations so no higher order differencing should be necessary. There is however evidence that a logarithmic random walk is the one best describing the process of prices and dividends. I will return to this in section 4.2.4.

The following equations are derived for both the differenced and undifferenced case. I choose to use the same parameters in both cases even though they are not the same. Although this to some extent may be confusing, I will do this since West himself does it this way and because using subscripts or different parameters can also confuse. In which cases the variables are differenced, will be clear from the context though.

If dividends follow an AR(q) process as described above, the dividend equation (4.6) can be written (undifferenced):

(4.10a) $$d_{t+1} = \mu + \phi_1 d_t + \ldots + \phi_q d_{t-q+1} + v_{t+1}$$

For the first difference:

(4.10b) $$\Delta d_{t+1} = \mu + \phi_1 \Delta d_t + \ldots + \phi_q \Delta d_{t-q+1} + v_{t+1}$$

Given that dividends follow such a process, the price equation (4.7) becomes:

(4.11a)
$$\begin{aligned}
p_{t+1} &= m + \delta_1 d_{t+1} + \ldots + \delta_q d_{t-q+2} + w_{t+1} \\
m + \delta_1 d_{t+1} + \ldots + \delta_q d_{t-q+2} &= \sum_1^{\infty} a^i E\left(d_{t+i+1} \middle| H_{t+1}\right) \\
w_{t+1} &= z_{t+1} + b_{t+1} \\
z_{t+1} &= \sum_1^{\infty} a^i \left[E\left(d_{t+i+1} \middle| I_{t+1}\right) - E\left(d_{t+i+1} \middle| H_{t+1}\right)\right]
\end{aligned}$$

b_i is as usual the bubble. For the differenced case, this becomes:

(4.11b)
$$\begin{aligned}
\Delta p_{t+1} &= m + \delta_1 \Delta d_{t+1} + \ldots + \delta_q \Delta d_{t-q+2} + w_{t+1} \\
m + \delta_1 \Delta d_{t+1} + \ldots + \delta_q \Delta d_{t-q+2} &= \sum_1^{\infty} a^i E\left(\Delta d_{t+i+1} \middle| H_{t+1}\right) \\
w_{t+1} &= z_t + b_{t+1} \\
z_{t+1} &= \sum_1^{\infty} a^i \left[E\left(d_{t+i+1} \middle| I_{t+1}\right) - E\left(d_{t+i+1} \middle| H_{t+1}\right)\right] - \sum_1^{\infty} a^i \left[E\left(\Delta d_{t+i+1} \middle| H_{t+1}\right)\right]
\end{aligned}$$

z_t is dated t since it is uncorrelated with any information in the information set H_t according to Assumption 4.1 (that is, z_t is orthogonal to H_t).

To sum up, the system that needs to be estimated is:

(3.2c) $$p_t = a\left(p_{t+1} + d_{t+1}\right) + u_{t+1}$$

(4.10a) $$d_{t+1} = \mu + \phi_1 d_t + ... + \phi_q d_{t-q+1} + v_{t+1}$$

(4.11a) $$p_{t+1} = m + \delta_1 d_{t+1} + ... + \delta_q d_{t-q+2} + w_{t+1}$$

And for the differenced case:

(3.2c) $$p_t = a\left(p_{t+1} + d_{t+1}\right) + u_{t+1}$$

(4.10b) $$\Delta d_{t+1} = \mu + \phi_1 \Delta d_t + ... + \phi_q \Delta d_{t-q+1} + v_{t+1}$$

(4.11b) $$\Delta p_{t+1} = m + \delta_1 \Delta d_{t+1} + ... + \delta_q \Delta d_{t-q+2} + w_{t+1}$$

Under the null hypothesis that the bubble is zero, the error term w_{t+1} defined in (4.11a) and (4.11b) will have a expectation of zero since z_t is uncorrelated with the information set. Therefore ordinary least squares is used to estimate the undifferenced equations (4.10a) and (4.11a) and the differenced equations (4.10b) and (4.11b).

West uses a two-step, two-stage least squares method to estimate the discount factor *a* in equation (3.2c). The first step is a standard two-stage least squares. The second stage is used to obtain an optimal heteroskedasticity-consistent estimate of *a*. Instruments used in the two-stage least square estimator are the variables on the right-hand side of the dividend equations (4.10a) and (4.10b).

4.2.4 Logarithmic difference

There is some evidence in financial research which suggest that logarithmic and not arithmetic differences are necessary to induce stationarity (Flood & Hodrick 1990). If dividends follow a log normal random walk, it is possible to obtain a closed-form solution for $\sum_1^\infty a^i E(d_{t+i}) \,|H_t$. In this case $\Delta(\log d_t)$ is assumed to be a log normal random variable such that $\Delta(\log d_t) \sim \mathrm{N}(\mu_l, \sigma^2)$ and $H_t = \{d_{t-1} \,|i \geq 0\}$. This implies that we can estimate the necessary parameters (according to West) as:

(4.12) $p_t = \sum_1^\infty a^i E(d_{t+i}) \,|H_t = \delta_l d_t$

where

(4.13) $\delta_l = \dfrac{\exp(\mu_l + \sigma^2/2)}{\frac{1}{a} - \exp(\mu_l + \sigma^2/2)}$

An estimate of δ_l by regressing p_t on d_t is then compared to that obtained from estimates of μ_l, σ^2 and a. μ_l and σ^2 is obtained as the sample mean and variance of $\Delta(\log d_t)$. The discount factor a was not estimated as in the other tests, instead a^{-1} was calculated as the average ex post return. In the data West had available, the estimate of μ_l was not significant from zero. He therefore also performed the test, assuming that μ_l=0. The test statistic is the variance of $\Delta(\log d_t)$, which is implied by (4.13).

(4.13b) $\sigma^2 = 2\log\{(1/a)[\delta_l/(1+\delta_l)]\} - 2\mu_l$

The variance is chi-square distributed, $\hat{\sigma}^2 \sim \chi^2(T)$, and if μ_l=0 is imposed, $\hat{\sigma}^2 \sim \chi^2(T-1)$. To check whether the coefficients are consistent with this variance, a 99 percent confidence interval is constructed around $\hat{\sigma}^2$ which is compared to the variance implied by δ_l, μ_l, and a.

4.2.5 Calculation of the variance-covariance matrix

The estimated parameter vector is defined as $\hat{\boldsymbol{\theta}} = \left(\hat{a}, \hat{\mu}, \hat{\phi}_1, \ldots, \hat{\phi}_q, \hat{m}, \hat{\delta}_1, \ldots, \hat{\delta}_q\right)$. This vector has dimension (2q+3) and has an asymptotic variance-covariance (2q+3) ×(2q+3) matrix **V**. **V** is calculated as follows, according to West (1986):

Let $\mathbf{D_t} = \left[1, \Delta^s d_t, \ldots, \Delta^s d_{t-q+1}\right]'$ be a (q+1)×1 vector consisting different lags of the dividends where s=0 or s=1 (no differencing and the first difference) and let these vectors be stacked into a T×(q+1) matrix $\mathbf{D} = \left[\mathbf{D_1}, \ldots, \mathbf{D_T}\right]'$. The right hand variables in the arbitrage equation are, $X_t = \left(d_{t+1} + p_{t+1}\right)$, which are stacked into a T×1 vector $\mathbf{X} = \left[X_1, \ldots, X_T\right]'$. A weighting matrix is defined as $\hat{\mathbf{A}} = \mathbf{X}'\mathbf{D}\left(\mathbf{T}\hat{\mathbf{S}}_\mathbf{d}\right)^{-1}$, where $\hat{\mathbf{S}}_\mathbf{d}$ is an estimate of $\mathbf{S_d} = E\left(\mathbf{D_t} u_t\right)\left(\mathbf{D_t} u_t\right)'$. u_t is as previously described the error terms of the arbitrage equation (3.2c). We now define a (2 ·q+3)×1 weighting matrix, $\mathbf{h_t}\left(\hat{\boldsymbol{\theta}}\right)$:

$$(4.14)\quad \mathbf{h_t}\left(\hat{\boldsymbol{\theta}}\right) = \begin{pmatrix} \hat{\mathbf{A}}\mathbf{D_t}\left(p_t - X_t\hat{b}\right) \\ \mathbf{D_t}\left(d_t - \mathbf{D_t'}\hat{\boldsymbol{\theta}}\right) \\ \mathbf{D_{t+1}}\left(d_{t+1} - \mathbf{D_{t+1}'}\hat{\boldsymbol{\theta}}\right) \end{pmatrix}$$

Let $\mathbf{h}_{t\theta}\left(\hat{\boldsymbol{\theta}}\right)$ be the partial derivatives of $\mathbf{h_t}\left(\hat{\boldsymbol{\theta}}\right)$ And $\mathbf{F} = \mathrm{diag}\left(T^{1/2}, \ldots, T^{1/2}\right)$. Then the variance-covariance matrix is:

$$(4.15)\quad \mathbf{V} = \left(\mathbf{F_T^{-1}}\mathbf{h_{t\theta}}\left(\hat{\boldsymbol{\theta}}\right)\mathbf{F_T^{-1}}\right)^{-1}\mathbf{S}\left(\mathbf{F_T^{-1}}\mathbf{h_{t\theta}}\left(\hat{\boldsymbol{\theta}}\right)\mathbf{F_T^{-1}}\right)^{-1}$$

Where **S** is the variance-autocovariance matrix of $\mathbf{h_t}\left(\hat{\boldsymbol{\theta}}\right)$. **S** is estimated as in Newey and West's (1987) proposition for a positive, semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix:

$$(4.16)\quad \hat{\mathbf{S}} = \hat{\boldsymbol{\Omega}}_0 + \sum_{j=1}^{m} w(j,m)\left[\hat{\boldsymbol{\Omega}}_j + \hat{\boldsymbol{\Omega}}_j'\right] \quad \text{where} \quad \hat{\boldsymbol{\Omega}}_j = \sum_{t=j+1}^{T} \frac{\mathbf{h}_t\left(\hat{\boldsymbol{\theta}}\right)\mathbf{h}_{t-j}\left(\hat{\boldsymbol{\theta}}\right)}{T}$$

$$\text{and} \quad w(j,m) = 1 - \left[j/(m+1)\right]$$

S is thus a weighted sum of the autocovariances from time t to $t+m$. The autocovariance order m, is usually determined as the number of non zero autocovariances..

