Space-Time Singularities and Raychaudhuri Equations

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Abstract

Raychaudhuri equations play important roles to describe the gravitational focusing and space-time singularities. Amal Kumar Raychaudhuri established it in 1955 to describe gravitational focusing properties in cosmology. When the star is heavier than a few solar masses, it could undergo an endless gravitational collapse without achieving any equilibrium state. The final outcome of gravitational collapse of a massive star must necessarily be a black hole which covers the resulting space-time singularity and causal message from the singularity cannot reach the external observer at infinity. In this article Raychaudhuri equations are derived with the help of general relativity and topological properties. An attempt has been taken here to describe gravitational focusing and space-time singularities in some detail with easier mathematical calculations.

Keywords: Gravitational focusing, Raychaudhuri equations, singularities.

1. Introduction

In this paper we describe how the matter fields with positive energy density affect the causality relations in space-time and cause focusing in the families of non-space like trajectories. Here the main phenomenon is that matter focuses the non-space like geodesics of the space-time into pairs of focal points or conjugate points.

General relativity models the physical universe as a four dimensional space-time manifold (M, g) which is differentiable, space-time orientable, connected, paracompact, Hausdorff and without boundary [1, 2, 3, 4]. Soon after the Einstein's field equations were discovered, Friedmann showed that the universe must have originated a finite time ago from an epoch of infinite density and curvatures where all the known physical laws break down which we call 'big bang'. So we cannot predict what was happened in the period of big bang and before big bang. Therefore, we can say that it is a singularity in space-time topology. Hence a space-time singularity marks the breakdown of all the laws of physics [5].

The derivation of the Raychaudhuri equations, presented by Raychaudhuri in his seminal paper in 1955, was different from which we derived here. At present the Raychaudhuri equations have been discussed and analyzed in a variety of frameworks of general relativity, quantum field theory, string theory, the theory of relativistic membranes and cosmology. Raychaudhuri pointed out the connection of his equations to the existence of singularities in his paper published in 1955 [6]. The definition of singularity first appeared in the works of Hawking and Penrose after a decade of derivation of the Raychaudhuri equations. In this article we also discuss the space-time singularities applying Raychaudhuri equations. Landau and Lifshitz [7] expressed that a singularity would always imply focusing of geodesics but focusing alone cannot imply a singularity.

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2. General Relativity and Einstein Equation

Einstein's field equation can be written as;

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu} . \qquad (1)$$

Where $G = 6.673 \times 10^{-11} m^3 kg^{-1} s^{-2}$ is the gravitational constant and $c = 10^8\, m/s$ is the velocity of light but in relativistic unit $G = c = 1$. Hence in relativistic units (1) becomes [8];

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi T_{\mu\nu} . \qquad (1a)$$

A perfect fluid is characterized by pressure $p = p(x^\mu)$, then the energy momentum tensor can be written as;

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu} \qquad (2)$$

where ρ is the scalar density of matter.

The covariant curvature tensor is defined by;

$$R_{\rho\mu\nu\sigma} = \frac{1}{2}\left(\frac{\partial^2 g_{\rho\sigma}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\nu}}{\partial x^\sigma \partial x^\rho} - \frac{\partial^2 g_{\mu\sigma}}{\partial x^\nu \partial x^\rho} - \frac{\partial^2 g_{\rho\nu}}{\partial x^\sigma \partial x^\mu} \right)$$

$$+ g_{\alpha\lambda}\left(\Gamma^\alpha_{\mu\nu} \Gamma^\lambda_{\rho\sigma} - \Gamma^\alpha_{\mu\omega} \Gamma^\lambda_{\rho\nu} \right). \qquad (3)$$

Ricci tensor is defined as;

$$R_{\mu\nu} = g^{\lambda\sigma} R_{\lambda\mu\sigma\nu} . \qquad (4)$$

Further contraction of (4) gives Ricci scalar;

$$\hat{R} = g^{\lambda\sigma} R_{\lambda\sigma} . \qquad (5)$$

For a perfect fluid $R_{\mu\nu}$ is defined as;

$$R_{\mu\nu} V^\mu V^\nu = -4\pi(\rho + 3p) \qquad (6)$$

where V^μ denotes the timelike tangent vector.

