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RANK AND ORDER CONDITIONS FOR IDENTIFICATION IN SIMULTANEOUS SYSTEM OF COINTEGRATING EQUATIONS WITH INTEGRATED VARIABLES OF ORDER TWO [1]

Rocco Mosconi[2] and and Paolo Paruolo[2,3]

This paper discusses identification of systems of simultaneous cointegrating equations with integrated variables of order two. Rank and order conditions for identification are provided for general linear restrictions, as well as for equation-by-equation constraints. As expected, the application of the rank conditions to triangular forms and other previous formulations for these systems shows that they are just-identifying. The conditions are illustrated on models of aggregate consumption with liquid assets and on system of equations for inventories.

Keywords: Identification, (Multi-)Cointegration, I(2), Stocks and flows, Inventory models.

1. INTRODUCTION

The identification problem of system of simultaneous equations (SSE) lies at the heart of classical econometrics, see e.g. Koopmans (1949). Rank (and order) conditions for identification of these systems provide necessary and sufficient (or simply necessary) conditions under which it is possible to estimate the structural form of economic interest. These conditions are well summarized in Fisher (1966) or Sargan (1988).

Simultaneous systems of cointegrating (CI) equations have revived interest on SSE over the last two decades, especially for variables integrated of order 1, I(1), see Engle and Granger (1987). Systems of CI equations provide statistical counterparts to the economic concept of equilibrium; identification conditions are hence important to distinguish when different equations correspond to different stable economic-equilibrium relations. For I(1) simultaneous systems of CI equations the rank and order conditions for identification coincide with the classical ones, see e.g. Saikkonen (1993), Davidson (1994) and Johansen (1995a).

More recently, simultaneous CI systems with variables integrated of order 2, I(2), have also been used to accommodate models with stock and flow variables, see Hendry and von Ungern-Sternberg (1981) and Granger and Lee (1989). The relevance of a coherent economic framework for both stock and flow variables is well documented in econometrics, see e.g. Klein (1950). Several flow variables,

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such as Gross Domestic Product, have been found to be well described as I(1) variables; if this is the case, then the corresponding stocks are I(2) by construction.

Inventory models, for example, involve stock and flow variables, see e.g. Arrow et al. (1951). Additional examples are given by models of aggregate consumption, income and and wealth, see Stone (1966, 1973), of public debt, government expenditure and revenues, of the (stock of) mortgage loans and periodic repayments.

A different rationale for I(2) models is provided by the literature on integral control mechanisms in economics initiated by Phillips (1954, 1956, 1957). Here, variables are cumulated on the way to build an 'integral stabilization policy', or integral control. Haldrup and Salmon (1998) discuss the relations of integral control mechanisms with I(2) systems. In the rest of the paper we indicate 'stock variables', or stocks – i.e. cumulated levels, as X_t and 'flow variables', or flows – i.e. levels, as ΔX_t, where Δ is the difference operator.

Stock and Watson (1993) and Johansen (1992) (henceforth SW and J92 respectively) gave contributions on the representation of I(2) systems. In I(2) systems, CI equations may involve both stock and flow variables, and they represent dynamic equilibrium relations. These equations are called 'multi-cointegrating' relations, see Granger and Lee (1989) and Engsted and Johansen (1999). They are a special case of 'polynomial-cointegration' relations, as introduced by Engle and Yoo (1991). A different type of CI equations consists of linear combinations of flow variables only; they represent balancing equations for flows, as in balanced growth models. Below, we label these CI equations as 'proportional control' relations.

The presence of multi-CI and proportional control CI relations gives a richer, albeit more complicated, structure of SSE. In fact, one key feature of I(2) systems is that the first differences of the multi-CI equations also contain proportional control terms; moreover, linearly combining multi-CI equations and proportional control equations result in alternative and equivalent multi-CI equations. As shown below, this gives rise to an identification problem that differs from the one encountered in I(1) systems.

Inference on systems of CI equations with I(2) variables has typically been based on reduced forms; see inter alia Boswijk (2000, 2010), Johansen (1995a, 1997, 2006), Kitamura (1995), Kurita et al. (2011), Paruolo (2000); Paruolo and Rahbek (1999), Rahbek et al. (1999) and Stock and Watson (1993). A notable exception is given by Johansen et al. (2010), who noted that the identification of the coefficients of the stock variables in multi-CI equations, β say, is identical to the identification of SSE with I(1) variables only, see their Section 4.2. The identification of the structural form of the remaining coefficients in the multi-CI equations and of the other CI equations has instead been left undiscussed.

The identification problem for simultaneous CI systems of equations with I(2) variables thus appears not to have been (fully) addressed in the literature; this is the purpose of the present paper. We provide rank and order conditions for identification for general linear restrictions. The leading special case of linear equation-by-equation constraints is also discussed. These rank and order conditions hold in general, irrespective of the subsequent estimation and inference method of choice, and type of model. In particular they apply to Vector AutoRegressive (VAR) processes, employed e.g. in Johansen (1996) or to linear processes, employed in SW.[1] Moreover, when the rank conditions are applied to the triangular form of SW or to similar formulations in line with the ones discussed in Johansen (1996), it is found that both are just-identifying.

The rank and order conditions provided here are also helpful in sequential identification schemes, as the one suggested in Johansen et al. (2010). There, the CI coefficients β multiplying the stock variables in the multi-CI relations are identified first, leaving all other CI coefficients unrestricted. Once β has been identified, (estimated) and fixed, one then has to check the identification of the other CI coefficients, θ say. The correct rank conditions for identification of θ are the ones given here.

The rest of the paper is organized as follows: Section 2 reports a motivating example; Section 3 introduces the relevant system of simultaneous equations; Section 4 discusses rank and order conditions for general affine restrictions. Section 5 presents results for equation-by-equation constraints. Sections 6 and 7 illustrate these results on selected special cases. Section 8 concludes. Proofs are placed in the Appendix.