4.2.6 The test statistic

West finds the relationship between the parameters in the three equations (analogous to equation (4.8) in the introduction) by using the formulas in Hansen and Sargent (1981). The constraints that are implied for stationary specifications (referred to later in this dissertation as the «specification constraints») are:

(4.17a)
$$\mathbf{R}' = \begin{pmatrix} m - a(1-a)^{-1}\Phi(a)^{-1}\mu \\ \delta_1 - \left[\Phi(a)^{-1} - 1\right] \\ \delta_j - \Phi(a)^{-1}\sum_{k=j}^{q} a^{k-j+1}\Phi_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$j = 2,\dots,q$$

For the differenced equations, these constraints are:

(4.17b)
$$\mathbf{R}' = \begin{pmatrix} m - \left[a(1-a)^{-1}\Phi(a)^{-1} + \Phi(a)^{-1} - 1\right]\mu \\ \delta_j - \left(\Phi(a)^{-1}\sum_{k=j+1}^{q} a^{k-j}\Phi_k + \left[\Phi(a)^{-1} - 1\right]\phi_j\right) \\ \delta_q - \left[\Phi(a)^{-1} - 1\right]\phi_q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$j = 2,\dots,q-1$$

Where $\Phi(a)^{-1} = \left[1 - \sum_{i=1}^{q} a^i \phi_i\right]$

$\mathbf{R} = \mathbf{R}(\boldsymbol{\theta})$, which is a (q+1)×1 vector. The null hypothesis is that $\mathbf{R}(\boldsymbol{\theta})$=0. The test statistic is then

(4.18)
$$\mathbf{R}(\hat{\boldsymbol{\theta}})' \left[\left(\frac{d\mathbf{R}}{d\hat{\boldsymbol{\theta}}}\right)\mathbf{V}\left(\frac{d\mathbf{R}}{d\hat{\boldsymbol{\theta}}}\right)'\right]^{-\mathbf{1}} \mathbf{R}(\hat{\boldsymbol{\theta}})$$

$\mathbf{R}(\hat{\boldsymbol{\theta}})'\,\mathbf{R}(\hat{\boldsymbol{\theta}})$ is a sum of square stochastic variables with expectation zero (the null hypothesis) and a variance equal to $\left(\frac{d\mathbf{R}}{d\hat{\boldsymbol{\theta}}}\right)\mathbf{V}\left(\frac{d\mathbf{R}}{d\hat{\boldsymbol{\theta}}}\right)'$. Expression (4.18) is therefore standard chi-square distributed with $q+1$ degrees of freedom. The derivatives of $\mathbf{R}(\hat{\boldsymbol{\theta}})$ are calculated analytically. This is done in the appendix.

4.2.7 Diagnostic tests

West uses diagnostic tests on the estimated equation to find whether the test is misspecified. The possible sources of misspecification that West tests for, are failure to allow for expectational irrationality and time variation in discount rates. Therefore four diagnostic checks are performed on the equations (4.10a), (4.10b) and (3.2c).

The first test is for serial correlation. Under rational expectations (according to Assumption 4.1) the expectational error u_{t+1} in the arbitrage equation (3.2c) should be serially uncorrelated. If the dividend equation is correctly specified, the error term v_{t+1} in the dividend equation (4.10a) should also be serially uncorrelated. One test is performed for first order serial correlation on both equations, and a calculation of the Box-Pierce Q statistic for the residuals in (3.2c) is done which tests first and higher order correlation.

A second test is performed only on equation (3.2c). This is, according to West, Hansen's (1982) test for instrumental residual orthogonality. This test detects a correlation between the residuals and the instruments used in the 2SLS procedure. If there is such a correlation, there is expectational irrationality and time variation in the discount rates. The null hypothesis is that (3.2c) is correctly specified. Under the null hypothesis, the test statistic is asymptotically distributed as a chi-squared random variable with q degrees of freedom.

One test is also applied to check the stability of the regression coefficients in (4.10a), (4.10b) and (3.2c). This way shifts in the discount rate or the dividends process are detected. This is done by dividing the test sample in two, and performing a test for a midsample shift of the coefficients. The test statistic is asymptotically distributed as a chi-squared random variable with one degree of freedom for (3.2c) and q+1 degrees of freedom for (4.10a) and (4.10b).

Finally there is diagnostic test implicit in the main test, since different lags and a differenced and undifferenced specification were used. If the test statistic is insensitive to these changes, it would be a indication that there is little chance that small changes in the specification of the dividend process will affect the results much.

4.2.8 Proposition: A modification of the test in the case of a small sample period

Often in these kind of tests, a small sample period is a problem. The data are registered yearly, which limits the number of cases severely. I propose to use some sub-indexes instead of one main index. These data are estimated separately in the model and then tested simultaneously in the test statistic.

First, for each test, the chi-square test statistics are calculated for all the sub-indexes. Each of these test statistics has $(q+1)$ degrees of freedom, as described previously. The calculated statistics from all the sub-samples are added together. This will yield a test statistic for n samples of $(q+1)\cdot n$. Formally, the technique can be presented as follows:

Assume that one has n samples $\{X_1, X_2, \ldots, X_n\}$ which yields the test statistics $\{Z_1, Z_2, \ldots, Z_n\}$, each having $(q+1)$ degrees of freedom (as described in 4.2.6). Then, due to the additive property of the chi-square statistic (Maddala 1992), one can add these together to form an aggregate test statistic $Z_T \sim \chi_{n(q+1)}$.

4.2.9 Comments about the test

In his article West presents three major points in his test:

1. To use the past history as a forecast of future dividends can be unrealistic; in particular if companies build up their capital stock by paying out no dividend at present, so that more dividends can be paid in the future. But even if the company is paying dividends that can indicate future dividends, expectations of other events can change the price. The market can receive information that is not revealed in the dividend process. But in the test West performed, the sample period was very long, so this would guard against this problem. West has also tested for stability of the dividend process.

2. The test is designed for the null hypothesis that there are no bubbles. If there are bubbles, this will influence the parameters in equation (4.11a) and (4.11b). He finds that the parameter of the dividend in these equations will be biased upwards in the case of a bubble.

3. The test has an advantage in that it does not require any additional information beyond prices and dividends.

Critics of the test (Flood and Hodrick, 1990) point out that West assumes the dividend forecasting equations to be stationary in either the levels of real dividends or their first difference. They argue that the logarithms of the real variables should be used. West does perform a test where logarithmic differences is used, as described in 4.2.4, but this is only possible if d_t follows a log normal random walk. Therefore only the constant average increase in the logarithm of the dividends is estimated, not the effect of previously paid dividends on expected dividends.

West is also criticised for assuming that the discount factor is constant throughout the sample period of 90 years and that the dividend process is assumed to be constant over such a long time horizon. However West does check this by testing for a midsample shift in the discount rate.

Since I am going to modify West's test, some comments on this are needed too. First, the sample period which I will analyse is very short compared to that used by West. The problems with companies paying no dividends or other factors influencing the price, will therefore be more apparent. Few companies in each index will enlarge this problem. The advantage is that I do not need to assume a constant dividend process and discount rate for a very long period.

5. Empirical results

5.1 General

I have used the statistics program SPSS for the estimation of the parameters and their significance levels. To work out the matrices in West's tests, indexes, tests and some test statistics I used the spread sheet Microsoft Excel. I also used the statistics program MicroFit to test for stationarity.

5.1.1 The data

The data on dividends and stock prices where collected from Oslo Bourse's «Børskurs listen» and the time series goes from1976 to 1997. Where there were «ordinary» and «free» stocks to choose between (respectively stocks that could only be sold to Norwegians and those without such restrictions), free stocks were chosen. In the same manner, when there were A stocks (stocks that pays dividends) and B stocks (stocks that does not), A stocks were chosen. At first data were collected data from 20 companies listed on the stock market. The selection was based on which companies that were registered for the full period. This span of the time series proved insufficient for mainly two reasons. First it seemed difficult to get stationarity, due to the few degrees of freedom at disposal. Second, there was much uncertainty related to the dividend process in the first sample. The inclusion of more cases into the test, certainly improved the results, as more series became stationary and the dividend process often more significant [12].

The inclusion of more cases into the sample introduced some problems. Half of the previously selected firms were not registered all the way back to 1976. These are denoted the *extended companies*. It was not possible to select new companies for the whole period since the ones previously selected were the only ones registered for the whole period 1982-97. I therefore had to extend these companies time series, by using other companies from the period prior to -

[12] Though, some of the specific parameters in the dividend process was affected in the opposite direction regarding significance.

82. This was done by sorting some selected companies, denoted as the *extension companies,* in the period 1976-82 by an implied beta (the covariance of the companies stocks and the market relative to the variance of the market), calculated using the official total index for this period as the market portfolio. The same beta was used to sort the extended companies. The companies were then matched by their corresponding ranking mate. The stock price of the extension companies was divided by their terminal price (the price overlapping the extended companies) and multiplied by the initial price of the extended companies, to smooth them in to the sample. It was taken into account that an introduction of many new companies at once could inflict a shock to the indexes. The extended companies were therefore implemented over a period of five years, so that only two companies were introduced at the same time. I also made sure that only one extended company was introduced at a time into the so called «*sub-indexes*» (sub groups of the total sample, explained in more detail later), and that there was at least two years between each introduction to these. This extension of the time series should not cause any problems, since this corresponds very well to how the Bourse itself handles inclusions and exclusions in their indexes.

The work of finding the correct prices required some effort, since at any time each of the companies stock prices had to be adjusted for any possible changes in their capital stock. 128 adjustments had to be made. This was easy for the cases after 1989, since for this period the adjustment factor (the number that previous stock prices has to be multiplied by to be comparable to present prices) was calculated by Oslo Bourse. For cases prior to this, the factors had to be calculated using a formula described in Bogstrand (1992), recommended by the Bourse. Prior to 1980, the relationship between new and old capital in emissions was not referred in the lists, so the factors had to be calculated by checking the level of capital for each year. This is not an accurate way to do it, so it was assumed that the adjustment factor should on averages be equal to the average in the period 1982-97 (0.916). By multiplying the changes in the capital with a certain constant, the relationship between the factors of different firms and different times would be the same, but the average could be adjusted so that it corresponded to the average in the1982-1997 period . This was done on the assumption that the boards in various firms keep to some rigid strategy when changes in capital is made. Whether huge or small capital adjustments are needed does not alter the per year capital adjustment, but rather how often (in how many subsequent years) it is done. This assumption is confirmed by my experience in calculating such factors. In addition, for the first year 1976, there were no capital changes to use in the calculation of the adjustment factor. For this year, it was therefore necessary to assume a fixed adjustment factor for firms that according to the list should have made a change in their capital stock in 1976. This factor was the 1982-1997 average. Such a procedure was allowed since it was only applied in the first year which means that it would not have large impacts on the series, and since the adjustment factors do not change a lot (standard deviation 1982-97, 0.089).

