Almost orthogonal outcomes under probabilistic voting: A cautionary example

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Abstract

I illustrate by example a way in which equilibria under probabilistic voting are fragile with respect to assumptions about the non-policy components of voter preferences. I also offer intuition for the fragility using the social welfare functions which also describe the equilibria.
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1 Introduction

In probabilistic voting, the non-policy terms in voters' preferences—the random disturbances from the viewpoint of the candidates—can be mod-

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eled as additive or multiplicative.[1] At least in 'macro' political economy models, one sees little justification offered for a particular choice between these alternatives.[2] How different can equilibrium voting outcomes be, depending on this choice? This note illustrates, by way of example, that the choice of additive or multiplicative disturbances can lead to outcomes that are not just a little different—we may, in fact, obtain equilibrium policy vectors that are nearly orthogonal to one another.

The example is a simple calibrated model of redistributive transfers financed with taxes on consumption, capital income, and labor income. In that sense it is a descendant of one the earliest applications of probabilistic voting, Lindbeck and Weibull (1987). The model also bears some similarities to Profeta (2007), a more recent application of probabilistic voting.[3] To put the difference in outcomes into some empirical context, the resulting "welfare states" under the alternative assumptions—transfers as a share of national income—are further apart than those of, for example, the US and Denmark.

Since certain fragilities of probabilistic voting have been pointed out before (Ball, 1999; Slutsky, 1986), it is worth emphasizing what this note is *not* about. It is not about non-existence of equilibria (equilibria exist), nor is it about non-uniqueness (equilibria are unique, given the structure of voter preferences). The caution it suggests is not for the theorist, but rather for the practitioner who, seeking to model voting outcomes over a multidimensional issue space, is considering an "off the shelf" version of probabilistic voting as a solution.[4]

In the next section, I give a thumbnail sketch of the model and present

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[1] In the language of Banks and Duggan (2005), these are the 'utility difference' and 'utility ratio' models.

[2] See, for example, Yang (1995), Hassler et al. (2005), Profeta (2007), or Alesina et al. (2013).

[3] In the model here and in Profeta (2007), individuals value consumption and leisure, and are taxed to finance redistributive transfers. In Profeta, a multi-dimensional issue space arises because income tax rates are individual-type-specific; here, taxes are paid on consumption and two types of factor income, and agents differ in their factor endowments.

[4] Probabilistic voting has long been used as a solution concept in models where the dimensionality of the issue space prevents application of the median voter theorem. The conditions for existence of a probabilistic voting equilibrium are typically much less stringent than the conditions sufficient to guarantee a Condorcet winner when the issue space is multi-dimensional (Enelow and Hinich, 1989; Coughlin, 1992).

the almost orthogonal outcomes we obtain under additive and multiplicative disturbances. Section 3 offers some intuition for the disparate equilibria, based on the relationship between probabilistic voting outcomes and the maximization of social welfare functions. Additive or multiplicative disturbances correspond to different social welfare functions (Banks and Duggan, 2005); in models of redistribution, the different objectives tilt optima either toward or away from the economy's less affluent members. In section 4, I give more details of the model, the calibration, and the calculations.

2 Almost orthogonal outcomes

Imagine a one-period economy with taxes on labor income, capital income and consumption, which generate revenue that is used to finance some exogenous government consumption and a lump-sum redistribution to agents in the economy. Agents differ in the productivity of their labor effort and in their shares in the economy's capital income.

The number of types is small (four), and no type constitutes a majority. Though static and simplistic, the model is calibrated in a realistic way. A more precise description of the economic environment appears later in Section 4 below.