In the following $a := b$ and $b =: a$ indicate that a is defined by b; $(a : b)$ indicates the matrix obtained by horizontally concatenating a and b. For any full column rank matrix H, $\bar{H}$ indicates $H(H'H)^{-1}$ and $H_\perp$ indicates a basis of the orthogonal complement of the space spanned by the columns of H. vec is the column stacking operator, $\otimes$ is the Kronecker product, $\text{blkdiag}(A_1, \ldots, A_n)$ a block-diagonal matrix with $A_1, \ldots, A_n$ as diagonal blocks.

We use the notation $X_t = I(d)$ to mean that X_t is a vector process integrated of order d, i.e. that $\Delta^d X_t$ is a stationary linear process with a non-zero moving average (MA) impact matrix, where d is an integer, and initial values are chosen appropriately.[2]

2. A MOTIVATING EXAMPLE

This section reports an example on sales and inventories, taken from Granger and Lee (1989); it is used here to show how the identification problem arises in models with stock and flows.

Let s_t and q_t represent sales and production of a (possibly composite) good. Sales s_t are market-driven and trending; in particular Granger and Lee assume that they are I(1). Production q_t is chosen to meet demand s_t, i.e. s_t and q_t

[1] SW considered a linear process setup and discuss the triangular form for systems with I(2) variables. Boswijk (2000) discussed the relation between the triangular form and other formulations of the error correction terms in I(2) VAR systems.

[2] This is in line with the definition of I(0) in Johansen (1996).

have the same trend. Hence $z_t := q_t - s_t$, the change in inventory, is I(0). This corresponds to $x_t := (q_t : s_t)'$ being CI with cointegrating vector $(1 : -1)'$, i.e.

$$(1 : -1)' x_t = I(0). \tag{2.1}$$

The stock of inventory $Z_t = \sum_{i=1}^t z_i + Z_0$ can be expressed in terms of the cumulated sales $S_t = \sum_{i=1}^t s_i + S_0$ and cumulated production, $Q_t = \sum_{i=1}^t q_i + Q_0$. Because q_t and s_t are assumed to be I(1), Q_t and S_t are I(2), and the previous CI relation of $x_t := (q_t : s_t)'$ in (2.1) corresponds to $X_t := (S_t : Q_t)'$ being CI with cointegrating vector $(1 : -1)'$, i.e. $(1 : -1)' X_t = I(1)$.

The principle of inventory proportionality anchors the inventory stock Z_t to a fraction of sales s_t, i.e. it satisfies $Z_t = a s_t + I(0)$. Because Z_t and s_t are I(1), the relation $Z_t = a s_t + I(0)$ is a CI relation; it involves the stock variables X_t and the flow variables $x_t = \Delta X_t$. In this case, Granger and Lee define $X_t := (S_t : Q_t)'$ to be multi-cointegrated, with 'multi-CI vector' $(1 : -1 : -a : 0)'$ i.e.

$$(1 : -1 : -a : 0) \begin{pmatrix} X_t \\ \Delta X_t \end{pmatrix} = I(0). \tag{2.2}$$

The present paper investigates the following question: is the multi-CI vector in (2.2) *unique* (with or without the 0 restriction in the last entry)? This is an instance of the identification problem addressed in the paper. In fact, the set of CI relations (2.1) and (2.2) forms a system of two equations,

$$\begin{pmatrix} 1 & -1 & -a & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} X_t \\ \Delta X_t \end{pmatrix} = I(0). \tag{2.3}$$

Pre-multiplication by the following 2×2 matrix with generic b

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

gives a system of equations with $(1 : -1 : -(a+b) : b)'$ in place of $(1 : -1 : -a : 0)'$ as multi-CI relation. Hence, without a 0 restriction on the last entry of this vector, the first equation, and hence the system, is not identified.

3. THE RELEVANT SIMULTANEOUS SYSTEM OF EQUATIONS

In this section we introduce the relevant SSE for general I(2) systems. Let Y_t be a $p \times 1$ vector of I(2) variables. Let also $X_t := (Y_t' : D_t')'$ be an $n \times 1$ vector where D_t is a $(n - p) \times 1$ vector of deterministic components; SW take $D_t = t^2$ with $n = p + 1$.[3]

[3]When there are no deterministic components we set $n = p$ and $X_t = Y_t$.

Next consider the set of possible CI relationships involving X_t and ΔX_t, both of which contain nonstationary variables; hence one needs to consider the variable vector $(X_t' : \Delta X_t')'$.[4] Consider first a set of $r < p$ equations of the form

$$\beta' X_t + \upsilon' \Delta X_t = \begin{pmatrix} \beta' & \upsilon' \end{pmatrix} \begin{pmatrix} X_t \\ \Delta X_t \end{pmatrix} = \mu_{1t} + I(0) \tag{3.1}$$

where β and υ are $n \times r$ and β is of full column rank r. Hereafter μ_{it}, $i = 1, 2$, is a deterministic vector, containing linear combinations of terms of the form $\Delta^u D_t$, $u \geq 2$. When $D_t = t^2$ as in SW, one has $\mu_t = \mu$, a constant vector.

When some rows in υ' are equal to 0, the corresponding rows in υ' describe CI relations that reduce the order of integration of X_t from 2 to 0, see (3.1). When, instead, some row in υ' is non-zero, the corresponding equation involves both stocks and flows, and it is hence a multi-CI equation. Remark that eq. (3.1) gives the structural form corresponding to eq. (5.1c) in SW's representation. Similarly it gives the structural form corresponding to eq. (2.4) in J92's representation.

The assumption that β' has full row rank r is similar to the requirement of classical SSE that there is a set of r variables for which the system can be solved for. In other words, if β' had not been of full row rank, one could have reduced the number of equations correspondingly. It is simple to observe that one needs $r < p$, because $r = p$ would contradict the assumption that $Y_t = I(2)$.