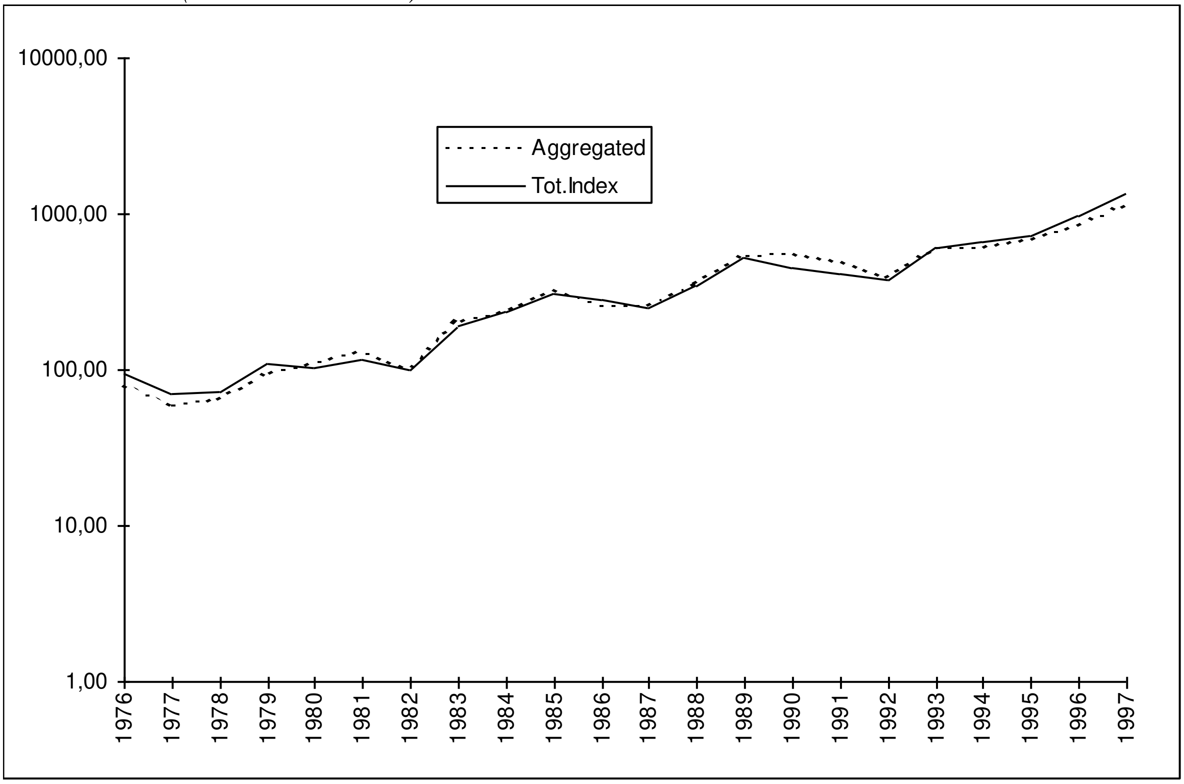
I found it better to make a justified estimate of the adjustments as described than not change the prices at all or leave the cases out, losing important degrees of freedom. It turned out that the adjustments must have been rather good, since a plot of the aggregate index based on the sample, and the total index from Oslo Bourse correspond very well (Figure 5.1)

Another thing that should be mentioned is that the 1997 prices are measured on November 20 th and not the last day of the year. I have chosen to do this, in a situation of relatively few observations in each time series, so that as many cases as possible where obtained. I assume that the difference between the current prices and prices at the 30 th of December does not differ so much that the results will be altered significantly.

The time series was deflated by the consumer price index. West (who got his time series from Shiller) uses the producer price index as deflator. The producer price index might be better suited for this purpose. However, I had to use what was available. The difference between these two indexes should not be very large, though. If deflation is left completely out, this will bias the results towards findings of a bubble. This happens because of the relationship between the parameters in the specification constrains, and the fact that the dividends in the dividend process equations (4.10a) and (4.10b) are lagged one period behind the dividends in the price process equations (4.11a) and (4.11b).

Figure 5.1: The aggregated index and Oslo Bourse's total index

Aggregated (the sample used in the test) and the total index, logarithmic scale. Yearly measurement (in nominal terms):



5.1.2 The Sub-indexes

Individual time series of stock prices and dividends tend to change too much and be too affected by different dividend policies applied by different owners. The time series were therefore put together to form indexes. First all companies were put into an *aggregated index*, which I then tested. This data set will be denoted *A*. I then constructed four *sub indexes* by adding five different companies together in each index. These data sets will be called X_n, where n=1,2,3,4. A test of all the sub indexes together (the expanded test proposition) requires that all the indexes must share the same dividend process. The series was therefor sorted into groups so that the common beta was approximately equal between the groups. By doing this, it was assumed that for equal risk (beta), an equal dividend process should be expected. To sort so few companies into groups so that the beta was approximately equal for each group, and at the same time take into account that the introduction of extended companies had to be spread within each sub-index, was difficult. The betas therefore varied a little between the groups. This should not be a problem, since the estimated beta must be expected to vary some around the true beta[13]. Two companies had such a high beta that their influence on the sub indexes made it difficult to get the sub-index betas equal. Their influence was therefore reduced by one quarter. The calculated betas where as follows:

Table 5.1: The calculated beta's for each sub-index

Sub-index 1	Sub-index 2	Sub-index 3	Sub-index 4
0,590465	0,586729	0,666076	0,637035

5.1.3 Stationarity

Due to a small sample, the test for stationarity was not possible to carry out in its full extent. I checked whether the data set was stationary or not by using an Augmented Dickey-Fuller test (ADF). When variables were stationarity for any higher order of serial correlation, they were mostly also stationary for all orders less than this. I did not test which lag length gave the ADF test most power. If such tests had revealed that the best specification of serial correlation of an order less or equal to the highest obtained, there would be no problem. If the order that gave best results was higher, one could claim stationarity due to the loss in degrees of freedom in the small sample. Therefore, when this is the case, the highest order of serial correlation assumed that gave stationarity is indicated in Table 5.2.

For the undifferenced price variable in the aggregate index, this was not the case. This was stationary for the eighth order of serial correlation, but not when any less orders were assumed. Due to the small sample, and the fact that a autocorrelation order of eight in yearly time series are rare, this variable was discarded as stationary.

If the unit root of a series was outside a 95% confidence interval of unity it was regarded as non stationary. Variables in levels, first difference and logarithmic difference were tested.

Table 5.2: Stationarity

[13] Thus, the estimated risk has some variance. Therefore, even if the beta for the groups was estimated as completely equal, we could not be sure that the risk for the different portfolios really was exactly the same.

The highest order serial correlation assumed that gave non-stationarity and was stationary when any lower orders of serial correlation was assumed. The order is indicated by a spot:

\Order of serial-correlation: *Variable:*	NONE	DF(0)	ADF(1)	ADF(2)	ADF(3)
Dividends Aggregated index, levels	●				
Dividends Sub-index 1, levels	●				
Dividends Sub-index 2, levels	●				
Dividends Sub-index 3, levels		●			
Dividends Sub-index 4, levels	●				
Dividends Aggregate index, differenced		●			
Dividends Sub-index 1, differenced		●			
Dividends Sub-index 2, differenced				●	
Dividends Sub-index 3, differenced				●	
Dividends Sub-index 4, differenced				●	
Dividends Aggregate index, logarithmic difference					●
Prices Aggregate index, levels	●				
Prices Aggregate index, differenced	●				

In the further discussion, I will treat series that are stationary when no autocorrelation is assumed as stationary. This is due to the small number of cases which makes it difficult do reject them as this. In addition, the series that are indicated as stationary for any higher order of autocorrelation in figure Table 5.2 will of course be regarded as that. All the differenced series are therefore regarded as stationary.

5.2 Shiller's variance test

In this test I have used only the aggregated index *A*, as the data set. As described in the previous chapter, the variance of the price and the variance of the constructed fundamental price is compared to see if the inequality (4.3) holds. The hypothesis used in the test are therefore (F is the F-test tests statistic):

$$(5.1) \qquad H_0: \quad F = \frac{Var(p_t)}{Var(p_t^*)} < 1 \qquad\qquad H_1: \quad F = \frac{Var(p_t)}{Var(p_t^*)} \geq 1$$

Where H_0 is the efficient market hypothesis (EMH). I have used an F-test to compare the two variances. The degrees of freedom are equal to the sample size for both the denominator and the numerator. As in Shiller, the time series was de-trended by dividing an the exponential growth factor (calculated by using the first and last price in the aggregated index sample). The discount factor was found to be 0.973, which yielded a test statistic of F=6.13. This results in a level of significance, the probability that H_0 is true if it is rejected, of approximately zero. Thus the EMH is be rejected in this test.

There has to be said though, as I mentioned in the previous chapter, that this test does not take account of possible non stationarity in the dividend or the price process since no differencing at all is done. The dividends and prices *was* found to be non stationary in levels but not in firs difference in the previously presented DF test. In performing the test, I also noticed that the results proved very sensitive to the discount factor. When the factor is calculated as Shiller suggests, the null is rejected. But if the discount factor is lower than 0,95, for example the ex post real return calculated in 5.3.7, the null hypothesis is not rejected.

5.3 West's specification test

5.3.1 Diagnostics tests

The short sample period limited the possible diagnostic tests significantly. I will therefore in some cases have to rest on West's findings, and assume that the evidence he found against different kinds of correlations and time varying parameters in the US stock market also will apply to the Norwegian market. Parameter stability over time should in fact also be plausible in this test, since the sample period is so short.

A mid sample shift test will not give much new information, since the number of cases in each part of the sample will vary from six to nine. This will therefore be omitted. The test on orthogonal residuals will also be omitted. The small number of cases also makes it difficult to test for serial correlation in the dividend equation (4.10a) and the arbitrage equation (3.2c) as West does. The large sample Lagrange Multiplier (LM) method for detecting serial correlation is certainly not suited for this small sample. The Box-Pierce Q statistic is not appropriate either for two reasons. First the small sample makes it difficult to test for higher order correlation. Second the test should not be used in model with auto regression (although it is frequently done (Maddala, 1992;p.540)). The Durbin-Watson (DW) test has the same weakness regarding the sample size and also shares a low power with the Q statistic. This test assumes that $\sum_{t=1}^{n} u_t \approx \sum_{t=1}^{n} u_{t-1}$, which cannot be assumed when the sample is bellow 20 cases as in this case. A DW test is still performed, since this test can be carried out when there is auto-regression without a problem. The test statistics are given in Table 5.3. This indicates that the statistic for all series is usually around 2 (no autocorrelation), with exemption of the undifferenced price equations (equ. (4.11a)). The fact that only these equations have test statistics below one is striking and suggests that they might be serially correlated to a certain degree. However, the previous argument still suggest that this is only an indication.

Table 5.3: Test for serial correlation:

The Durbin-Watson statistics for various models:

Model:	**Simple**	**2 lag**	**2 lag, differenced**	**4lag**	**4 lag, differenced**
dividend equation:	2.130	1.953	1.967	2.006	2.179
Price equation:	0.289	0.863	2.249	1.517	2.151

Model:	**2 lag 4 series**				**2 lag, 4 series differenced**				**4 lag 4 series**				**4 lag, 4 series differenced**			
Index:	**X_1**	**X_2**	**X_3**	**X_4**	**X_1**	**X_2**	**X_3**	**X_4**	**X_1**	**X_2**	**X_3**	**X_4**	**X_1**	**X_2**	**X_3**	**X_4**
dividend equation	2.540	1.939	1.860	1.877	1.913	2.220	2.283	2.198	1.246	1.892	1.945	1.564	2.111	1.991	1.933	1.760
Price equation	0.586	0.833	0.587	0.595	2.584	2.832	2.007	1.323	0.805	0.927	0.844	0.755	1.780	2.987	2.126	1.433

The dividend equations are equ. (4.10a) and (4.10b). The price equations are equ.(4.11a) and (4.11b).

As previously presented stationarity tests were also performed on the series. In addition, there is a implicit diagnostic test since the test is specified in many different ways. In addition to various specifications of the dividend process, four sub-samples of the main index were estimated as a modification of West's test. The results proved not to be very sensitive to the different specifications, so it can be assumed that small changes in the specification of the dividends would affect the results little.

One diagnostic test was also carried out to check whether the samples in the four-sample model could be used together. This test will be presented in section 5.3.6.

5.3.2 Statistical deviation from West's method

The two step procedure used by West to estimate the discount rate, was not used. Instead only 2 stage least squares was used, possibly causing heteroskedasticity. This was mainly done because it was not assumed to alter the results significantly and statistics programs that could handle such a procedure was difficult to find[14].

