Is data interpretation in utility and prospect theories unquestionably correct?

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Contents

Introduction

This is a very draft version (or extended abstract) of the report "The random-lottery incentive system. Can $p\sim 1$ experiments deductions be correct?". It is published to extend the abstract of the report and to facilitate its perception.

This is a part of the whole research. The research was motivated by the paradoxes of the utility and prospect theories. The analysis of such paradoxes was started in 1738 by Bernoulli in [1]. Examples of these paradoxes include the Allais paradox [2], the Ellsberg paradox [3], the "four-fold pattern" paradox (see, e.g., [4]), etc. In 2002 Kahneman received the Prize in Economic Sciences in Memory of Alfred Nobel for research in this field. In 2006, Kahneman and Thaler [5] pointed out that preferences inconsistencies in the paradoxes have still not been adequately overcome.

One possible way to solve the paradoxes of utility and prospect theories was widely discussed, e.g., in [6]-[8]. The essence of this way consists in a proper attention to noise, imprecision, and other reasons that might cause dispersion, scattering, or spread of the data.

The second possible way (see [9]) is to consider the zones near the boundaries of the probability scale, e.g. at $p\sim 1$.

The research (see, e.g., [10]-[18]) partially presented in this article combines these two ways. That is, it considers a dispersion of the data near the boundaries (or influence of a dispersion of the data near the boundaries).

Simple considerations of the research have an applied mathematical character but a significant practical importance. The ultimate aims are to provide pure mathematical support both for works that are based on the dispersion of data and for works those consider the zones near the boundaries of the probability scale.

1. The Aczél-Luce question

1.1. The question

Aczél and Luce [9] emphasized a fundamental question: whether $W(1)=1$ (whether the Prelec weighting function is equal to 1 at $p=1$). From now on, we refer to this question as the Aczél-Luce question (or Luce question).

1.2. The importance of the question

The Aczél-Luce question means that the Prelec weighting function has a discontinuity near $p=1$. This is not quantitative but a topological feature. One of possible answers to the question can cardinally change the situation in the utility and prospect theories.

2. Purely mathematical restrictions

The research, partially presented in this article, considers a dispersion of the data near the boundaries (or influence of a dispersion of the data near the boundaries). This has been expressed in a form of a sequence of lemmas and theorems (more detailed see below in the Appendices).

2.1. The theorem

A purely mathematical theorem proves $W(1)<1$.

The theorem is based on a sequence of lemmas and theorems (more detailed see below in the Appendices):

For a finite non-negative function on a finite interval, the analog of the dispersion is proved to tend to 0, when the mean of the function tends to a boundary of the interval. Hence, if the analog of the dispersion is not less than a non-zero value, then the non-zero restrictions exist on the mean. Namely, the mean cannot be closer to a boundary of the interval than by another non-zero value.

As far as the probability estimation corresponds to such a function and a non-zero dispersion of data takes place, then the non-zero restrictions exist on the probability estimation.

As far as the probability is the limit of the probability estimation and a non-zero minimal dispersion of data takes place, then the non-zero restrictions exist on the probability. Namely, the probability cannot be closer, than by the non-zero value, to a boundary of the probability scale.

If there is a non-zero restriction on the probability at $p=1$, then $W(1)<1$.

As a matter of fact, the non-zero minimal dispersion of data can be caused, e.g., by non-zero noises.

2.2. Experimental evidence. A seemed contradiction

At present, the experiments at $p\sim1$ seem to not support or disprove the theorem. Nevertheless, fine details of the experiments should be analyzed.

3. Analysis of fine details of the experiments.

Let us analyze the experiments. To analyze fine details, let us confine ourselves to gain at high probabilities.

The journals QJE, Econometrica, AER, JEL and JEP have been analyzed for the period 2003-2013.

Let us consider some typical descriptions (here only two) of the experiments.

In [19] at page 1402 we see "At the beginning of the experiment, stakeholders were told that the computer would randomly choose one of the situations and one of the choices in this situation to determine their final outcome."

In [20] at page 3365 we see "One choice for each subject was selected for payment by drawing a numbered card at random."

We see that subjects are stimulated and paid by the choice of one from a number of situations. Let us consider this feature more closely.

4. Certainty and lottery. Correctness of data interpretation

First, let us note that the stimulation by the payment for the choice of one from a number of situations may be named as the uncertain stimulation. We may name it also as the stimulation by uncertain incentives.

4.1. Inconsistency between the certain outcomes and uncertain incentives

Suppose, that the subjects choose the uncertain choice, that is the choice, which probability is strictly less than *1* (and strictly more than *0*). In this case, the choice and the incentive are of the same type.

Suppose, that the subjects choose the certain choice, that is the choice, which probability is strictly equal to *1* (or strictly equal to *0*). In this case, the choice and the incentive are of different types. Moreover, this uncertain incentive calls the certain outcome into question.