The first differences of eq. (3.1), $\beta' \Delta X_t + \upsilon' \Delta^2 X_t$, are also stationary; this implies that $\beta' \Delta X_t$ is stationary, i.e. that $\beta' \Delta X_t$ contains r CI relations. Moreover, other CI relations involving only ΔX_t can be present in the form

$$\gamma' \Delta X_t = \begin{pmatrix} 0 & \gamma' \end{pmatrix} \begin{pmatrix} X_t \\ \Delta X_t \end{pmatrix} = \mu_{2t} + I(0), \tag{3.2}$$

where γ is $n \times s$ and of full column rank. This gives the structural form corresponding to eq. (5.1b) in SW's representation and to eq. (2.2) in J92's representation.

Similarly to β' above, we assume that γ' has been reduced to be of full row rank s, which must be less than p. Moreover, γ' needs to be linearly independent from β', otherwise the SSE composed of $\beta' \Delta X_t$ and $\gamma' \Delta X_t$ would contain some redundant equations; this implies that s must be less than $p - r$. SW and J92 proved that, for the system to contain I(2) variables, one needs $s < p - r$.

Collecting terms, we find that the following system of $2r + s$ stationary SSE is relevant for the discussion of identification in I(2) cointegrated system

$$\zeta' \begin{pmatrix} X_t \\ \Delta X_t \end{pmatrix} = \begin{pmatrix} \beta' & \upsilon' \\ 0 & \gamma' \\ 0 & \beta' \end{pmatrix} \begin{pmatrix} X_t \\ \Delta X_t \end{pmatrix} = \mu_t + I(0), \tag{3.3}$$

where ζ' indicates the matrix of CI SSE coefficients and $\mu_t := (\mu_{1t}' : \mu_{2t}' : \Delta \mu_{1t}')'$. Remark that ζ' contains cross-equation restrictions given by the presence of β' in the first and third block of rows.

[4] Note that deterministic components are included in both X_t and ΔX_t.

The identification problem can now be presented as follows. An equivalent stationary SSE is obtained by pre-multiplying eq. (3.3) by Q' with

$$Q := \begin{pmatrix} \underset{r\times r}{Q_{11}} & 0 & 0 \\ Q_{21} & \underset{s\times s}{Q_{22}} & 0 \\ Q_{31} & Q_{32} & Q_{11} \end{pmatrix}. \tag{3.4}$$

where Q_{jj} are non-singular matrices, $j = 1, 2$. In fact the coefficient matrices in

$$Q' \begin{pmatrix} \beta' & \upsilon' \\ 0 & \gamma' \\ 0 & \beta' \end{pmatrix} \begin{pmatrix} X_t \\ \Delta X_t \end{pmatrix} = \begin{pmatrix} \beta^{\star\prime} & \upsilon^{\star\prime} \\ 0 & \gamma^{\star\prime} \\ 0 & \beta^{\star\prime} \end{pmatrix} \begin{pmatrix} X_t \\ \Delta X_t \end{pmatrix}$$

have the same 0 and cross-equation restrictions, where $\beta^\star = \beta Q_{11}$, $\gamma^\star = \gamma Q_{22} + \beta Q_{32}$, $\upsilon^\star = \upsilon Q_{11} + \gamma Q_{21} + \beta Q_{31}$. This gives rise to the identification problem for SSE with I(2) variables.

We observe that this identification problem is different from the one encountered in I(1) system; in that case the observationally equivalent structures are induced by any non-singular matrix Q, see Saikkonen (1993), Davidson (1994) and Johansen (1995a). Here, instead, the 0 and cross-equation restrictions imply that Q satisfies the structure in (3.4). Hence, unlike in I(1) system, one need to discuss appropriate rank and order conditions for identification that differ from the classical ones; this is the contribution of the present paper.

Now consider a sequential identification procedure that first aims at identifying β through affine restrictions of the type $R'_\circ \operatorname{vec}\beta = c_\circ$, and subsequently consider identification of $(\upsilon : \gamma)$. The identification of β is standard, and the associated rank condition is $\operatorname{rank} R'_\circ(I_r \otimes \beta) = r^2$, see Sargan (1988) and Johansen (1995b). When β is identified, the identification problem for $(\upsilon : \gamma)$ is of the type (3.4) with $Q_{11} = I_r$; the identification conditions for $(\upsilon : \gamma)$ are a special instance of the rank conditions given in the next section, where the restriction on $(\upsilon : \gamma)$ are joined with the restrictions in $R'_\circ \operatorname{vec}\beta = c_\circ$.

4. GENERAL AFFINE RESTRICTIONS

In this section we consider the SSE (3.3) under general linear restrictions on ζ. Generic affine restrictions are defined first, and next rank and order conditions for identification are given. These results are specialized to equation-by-equation restrictions in the next section.

We now introduce additional notation. Recall the form in (3.4) of the class of matrices that gives rise to lack of identification; let $Q_{00} := (Q_{ij})_{i=2,3;j=1,2}$ be the square matrix of dimension $r + s$ in the lower left corner of Q, and note that $\operatorname{vec} Q = N((\operatorname{vec} Q_{11})' : (\operatorname{vec} Q_{00})')'$ for an appropriate $(2r + s)^2 \times (r^2 + (r + s)^2)$ matrix N that maps the free elements of Q to $\operatorname{vec} Q$. The precise form of N is given in Lemma 5 in the Appendix.

We next wish to partition the matrix ζ into appropriate sub-blocks. To this purpose, define $(L_1 : L_2 : L_3) := I_{2r+s}$, $L_{ij} := (L_i : L_j)$, $i, j = 1, 2, 3$, where L_1 and L_3 have r columns and L_2 has s columns; these matrices act on the columns of ζ; for instance we indicate $\zeta^i := \zeta L_i$ and $\zeta^{ij} := \zeta L_{ij}$. Remark that ζ^{12} contains all parameters in ζ. Similarly, define $(J_1 : J_2) := I_{2n}$, where J_1 and J_2 have n columns each; these matrices act on the rows of ζ. In particular $J_2'\zeta = (\upsilon : \gamma : \beta)$, which again contains all parameters in ζ.