The order of autocorrelation used in calculating the variance-covariance matrices, was set to the one, yielding the highest variance-covariance. This was done since the number of cases limited the possibility of testing for autocorrelation.

5.3.3 The dividend process

The large variance caused by a slight misspecification of the dividend process, will be present in the variance-covariance matrix used to estimate the main test statistic. A low level of explained variance is therefore not a problem in this context. However, if there exists no relationship at all between current and lagged dividends, the specification constrains will not hold.

However, one should not require a conventional significance level of the dividend process in this case, for the process to be assumed. If this is done for all the variables, the whole test would be discarded at this early stage. It should be tolerated that the dividend process can be insignificant at conventional levels, due to the small sample size. Even if the process is correctly specified, the sample may be so small that white noise makes a conventional level of significance impossible to obtain. Since it is not the dividend process in itself that is tested in this model, a higher level of significance could be used to justify an assumption of a correctly specified process. In Maddala (1992) the problem of «pre-testing» is discussed and it is argued that conventional levels are inappropriate for small (such as this) and large samples. Test with small samples should use higher levels of significance, and tests with large samples lower (Maddala 1992). It can in some cases be appropriate to use levels as high as 95% (!) for pre-tests.

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[14] MicroFit, TSP and SPSS did not have such built in procedures that could be used, even if the estimation was done in several steps. The main problem is that these programs, as far as I now, do not have any weighted 2SLS procedure.

Below in Table 5.4 and 5.5, a significance level of 10 and 50 percent is tested on the parameters. At such high levels of significance as 50 percent, the results will only tell if the dividend process is reasonable to *assume*, thus no conclusions can be drawn. The table can be interpreted as whether there is more than a 50 percent chance that the dividend process is correctly specified. If the answer is yes, there is reason to assume the process. The table shows that for most parameters, the specified dividend process *can* be assumed according to the described criteria. The purpose of this discussion is not to argue that the dividend process is correctly specified, but rather to emphasise that there is no reason for not inferring it. The 10% test, shows that some of the parameters are significant at a conventional level. These are mainly the parameters of the undifferenced, non-stationary (according to the DF test) processes. An ANOVA test of the hypothesis that the dividend process is wrongly specified is rejected for most series at a 20 percent level of significance (Table 5.6). A further support to the assumption that the dividend process is correctly specified, is the results of West's larger sample tests, that revealed mostly significant parameters.

Table 5.4: Significance of the dividend process at a 10% level of significance

Significance at a 10% level, Y indicates significance, N not significance:

	Original model					2 lag, undifferenced, four series				2 lags, differenced, four series				4 lags, undifferenced, four series				4 lags, differenced, four series			
	Simple	**2-lags**	**2-lags, diff.**	**4-lags**	**4 -lags, diff.**	X_1	X_2	X_3	X_4	X_1	X_2	X_3	X_4	X_1	X_2	X_3	X_4	X_1	X_2	X_3	X_4
μ		**Y**	N	**Y**	N	**Y**	**Y**	**Y**	**Y**	N	N	N	N	**Y**	**Y**	**Y**	**Y**	N	N	N	N
ϕ_1	**Y**	**Y**	N	**Y**	N	**Y**	**Y**	N	**Y**	N	N	**Y**	N	**Y**	**Y**	N	**Y**	N	N	**Y**	N
ϕ_2		N	N	N	N	N	N	N	N	N	N	**Y**	N	N	N	N	N	N	N	**Y**	N
ϕ_3				N	N									N	N	N	N	N	**Y**	N	N
ϕ_4				N	N									N	**Y**	N	N	**Y**	N	N	N

Table 5.5: Significance of the dividend process at a 50% level of significance

Significance at a 50% level, Y indicates significance, N not significance:

	Original model					2 lag, undifferenced, four series				2 lags, differenced, four series				4 lags, undifferenced, four series				4 lags, differenced, four series			
	Simple	**2-**	**2-**	**4-**	**4 -**	X_1	X_2	X_3	X_4	X_1	X_2	X_3	X_4	X_1	X_2	X_3	X_4	X_1	X_2	X_3	X_4

		lags	lags, diff.	lags	lags, diff.																
μ		Y	N	Y	N	Y	Y	Y	Y	N	N	N	N	Y	Y	Y	Y	N	N	N	Y
ϕ_1	Y	Y	N	Y	N	Y	Y	Y	Y	Y	N	Y	N	Y	Y	Y	Y	Y	N	Y	Y
ϕ_2		N	N	N	N	Y	N	N	N	N	Y	Y	Y	Y	N	N	N	N	Y	Y	Y
ϕ_3				N	Y									N	Y	N	N	N	Y	Y	Y
ϕ_4				Y	Y									N	Y	Y	Y	Y	N	N	N

Table 5.6: ANOVA test on the dividend process:

Significance at a 50% level, Y indicates significance, N not significance:

	Original model					2 lag, undifferenced, four series				2 lags, differenced, four series				4 lags, undifferenced, four series				4 lags, differenced, four series			
	Simple	**2-lags**	**2-lags, diff.**	**4-lags**	**4 -lags, diff.**	X_1	X_2	X_3	X_4	X_1	X_2	X_3	X_4	X_1	X_2	X_3	X_4	X_1	X_2	X_3	X_4
ANOVA D	Y	Y	N	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	Y	Y	Y	N	Y	Y	N

5.3.4 The simple test

In performing the simplest case, the calculation of the test statistic and variance was conducted as described in relation to the advanced test (section 4.2.5 and 4.2.6). The relationship between the parameters that correspond to the specification constraints, **R**, in (4.17a) and (4.17b) are:

(5.2) $$R = \delta_1 - \frac{a\phi}{1 - a\phi} = 0$$

The partial derivatives of this forms a vector:

(5.3) $$\frac{dR}{d\hat{\boldsymbol{\theta}}} = \left(\frac{dR}{d\delta_1}, \frac{dR}{d\phi}, \frac{dR}{da} \right)$$

which is calculated analytically. The test statistic is therefore:

(5.4) $$R^2 \cdot \left(\frac{dR}{d\hat{\boldsymbol{\theta}}} \right) \mathbf{V} \left(\frac{dR}{d\hat{\boldsymbol{\theta}}} \right)' \sim \chi(1)$$

Where **V** is the variance-covariance matrix described in the previous chapter. The test statistic was found to be 0.48, and hence the EMH of no bubbles is rejected. The corresponding level of significance is 49%, so there is no support for the alternative hypothesis.

It has to be noted though that this is a fairly simple model, with its limitations. The rejection of the alternative hypothesis is probably caused more by large residuals than a specification constraint relatively close to zero. While the δ_1 implied by the dividend process and the discount factor relative to the estimated δ_1 is approximately twenty five percent, the standard deviation of the residuals in the price equation (the deviation from a mean of zero) is about three times the ones in for instance the more advanced four lag, differenced model. In addition, the Durbin-Watson test strongly indicated a serial correlation, displaying a test statistic of 0.289.

5.3.5 The two and four lags model, differenced and undifferenced, with one sample

The test was performed using the aggregated index (*A*), and the method used is described in the previous chapter. In these and the four sample models, *two and four* lags are used as in most models of West. West uses a method developed by Hannah and Quinn (1979) to decide the lag lengths. I will however set them a priori to two and four. This is done because it gives a good combination of a substantial difference in the specifications, which is an implicit diagnostic check, and a not too complicated model.

It turns out that very high test statistic rejects the null completely for all four models . Thus, even if the estimated parameters should differ some from their true value, the difference would have to be substantial for the null to be verified. It must be noted though, that dividends are not significant for the differenced models as discussed earlier.

Table 5.7: Final results for West's specification test, one sample

	2 lags, undifferenced	2 lags, differenced	4 lags, undifferenced	4 lags, differenced
Test statistic	1444.55	692.7042	16570.22	8375.302
Sign.	0.00	0.00	0.00	0.00

5.3.6 The two and four lags model, differenced and undifferenced, with four samples

In this model, *four* samples are used instead of the *one* for which the original test was developed. The reason for doing this is as follows: Assume that each estimated parameter is normally distributed around its true value and that these parameters can be estimated from more than one sample. If the parameters are estimated from all the samples available, one would expect that they on average would be closer to the true parameter, than a single estimate. The best solution if one has more than one sample, will usually be to pool them. But this procedure does not apply to time series, since each observation is connected to a specific moment in time. An alternative procedure is to add the observations for each moment in time together, and then estimate one parameter (as is done in the aggregate index). One will however lose a considerable amount of information by doing this, since the argument for using more than one sample does not apply when only one estimate is done. A better procedure is to use the estimates in the available samples together.

Since arbitrage implies that the parameters *should* be similar (since the sub-indexes are assumed to have the same risk, due to a similar beta), the a test should be applied to check whether the coefficients are representative. If they are not, it suggests that the process is affected by to much noise. In this case, such noise will probably be originating from few stocks in each index. Companies that for some periods have an unusual dividend policy, or the fact that particular information may not be revealed by the dividend process can induce such noise, as discussed in section 4.2.9. In the aggregated sample, such factors will not be that apparent.

An ANOVA test for independent samples is therefore performed to check whether the samples are homogenous. In Maddala (1992; p.170), this test is used to check whether two independent samples can be pooled (e.g. if the hypothesis that the coefficients in the linear regression model are equal in the two samples). It should be no problem though, to use this test to check whether the four sub-indexes I have available can be assumed to yield the same parameters. If this is not rejected, I will assume that the estimated parameters from the four samples are normally distributed around the true ones, and that these are the same for all the samples.

In the ANOVA test presented by Maddala 1992, a null hypothesis stating that two sample shares the same parameters are tested against an alternative hypothesis that they do not. The test statistic is calculated as:

$$F = \frac{(RRSS - URSS)/k}{URSS/\left(n_1 + n_2 - 2k\right)} \tag{5.5}$$

Which is F-distributed with k (the number of restrictions, that is the number of parameters) and ($n_1 + n_2 + 2k$) degrees of freedom. RRSS is the restricted sum of squares, the sum of squares resulting from pooling the samples. URSS is the sum of the two samples sum of squares, n_i is the number of observations in sample i and k is the number of parameters in the model. I have imposed the restriction on index two to four that the parameters are equal to the ones estimated from index one. I then used this test statistic:

$$F = \frac{\left(\sum_{i=2}^{4} RRSS_i - URSS_i\right)/k}{\left(\sum_{i=2}^{4} URSS_i\right)/\left(\sum_{i=2}^{4} n_i - 3k\right)} \tag{5.6}$$

Where RRSS is the restricted residual sum of squares of sub-index i after imposing the restriction that the parameters are equal to the ones estimated from index one. URSS is the unrestricted residual sum of square from sub-index i. This test statistic is F-distributed with k and $\sum_{i=2}^{4} n_i - 3k$ degrees of freedom.

Table 5.8: Shared parameters

The test results show whether the sub-indexes can be used in the same tests; the probability that the samples do not share the same parameters are given as the level of significance:

	2 lag	2 lag, differenced	4 lag	4 lag, differenced
F	3,221	1,878	4,422	5,753
Sign.	0,030	0,146	0,003	0,001

H_0, that the samples share the same parameters, is not rejected if the significance is above a conventional level. At a 5 percent level of significance, the samples in the two four lag models and the on lag undifferenced model are rejected as similar, and can therefore not be used together. I will however refer the resulting test statistics of these tests too, for comparison.