4.2. The role of incentives

Do incentives affect the choice of the subjects?

The correct answer to this question needs a special research.

However, we may be sure, that if incentives would not have any influence on the choice of the subjects, then there would no reason to use such incentives.

Indeed, in [20] at page 3365 we see "Subjects were told to treat each decision as if it were to determine their payments."

So, we cannot exclude that an incentive affects a choice of a subject, at least partially.

So, the correctness of the use of uncertain incentives for certain outcomes is questionable. We may name this problem as a "certain-uncertain" inconsistency.

5. The random-lottery incentive system

In [20] at page 3365 we see "This random-lottery mechanism, which is widely used in experimental economics, …"

In [21] we see:

"… the random-lottery incentive system has become the almost exclusively used incentive system for individual choice, and numerous studies have used and tested it. It is used by people well recognized in experimental economics".

"In the random-lottery incentive system, a subject is asked to choose in several choice situations. At the end, a random procedure selects only one of those several situations to be played for real. The other situations are not played for real. …"

"Holt (1986) described a problem for the random-lottery incentive system that might arise theoretically. Subjects may not perceive every choice situation as isolated, but may perceive their situation as a grand meta-lottery over many choice situations … … Starmer & Sugden (1991) were the first to empirically test whether the problem pointed out by Holt arises in experiments. It did not in their study. Subjects treated every choice situation as isolated and did not treat them as a grand meta-lottery."

So, we may conclude:

1) The random-lottery incentive system is widespread in the utility and prospect theories. Moreover there are no widespread mentions about differences between the results of the random-lottery incentive system and other systems.

2) The essence of the random-lottery incentive system does correspond to the name of the system.

3) The question of considering an isolated situation as a grand meta-lottery over many choice situations has been already brought up. Nevertheless, the specific "certain-uncertain" inconsistency question has not been considered.

6. The importance of the correctness of deductions

In any case, the correctness of the interpretation and deductions is important. So, the evident "certain-uncertain" inconsistency should be eliminated.

In any case, the Aczél-Luce question is important also. In the case if, due to the theorem, $W(1)<1$, then a lot of results and even concepts of the utility and prospect theories may be modified or changed.

7. About the present situation and possible measures to attain correct deductions

Possible measures to attain correct deductions should be elaborated by teams of researchers. At present, one may suppose the following situation and possible measures to attain correct deductions:

In the middle of the probability scale, the deductions may be the same or slightly corrected.

When the probability tends to the restriction $p \rightarrow 1-r_{Rand\text{-}Lott}$ due to non-zero dispersion of the random-lottery of the incentive system, then the results are affected by non-linear functions, e.g., power and exponential ones. So, to be correct, the deductions should be recalculated by the inverse functions.

At the probabilities those are in the forbidden zone $p \geq 1-r_{Rand\text{-}Lott}$, a new approach needs to make the deductions correct.

The situation at $p \sim 0$ is the same but (due to the first consequence of the hypothesis of uncertain future) is shifted to $p=0$ and inversed, that is, when at $p \sim 1$ the results are lower than the line $W(p)=p$, then at $p \sim 0$ the results are higher than the line $W(p)=p$.

Conclusions

Aczél and Luce (2007) emphasized a fundamental question: whether *W(1)=1* (whether the Prelec weighting function equals *1* at *p=1*).

A purely mathematical theorem proves *W(1)<1*.

The experiments at *p~1* seem to disprove the theorem.

However, in the prevailing random-lottery incentive system, the choices of certain outcomes are stimulated by uncertain lotteries.

Because of this evident "certain-uncertain" inconsistency, the deductions from the random-lottery incentive experiments, those include the certain outcomes, cannot be unquestionably correct. So, these deductions need an additional proof, or an amendment, or a new approach.

At that, they may need different measures for different *p*. Namely:

In the middle of the probability scale, the deductions may be the same or slightly corrected.

When the probability tends to the restriction $p \rightarrow 1-r_{Rand\text{-}Lott}$ the deductions should be recalculated by non-linear functions, e.g., by functions those are inverse to power and exponential ones.

At the probabilities those are in the forbidden zone $p \geq 1-r_{Rand\text{-}Lott}$, a new approach needs to make the deductions correct.

References

[1] Bernoulli D. (1738), "Specimen Theoriae Novae de Mensura Sortis", Commentarii Academiae Scientiarum Imperialis Petropolitanae, 5 (1738): 175-192. English translation: "Exposition of a New Theory on the Measurement of Risk", *Econometrica*, 22(1): 23-36.

[2] Allais, M. (1953), "Le comportement de l'homme rationnel devant le risque: Critique le postulats et axioms de L'École Américaine", *Econometrica*, 21: 503-546.