Affine restrictions on ζ can be stated directly on ζ, or on ζ^{12} or on $J_2'\zeta$, because all these matrices contain the nonzero coefficients matrices υ, γ, β. As a consequence, it is simple to verify that the following identities hold

$$\text{vec}\,\zeta = A\,\text{vec}\,\zeta^{12}, \qquad \text{vec}\,\zeta^{12} = B\,\text{vec}\,(J_2'\zeta), \tag{4.1}$$

where A and B are appropriate 0-1 matrices, described in detail in Lemma 6 in the Appendix.

Affine restrictions on ζ can be written in terms of ζ, ζ^{12} and $J_2'\zeta$ as follows:

$$\underset{m\times u}{R'}\,\text{vec}\,\zeta = c, \qquad \underset{m_\dagger\times v}{R_\dagger'}\,\text{vec}\,\zeta^{12} = c_\dagger, \qquad \underset{m_\diamond\times f}{R_\diamond'}\,\text{vec}\,(J_2'\zeta) = c_\diamond \tag{4.2}$$

where $u := 2n(2r+s)$, $v := 2n(r+s)$, $f := n(2r+s)$. Due to (4.1), the restrictions design matrices R, $R_\dagger$ and $R_\diamond'$ in (4.2) need to satisfy the following identities

$$R_\dagger = (\bar{B}R_\diamond : B_\perp), \tag{4.3}$$

$$R = (\bar{A}R_\dagger : A_\perp) = (\bar{A}\bar{B}R_\diamond : \bar{A}B_\perp : A_\perp), \tag{4.4}$$

and $c_\dagger = (c_\diamond' : 0')'$, $c = (c_\dagger' : 0')' = (c_\diamond' : 0' : 0')'$, see Lemma 7 in the Appendix.

We are now in the position to state the rank and order conditions.

THEOREM 1 (Identification, general case) *Let $j := \text{rank}\,(R'(I_{2r+s}\otimes\zeta)N)$; one has*

$$j = \text{rank}\left(R_\dagger'\bar{A}'(I\otimes\zeta)N\right) = \text{rank}(R_\diamond'\bar{B}'\bar{A}'(I\otimes\zeta)N). \tag{4.5}$$

A necessary and sufficient condition (*rank condition*) *for the restrictions* (4.2) *to identify ζ is given by*

$$j = r^2 + (r+s)^2. \tag{4.6}$$

A necessary but not sufficient condition (*order condition*) *for* (4.6) *is*

$$m_\diamond \geq r^2 + (r+s)^2. \tag{4.7}$$

The rank condition in (4.6) can be compared with the one obtained for standard SSE, see e.g. Sargan (1988), Chapter 3, Theorem 1. The matrix $R'(I_{2r+s}\otimes\zeta)N$ in the rank condition here is very similar to the matrix $R'(I_{2r+s}\otimes\zeta)$ in the standard case, the only difference being the additional factor N here. This is due to the different class of matrices Q in (3.4) that give rise to observationally equivalent structures.

5. Equation-by-equation restrictions

In this section we specialize the rank and order conditions to the case of equation-by-equation constraints. We indicate the i-th column of ζ as ζ_i. Note that the coefficients in ζ' can be expressed as a function of the first $r+s$ columns in ζ, i.e. by ζ^{12}.

Equation-by-equation restrictions can be hence formulated on the first $r+s$ columns ζ_i in ζ^{12} as follows

$$\underset{m_i \times 2n}{R_i'} \zeta_i = c_i, \qquad i = 1, \ldots, r+s. \tag{5.1}$$

These restrictions are a special case of (4.2), with

$$R_\dagger = \text{blkdiag}(\mathcal{R}_1, \mathcal{R}_2),$$

where $\mathcal{R}_1 = \text{blkdiag}(R_1, \ldots, R_r)$ collects the first r equations (in ζ^1) and $\mathcal{R}_2 = \text{blkdiag}(R_{r+1}, \ldots, R_{r+s})$ the second s equations (in ζ^2). For the latter equations in ζ^2, the zero restrictions on the first n entries imply that

$$R_i = \begin{pmatrix} I_n & 0 \\ 0 & \underset{n \times m_i - n}{R_i^\gamma} \end{pmatrix}, \qquad c_i = \begin{pmatrix} 0_n \\ c_i^\gamma \end{pmatrix}, \qquad i = r+1, \ldots, r+s, \tag{5.2}$$

where 0_u indicates an $u \times 1$ vector of zeros. The following theorem gives rank and order conditions for the case of equation-by-equation restrictions using the notation in (5.2).

Theorem 2 (Identification, equation-by-equation restrictions) *Let the restrictions be given as in (5.1), (5.2); then the i-th column of ζ, $i = 1, \ldots, r$ (i.e. column i in ζ^1) is identified if and only if*

$$\text{rank}\,(R_i'\zeta) = 2r + s, \qquad i = 1, \ldots, r; \tag{5.3}$$

the i-th column in ζ, $i = r+1, \ldots, r+s$ (i.e. column number $i - r$ in ζ^2) is identified if and only if:

$$\text{rank}\left(R_i^{\gamma\prime}(\gamma : \beta)\right) = r + s, \qquad i = r+1, \ldots, r+s. \tag{5.4}$$

The joint validity of rank conditions (5.3) for $i = 1, \ldots, r$ and (5.4) for $i = r+1, \ldots, r+s$ is equivalent to the rank condition (4.6), which, in this case, can be expressed equivalently as follows

$$\text{rank}\,(\mathcal{R}_1'\,(I_r \otimes \zeta)) = r\,(2r+s) \quad \textit{and} \quad \text{rank}\left(\mathcal{R}_2'\left(I_s \otimes \zeta^{23}\right)\right) = s\,(r+s)\,. \tag{5.5}$$

A necessary but not sufficient condition (order condition) for (5.3) is

$$m_i \geq 2r + s, \qquad i = 1, \ldots, r. \tag{5.6}$$

Similarly, a necessary but not sufficient condition (order condition) for (5.4) is

$$m_i - n \geq r + s, \qquad i = r+1, \ldots, r+s. \tag{5.7}$$

In practice, conditions (5.3) and (5.4) can be controlled prior to estimation by generating random numbers for the parameters and substituting them into (5.3) and (5.4), as suggested by Boswijk and Doornik (2004). Alternatively one could modify the generic identification approach of Johansen (1995a), Theorem 3, to obtain conditions that do not depend on specific parameter values.