The policy decision is a vector of taxes and an associated transfer payment. The issue space is three-dimensional, since a choice of the three tax rates implies a transfer level, via the government's budget constraint. Consider the following two very different outcomes (where the τ_C, τ_N and τ_K are tax rates on consumption, labor income and capital income, and T is the associated transfer as a fraction of aggregate output):

$$\begin{bmatrix} \tau_C \\ \tau_N \\ \tau_K \end{bmatrix} = \begin{bmatrix} 0.00 \\ 0.25 \\ 0.00 \end{bmatrix} \tag{A}$$

which yields a transfer of

$$T = 0$$

versus

$$\begin{bmatrix} \hat{\tau}_C \\ \hat{\tau}_N \\ \hat{\tau}_K \end{bmatrix} = \begin{bmatrix} 0.51 \\ 0.03 \\ 1.00 \end{bmatrix} \tag{M}$$

which yields a transfer of

$$\hat{T} = 0.34.$$

These orthogonal outcomes were obtained as probabilistic voting equilibria from a single model with the structure described above, the only difference being whether we assume the non-policy elements in voters' preferences enter additively (A) or multiplicatively (M).

Obviously, the outcomes are far apart. Using the Euclidean distance between tax vectors as a metric, cases (A) and (M) are further apart than those of the most and least generous welfare states among advanced economies.[5]

3 Intuition: Additive disturbances, multiplicative disturbances and social welfare functions

The intuition for the disparate outcomes under additive or multiplicative disturbances can be seen by considering the equivalence between equilibrium outcomes under probabilistic voting and the maximization of a social welfare function. The additive or multiplicative cases correspond to social welfare functions that differ in the weight they place on the utility of the less affluent.

The political environment consists of candidates from two parties vying for election. Candidates espouse policy platforms—tax vectors, in our case—and are assumed committed to enacting their platforms, if elected. In the probabilistic voting framework, voters' preferences over election outcomes depend on more than just candidates' policy platforms.[6]

In the additive case, the utility a voter of type i gets from candidate A winning the election and enacting policy τ^A is

$$V_i^A\left(\tau^A\right) = \xi_i^A + v_i\left(\tau^A\right),$$

where $v_i \geq 0$ is the voter's indirect utility function over the choice of tax vector. In a two-party election, a voter of type i votes for candidate A over

[5]The corresponding tax vectors for the U.S. and Denmark, for example, are roughly $\tau_{\text{US}} = [0.06, 0.23, 0.27]$ and $\tau_{\text{DK}} = [0.21, 0.40, 0.40]$ (Carey and Rabesona, 2002), for a Euclidean distance of 0.26, or 26 percentage points. The distance between τ and $\hat{\tau}$ above is 1.14, or 114 percentage points.

[6]For example, voting intentions may depend on voters' perception of the leadership qualities of the candidates, as in models of valence (Schofield, 2004).

candidate B if

$$v_i\left(\tau^A\right) - v_i\left(\tau^B\right) > \xi_i^B - \xi_i^A \equiv \psi_i.$$

The ψ_i's—which represent aspects of voters' party preferences apart from the explicit policies—are taken as random from the two candidates' standpoints. Given a distribution of the ψ_i's in the population, assume that each candidate chooses his policy platform to maximize his expected plurality, given the policy choice of the other candidate. Assuming that $F(\psi_i)$, the CDF of ψ_i, has the logistic form (independent of i), candidate a's expected plurality is then

$$2\sum_i f_i\left(\frac{\exp\left[v_i\left(\tau^A\right)\right]}{\exp\left[v_i\left(\tau^A\right)\right]+\exp\left[v_i\left(\tau^B\right)\right]}\right) - 1, \tag{1}$$

where f_i is the fraction of type i agents in the electorate.

In contrast to the additive case just described, suppose instead that the utility a voter of type i gets from candidate A winning the election and enacting policy τ^A is

$$V_i^A\left(\tau^A\right) = \exp\left(\xi_i^A\right) v_i\left(\tau^A\right),$$

with $\psi_i \equiv \xi_i^B - \xi_i^A$ still distributed logistically. In this case, A's expected plurality takes the same form as (1) above, but with $\exp\left(v_i\left(\cdot\right)\right)$ replaced by $v_i\left(\cdot\right)$ —*i.e.*,

$$2\sum_i f_i\left(\frac{v_i\left(\tau^A\right)}{v_i\left(\tau^A\right)+v_i\left(\tau^B\right)}\right) - 1. \tag{2}$$