[3] Ellsberg L. (1961), "Risk, Ambiguity, and the Savage Axioms", *The Quarterly Journal of Economics*, 75 (4): 643-669.

[4] Tversky, A. and Wakker, P. (1995) "Risk attitudes and decision weights", *Econometrica*, 63: 1255-1280.

[5] Kahneman, D. and Thaler, R. (2006) "Anomalies: Utility Maximization and Experienced Utility", *Journal of Economic Perspectives*, 20, #1: 221-234.

[6] Schoemaker, P., and Hershey J. (1992), "Utility measurement: Signal, noise, and bias", *Organizational Behavior and Human Decision Processes*, 1992, vol. 52 (3): 397-424.

[7] Hey, J., and Orme C. (1994), "Investigating Generalizations of Expected Utility Theory Using Experimental Data", *Econometrica*, 62 (6): 1291-1326.

[8] Butler, D. J., and Loomes G. C. (2007), "Imprecision as an Account of the Preference Reversal Phenomenon", *American Economic Review*, 97 (1): 277-297.

[9] Aczél, J. and Luce R. D. (2007), "A behavioral condition for Prelec's weighting function on the positive line without assuming W(1)=1", *Journal of Mathematical Psychology*, Volume 51, Issue 2, April 2007, Pages 126–129.
[10] Harin, A. (2004) "Arrangement infringement possibility approach: some economic features of large-scale events" Research Announcements, Economics Bulletin, 2004, vol. 28, issue 11, pages A0.
[11] Harin A. (2010), "РАЗРЫВЫ В ШКАЛЕ ВЕРОЯТНОСТЕЙ. ИХ ПРОЯВЛЕНИЯ В ЭКОНОМИКЕ И ПРОГНОЗИРОВАНИИ", *EconStor Open Access Articles*, 2010.
[12] Harin A. (2011), "ТЕОРЕМЫ О СУЩЕСТВОВАНИИ РАЗРЫВОВ НА ЧИСЛОВЫХ ОТРЕЗКАХ И В ШКАЛЕ ВЕРОЯТНОСТЕЙ И НЕКОТОРЫЕ ВОЗМОЖНОСТИ ИХ ПРИМЕНЕНИЯ", *EconStor Open Access Articles*, 2011.
[13] Harin A. (2012), "Data Dispersion in Economics (I) --- Possibility of Restrictions", *Review of Economics & Finance*, 2(3): 59-70.
[14] Harin A. (2012), "Data Dispersion in Economics(II)--- Inevitability and Consequences of Restrictions", *Review of Economics & Finance*, 2(4): 24-36.
[15] Harin, A. (2005) "A Rational Irrational Man" Public Economics from Economics Working Paper Archive at WUSTL, 0511005, 2005.
[16] Harin, A. (2007) "Principle of uncertain future and utility" MPRA Paper from University Library of Munich, Germany, 2007.
[17] Harin, A. (2008) "Solution of the Ellsberg paradox by means of the principle of uncertain future" MPRA Paper from University Library of Munich, Germany, 8168, 2008.
[18] Harin, A. (2009) "General correcting formula of forecasting?" MPRA Paper from University Library of Munich, Germany, 15746, 2009.
[19] Cappelen, Alexander W., James Konow, Erik Ø. Sørensen, and Bertil Tungodden. 2013. "Just Luck: An Experimental Study of Risk-Taking and Fairness." American Economic Review, 103(4): 1398-1413.
[20] Andreoni, James, and Charles Sprenger. 2012. "Risk Preferences Are Not Time Preferences." American Economic Review, 102(7): 3357-76.
[21] Peter P. Wakker, January, 2007, "Message to referees who want to embark on yet another discussion of the random-lottery incentive system for individual choice", http://people.few.eur.nl/wakker/miscella/debates/randomlinc.htm
[22] Harin, A. (2013) "Data dispersion near the boundaries: can it partially explain the problems of decision and utility theories?" Working Papers from HAL, 00851022, 2013.
[23] Harin A. (2010), "Theorem of existence of ruptures in the probability scale", *9th International conference "Financial and Actuarial Mathematics and Eventoconverging Technologies"*, (2010).
[24] Harin A. (2010), "Theorem of existence of ruptures in probability scale. Preliminary short version", MPRA Paper from University Library of Munich, Germany, 23319, (2010).
[25] Harin, A. (2009) "Ruptures in the probability scale? Calculation of ruptures' dimensions" MPRA Paper from University Library of Munich, Germany, 19348, 2009.

Appendices (see [13], [14], [22])

A1. An illustrative example of restrictions for mean

A.1.1. Two points

Let us suppose given an interval *[A, B]* (see Figure 1). Let us suppose that two points are determined on this interval: a left point x_{Left} and a right point x_{Right} : $x_{Left}<x_{Right}$. The coordinates of the middle, mean point may be calculated as $M=(x_{Left}+x_{Right})/2$.