6. ILLUSTRATION

In this section we illustrate the use of the rank and order conditions. The discussion is based on models of aggregate consumption with liquid assets discussed in Hendry and von Ungern-Sternberg (1981), henceforth HUS, and on the model of inventories introduced in Section 2.

HUS discuss the role of liquid assets in integral control mechanisms for the aggregate consumption function. As pointed out in HUS, when flow variables (as income and consumption) are I(1), the corresponding stock variables (as components of wealth), are in general I(2). Define c_t as real consumption of nondurables and services, y_t as real income, and W_t as real liquid assets. Consider the time series

$$r_t := \Delta W_t - y_t + c_t. \tag{6.1}$$

which includes expenditure for durable goods, investment in other assets, and an 'inflation tax' on liquid assets. Define now, for $t = 1, \ldots, T$, the stock variables $Y_t = \sum_{j=1}^{t} y_j$, $C_t = \sum_{j=1}^{t} c_j$, $R_t = \sum_{j=1}^{t} r_j$ with $W_t = W_0 + Y_t - C_t - R_t$, see (6.1).

Next consider the following 5-dimensional vector $X_t = (Y_t : C_t : R_t : \pi_t : i_t)'$, where π_t is inflation and i_t is the real interest rate. Assuming as in HUS that y_t and c_t are I(1), one finds that X_t is I(2) because it contains the cumulation of y_t and c_t. In line with HUS, we also posit that liquidity W_t, inflation π_t and the interest rate i_t are I(1). This implies that, defining e_j as the j-th column of I_5, the vectors $(e_1 - e_2 - e_3)$, e_4 and e_5 belong to the span of $(\beta : \gamma)$.

HUS postulate the existence of an 'integral control' CI relation $W_t = \tau y_t + I(0)$ and a 'proportional control' CI relation $c_t = \theta y_t + I(0)$. Given that r_t contains expenditure for durable goods, investment in other assets, and an inflation tax on liquid assets, r_t should depend on the real interest rate i_t for the non-liquid assets component, and on inflation π_t and past liquidity W_t for the inflation tax component. If this dependence can be represented as a linear relation, one could postulate the existence of a CI relation of the type

$$r_t = \psi \pi_t + \varphi i_t + \xi W_t + I(0) \tag{6.2}$$

where we have used the fact that $W_{t-1} = W_t - \Delta W_t$ (with $\Delta W_t = I(0)$) can be replaced by W_t in the CI relation (6.2).

Placing together (i) the 'integral control' CI relation $W_t = \tau y_t + I(0)$, (ii) the 'proportional control' CI relation $c_t = \theta y_t + I(0)$, (iii) the CI relation (6.2), and

(iv) the requirement that $(e_1 - e_2 - e_3)$, e_4 and e_5 belong to the span of $(\beta : \gamma)$, one deduces that $r = 2$, $s = 2$.[5]

Moreover, the above implies the following CI SSE

$$\zeta' = \begin{pmatrix} \beta' & \upsilon' \\ & \gamma' \\ & \beta' \end{pmatrix} = \left(\begin{array}{ccccc|ccccc} 1 & -1 & -1 & 0 & 0 & -\tau & 0 & 0 & 0 & 0 \\ -\xi & \xi & \xi & -\psi & -\varphi & 0 & 0 & 1 & 0 & 0 \\ \hline & & & & & -\theta & 1 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 & 1 \\ \cline{6-10} & & & & & 1 & -1 & -1 & 0 & 0 \\ & & & & & -\xi & \xi & \xi & -\psi & -\varphi \end{array}\right).$$

Notice that the second vector in γ is here represented as e_4; this follows from the assumption that the vectors $(e_1 - e_2 - e_3)$, e_4 and e_5 belong to the span of $(\beta : \gamma)$ and the form of β and of the first column of γ. This implies that, provided $\psi \neq 0$, e_4 can be obtained as a linear combination of the columns in β and γ.

We can now check whether this specification is identified or not. Let f_i denote the i-th column in I_{10}, e_j the j-th column of I_5 and 0_u the $u \times 1$ vector of zeros. The equation-by-equation restriction matrices, see (5.1) and (5.2), are given by

$$\begin{aligned} R_1 &= (f_1 : \cdots : f_5 : f_7 : \cdots : f_{10}), & c_1 &= (1 : -1 : -1 : 0_6')', & m_1 &= 9, \\ R_2 &= (f_1 + f_2 : f_1 + f_3 : f_6 : \cdots : f_{10}), & c_2 &= (0_4' : 1 : 0_2')', & m_2 &= 7, \\ R_3^\gamma &= (e_2 : \cdots : e_5), & c_3^\gamma &= (1 : 0_3')', & m_3 &= 9, \\ R_4^\gamma &= I_5, & c_4^\gamma &= (0_4' : 1)', & m_4 &= 10. \end{aligned}$$

The first two equations meet therefore the order condition (5.6), since $m_i \geq 2r + s = 6$ for $i = 1, 2$. Also, the third and fourth equations meet the order condition (5.7), since $m_i - 5 \geq r + s = 4$ for $i = 3, 4$.