It's easy to verify that the first-order conditions for maximizing (1), evaluated at a symmetric equilibrium ($\tau^A = \tau^B$) are identical to the first-order conditions for maximizing a utilitarian social welfare function:

$$S(\tau) = \sum_i f_i v_i(\tau). \tag{3}$$

Likewise, the first-order conditions for maximizing expected plurality with multiplicative disturbances (2) are (at $\tau_A = \tau_B$) identical to the first-order conditions for maximizing a social welfare function of the form

$$\hat{S}(\tau) = \sum_i f_i \log\left(v_i(\tau)\right). \tag{4}$$

These results are essentially Corollary 3 and Corollary 3′ from Banks and Duggan (2005).

Note that in comparison to (3)—in which voter utilities are perfect substitutes—the curvature present in (4) offers greater gains from the transfer of utils from the relatively well-off to the relatively worse-off. As a result, the preferences of poorer agents will receive effectively more weight under (4) than under (3).

To see this, suppose that $\tau^\star$ maximizes the utilitarian social welfare function (3). Then, for τ near $\tau^\star$, (4) can, to a first-order approximation, be written as[7]

$$\hat{S}(\tau) \cong \eta + \sum_i \left(\frac{f_i}{v_i(\tau^\star)} \right) v_i(\tau),$$

which has the same form as the utilitarian social welfare function (3)—up to the constant η—but gives relatively more weight to types with lower values of $v_i(\tau^\star)$.

If the preferred policy vectors of our model economy's poor and rich are far apart, the probabilistic voting outcomes will be far apart. This is the intuition for the disparate outcomes shown in the introduction.

4 The model behind the example

The model is a simple static model of an aggregate economy, with preferences and technology that are standard in much of macroeconomics. Agents maximize a utility function

$$u(c,n) = c(1-n)^\phi$$

subject to

$$(1-\tau_N)\, wen + (1-\tau_K)\, s\Pi + T = (1+\tau_C)\, c,$$

where c is consumption, n is labor effort, w is the wage rate, Π is aggregate profits, and T is the lump-sum transfer. The agent's type is a pair (e,s), where e is the type's labor productivity, and s is the type's share of aggregate profits. There is a distribution $f(e,s)$ of types, with $\sum_{(e,s)} f(e,s) = 1$. Let $n(e,s)$ and $c(e,s)$ denote the consumption and work effort of agent type (e,s).

[7] Using the fact that $\log(v_i(\tau)) \cong \log(v_i(\tau^\star)) + (1/v_i(\tau^\star))(v_i(\tau) - v_i(\tau^\star))$.

The technology for producing output is Cobb-Douglas, $Y = N^{\alpha}$, where Y is aggregate output and $N \equiv \sum_{(e,s)} f(e,s) n(e,s) e$ is the aggregate effective labor input. Aggregate profits are given by $\Pi = (1-\alpha) Y$ and the wage w obeys $w = \alpha Y / N$.

Exogenous government consumption is specified as a fraction g of aggregate output, and generates no utility for individuals. The aggregate resource constraint is thus

$$C = Y - G = (1-g)Y,$$

where $C = \sum_{(e,s)} f(e,s) c(e,s)$.

The government runs a balanced budget—

$$T + gY = \tau_N w N + \tau_K \Pi + \tau_C C$$

—so the issue space can be taken as the set of three-dimensional tax vectors $\tau = (\tau_C, \tau_N, \tau_K)$. The lump-sum transfer can be used only for redistribution, not as a lump-sum tax to finance government consumption $(T \geq 0)$.

While the model is too simple to be taken as a good description of a real-world economy, I nevertheless try to calibrate it to be roughly consistent with U.S. data. I assume that α, which corresponds to labor's share of national income, is 0.6. I choose ϕ to be consistent with the average agent devoting 30% of his time to work when the tax vector is $\tau_0 = (0.05, 0.25, 0.25)$ (a very rough approximation to the U.S. tax system).