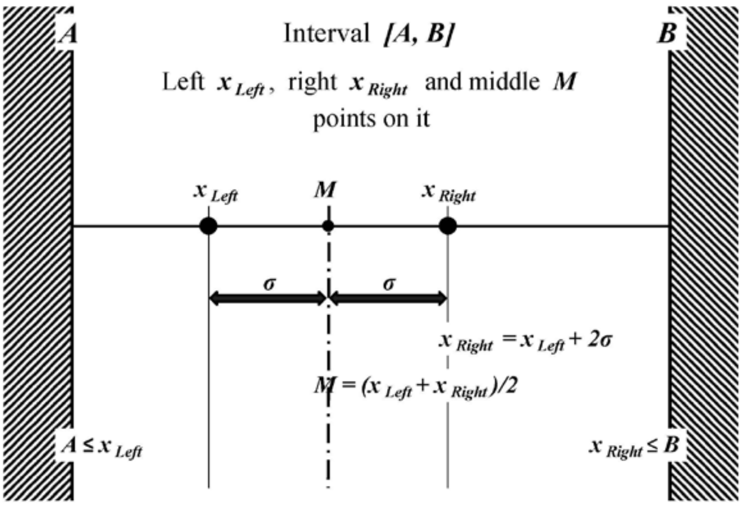

Figure 1. An interval *[A, B]*. Left x_{Left}, right x_{Right} and middle, mean M points on it

Let us suppose that $x_{Right}-x_{Left}\geq 2\sigma=2Const_{\sigma}>0$. So, of course, $x_{Right}\geq x_{Left}+2\sigma$ and $x_{Left}\leq x_{Right}-2\sigma$. For the sake of simplicity, Figures 1-3 represent the case of the equality $x_{Right}-x_{Left}=2\sigma$ and also, of course, $x_{Right}=x_{Left}+2\sigma$, $x_{Left}=x_{Right}-2\sigma$ and $M-x_{Left}=x_{Right}-M=\sigma=Const_{\sigma}>0$.

So, $M=x_{Left}+\sigma>x_{Left}$ and $M=x_{Right}-\sigma<x_{Right}$.

Suppose further that $x_{Left}\geq A$ and $x_{Right}\leq B$.

One can easily see that two types of zones for M can exist in the interval:

1) The mean point M can be located only in the zone which will be referred to as "allowed" (see Figure 2).

2) The mean point M cannot be located in the zones which will be referred to as "forbidden" (see Figure 3).

A.1.2. Allowed zone

Due to the conditions of the example, the left point x_{Left} may not be located further left than the left border of the interval $x_{Left} \geq A$ and the right point x_{Right} may not be located further right than the right border of the interval $x_{Right} \leq B$.

For M, we have $M = x_{Left} + \sigma \geq A + \sigma > A$ and $M = x_{Right} - \sigma \leq B - \sigma < B$ (see Figure 2).

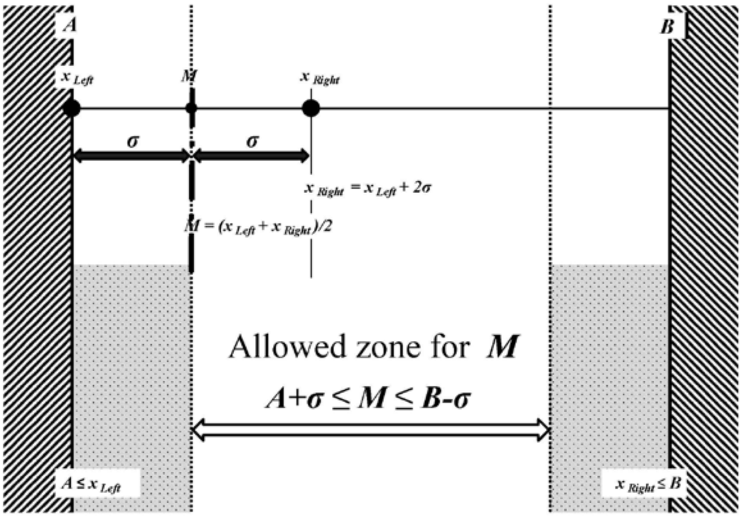

Figure 2. Allowed zone for M

The width of the allowed zone for M is equal to

$$B - \sigma - (A + \sigma) = (B - A) - 2\sigma.$$

It is less than the width $(B-A)$ of the total interval $[A, B]$ by 2σ. Also, the allowed zone is a proper subset of the total interval.

If the distance 2σ between the left x_{Left} and right x_{Right} points is non-zero, then the difference between the width of the allowed zone and the width of the interval is non-zero also. If the distance is greater than 2σ, then the difference is greater than 2σ also.