The estimation is done as described earlier, with the modification presented in 4.2.8. The results are presented in this table:

Table 5.9: Final results for West's specification test, four samples

	2 lags, undifferenced	**2 lags, differenced**	**4 lags, undifferenced**	**2 lags, differenced**
Test statistic	931,46249	856,33402	6412,7942	3410,8569
Sign.	0.00	0.00	0.00	0.00

Thus, all the tests reject the hypothesis of no bubbles.

5.3.7 The log-normal random walk model

The log-normal random walk model was performed as described in section 4.2.4. The resulting estimated parameters were as follows:

Table 5.10: Final results in the log-normal random walk model:

The discount factor ***a***	0.914
μ_l	0,016423
σ^2 estimated by the mean, μ_l, and discount rate, ***a***	0,096
$\hat{\sigma}^2$	0,09739
$\hat{\sigma}^2$, confidence interval	0,182928 **To** 0,040224

As the results show, the σ^2 estimated by the mean and the discount rate, does lie inside the confidence interval, so the null hypothesis that there are no bubbles is *not rejected* in this test.

The simplicity of this test is however weak point, since it requires the expectations of the dividends to be formed according to a log normal random walk process, with a average increase of μ. In addition, the price-dividend process is the same as in the simple model, where autocorrelation was indicated by the Durbin-Watson test.

5.4 Discussion

Like most of West's tests, the Shiller test rejects the EMH. The reliability of the results can be discussed though. The assumption of non stationarity and the possible inclusion of the bubble in both the calculated variances suggests that the test can at best be an *indication* of bubbles. The variance test has one obvious advantage over the specification test that has been performed here though, since it does not require certain regression equations to be correct, it rests on only one test statistic. This does not change the conclusion of the appropriateness of this test though. The log normal random walk test used by West would perhaps have less serious faults than the variance test, and is still fairly uncomplicated to calculate with only one main test statistic. This test did *not* reject the EMH.

5.4.1 The test statistics

The test statistics are larger than the ones found by West. There can be several interpretations of this. A closer look at the R vector can be useful in this context. (only the one-sample models are shown, since the argument would basically be the same if the four sample models where also used):

Table 5.11: The R vector

Estimates of the elements in the R vector for the various one-sample models:

\Model R	**Simple**	**2 lag**	**2lag, differenced**	**4 lag**	**4 lag, differenced**
m		299,5976	26,5692	384,102	17,97027
δ_1	29,03549	24,08897	9,091431	23,07088	6,771849
δ_2		-35,54146	17,808	21,40927	19,57344
δ_3				-44,0837	-22,82651
δ_4				-24,43782	-2,89215

Compared to:

Table 5.12: The parameters in the price process:

Estimates of the parameters in the price equations (4.11a) and (4.11b) for the various one-sample models:

\Model Parameter	**Simple**	**2 lag**	**2lag, differenced**	**4 lag**	**4 lag, differenced**
m	0	333,8113	26,42373	416,092	17,38339
δ_1	38,20853	26,02214	9,096762	25,21474	6,723797
δ_2	0	-35,63632	17,76733	21,33618	19,55118
δ_3	0	0	0	-44,05787	-22,59265
δ_4	0	0	0	-23,67378	-2,890705

As one remembers from the calculation of the specification constraints, the R vector ((4.17a) and (4.17b)) is the difference between the estimated price equation parameters and these parameters implied by the discount factor and the dividend process parameters. The discrepancies between these two estimates are so large (the R vector is close to the price equation parameters) that it is unlikely that the difference would be altered much by changes in the dividend equations. Also, as mentioned previously, West points out that *a bubble will bias the coefficients upwards*. This is just what we see in the above tables.

As previously described, the dividend process must on average describe the markets estimation of future return. A reason for large test statistics, is thus that the specification constraints (the R vector) are not expected to be zero. If the market do not base their expectation of income of an asset on past dividends, one cannot derive these constraints. We can draw no certain conclusions here, since the data material is inconclusive at this point. The discussion in 5.3.3 still suggests that the dividend process *does* explain the expectations of future dividends.

It might be that the market works very inefficiently, and more inefficiently in Norway than in the USA for which West's test is performed, thus causing high test statistics. In the seventies and early eighties, trading on the stock market was considerably less in Norway than in the USA, and much less than in the present Norwegian stock market. On the 4 th of October 1997, stocks for NOK 1.780.140.000 was traded . In comparison, at the 28 th of December 1978, total trading amounted to NOK 2.556.834. The numbers are in 1997 currency, so the market was 696 times larger in 1997 on the respective dates. In addition the Norwegian market was very protected, due to national legislation. Much of these governmental constraints was removed during the eighties and nineties. A small restricted market creates opportunities for arbitrage and speculation so that bubbles can be expected. This explanation seems plausible as a cause for the strong rejection of the null hypothesis.

5.4.2 The proposed modification of the test

Only one of the expanded tests did not fail in any diagnostic tests (see next section). Common parameters in the two four-lag tests (in levels and first difference) was rejected at a five percent level of significance. The two lag, undifferenced test, was on the limit to being rejected. Three of the sub-indexes failed however at the stationarity test when they were not differenced, so the undifferenced tests was therefore not accepted. In addition the price equation of these tests, (4.11a), failed on the DW test. The remaining test rejects the null hypothesis strongly, though. If a better data material had existed however (more companies in each index), the parameters in the four sub-indexes might been more similar and the rejection of this could have been avoided.

The two lag differenced test displayed a test statistic similar to the ones for the one sample tests when the degrees of freedom is taken in to consideration. This suggests that there is no serious problems with the proposed procedure. Since it is based on different samples assumed to share the same properties, it *should* give a more certain result than the one sample test. In this particular test, this is however not necessarily true since the sub-indexes could be more similar (e.g. a higher significance level on the F test performed to test this). All in all, I will conclude though, that this test definitely strengthens the alternative hypothesis, and have been useful in testing for bubbles.

This attempt to use many samples simultaneously in a test for bubbles can be used as an example of how this can be done, so that future improvements in data material and technique can be made. I started off using a method where all vectors and matrices were expanded by the number of sub-indexes. This proved not successful because the variance-covariance matrix became negative for some autocorrelation orders. In addition, for the four lag tests, this matrix became close to zero. This suggested that such a procedure had faults, so the current procedure, adding the test statistics together in an aggregate statistic, was used instead. I have not discovered any problems using the revised technique, so this is recommended. The diagnostic test of checking whether the parameters was similar was originally designed for the expansion method. In the test that was eventually used, this diagnostic test is not necessary to make sure that consistent results are found, since an individual variance-covariance matrix is applied to each sub-index. However, since arbitrage implies that the parameters should be similar, as described in section 5.3.6, the test is applied to check whether the coefficients are representative.

5.4.3 Conclusion

Of the tests based on West and Shiller that were performed, bubbles were found in all but two test. These was the ones assuming the simplest dividend processes. Thus, the tests should not be regarded as equally strong. Some of them must be said to be less powerful for revealing bubbles due to the results of the diagnostic tests. The undifferenced tests are probably not stationary, as found in the DF tests. These tests can therefore not be regarded as strong evidence for or against bubbles. This includes the Shiller test. The DW test also revealed that the undifferenced West tests had a suspicious price process which could be auto correlated of the first order. In particular, the simple test had a low DW statistic. Since the log-normal walk model uses the same price process parameter as the one estimated in the simple test, this also fail in the DW test. Finally, the two four lag models in the four sample test did not give correct test statistics since the sample was too different due to unequal parameters.

The test results from these tests can only be regarded as indications, since they failed the diagnostic tests. Some of the discarded West tests serve as implicit diagnostic tests themselves, since common test results indicate that small changes in the specification is not of vital importance. The simple test and the log normal random walk model is rather different from the two and four lag models. They are therefore not so well suited to be used as implicit diagnostic checks. This check mainly has its power in detecting whether *small changes* in the model influence on the result. Small changes in the *two and four lag models* should therefor not alter the test results much, since all the two and four lag models clearly rejects the null. The most trustworthy tests that remains are then:

Table 5.13: The remaining tests

Final results for the most trustworthy tests:

	2 lag, differenced	**4 lag, differenced**	**2 lag, 4samples, differenced**
Test statistic	692.7042	8375.302	856,33402
Sign.	0.00	0.00	0.00

All these tests strongly rejects the efficient market hypothesis of no bubbles. It is therefore concluded that there is signs of bubbles in the Norwegian stock market, making the reasonable assumption based on the discussion in 5.3.3 that the dividend process is correctly specified.

Although signs of bubbles definitely was found, the tests have their limitations. I have earlier in this section described the problems of the Shiller tests, and the two simplest West tests. However, there are limitations to the remaining tests as well. As mentioned, it rests on the dividend process. The specified process must be correct, which is not indisputable. It is suggested that the logarithm of the dividend in first difference should be used. This is not possible in West's test, except form when a random walk process is assumed, which brings other problems (such as a low DW statistic). In addition, it is assumed constant parameters throughout the whole period, which may not be the case, even though this test has been conducted using data from a comparatively short time period. Also two of the tests did not reject the EMH, but these had probably more serious faults than the remaining tests as explained previously.

In the tests I have performed, the test results do suffer from few cases in the samples. This had direct consequences for the diagnostic tests. Some could not be conducted, and others was weakened by the few number of cases. It was therefore necessary to rest on West's evidence in some cases.

6. Concluding remarks

6.1 Rational bubbles

Rational deterministic bubbles are often no problem for the economy. On the contrary they can often contribute to stability and autonomy for countries as well as individuals. In particular this is the case for bubbles in a general equilibrium. Deterministic bubbles do not burst, so they do not contribute to uncertainty in the economy. However, since deterministic bubbles can only be regarded as a special case of stochastic bubbles (where q=1), one can assume that there are few such certain bubbles. On example though, is gold as mentioned in section 3.5.8. At least historically, this asset has displayed a remarkable stability. National banks hold amounts of it basically to support their currency. During economic fluctuations, the gold reserves give the government the credibility needed to hold a certain exchange rate or avoid dramatic changes in the interest rate. However, it is presently signs of this bubble bursting («The Economist» week 48, 1997). In contrast to such deterministic bubbles, stochastic bubbles will possibly describe *the stock market* better. In the case of such bubbles, it is implied by the model that fluctuations can be large, since the bubble can vanish at any time. They differ from non-rational bubbles in that the buyers know the risk, and value the asset accordingly. The participants know that buying the bubble asset will be beneficial for them, or else they would not enter the market. The explanation for this behaviour is that it can be beneficial when saving exceeds investments (the bubble) since there may be no *additional* investment opportunities that yield sufficient returns (because of decreasing marginal returns on investment). Therefore a risky investment which pays a risk premium can be preferred to a relatively small return on investment. Although this kind of bubble causes greater uncertainty relative to a no bubble economy, this might be Pareto efficient since stochastic bubbles can also lead the economy to a state of dynamic efficiency (as can deterministic bubbles). If the bubble pulls the economy from a state of dynamic inefficiency, the stochastic bubble is Pareto efficient. An economy in dynamic inefficiency can be described as a society where people save «too much». A bubble would reduce the capital stock and increase consumption, and therefore help the economy to a Pareto improvement [15].