The equation for geodesic is;

$$\frac{du^\mu}{dt} + \Gamma^\mu_{\nu\lambda} u^\mu u^\nu = 0 . \qquad (7)$$

The separation between nearby time like geodesics is given by geodesic deviation equation or Jacobi equation;

$$D^2 V^\mu = -R^\mu_{\nu\sigma\lambda} T^\nu V^\lambda T^\sigma \qquad (8)$$

where Riemann curvature tensor;

$$R^{\alpha}_{\mu\nu\sigma} = \Gamma^{\alpha}_{\mu\sigma;\nu} - \Gamma^{\alpha}_{\mu\nu;\sigma} + \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\sigma} - \Gamma^{\alpha}_{\beta\sigma}\Gamma^{\beta}_{\mu\nu} \quad (9)$$

is a tensor of rank four.

If $R^{\mu}_{\nu\sigma\lambda} = 0$ then $D^2V^{\mu} = 0$, if $R^{\mu}_{\nu\sigma\lambda} \neq 0$ then the neighboring non-spacelike geodesics will necessarily accelerated towards or away from each other.

3. Definitions of Singularity

Let us consider a space-time manifold $M = \Re^4$. Einstein's empty space equation is $R_{\mu\nu} = 0$. From this we have Schwarzschild metric;

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \quad (10)$$

where there are two singularities at $r = 0$ and $r = 2m$, because one of the $g^{\mu\nu}$ or $g_{\mu\nu}$ is not continuously defined. Here $r = 0$ is a real singularity in the sense that along any non-spacelike trajectory falling into the singularity as $r \to \infty$, the Kretschman scalar $\alpha = R^{\mu\nu\gamma\sigma} R_{\mu\nu\gamma\sigma}$ tends to infinity and $r = 2m$ is a coordinate singularity and could be removed by the coordinate transformation. After some efforts it is realized that $r = 2m$ is not a genuine space-time singularity but merely a coordinate defect, and what was really needed was an extension of the Schwarzschild manifold. Such an extension of the space-time was obtained by Kruskal and Szekeres [9, 10] and this may be regarded as an important insight involving a global approach. The Friedmann-Robertson-Walker model is;

$$ds^2 = -dt^2 + S^2(t)\left[\frac{dr^2}{1 - kr^2} + + r^2\left(d\theta^2 + \sin^2\theta\ d\phi^2\right)\right] \quad (11)$$

where $S(t)$ is the scale factor and k is a constant which denotes the spatial curvature of the three-space and could be normalized to the values +1, 0, –1.

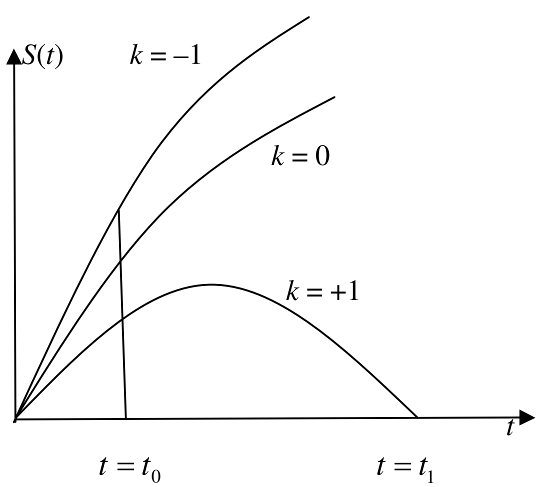

Figure 1. The behavior of the curve $S(t)$ for the three values k = –1, 0, +1; the time $t = t_0$ is the present time and $t = t_1$ is the time when $S(t)$ reaches zero again for k = +1 .