So, the mean point M can be located only in the allowed zone of the interval.

A.1.3. Restrictions, forbidden zones

Let us define the term "restriction" for the purposes of this article:

Definition. A **restriction** (more exactly, a **restriction on the mean**) signifies the impossibility for the mean to be located closer to a border of the interval than some fixed distance. In other words, a restriction implies here a forbidden zone for the mean near a border of the interval.

The value of a restriction or the width of a forbidden zone signifies the minimal possible distance between the mean and a border of the interval. For brevity, the term "the value of a restriction" may be shortened to "restriction."

If $A \le x_{Left}$, $x_{Right} \le B$ and $x_{Right}-x_{Left}=2\sigma$, then restrictions, forbidden zones with the width of one sigma σ exist between the mean point and the borders of the interval (see Figure 3). So there are two forbidden zones, located near the borders of the interval. The mean point M can not be located in these forbidden zones.

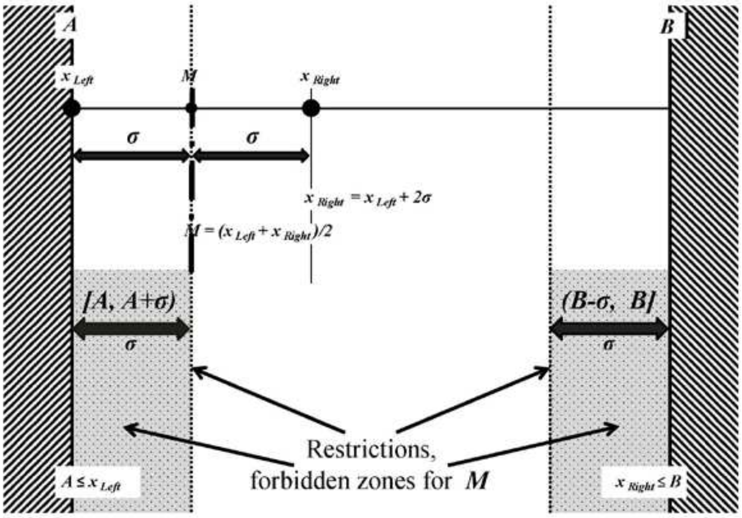

Figure 3. Forbidden zones, restrictions on M

The restrictions, the forbidden zones are shown by two dotted lines and by painting in the bottom part of Figure 3.

As we can easily see, restrictions on the mean or forbidden zones exist between the allowed zone of the mean M and the borders A and B of the interval $[A; B]$. The value of the restriction, or, equivalently, the width of the forbidden zone, is equal to σ.

So, the restrictions of the value σ on the mean point M exist near the borders of the interval

A2. An illustrative example of restrictions for probability

Consider a classical example: an aiming firing at a target.
Suppose a round target (Figure 4) of the diameter *2L*.

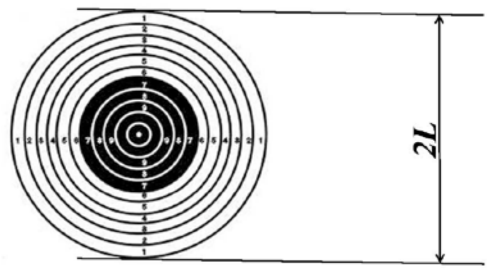

Figure 4. Target for firing

For the obviousness suppose (Figure 5) the dispersion of hits is uniformly distributed in a zone of the diameter 2σ (See an example of the normal distribution, e.g., in [23]).

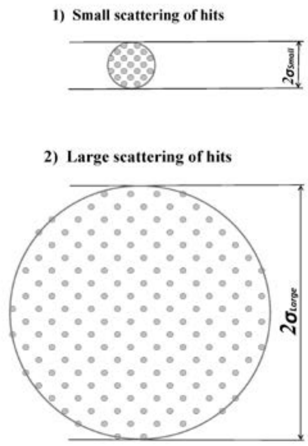

Figure 5. Dispersion of hits is uniformly distributed in a zone of the diameter 2σ

Notes about this figure:

Note 1: This is only a simplified example (See an example of the normal distribution, e.g., in [23]).

Note 2: The case 1) represents the case of small diameter $2\sigma_{Small}$ of the zone of dispersion of hits.

The case 2) represents the case of large diameter $2\sigma_{Large}$ of the zone of dispersion of hits.

Suppose the point of aiming may be varied between the center of the target and a point which is outside the target.

The case, when the diameter $2\sigma_{Small}$ of the zone of dispersion of hits is considerably less than the diameter $2L$ of the target, is drawn on the figure 6.