6.2 Non-rational bubbles

If there are speculative non-rational bubbles, it will be expected that these also contribute to a lower level of capital in the economy, as mentioned in section 3.6. In the case of speculative bubbles, the participants in the market (or at least some of them) do not have rational expectations about the future dividends or prices of the assets. Investment is not undertaken solely on the basis of knowledge about the risk and return. The market is instead driven by accidental reactions and psychological mechanisms as described in section 3.6. As mentioned there, the equations (3.11b) and (3.13b) will not hold in the case of non-rational speculative bubbles. The return on these assets can therefore be very high, which should be expected to drive out some investment.

Bubbles which are not fully rational may cause the market to be more volatile than if such bubbles did not exist. A relationship between volatility and speculative bubbles is exactly what Shiller's (1981) variance test is based on, as described in section 4.1. It is also the case that in markets where bubbles are more likely to arise (such as the stock market), volatility is greater than in markets where assets are more likely to be fundamentally priced (such as the bond market). It can be claimed though that this has to do with risk, but then again high risk in these markets may be an indication of bubbles.

Fluctuations and higher risk in asset markets caused by bubbles need not be a problem, but if bubbles are not rational, the participants will not acknowledge the true risks and the return which it is rational to expect. The portfolio held by market participants may therefore be inconsistent with the one corresponding to the individual's utility function and thus not Pareto optimal.

If the economy is already dynamically efficient, a further decrease in the capital stock will reduce steady state consumption (equation (2.16)). This may not be Pareto efficient. The

[15] This is based on the assumption that an increase in consumption is spread all over the population since everybody shares the same utility function and participates in the asset market, so the bubble does not cause any

reduction in the capital stock that a bubble (rational or non-rational) bring can therefore be a problem for an economy if it already is dynamically efficient. If bubbles arise in countries where capital is relatively scarce, they can make the situation worse by reducing steady state consumption.

6.3 The situation in Norway and implications for the global financial markets

As the test results suggests, there are signs of bubbles in the Norwegian stock market. As pointed out in section 3.7, the relatively low level of investments in Norway during the eighties, is consistent with a bubble being present in the economy. It is difficult though, to conclude whether investments in Norway were «too high» in the seventies. The bubble might have driven the economy into dynamic efficiency, but it may also have decreased the investment from an already dynamically efficient level. It is clear that investment in the seventies was very high due to expansion of the oil-sector. This might not, however, be a result of dynamic inefficiency (which arises when capital yields a low rate of return relative to economic growth), but rather a very high rate of return on capital in these years.

Whether rational bubbles or non-rational bubbles has been present, is of course difficult to be sure about. But the small and protected capital market in the late seventies and early eighties can certainly not be expected to be more rational than the asset markets in USA. This change in openness of the Norwegian capital market is also likely to have had consequences for the probability of dynamic efficiency or inefficiency. The worlds current financial market is probably more homogenous now than before, after a number of reforms in many countries since the seventies. Such properties as dynamic efficiency or inefficiency are probably shared by most countries. It will therefore make little difference to what extent the small Norwegian economy contributes to this. The theory assumes a closed economy. To explain this in a theoretical context, one can therefore think of the Norwegian financial market in the early eighties as rather closed due to restrictions imposed by the government. Today on the other hand, one has to assume that all the countries participating in the global financial market comprise this closed economy.

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redistribution.

A conclusion can be drawn on the basis of the theoretical argument, no matter whether bubbles are rational or non-rational. The indication of bubbles in the Norwegian economy suggests important consequences for the capital accumulation in Norway during the last two decades. Whether the bubbles have driven the economy in the direction of dynamic efficiency or not, is impossible to tell though. Since the capital market has become more global, the stock market in Norway today probably make little difference on whether the global economy has become dynamically efficient or not. However, even though the Norwegian market have little influence on the global one, the global market has probably strong influence on Norway. Thus, a bubble in the Norwegian financial market today might imply that most other countries also have inefficient asset markets. This might indicate that the world capital accumulation is less than it would be if bubbles did not exist.

7. Appendix

7.1 Derivation of the specification constraint vector R

7.1.1 The undifferenced case

Let R_i denote the i'th element of the (q+1)×1 vector **R**, the constraints of the parameters in the model. Then the partial derivative matrix of **R** with respect to the parameters is:

\With resp. to: For element:	μ	ϕ_i	m
R_1:	$\frac{dR_1}{d\mu} = -\frac{a}{1-a}\Phi(a)^{-1}$	$\frac{dR_1}{d\phi_i} = -\frac{a}{1-a}\left(\Phi(a)^{-1}\right)^2 a^i \mu$	$\frac{dR_1}{dm} = 1$
R_2:	$\frac{dR_2}{d\mu} = 0$	$\left.\frac{dR_2}{d\phi_i}\right\|_{i>1} = -\left(\Phi(a)^{-1}\right)^2 a^i \phi_1$ $\left.\frac{dR_2}{d\phi_1}\right\|_{i=1} = -\left[\Phi(a)^{-1} - 1\right] - \left(\Phi(a)^{-1}\right)^2 a\phi_1$	$\frac{dR_2}{dm} = 0$
R_{j+1}:	$\frac{dR_{j+1}}{d\mu} = 0$	$\left.\frac{dR_{j+1}}{d\phi_i}\right\|_{i<j} = -\left(\Phi(a)^{-1}\right)^2 a^i \sum_{k=j}^{q} a^{k-j+1}\phi_k$ $\left.\frac{dR_{j+1}}{d\phi_i}\right\|_{i\geq j} = -\left(\Phi(a)^{-1}\right)^2 a^i \sum_{k=j}^{q} a^{k-j+1}\phi_k - \Phi(a)^{-1} a^{i-j+1}$	$\frac{dR_{j+1}}{dm} = 0$

\With resp. to: For element:	δ_i	a
R_1:	$\frac{dR_1}{d\delta_1}=0$ $\frac{dR_1}{d\delta_j}=0$	$\frac{dR_1}{da}=-\frac{\Phi(a)^{-1}\mu}{1-a}-\frac{a\Phi(a)^{-1}\mu}{(1-a)^2}-\frac{a}{1-a}\left(\Phi(a)^{-1}\right)^2\left(\phi_1+2a\phi_2\cdots qa^{q-1}\phi_q\right)\mu$
R_2:	$\frac{dR_2}{d\delta_1}=1$ $\left.\frac{dR_2}{d\delta_j}\right\vert_{j>2}=0$	$\frac{dR_2}{da}=-\left(\Phi(a)^{-1}\right)^2\left(\phi_1+2a\phi_2\cdots qa^{q-1}\phi_q\right)\phi_1$
R_{j+1}:	$\frac{dR_{j+1}}{d\delta_1}=0$ $\frac{dR_{j+1}}{d\delta_j}=1$	$\frac{dR_{j+1}}{da}=-\left(\Phi(a)^{-1}\right)^2\left(\phi_1+2a\phi_2\cdots qa^{q-1}\phi_q\right)\sum_{k=j}^{q}a^{k-j+1}\phi_k$ $-\Phi(a)^{-1}\sum_{k=j}^{q}\left[k-j+1\right]a^{k-j}\phi_k$

For j=2,3,...,q and for i=1,2,,...,q

The derivative of $\Phi(a)^{-1}$ with respect to ϕ_i and the discount factor a is:

$\frac{d\Phi(a)^{-1}}{d\phi_i}=\left(\Phi(a)^{-1}\right)^2a^i$ and $\frac{d\Phi(a)^{-1}}{da}=\left(\Phi(a)^{-1}\right)^2\left(\phi_1+2a\phi_2\cdots qa^{q-1}\phi_q\right)$ which is calculated directly in the matrix above.

7.1.2 The differenced case

Let R_i denote the i'th element of the (q+1)×1 vector **R**, the constraints of the parameters in the model. Then the partial derivative matrix of **R** with respect to the parameters is:

\With resp. to: For element:	μ	ϕ_i
R_1:	$\frac{dR_1}{d\mu}=-\frac{1}{1-a}\Phi(a)^{-1}+1$	$\frac{dR_1}{d\phi_i}=-\frac{1}{1-a}\left(\Phi(a)^{-1}\right)^2 a^i\mu$
R_{j+1}:	$\frac{dR_{j+1}}{d\mu}=0$	$\left.\frac{dR_{j+1}}{d\phi_i}\right\rvert_{i>j}=-\left(\Phi(a)^{-1}\right)^2 a^i\left(\sum_{k=j+1}^{q}a^{k-j}\phi_k+\phi_j\right)-\Phi(a)^{-1}a^{k-j}$ $\left.\frac{dR_{j+1}}{d\phi_i}\right\rvert_{i=j}=-\left(\Phi(a)^{-1}\right)^2 a^j\left(\sum_{k=j+1}^{q}a^{k-j}\phi_k+\phi_j\right)-\left[\Phi(a)^{-1}-1\right]$ $\left.\frac{dR_{j+1}}{d\phi_i}\right\rvert_{i<j}=-\left(\Phi(a)^{-1}\right)^2 a^j\left(\sum_{k=j+1}^{q}a^{k-j}\phi_k+\phi_j\right)$
R_q:	$\frac{dR_q}{d\mu}=0$	$\left.\frac{dR_q}{d\phi_i}\right\rvert_{i<q}=-\left(\Phi(a)^{-1}\right)^2 a^i\phi_q$ $\left.\frac{dR_q}{d\phi_i}\right\rvert_{i=q}=-\left(\Phi(a)^{-1}\right)^2 a^q\phi_q-\left[\Phi(a)^{-1}-1\right]$

\With resp. to: For element:	m	δ_i	a
R_1:	$\frac{dR_1}{dm}=1$	$\frac{dR_1}{d\delta_1}=0$ $\frac{dR_1}{d\delta_j}=0$	$\frac{dR_1}{da}=-\frac{\Phi(a)^{-1}\mu}{(1-a)^2}-\frac{1}{1-a}\left(\Phi(a)^{-1}\right)^2\left(\phi_1+2a\phi_2\cdots qa^{q-1}\phi_q\right)\mu$
R_{j+1}:	$\frac{dR_{j+1}}{dm}=0$	$\frac{dR_{j+1}}{d\delta_j}=1$ $\left.\frac{dR_{j+1}}{d\delta_i}\right\vert_{j\neq i}=0$	$\frac{dR_{j+1}}{da}=-\left(\Phi(a)^{-1}\right)^2\left(\phi_1+2a\phi_2\cdots qa^{q-1}\phi_q\right)\left(\sum_{k=j+1}^{q}a^{k-j}\phi_k+\phi_j\right)$ $-\Phi(a)^{-1}\sum_{k=j+1}^{q}[k-j]a^{k-j-1}\phi_k$
R_q:	$\frac{dR_q}{dm}=0$	$\left.\frac{dR_q}{d\delta_j}\right\vert_{j\neq q}=0$ $\frac{dR_q}{d\delta_q}=1$	$\frac{dR_q}{da}=-\left(\Phi(a)^{-1}\right)^2\left(\phi_1+2a\phi_2\cdots qa^{q-1}\phi_q\right)\phi_q$

For j=1,2,...,q and for i=1,2,,...,q-1

The derivative of $\Phi(a)^{-1}$ with respect to ϕ_i and the discount factor a is:

$\frac{d\Phi(a)^{-1}}{d\phi_i}=\left(\Phi(a)^{-1}\right)^2 a^i$ and $\frac{d\Phi(a)^{-1}}{da}=\left(\Phi(a)^{-1}\right)^2\left(\phi_1+2a\phi_2\cdots qa^{q-1}\phi_q\right)$ which is calculated directly in the matrix above.