Consider now the rank condition (5.3) for the first two equations. One finds

$$\zeta' R_1 = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\xi & \xi & \xi & -\psi & -\varphi & 0 & 1 & 0 & 0 \\ & & & & & 1 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 1 \\ & & & & & -1 & -1 & 0 & 0 \\ & & & & & \xi & \xi & -\psi & -\varphi \end{pmatrix} \tag{6.3}$$

which has rank $2r + s = 6$ if and only if ψ is different from 0. Hence, for all parameter values except for a set of Lebesgue measure 0, the rank condition (5.3) is satisfied by the first equation, which is hence (generically) identified. Because $m_1 - 2r - s = 3$, there are 3 over-identifying restrictions.

[5] If instead one assumes that π_t and i_t are I(0) and r_t is I(1) and does not cointegrate, one would expect $e_4' X_t$ and $e_5' X_t$ to be stationary. This would imply $r = 3$ and $s = 1$, with e_4, e_5 included in column span of β.

For the second equation we find

$$\zeta' R_2 = \begin{pmatrix} 0 & 0 & -\tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ & & -\theta & 1 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 1 \\ & & 1 & -1 & -1 & 0 & 0 \\ & & -\xi & \xi & \xi & -\psi & -\varphi \end{pmatrix}$$

which has rank $5 < 2r+s = 6$. Hence, the rank condition (5.3) fails for the second equation, which therefore is not identified. If one added the further restriction $\xi = 0$, corresponding to the addition of f_1 as an additional column in R_2 (the first one, say), one would find instead

$$\zeta' R_2 = \begin{pmatrix} 1 & 0 & 0 & -\tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ & & & -\theta & 1 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 1 \\ & & & 1 & -1 & -1 & 0 & 0 \\ & & & 0 & 0 & 0 & -\psi & -\varphi \end{pmatrix}$$

which has rank $2r + s = 6$ if and only if ψ is different from 0. Hence the rank condition (5.3) is satisfied by the second equation when $\xi = 0$, for all parameter values except for a set of Lebesgue measure 0. The equation is hence (generically) identified.[6] Because in this case $m_2 - 2r - s = 1$, there is 1 over-identifying restriction.

Consider finally the third and fourth equations. One has $r+s = 4$ in (5.4) and

$$(\gamma : \beta)' R_3^\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 \\ \xi & \xi & -\psi & -\varphi \end{pmatrix},$$

$$(\gamma : \beta)' R_4^\gamma = \begin{pmatrix} -\theta & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & -1 & 0 & 0 \\ -\xi & \xi & \xi & -\psi & -\varphi \end{pmatrix}.$$

The matrix for the third equation has rank 4 if and only if $\psi \neq 0$. The matrix for the fourth equation has rank 4 if and only if $\psi \neq 0$ or $\theta \neq 1$ or both. Hence, for all parameter values, except for a set of Lebesgue measure 0, the rank condition (5.4) is satisfied by both equations, which are hence (generically) identified.

Because $m_3 - n - r - s = 0$ the third equation is just-identified while, being $m_4 - n - r - s = 1$, there is 1 over-identifying restriction on the fourth equation. Note that the identification of equations 1, 3 and 4 does not depend on the additional restriction $\xi = 0$ required for identification of the second equation.

[6]Note that the first equation remains identified also under $\xi = 0$, because the rank in (6.3) does not depend on the value of ξ.

Consider now the model of inventories in Section 2, and the specification in eq. (2.3). One has $r=1$, $s=0$; for the first equation

$$\zeta' R_1 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

which has rank 2, so that the rank condition (5.3) is satisfied. The number of restrictions is $m_1 = 3$, and the number of overidentifying restrictions is $m - 2r - s = 1$.

When one makes the element in position $1,4$ of ζ' unrestricted, however, one finds

$$\zeta' R_1 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

which has rank 1, so that the rank condition (5.3) fails. Note that the order condition is met in this case, see (5.6), because $m_1 = 2r = 2$.

7. TRIANGULAR FORMS AND OTHER SCHEMES

In this section we consider triangular forms and other similar identification schemes on ζ. Using the rank condition above, we show that these restrictions are just-identifying. More specifically, we first consider a normalization similar in spirit to Johansen (1991) for I(1) systems; next we discuss the triangular form of SW, which is shown to be a special case of the former restrictions.

In I(1) models, the CI parameters are contained in β; in that context, one popular just-identified normalized version of β is $\beta_c := \beta(c'\beta)^{-1}$, where c is a matrix of the same dimensions of β with the property that $c'\beta$ is square and nonsingular, see Johansen (1991). In the special case of c chosen as $c := (I_r : 0)$, the normalized CI vectors β_c' take the form of $\beta_c' := (I : -A)$, which gives the triangular form, see Phillips (1991). The choice of $\beta_c' := (I : -A)$ accommodates any reduced form of the simultaneous systems of equations associated with β.[7]

Consider now a CI SSE with I(2) variables, associated with ζ. Assume one can specify a matrix c_0 of the same dimension of β and a matrix c_1 of the same dimension of γ, with the property that $c_0'\beta$ and $c_1'\gamma$ are square and nonsingular and that $(c_0 : c_1)$ has full column rank.

Given the matrices c_0 and c_1, we consider the following restrictions on ζ, i.e. on β, υ and γ:

$$c_0'\beta = I_r, \qquad (c_0 : c_1)'(\upsilon : \gamma) = \text{blkdiag}\left(0_{r,r}, I_s\right), \tag{7.1}$$

where the first equality includes r^2 constraints and the second one $(r+s)^2$, with a total of $m_\diamond = r^2 + (r+s)^2$ restrictions. The following proposition applies.

[7] See Johansen (1995a) for a discussion of the corresponding structural forms and their identification.