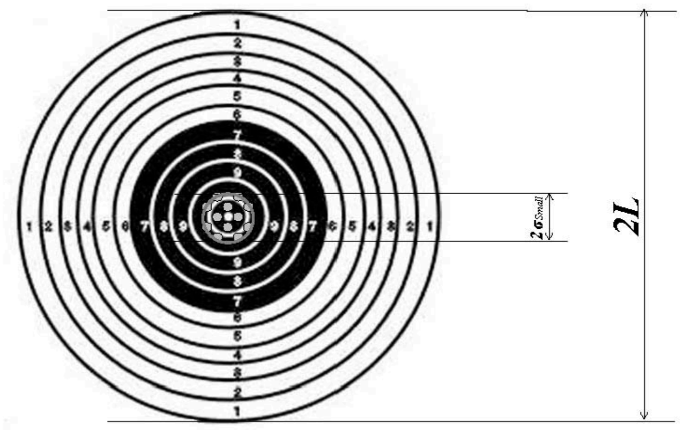

Figure 6. Firing for the small dispersion of hits

Note: The diameter $2\sigma_{Small}$ of the zone of dispersion of hits is considerably less than the diameter $2L$ of a target.

At the condition of the small dispersion of hits, the maximum possible probability of hit in the target can be equal to *1* (can reach the boundary of the probability scale).

When the point of aiming is varied between the center of the target and a point which is outside the target, the probability of hit in the target is varied from *1* to *0*. There are no restrictions in the probability scale.

The case, when the diameter $2\sigma_{Large}$ of the zone of dispersion of hits is considerably more than the diameter $2L$ of the target, is drawn on the figure 7.

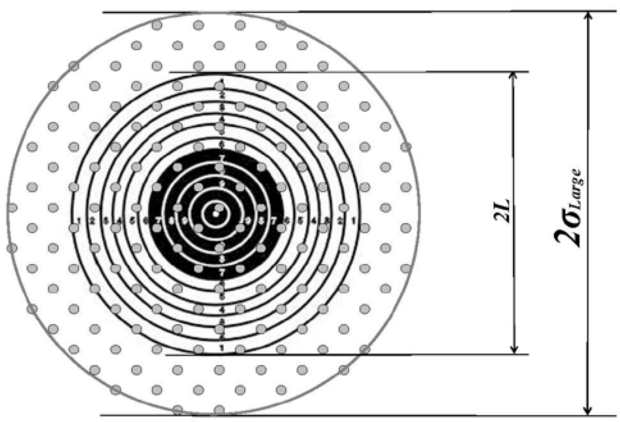

Figure 7. Firing for the large dispersion of hits

Note: The diameter $2\sigma_{Large}$ of the zone of dispersion of hits is considerably more than the diameter $2L$ of the target.

At the condition of the large dispersion of hits (exactly speaking at the condition the diameter $2\sigma_{Large}$ of the zone of dispersion of hits is more than the diameter $2L$ of a target), the maximum possible probability of hit in the target can not be equal to *1*.

So, the situatiuon for the probability for this case is drawn on the figure 8.

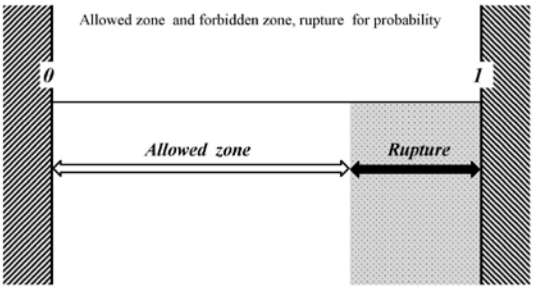

Figure 8. Restriction for the probability: Allowed zone and forbidden zone
Note: See the example of two restrictions for two boundaries in [24]).

The value $P_{AllowedMax}$ of the maximal allowed probability of the allowed zone $[0, P_{AllowedMax}]$ may be estimated as the ratio of the mean number of the hits in the target to the total number of the hits. In this particular case, when the distribution of hits is supposed to be uniform, this ratio equals to the ratio of the area of hits scattering to the area of the target

$$P_{AllowedMax} = S_{Target} / S_{HitsLarge} = \pi L^2 / \pi\sigma^2{}_{Large} = L^2 / \sigma^2{}_{Large} \ .$$

If

$$L < \sigma_{Large} \ ,$$

then

$$P_{AllowedMax} < 1 \ .$$

In this particular case, the probabilities of hit in the target, that are larger than $P_{AllowedMax}$, are impossible. The allowed probabilities of hit in the target belong to the allowed zone $[0, P_{AllowedMax}]$.

The value of the restriction $R_{Restriction}$ may be estimated as the difference between unit and the maximal allowed probability $P_{AllowedMax}$ of hit in the target

$$R_{Restriction} = 1 - P_{AllowedMax} > 0 \ ,$$

and, if $L < \sigma_{Large}$, then $R_{Restriction}$ is a positive nonzero quantity. At the conditions of the figure 7, it is evident the probability $P_{AllowedMax}$ can not be more, then *0.5-0.7* (*50%-70%*) and the restriction $R_{Restriction}$ is as more as *0.3-0.5* (*30%-50%*).