When $k = 0$ the three-space is flat and (11) is called Einstein de-Sitter static model, when $k = +1$ and $k = -1$ the three-space are of positive and negative constant curvature; these incorporate the closed and open Friedmann models respectively (figure 1).

The Einstein equations imply that $\rho + 3p > 0$ at all times, where ρ is the total density and p is the pressure, there is a singularity at $t = 0$, since $S^2(t) \to 0$ when $t \to 0$ in the sense that curvature scalar $\hat{R} = R^{\mu\nu} R_{\mu\nu}$ bends to infinity. Here we consider the time $t = 0$ is the beginning of the universe. Thus there is an essential curvature singularity at $t = 0$ which cannot be transformed away by any coordinate transformation. The existences of real singularities where the curvature scalars and densities diverge imply that all the physical laws break down. Let us consider the metric;

$$ds^2 = -\frac{1}{t^2} dt^2 + dx^2 + dy^2 + dz^2 \qquad (12)$$

which is singular on the plane $t = 0$. If any observer starting in the region $t > 0$ tries to reach the surface $t = 0$ by traveling along timelike geodesics, he will not reach at $t = 0$ in any finite time, since the surface is infinitely far into the future. If we put $t' = \log(-t)$ in $t < 0$ then (12) becomes;

$$ds^2 = -dt'^2 + dx^2 + dy^2 + dz^2 \qquad (13)$$

With $-\infty < t' < \infty$ which is Minkowski metric and there is no singularity at all, which is a removable singularity like Schwarzschild singularity at $r = 2m$ [11]. Let us consider a non-space like geodesic which reaches the singularity in a proper finite time. Such a geodesic will have not any end point in the regular part of the space-time. A timelike geodesic which, when maximally extended, has no end point in the regular space-time and which has finite proper length, is called time like geodesically incomplete. Now we shall discuss some definitions related with singularity [12].

Definition: The generalized affine parameter (g.a.p.) length of a curve $\gamma : [0, a) \to M$ with respect to a frame,

$$\mathbf{E} = \left(\underset{a}{\mathbf{E}}, a = 0,1,2,3\right)$$

at $\gamma(0)$ is given by;

$$\ell_{\mathbf{E}}(\gamma) = \int_0^a \left(\sum_{i=0}^{3} g\left(\dot{\gamma}\, \underset{i}{E}(s)\right) \right)^{1/2} ds$$

where $\dot{\gamma} = \frac{d\gamma}{ds}$ is tangent vector and $\mathbf{E}(s)$ is defined by parallel propogation along the curve, starting with an initial value $\mathbf{E}(0)$.

Definition: A curve $\gamma : [0, a) \to M$ is incomplete if it has finite g.a.p. length with respect to some frame $\mathbf{E}$ at $\gamma(0)$. If $\ell_E(\gamma) < \infty$, then if we take any other frame $\mathbf{E}'$ at $\gamma(0)$ we have $\ell_{E'}(\gamma) < \infty$. This is because the corresponding parallel propogated frames satisfy;

$$\underset{i}{E'} = L_i^j E_j$$

for a constant Lorentz matrix **L** and hence;

$\ell_{E'} \leq \|\mathbf{L}\| \ell_E$,

where $\|\mathbf{L}\| = Sup\left(\left(L_i^j X^i\right)\right)^{1/2}$.

Definition: A curve $\gamma : [0, a) \to M$ is termed inextensible if there is no curve $\gamma' : [0, b) \to M$ with $b > a$ such that $\gamma' \| [0, a) = \gamma$. This is equivalent to saying that there is no point *p* in *M* such that $\gamma(s) \to p$ as $s \to a$ i.e., γ has no end point in *M*.