For the distribution of agent types, I use two e values and two s values, for a total of four agent types. I calibrate the marginal distribution of e to match the cross-sectional standard deviation of log real wages from Katz and Autor (1999). I calibrate the marginal distribution of s to match the U.S. distribution of wealth by quintiles reported in Budría-Rodríguez et al. (2002). I set the correlation between e and s based on the wealth–earnings correlation reported in Budría-Rodríguez et al. (2002). None of the four types constitutes a majority, of course, though the low-e, low-s type comes close at 47 percent of the population.

I solve the model numerically on a $101 \times 101 \times 101$ grid of tax vectors in $[0,1] \times [0,1] \times [0,1]$—so each τ takes values in $\{0.00, 0.01, 0.02, \ldots, 1.00\}$. Calculating the competitive equilibrium at each tax vector τ yields indirect utility functions $v_{(e,s)}(\tau)$ for each voter type. I solve for voting equilibria by solving the equivalent social welfare maximizations described in section 3.[8]

Case A in section 2 is the symmetric equilibrium of the two-candidate, normal form game with payoffs implied by the expected plurality function (1). Case M is the equivalent object for payoffs given by (2). For the reasons described in 3, the voting equilibrium in case A (the additive disturbance case) leans toward the favorite outcome of the wealthier agents, who prefer zero redistributive transfers and a tax on labor income just sufficient to pay for the exogenous government consumption. The voting equilibrium in case M (the multiplicative disturbance case) tilts toward the favorite outcome of the poorest agents, who prefer large transfers financed by taxes on capital and consumption.

References

Alesina, A., G.-M. Angeletos, and G. Cozzi (2013). Fairness and redistribution: Reply. *American Economic Review 103*(1), 554–61.

Ball, R. (1999). Discontinuity and non-existence of equilibrium in the probabilistic spatial voting model. *Social Choice and Welfare 16*(4), 533–555.

Banks, J. S. and J. Duggan (2005). Probabilistic voting in the spatial model of elections: The theory of office-motivated candidates. In D. Austen-Smith and J. Duggan (Eds.), *Social choice and strategic decisions*, Studies in Social Choice and Welfare, pp. 15–56. Springer.

Budría-Rodríguez, S., J. Díaz-Giménez, V. Quadrini, and J.-V. Ríos-Rull (2002, Summer). Updated facts on the U.S. distributions of earnings, income, and wealth. *Federal Reserve Bank of Minneapolis Quarterly Review 26*(3), 2–35.

Carey, D. and J. Rabesona (2002). Tax ratios on labour and capital income and on consumption. *OECD Economic Studies 2002*(2), 129–174.

Coughlin, P. J. (1992). *Probabilistic voting theory*. Cambridge University Press.

[8] The programs for these computations are available at `http://www.jimdolmas.net/economics/current-work`.

Enelow, J. M. and M. J. Hinich (1989). A general probabilistic spatial theory of elections. *Public Choice 61*(2), 101–113.

Hassler, J., P. Krusell, K. Storesletten, and F. Zilibotti (2005). The dynamics of government. *Journal of Monetary Economics 52*(7), 1331–1358.

Katz, L. F. and D. H. Autor (1999). Changes in the wage structure and earnings inequality. In O. Ashenfelter and D. Card (Eds.), *Handbook of Labor Economics*, Volume 3A, Chapter 26, pp. 1463–1555. Elsevier.

Lindbeck, A. and J. W. Weibull (1987). Balanced-budget redistribution as the outcome of political competition. *Public choice 52*(3), 273–297.

Profeta, P. (2007). Political support and tax reforms with an application to italy. *Public Choice 131*(1-2), 141–155.

Schofield, N. (2004). Equilibrium in the spatial 'valence' model of politics. *Journal of Theoretical Politics 16*(4), 447–481.

Slutsky, S. M. (1986). Elections with incomplete information: Comments on the papers of Coughlin and Ferejohn. *Public Choice 50*(1/3), 105–129.

Yang, C.-C. (1995). Endogenous tariff formation under representative democracy: A probabilistic voting model. *The American Economic Review 85*(4), 956–963.