A3. Theorems of existence of restrictions

A.3.1. Preliminary notes

Let us suppose given a finite interval, $X=[A, B] : 0<Const_{AB}\leq(B-A)<\infty$, a set of points $\{x_k\} : k=1, 2, \ldots K : 2\leq K\leq\infty$, and a finite non-negative function $f_K(x_k)$ such that for $x_k<A$ and $x_k>B$ the statement $f_K(x_k)\equiv 0$ is true; for $A\leq x_k\leq B$ the statement $0\leq f_K(x_k)<\infty$ is true, and

$$\sum_{k=1}^{K} f_K(x_k) = W_K \ ,$$

where W_K (the total weight of $f_K(x_k)$) is a constant such that

$$0 < W_K < \infty \ .$$

Without loss of generality, the function $f_K(x_k)$ may be normalized so that $W_K=1$.

Definition 1.1. Let us define an analog of the moment of n-th order of the function $f_K(x_k)$ relative to a point x_0:

$$E(X - X_0)^n = \frac{1}{W_K}\sum_{k=1}^{K}(x_k - x_0)^n f_K(x_k) = \sum_{k=1}^{K}(x_k - x_0)^n f_K(x_k) \ .$$

From now on, for brevity, we refer to this analog of the moment of n-th order as simply the moment of n-th order.

Let us suppose the mean $M\equiv E(X)$ of the function $f_K(x_k)$ exists

$$E(X) \equiv \frac{1}{W_K}\sum_{k=1}^{K} x_k f_K(x_k) = \sum_{k=1}^{K} x_k f_K(x_k) \equiv M \ .$$

Let us suppose at least one central moment $E(X-M)^n : 2\leq n<\infty$, of the function $f_K(x_k)$ exists

$$E(X - M)^n = \frac{1}{W_K}\sum_{k=1}^{K}(x_k - M)^n f_K(x_k) = \sum_{k=1}^{K}(x_k - M)^n f_K(x_k) \ .$$

One may prove (see, e.g., [22]), that a function, which attains the maximal possible central moment, is concentrated at the borders of the interval. At that, the moduli of the central moments of such a function are not greater than the estimate

$$Max(|E(X - M)^n|) \leq (M - A)^n \frac{B - M}{B - A} + (B - M)^n \frac{M - A}{B - A} \ .$$

A.3.2. General lemma for the mean

Lemma 3.2. If, for the function $f_K(x_k)$ defined in Section A.3.1, $M \equiv E(X)$ tends to A or to B, then, for $n : 2 \le n < \infty$, $E(X-M)^n$ tends to zero.

Proof. For $M \to A$, the estimate gives

$$\begin{aligned} |E(X-M)^n| &\le (M-A)^n \frac{B-M}{B-A} + (B-M)^n \frac{M-A}{B-A} = \\ &= [(M-A)^{n-1} + (B-M)^{n-1}] \frac{(M-A)(B-M)}{B-A} < \\ &< [(B-A)^{n-1} + (B-A)^{n-1}] \frac{(M-A)(B-M)}{B-A} \le \\ &\le 2(B-A)^{n-1}(M-A) \xrightarrow[M \to A]{} 0 \end{aligned}.$$

This rough estimate is already sufficient for the purpose of this article. But a more precise estimate may be obtained:

$$|E(X-M)^n| \le (B-A)^{n-1}(M-A) \xrightarrow[M \to A]{} 0 \ .$$

For $M \to B$, the proof is similar and gives

$$|E(X-M)^n| \le (B-A)^{n-1}(B-M) \xrightarrow[M \to B]{} 0 \ .$$

So, if $(B-A)$ and n are finite and $M \to A$ or $M \to B$, then $E(X-M)^n \to 0$.

A.3.3. General theorem for the mean

Let us define two terms for the purposes of this article:

Definition 3.3.1. A **restriction on the mean** $\boldsymbol{r_{Mean}}$ (or, simply, a **restriction**) signifies the impossibility for the mean to be located closer to a border of the interval than some fixed distance. In other words, a restriction implies here a forbidden zone for the mean near a border of the interval.

The value of a restriction or the width of a forbidden zone signifies the minimal possible distance between the mean and a border of the interval. For brevity, the term "the value of a restriction" may be shortened to "restriction."

Definition 3.3.2. Let us define "**restriction on dispersion** of the n-th order" $r^n_{Dispersion.n} \equiv \boldsymbol{r^n_{Disp.n}} : r_{Disp.n} > 0$ (where dispersion is taken in the broad sense, as scattering, spread, variation, etc.) to be the minimal absolute value of the analog of the n-th order central moment $E(X-M)^n$ such that $|E(X-M)^n| \ge r^n_{Disp.n} > 0$.