7.2 The companies used in the samples

The calculated betas for the extended companies (the original ones for the period 1982-97) and the extension companies (the ones used to extend the time series that did not go all the way back to 1977). The table is to understand so that for example Norsk Skog extended Helly-Hansen from 1984.The betas for the extension companies are much higher, probably because they are calculated based on prices and dividends in the period 1976-1983. A short period in which the stock market was rather small and therefor probably more inefficient (see discussion in section 6.3).

Extended companies	**Beta**	**Extension companies**	**Beta**	**Year of inclusion of the extended companies**
Norsk Hydro	1.06			
Norsk Skog	0.46	Helly-Hansen	6.53	1984
Bonheur	0.38	Det Nordenfjeldske DS	5.42	1982
Kværner	0.23			
Saga	0.30	Follum Fabrikker	6.20	1980
Bjølve Fossen	0.38			
Elkem	0.08			
Det Sønderfjelske DS	0.28	Nobø Fabrikker	1.92	1982
Belships & Co	3.61	Det Bergenske DS	20.67	1980
Gyldendal	17.22	Bjølsen Valsemølle	11.30	1984
Nydalen Compagniet	0.25	Elektrisk Bureau	1.07	1981
Moelven	0.29	Norema Industrier	3.56	1983
Ganger Rolf	11.14			
Hafslund	0.11			
Nordlandsbanken	0.07			
Storebrand	0.07			
Borgestad	0.03			
DNO	0.01	Bergen Bank	1.06	1983
Skiens Aktiemølle	0.01	Kredittkassen	1.05	1981
Orkla	27.25			

7.3 The estimated parameters

In these tables, the estimated parameters, their test statistics, t, and significance(the probability that they are different from zero) and the ANOVA tests on the dividend process (in the bottom row) are quoted:

7.3.1 The original model

	1 lags, no constant		**2 lags**		**2 lags, differenced**		**4 lags**		**4 lags, differenced**	
	Est.	**Sign./t**	**Est.**	**Sign./t**	**Est.**	**Sign./t**	**Est.**	**Sign./t**	**Est.**	**Sign./t**
a	0.91	0.00 *15.45*	0.88	0.00 *16.12*	0.55	0.10 *1.71*	0.89	0.00 *15.74*	0.86	0.00 *7.10*
μ			1.43	0.00 *5.16*	-0.13	0.71 *-0.38*	1.13	0.00 *4.40*	-0.10	0.76 *-0.31*
ϕ_1	0.99	0.00 *48.39*	0.83	0.00 *3.66*	-0.08	0.73 *-0.35*	0.84	0.00 *3.58*	-0.02	0.94 *-0.08*
ϕ_2			-0.03	0.89 *-0.14*	-0.08	0.78 *-0.28*	-0.03	0.93 *-0.09*	-0.04	0.90 *-0.13*
ϕ_3							-0.21	0.57 *-0.57*	-0.27	0.35 *-0.96*
ϕ_4							0.24	0.38 *0.90*	0.28	0.39 *0.88*
m			333.81	0.00 *3.69*	26.42	0.12 *1.63*	416.09	0.00 *6.21*	17.38	0.36 *0.95*
δ_1	38.21	0.00 *7.16*	26.02	0.23 *1.23*	9.10	0.41 *0.84*	25.21	0.07 *1.96*	6.72	0.59 *0.55*
δ_2			-35.64	0.08 *-1.85*	17.77	0.12 *1.63*	21.34	0.19 *1.37*	19.55	0.12 *1.64*
δ_3							-44.06	0.03 *-2.50*	-22.59	0.12 *-1.69*
δ_4							-23.67	0.10 *-1.77*	-2.89	0.84 *-0.20*
ANOVA D	**338.20**		**165.02**		**0.06**		**117.42**		**1.21**	
Sign.	**0.00**		**0.00**		**0.81**		**0.00**		**0.29**	

7.3.2 The sub-index tests:

The two lags, undifferenced model:

	Sub Index 1		Sub Index 2		Sub Index 3		Sub Index 4	
	Est.	**Sign./t**	**Est.**	**Sign./t**	**Est.**	**Sign./t**	**Est.**	**Sign./t**
a	0.94	0.00	0.90	0.00	0.89	0.00	0.85	0.00
		15.18		*12.22*		*10.60*		*13.78*
μ	1.90	0.00	2.71	0.00	4.29	0.00	1.69	0.00
		4.08		*5.33*		*7.02*		*5.34*
ϕ_1	0.58	0.02	0.69	0.01	0.38	0.12	0.79	0.00
		2.60		*3.02*		*1.63*		*3.49*
ϕ_2	0.17	0.48	-0.08	0.72	-0.01	0.98	-0.04	0.86
		0.73		*-0.37*		*-0.02*		*-0.18*
m	203.25	0.01	293.12	0.00	336.91	0.01	364.00	0.00
		2.93		*3.76*		*3.20*		*4.04*
δ_1	18.24	0.15	15.18	0.26	-1.20	0.95	7.72	0.70
		1.49		*1.15*		*-0.07*		*0.39*
δ_2	-5.12	0.68	-20.57	0.11	-7.49	0.65	-28.36	0.13
		-0.43		*-1.68*		*-0.46*		*-1.57*
ANOVA D	**57.95**		**63.05**		**52.46**		**130.39**	
Sign.	**0.00**		**0.00**		**0.00**		**0.00**	

The two lags, first difference model:

	Sub Index 1		Sub Index 2		Sub Index 3		Sub Index 4	
	Est.	**Sign./t**	**Est.**	**Sign./t**	**Est.**	**Sign./t**	**Est.**	**Sign./t**
a	0.68	0.05	0.45	0.38	0.46	0.36	1.22	0.45
		2.15		*0.90*		*0.94*		*0.78*
μ	-0.02	0.97	-0.20	0.75	-0.28	0.60	-0.22	0.57
		-0.03		*-0.32*		*-2.06*		*-0.58*
ϕ_1	-0.29	0.23	-0.12	0.61	-0.45	0.05	-0.13	0.56
		-1.24		*-0.52*		*-2.06*		*-0.59*
ϕ_2	0.01	0.96	-0.20	0.40	-0.48	0.08	-0.28	0.23
		0.05		*-0.85*		*-1.88*		*-1.24*
m	15.41	0.42	24.85	0.22	31.32	0.22	27.74	0.13
		0.82		*1.26*		*1.27*		*1.57*
δ_1	5.88	0.44	4.38	0.57	15.04	0.15	-4.81	0.65
		0.79		*0.58*		*1.49*		*-0.46*
δ_2	7.07	0.36	7.71	0.31	16.96	0.13	4.04	0.71
		0.95		*1.05*		*1.60*		*0.38*
ANOVA D	**0.86**		**0.43**		**2.43**		**0.78**	
Sign.	**0.37**		**0.52**		**0.14**		**0.39**	

The four lags, undifferenced model:

	Sub Index 1	Sub Index 2	Sub Index 3	Sub Index 4

	Est.	Sign./t	Est.	Sign./t	Est.	Sign./t	Est.	Sign./t
a	0.95	0.00	0.89	0.00	0.88	0.00	0.85	0.00
		15.42		*11.75*		*10.76*		*13.45*
μ	2.83	0.03	1.40	0.00	2.20	0.00	1.18	0.00
		2.42		*3.71*		*5.09*		*4.31*
ϕ_1	0.56	0.01	0.69	0.00	0.35	0.16	0.74	0.01
		2.96		*3.41*		*1.46*		*3.08*
ϕ_2	0.26	0.30	-0.07	0.76	-0.13	0.63	-0.17	0.57
		1.07		*-0.31*		*-0.49*		*-0.58*
ϕ_3	-0.02	0.66	-0.25	0.34	0.16	0.62	0.07	0.83
		-0.45		*-0.99*		*0.51*		*0.22*
ϕ_4	-0.22	0.88	0.46	0.04	0.22	0.44	0.20	0.42
		-0.15		*2.18*		*0.79*		*0.82*
m	242.31	0.55	402.28	0.00	384.38	0.01	411.85	0.00
		0.61		*5.02*		*3.14*		*3.87*
δ_1	20.95	0.05	12.27	0.30	9.89	0.58	14.01	0.52
		2.16		*1.09*		*0.57*		*0.67*
δ_2	15.28	0.12	2.82	0.84	19.25	0.29	-9.72	0.70
		1.67		*0.21*		*1.09*		*-0.39*
δ_3	-1.15	0.69	-23.15	0.11	-28.86	0.15	-4.70	0.85
		0.40		*-1.70*		*-1.54*		*-0.19*
δ_4	-25.27	0.45	-13.21	0.23	-13.80	0.44	-25.75	0.20
		-0.77		*-1.25*		*-0.79*		*-1.36*
ANOVA D	**20.18**		**25.39**		**20.47**		**52.12**	
Sign.	**0.00**		**0.00**		**0.00**		**0.00**	

The four lags, first difference model:

	Sub Index 1 Est.	Sign./t	Sub Index 2 Est.	Sign./t	Sub Index 3 Est.	Sign./t	Sub Index 4 Est.	Sign./t
a	0.73	0.02	0.74	0.01	0.84	0.00	1.24	0.03
		2.69		*2.93*		*4.06*		*2.39*
μ	-0.07	0.90	-0.33	0.56	-0.35	0.54	-0.31	0.44
		-0.12		*-0.59*		*-0.63*		*-0.79*
ϕ_1	-0.29	0.22	-0.18	0.51	-0.55	0.04	-0.24	0.38
		-1.29		*-0.68*		*-2.20*		*-0.91*
ϕ_2	-0.01	0.96	-0.26	0.26	-0.59	0.10	-0.38	0.18
		-0.05		*-1.17*		*-1.75*		*-1.39*
ϕ_3	-0.08	0.76	-0.50	0.04	-0.34	0.30	-0.28	0.31
		-0.31		*-2.18*		*-1.06*		*-1.05*
ϕ_4	-0.46	0.08	0.10	0.72	0.00	1.00	-0.05	0.85
		-1.88		*0.37*		*0.01*		*-0.19*
m	9.29	0.60	22.16	0.31	25.10	0.45	33.55	0.14
		0.54		*1.05*		*0.84*		*1.55*
δ_1	4.63	0.50	2.46	0.80	16.05	0.19	-4.16	0.77
		0.69		*0.26*		*1.15*		*-0.30*
δ_2	9.65	0.18	6.21	0.45	21.19	0.02	6.72	0.64
		1.43		*0.77*		*1.34*		*0.48*
δ_3	2.39	0.74	-15.95	0.08	-6.78	0.76	6.74	0.62
		0.34		*-1.89*		*-0.42*		*0.51*
δ_4	-17.97	0.02	2.32	0.81	6.04	0.54	5.36	0.70
		-2.65		*0.24*		*0.40*		*0.40*
ANOVA D	**0.23**		**1.70**		**1.08**		**0.15**	
Sign.	**0.64**		**0.22**		**0.32**		**0.70**	

7.4 The ADF tests

95% confidence intervals in brackets. A d before the variable name, indicated that it has been differenced.