PROPOSITION 3 (Normalizations) *The $m_\diamond = r^2 + (r+s)^2$ restrictions (7.1) are just-identifying.*

We next show that constraints (7.1) contain the triangular form of SW as a special case. Let c_0 and c_1 be chosen to satisfy (7.1) and also orthogonal to one another, $c_0'c_1 = 0$. Define c_2 to be any choice of $(c_0 : c_1)_\perp$; this implies that the matrix $C := (c_0 : c_1 : c_2)$ is square and nonsingular, because it is composed of orthogonal blocks. Next consider the following rotation of X_t, given by the variables $\widetilde{X}_t := (X_{0t}' : X_{1t}' : X_{2t}')' := \bar{C}'X_t$, where X_{it} is defined as $\bar{c}_i'X_t$ and where $\bar{C} := (\bar{c}_0 : \bar{c}_1 : \bar{c}_2)$ by block-orthogonality.[8] The following proposition applies.

COROLLARY 4 (Triangular form) *Let β, υ and γ be the parameters in ζ normalized by the just-identifying restrictions (7.1), where c_0, c_1 and c_2 are chosen orthogonal to one another, $c_i'c_j = 0$, $i \neq j$. Then the cointegrated SSE associated with ζ can be expressed equivalently in terms of the rotated variables $\widetilde{X}_t := (X_{0t}' : X_{1t}' : X_{2t}')' := \bar{C}'X_t$, $C := (c_0 : c_1 : c_2)$, as follows*

$$\begin{pmatrix} \beta' & \upsilon' \\ 0 & \gamma' \\ 0 & \beta' \end{pmatrix} \begin{pmatrix} X_t \\ \Delta X_t \end{pmatrix} =$$

$$\begin{pmatrix} I_r & -A_1 & -A_2 & 0 & 0 & -A_3 \\ 0 & 0 & 0 & 0 & I_s & -A_4 \\ 0 & 0 & 0 & I_r & -A_1 & -A_2 \end{pmatrix} \begin{pmatrix} \widetilde{X}_t \\ \Delta \widetilde{X}_t \end{pmatrix}. \tag{7.2}$$

The r.h.s. of (7.2) corresponds to the triangular form in SW, see eq. (4.2) *in Theorem* 4.1 *in Boswijk (2000). Because this is a special case of Proposition 3, ζ in (7.2) is just-identified.*

8. CONCLUSIONS

This paper provides order and rank conditions for general linear hypotheses on the cointegrating vectors in I(2) systems; results are illustrated for the case of equation-by-equation restrictions on models of aggregate consumption with liquid assets and on models of inventories.

If one has in mind an equation-by-equation identification scheme, the main implications of our results, stemming from the block triangularity of the matrix Q in (3.4), are the following.

(i) The coefficients in β, i.e. the coefficients involving the stock variables in the multi-CI equations, might be identified separately following the standard approach adopted for I(1) systems, see Johansen et al. (2010).

[8] For instance C can be chosen as the identity matrix or any permutation of the columns of the identity matrix so that, as in SW, the variables in $\widetilde{X}_t$ coincide with the variables in X (or permutations thereof).

(ii) In this case, however, identification of the coefficients in γ, i.e. the coefficients of the proportional control relations, has to be analysed jointly with β in a non-standard way, since each column of γ can be replaced with a linear combination of columns of γ and β, but not the other way round.

(iii) Even more relevantly, the identification analysis of the coefficients in υ, i.e. the coefficients involving the flow variables in the multi-CI equations, requires a joint non-standard analysis of υ, γ and β, since each column of υ could be replaced with a linear combination of columns in υ, γ and β, but not the other way round.

Therefore, even if the purpose of identification is to check whether multi-CI equations are identified or not, one has to resort to the identification results discussed in this paper; in fact, applying standard identification results to multi-CI equations, i.e. placing at least r restrictions on each equation and then checking for the usual rank condition is incorrect. Another advantage of the approach discussed in this paper is that alternative identification schemes are also covered, involving for example (linear) cross-equation restrictions.

The present approach is based on algebraic conditions similar to the ones employed in Sargan (1988) for classical SSE. These rank and order conditions can be linked to quantities related to the likelihood, as in Rothenberg (1971) for the case of classical SSE. This link is analysed in a separate paper to which we refer for further details, see Mosconi and Paruolo (2014).

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A. APPENDIX

LEMMA 5 (Matrix N) *One has*

$$N = (L_1 \otimes L_1 + L_3 \otimes L_3 : L_{12} \otimes L_{23}) \,. \tag{A.1}$$

PROOF OF LEMMA 5: Eq. (A.1) is proved observing that $Q = L_1 Q_{11} L_1' + L_3 Q_{11} L_3' + L_{23} Q_{00} L_{12}'$, and applying the vec operator. *Q.E.D.*

LEMMA 6 (Structure of connecting matrices A and B) *The A and B matrices in (4.1) are given by*

$$\underset{u\times v}{A} = \begin{pmatrix} I_{2n(r+s)} \\ U_1' \otimes J_2 J_1' \end{pmatrix} = (L_{12} \otimes I_{2n}) + (L_3 \otimes J_2)(U_1' \otimes J_1'), \tag{A.2}$$

$$\underset{v\times f}{B} = (I_{r+s} \otimes J_2 : U_1 \otimes J_1), \tag{A.3}$$

where B has ortho-normal columns, $\bar{B} = B$, and U_1, U_2, with r and s columns respectively, are defined as $(U_1 : U_2) := I_{r+s}$.