For $n=2$ the restriction on the dispersion of second order is the minimal possible dispersion (in the particular sense) $r^2_{Disp.2} = \sigma^2_{Min}$.

Theorem 3.2. If, for the finite non-negative discrete function $f_K(x_k)$ defined in Section 1, with the mean $M \equiv E(X)$ and the analog of an n-th ($2 \le n < \infty$) order central moment $E(X-M)^n$ of the function, a non-zero restriction on dispersion of the n-th order $r^n_{Disp.n} = Const_{Disp.n} > 0 : |E(X-M)^n| \ge r^n_{Disp.n}$, exists, then the non-zero restriction $r_{Mean} > 0$ on the mean $E(X)$ exists and $A < (A + r_{Mean}) \le M \equiv E(X) \le (B - r_{Mean}) < B$.

Proof. From the conditions of the theorem and from the lemma for $M \to A$, we have

$$0 < r^n{}_{Disp.n} \leq | E(X-M)^n | \leq (B-A)^{n-1}(M-A)$$

and

$$0 < \frac{r^n{}_{Disp.n}}{(B-A)^{n-1}} \leq (M-A) \ .$$

So,

$$(M-A) \geq r_{Mean} \equiv \frac{r^n{}_{Disp.n}}{(B-A)^{n-1}} > 0 \ .$$

For $M \to B$, the proof is similar and gives

$$(B-M) \geq r_{Mean} \equiv \frac{r^n{}_{Disp.n}}{(B-A)^{n-1}} > 0 \ .$$

So, as long as $(B\text{-}A)$ and n are finite and $r^n{}_{Disp.n}=Const_{Disp.n}>0$, then $r_{Mean}=Const_M>0$ and $A<(A+r_{Mean})\leq M\leq(B\text{-}r_{Mean})<B$.

Note

This estimate is an ultra-reliable one. It is, in a sense, as ultra-reliable as the Chebyshev inequality. Preliminary calculations [25] which were performed for real cases, such as the normal, uniform and exponential distributions with the minimal values $\sigma^2{}_{Min}$ of the analog of the dispersion (in the particular sense), gave the restrictions r_{Mean} on the mean of the function, which are not worse than

$$r_{Mean} \geq \frac{\sigma_{Min}}{3} \ .$$

A.3.4. Lemma for the probability estimation

Lemma: If $f_K(x_k)$ is defined as in section A3.1, and either $E[X]\to 0$ or $E[X]\to 1$, then, for $1<n<\infty$,

$$| E(X-M)^n | \to 0 \ .$$

Proof: As long as the conditions of this lemma satisfy the conditions of the lemma in section A3.2, then the statement of this lemma is as true as the statement of the lemma in section A3.2.

A.3.5. Theorem for probability estimation

Theorem: If a probability estimation, frequency F_K, and $\{x_k\}$ are defined as in section A3.1, such that $M \equiv E[X] \equiv F_K$, there are $n : 1<n<\infty$, and $r_{dispers}>0 : E[(X-M)^n] \geq r_{dispers}>0$, then, for the probability estimation, frequency $F_K \equiv M \equiv E[X]$, a restriction r_{mean} exists such as $0<r_{mean} \leq F_K \leq (1-r_{mean})<1$.

Proof: As long as the conditions of this theorem satisfy the conditions of the theorem in section A3.3, then the statement of this theorem is as true as the statement of the theorem in section A3.3.

A.3.6. Theorem for probability

Theorem: If, for the probability scale [$0; 1$], a probability P and the probability estimation, frequency F_K, for a series of tests of number $K : K>>1$, are determined such that when the number K of tests tends to infinity, the frequency F_K tends at that to the probability P, that is

$$P = \lim_{K \to \infty} F_K \ ,$$

non-zero restrictions $r_{mean} : 0<r_{mean} \leq F_K \leq (1-r_{mean})<1$ exist between the zone of the possible values of the frequency and every boundary of the probability scale, then the same non-zero restrictions $r_{mean} : 0<r_{mean} \leq P \leq (1-r_{mean})<1$ exist between the zone of the possible values of the probability P and every boundary of the probability scale.

Proof: Consider the left boundary 0 of the probability scale [$0; 1$]. The frequency F_K is not less than r_{mean}:

$$F_K \geq r_{mean} \ .$$

Hence, we obtain for P:

$$P = \lim_{K \to \infty} F_K \geq \lim_{K \to \infty} r_{mean} = r_{mean} \ .$$

So, $P \geq r_{mean}$. Note that this is true for both monotonous and dominated convergence. The reason is the fixation of the minimal value of all the F_K by the conditions of the theorem.

For the right boundary 1 of the probability scale the proof is similar to that above.