X_1

Statistic	Observations	Without trend	With trend
DF	21	-2.1076(-3.0115)	-1.9107(-3.6454)
ADF(1)	20	-1.6235(-3.0199)	-1.2684(-3.6592)
ADF(2)	19	-1.6487(-3.0294)	-1.2733(-3.6746)
ADF(3)	18	-2.7565(-3.0401)	-2.8039(-3.6921)
ADF(4)	17	-.76208(-3.0522)	-2.0279(-3.7119)
ADF(5)	16	-.51794(-3.0660)	-1.5724(-3.7347)
ADF(6)	15	-.66823(-3.0819)	-1.9597(-3.7612)
ADF(7)	14	-.92027(-3.1004)	-3.0186(-3.7921)
ADF(8)	13	-.98161(-3.1223)	-2.7685(-3.8288)

X_2

Statistic	Observations	Without trend	With trend

DF	21	-2.5935(-3.0115)	-2.1983(-3.6454)
ADF(1)	20	-2.1364(-3.0199)	-1.9963(-3.6592)
ADF(2)	19	-1.9526(-3.0294)	-1.6443(-3.6746)
ADF(3)	18	-1.8131(-3.0401)	.49283(-3.6921)
ADF(4)	17	-1.6839(-3.0522)	1.2105(-3.7119)
ADF(5)	16	-1.7930(-3.0660)	1.3258(-3.7347)
ADF(6)	15	-.75403(-3.0819)	1.7628(-3.7612)
ADF(7)	14	-.35385(-3.1004)	2.0162(-3.7921)
ADF(8)	13	-.77830(-3.1223)	.58433(-3.8288)

X_3

Statistic	Observations	Without trend	With trend
DF	21	-3.2154(-3.0115)	-3.5132(-3.6454)
ADF(1)	20	-3.3635(-3.0199)	-3.3349(-3.6592)
ADF(2)	19	-1.9516(-3.0294)	-1.9736(-3.6746)
ADF(3)	18	-1.5219(-3.0401)	-1.1973(-3.6921)
ADF(4)	17	-1.7756(-3.0522)	-1.5924(-3.7119)
ADF(5)	16	-1.6657(-3.0660)	-2.7689(-3.7347)
ADF(6)	15	-1.1797(-3.0819)	-2.2105(-3.7612)
ADF(7)	14	-.76158(-3.1004)	-2.6136(-3.7921)
ADF(8)	13	-.56329(-3.1223)	-1.4014(-3.8288)

X_4

Statistic	Observations	Without trend	With trend
DF	21	-2.3498(-3.0115)	-1.8361(-3.6454)
ADF(1)	20	-2.3006(-3.0199)	-1.7177(-3.6592)
ADF(2)	19	-1.4715(-3.0294)	-.65871(-3.6746)
ADF(3)	18	-1.1114(-3.0401)	.16127(-3.6921)
ADF(4)	17	-1.9169(-3.0522)	.42772(-3.7119)
ADF(5)	16	-1.7996(-3.0660)	.21400(-3.7347)
ADF(6)	15	-3.4231(-3.0819)	.30415(-3.7612)
ADF(7)	14	-1.4971(-3.1004)	-.45129(-3.7921)
ADF(8)	13	-2.8068(-3.1223)	-.82318(-3.8288)

A

Statistic	Observations	Without trend	With trend
DF	21	-2.2775(-3.0115)	-1.4663(-3.6454)
ADF(1)	20	-2.2805(-3.0199)	-.98536(-3.6592)
ADF(2)	19	-1.6043(-3.0294)	-.26893(-3.6746)
ADF(3)	18	-2.0397(-3.0401)	.76230(-3.6921)
ADF(4)	17	-1.7765(-3.0522)	.95863(-3.7119)
ADF(5)	16	-1.0319(-3.0660)	1.1133(-3.7347)
ADF(6)	15	-.71856(-3.0819)	1.2346(-3.7612)
	14	0.96133(-3.1004)	2.0047(-3.7921)
	13	2.5158(-3.1223)	2.3054(-3.8288)

dX_1

Statistic	Observations	Without trend	With trend
DF	20	-5.8280(-3.0199)	-6.5425(-3.6592)
ADF(1)	19	-3.1642(-3.0294)	-4.0063(-3.6746)
ADF(2)	18	-2.2084(-3.0401)	-4.3213(-3.6921)
ADF(3)	17	-2.8995(-3.0522)	-4.6795(-3.7119)

ADF(4)	16	-1.7793(-3.0660)	-2.5625(-3.7347)
ADF(5)	17	-1.1216(-3.0819)	-1.6868(-3.7612)
ADF(6)	18	-.86850(-3.1004)	-1.0242(-3.7921)
ADF(7)	19	-1.5832(-3.1223)	-.36987(-3.8288)
ADF(8)	20	-2.1281(-3.1485)	-10.8296(-3.8731)

dX_2

Statistic	Observations	Without trend	With trend
DF	20	-4.8265(-3.0199)	-4.8921(-3.6592)
ADF(1)	19	-3.6418(-3.0294)	-3.8542(-3.6746)
ADF(2)	18	-4.6997(-3.0401)	-5.8636(-3.6921)
ADF(3)	17	-2.1641(-3.0522)	-3.3469(-3.7119)
ADF(4)	16	-.69919(-3.0660)	-2.0970(-3.7347)
ADF(5)	15	.082419(-3.0819)	-.85641(-3.7612)
ADF(6)	14	.54963(-3.1004)	-.17976(-3.7921)
ADF(7)	13	1.2693(-3.1223)	.14181(-3.8288)
ADF(8)	12	.62619(-3.1485)	.14334(-3.8731)

dX_3

Statistic	Observations	Without trend	With trend
DF	20	-6.0352(-3.0199)	-6.3124(-3.6592)
ADF(1)	19	-4.9257(-3.0294)	-4.9418(-3.6746)
ADF(2)	18	-3.9535(-3.0401)	-4.1023(-3.6921)
ADF(3)	17	-2.0286(-3.0522)	-2.3447(-3.7119)
ADF(4)	16	-.91532(-3.0660)	-1.0608(-3.7347)
ADF(5)	15	-1.6158(-3.0819)	-1.5221(-3.7612)
ADF(6)	14	-1.3642(-3.1004)	-1.1960(-3.7921)
ADF(7)	13	-1.8407(-3.1223)	-1.5040(-3.8288)
ADF(8)	12	-1.5307(-3.1485)	-1.1725(-3.8731)

dX_4

Statistic	Observations	Without trend	With trend
DF	20	-4.6312(-3.0199)	-4.9657(-3.6592)
ADF(1)	19	-4.1469(-3.0294)	-4.4185(-3.6746)
ADF(2)	18	-3.5962(-3.0401)	-3.9763(-3.6921)
ADF(3)	17	-1.9886(-3.0522)	-3.2389(-3.7119)
ADF(4)	16	-.84305(-3.0660)	-2.1255(-3.7347)
ADF(5)	15	-.39014(-3.0819)	-3.9939(-3.7612)
ADF(6)	14	.60654(-3.1004)	-1.0482(-3.7921)
ADF(7)	13	-.10175(-3.1223)	-5.1835(-3.8288)
ADF(8)	12	-.34190(-3.1485)	-2.2899(-3.8731)

dA

Statistic	Observations	Without trend	With trend
DF	20	-4.5782(-3.0199)	-5.4399(-3.6592)
ADF(1)	19	-3.1878(-3.0294)	-3.8971(-3.6746)
ADF(2)	18	-2.9629(-3.0401)	-5.0748(-3.6921)
ADF(3)	17	-1.1520(-3.0522)	-3.3572(-3.7119)
ADF(4)	16	-.27994(-3.0660)	-2.0776(-3.7347)
ADF(5)	15	.11628(-3.0819)	-2.2935(-3.7612)
ADF(6)	14	.44370(-3.1004)	-1.2367(-3.7921)

ADF(7)	13	1.3100(-3.1223)	.36386(-3.8288)
ADF(8)	12	1.0201(-3.1485)	.16825(-3.8731)

Logarithm of dividends, aggregated index, differenced

Statistic	Observations	Without trend	With trend
DF	20	-5.5746(-3.0199)	-5.4675(-3.6592)
ADF(1)	19	-4.1099(-3.0294)	-4.0165(-3.6746)
ADF(2)	18	-5.2618(-3.0401)	-5.3239(-3.6921)
ADF(3)	17	-3.6501(-3.0522)	-3.6362(-3.7119)
ADF(4)	16	-1.6932(-3.0660)	-1.6531(-3.7347)
ADF(5)	15	-3.0912(-3.0819)	-2.9713(-3.7612)
ADF(6)	14	-1.9517(-3.1004)	-1.1337(-3.7921)
ADF(7)	13	-1.3802(-3.1223)	-2.3093(-3.8288)
ADF(8)	12	-.92514(-3.1485)	-2.1730(-3.8731)

P, Prices, aggregated index, levels

Statistic	Observations	Without trend	With trend
DF	21	.060490(-3.0115)	-2.4782(-3.6454)
ADF(1)	20	.062301(-3.0199)	-2.4892(-3.6592)
ADF(2)	19	.32676(-3.0294)	-2.5311(-3.6746)
ADF(3)	18	1.4027(-3.0401)	-1.3053(-3.6921)
ADF(4)	17	.79788(-3.0522)	-2.2408(-3.7119)
ADF(5)	16	.86634(-3.0660)	-3.7390(-3.7347)
ADF(6)	15	.75658(-3.0819)	-2.3887(-3.7612)
ADF(7)	14	.54108(-3.1004)	-2.6638(-3.7921)
ADF(8)	13	-3.8512(-3.1223)	2.5983(-3.8288)

dP, Prices aggregated index, differenced

Statistic	Observations	Without trend	With trend
DF	20	-4.4027(-3.0199)	-4.5787(-3.6592)
ADF(1)	19	-3.2855(-3.0294)	-3.4823(-3.6746)
ADF(2)	18	-4.0360(-3.0401)	-4.6281(-3.6921)
ADF(3)	17	-1.8426(-3.0522)	-2.1575(-3.7119)
ADF(4)	16	-1.4753(-3.0660)	-1.7481(-3.7347)
ADF(5)	15	-2.9108(-3.0819)	-2.9696(-3.7612)
ADF(6)	14	-2.2981(-3.1004)	-2.2001(-3.7921)
ADF(7)	13	-4.2830(-3.1223)	-10.1104(-3.8288)
ADF(8)	12	.87341(-3.1485)	-.49214(-3.8731)

7.5 Test statistic for the individual sub-indexes:

The 2 lags, undifferenced model:

X1	X2	X3	X4
511.2103	7248.8795	22282.4168	20930.05
0.00	0.00	0.00	0.00

The 2 lags. differenced model:

X1	X2	X3	X4
142.9409	734.586904	4432.29909	420.9949
0.00	0.00	0.00	0.00

The 4 lags. differenced model:

X1	**X2**	**X3**	**X4**
1107.452	43240.3327	186852.166	32740.03
0.00	0.00	0.00	0.00

The 4 lags. undifferenced model:

X1	**X2**	**X3**	**X4**
727.0997	1783.76044	2642.81061	716.0794
0.00	0.00	0.00	0.00

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