PROOF OF LEMMA 6: To find the A matrix, observe that $\operatorname{vec}\zeta = ((\operatorname{vec}\zeta^{12})' : (\operatorname{vec}\zeta^{3})')'$, and that $\zeta^3 = J_2 J_1' \zeta^{12} U_1$. Vectorizing, one finds $\operatorname{vec}\zeta^3 = (U_1' \otimes J_2 J_1') \operatorname{vec}\zeta^{12}$ from which (A.2) follows. To find the B matrix, write $\zeta^{12} = J_2(\upsilon : \gamma) + J_1 \beta U_1'$ and vectorize. *Q.E.D.*

LEMMA 7 (Relation among affine restrictions design matrices) *Due to (4.1), the design matrices R, $R_\dagger$ and $R_\diamond$ in (4.2) need to satisfy the identities (4.3), (4.4).*

PROOF OF LEMMA 7: Eq. (4.1) shows that $A_\perp' \operatorname{vec}\zeta = 0$, $B_\perp' \bar{A}' \operatorname{vec}\zeta = 0$, $\operatorname{vec}\zeta^{12} = \bar{A}' \operatorname{vec}\zeta$ and that $\operatorname{vec}(J_2'\zeta) = \bar{B}' \bar{A}' \operatorname{vec}\zeta$. Together, these equality imply (4.3), and (4.4). *Q.E.D.*

PROOF OF THEOREM 1: We first prove (4.5). Observe that the rows of R' corresponding to $A_\perp$, see (4.3), give zero rows in $R'(I_{2r+s} \otimes \zeta)N$ hence

$$j = \operatorname{rank} R_\dagger' \bar{A}'(I \otimes \zeta)N = \operatorname{rank} R_\dagger' \left((U_1 \otimes \zeta^1) : (I_{r+s} \otimes \zeta^{23})\right) \tag{A.4}$$

because $\bar{A}'(I \otimes \zeta)N = ((U_1 \otimes \zeta^1) : (I_{r+s} \otimes \zeta^{23}))$. Similarly observe that $B_\perp' \bar{A}'(I_{2r+s} \otimes \zeta)N = 0$, which implies (4.5) by (4.4).

We next wish to show that, if ζ and $\zeta^\star := \zeta Q$ both satisfy (4.2), then $Q = I_{2r+s}$ iff (4.6) holds. Hence assume $R' \operatorname{vec} \zeta = c$ and $R' \operatorname{vec} \zeta^\star = c$; subtract these two equations term by term and set $G := I_{2r+s} - Q$. Partition G conformably with Q and write $\operatorname{vec} G = Ng$ where $g := ((\operatorname{vec} G_{11})' : (\operatorname{vec} G_{00})')'$ and N is defined in (A.1); one finds

$$0 = R' \operatorname{vec}(\zeta G) = R'(I_{2r+s} \otimes \zeta) \operatorname{vec} G = R'(I_{2r+s} \otimes \zeta) N g \tag{A.5}$$

Observe that $G = 0$ iff $g = 0$; this proves the rank condition (4.6). The order condition (4.7) is proved observing that $m_\diamond$ and $r^2 + (r+s)^2$ are the number of rows and columns of $R'_\diamond B'_\perp \bar{A}'(I_{2r+s} \otimes \zeta)N$. *Q.E.D.*

Proof of Theorem 2: Condition (5.5) follows from (A.4) by substituting $R_\dagger = \operatorname{blkdiag}(\mathcal{R}_1, \mathcal{R}_2)$. *Q.E.D.*

Proof of Proposition 3: Observe that one can write (7.1) as $R'_\diamond \operatorname{vec}(J'_2 \zeta) = c_\diamond$ with $R_\diamond = \operatorname{blkdiag}(I_{r+s} \otimes (c_0 : c_1) : I_r \otimes c_0)$. Recall that $\bar{A}'(I \otimes \zeta)N = ((U_1 \otimes \zeta^1) : (I_{r+s} \otimes \zeta^{23}))$ from the proof of Theorem 1, so that

$$\begin{aligned} \bar{B}'\bar{A}'(I \otimes \zeta)N &= (I_{r+s} \otimes J_2 : U_1 \otimes J_1)' \left((U_1 \otimes \zeta^1) : (I_{r+s} \otimes \zeta^{23})\right) \\ &= \begin{pmatrix} U_1 \otimes \upsilon & I_{r+s} \otimes (\gamma : \beta) \\ I_r \otimes \beta & 0 \end{pmatrix}. \end{aligned}$$

Hence one finds, when (7.1) are satisfied,

$$\begin{aligned} R'_\diamond \bar{B}'\bar{A}'(I \otimes \zeta)N &= \begin{pmatrix} U_1 \otimes g'\upsilon & I_{r+s} \otimes (c_0 : c_1)'(\gamma : \beta) \\ I_r \otimes c'_0\beta & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & I_{r+s} \otimes \begin{pmatrix} 0 & I_s \\ I_r & c'_1\beta \end{pmatrix} \\ I_{r^2} & 0 \end{pmatrix} \end{aligned}$$

which is square and of full rank $r^2 + (r+s)^2$. Thus the normalisations (7.1) satisfy the rank condition for identification (4.6); moreover, the order condition (5.6) is satisfied with an equal sign, i.e. the restrictions are just-identifying. *Q.E.D.*

Proof of Corollary 4: Define $D = \operatorname{blkdiag}(C, C)$, and observe that $C^{-1} = \bar{C}$ by block-orthogonality. Hence one has $\zeta' (X'_t : \Delta X'_t)' = \zeta' D \bar{D}' (X'_t : \Delta X'_t)' = \zeta' D \left(\widetilde{X}'_t : \Delta \widetilde{X}'_t\right)'$, where we have used orthogonal projections in the form $I = C\bar{C}'$. It remains to show that $\zeta' D$ has the form on the r.h.s. of (7.2). First observe that

$$\zeta' D = \begin{pmatrix} \beta' C & \upsilon' C \\ 0 & \gamma' C \\ 0 & \beta' C \end{pmatrix}$$

where $\beta' C = (I_r : -A_1 : -A_2)$ and $\gamma' C = (0 : I : -A_4)$ by (7.1), where A_i indicate generic matrices of appropriate dimensions. Finally, because $(c_0 :$

$c_1)'\upsilon = 0$, applying orthogonal projections of the form $I = \bar{C}C'$, one finds $\upsilon = \bar{c}_2 c_2' \upsilon = -\bar{c}_2 A_3$, with $A_3 := -c_2'\upsilon$. This completes the proof. *Q.E.D.*