Definition: A space-time is incomplete if it contains an incomplete inextensible curve. By the above definitions we can say that a space-time is called incomplete if it contains an incomplete timelike inextensible curve. The Friedman 'big bang' models are geodesically incomplete, since the curve defined by;

$\gamma(s)^0 = S(t) - s$

$\gamma(s)^i =$ Constant, $i = 1, 2, 3$

is a geodesic which is incomplete, having no endpoint in *M* as $s \to S(t)$. Minkowski space is not incomplete. The region $r > 2m$ in the Schwarzschild metric is incomplete, while region $0 < r < 2m$ is not a space-time, since the metric is not defined at $r = 2m$.

Definition: An extension of a space-time (M, g) is an isometric embedding $\theta : M \to M'$ where (M', g') is a space-time and θ is onto a proper subset of M'. By the above definition Schwarzschild metric is not singular at $r = 2m$ by Kruskal-Szekeres extension. A space-time is termed extensible if it has an extension.

Definition: A space-time is singular if it contains an incomplete curve $\gamma : [0, a) \to M$ such that there is no extension $\theta : M \to M'$ for which $\theta \circ \gamma$ is extensible.

4. Derivation of Raychaudhuri Equations

Let V^i denotes the timelike tangent vector in the manifold to the congruence. Let us choose the parameter to be the proper time along such timelike trajectories, this can be normalized to be a unit tangent vector [4, 13], i.e.,

$$V^i V_i = -1. \qquad (14)$$

The spatial part h_{ij} of the metric tensor is defined as;

$$h_{ij} = g_{ij} + V_i V_j. \qquad (15)$$

Therefore, we can write;

$$g^{ik} h_{kj} = g^{ik} g_{kj} + g^{ik} V_k V_j \qquad (16)$$

$h^i_j = \delta^i_j + V^i V_j$

where δ^i_j is a Kronecker delta of rank two and is defined by;

$$g_{jk} g^{ki} = \delta^i_j = \begin{cases} 1 \text{ if } i = j \text{ (no summation)} \\ 0 \text{ if } i \neq j. \end{cases}$$

$h_{ij} V^i = g_{ij} V^i + V_i V^i V_j$

$= V_j - V_j = 0$.

Similarly, we can write;

$$h_{ij} V^j = h^i_j V_i = h^i_j V_j = 0. \qquad (17)$$

So, h^i_j can be called the projection operator onto the subspace of T_p, orthogonal to the vector V^i. Now we have,

$h_{ij} h^j_k = \left(g_{ij} + V_i V_j\right)\left(g^j_k + V^j V_k\right)$

$= g_{ij} g^j_k + g_{ij} V^j V_k + V_i V_j g^j_k + V_i V^j V_j V_k$

$= g_{ik} + V_i V_k + V_i V_k - V_i V_k$

$$= g_{ik} + V_i V_k = h_{ik}. \qquad (18)$$

Again we have,

$$h^{ij} h_{ij} = h^i_i = \delta^i_i + V^i V_i = 3. \qquad (19)$$

For the given congruence of timelike geodesics, the expansion, shear and rotation tensors are respectively defined as follows:

i) The volume expansion θ is defined by;

$$\theta_{ij} = V_{(k;l)} h^k_i h^l_j. \qquad (20a)$$

Hence we can write;

$\theta = \theta_{ij} h^{ij} = V_{(k;l)} h^k_i h^l_j h^{ij}$

$= \frac{1}{2}\left(V_{k;l} + V_{l;k}\right) h^{lk}$

$= \frac{1}{2}\left(V_{k;l} + V_{l;k}\right)\left(g^{lk} + V^l V^k\right)$

$$=\frac{1}{2}\left(V^{k}_{;k}+V^{k}_{;k}\right)=V^{k}_{;k}.$$

ii) Shear, distortion in shape without change in volume, which is trace free and orthogonal to V^i and is given by a symmetric tensor;

$$\sigma_{ij}=\theta_{ij}-\frac{1}{3}h_{ij}\theta \qquad (20b)$$

iii) Here rotation or vorticity means rotation without change in shape and it is given by an anti-symmetric tensor orthogonal to V^i as;

$$\omega_{ij}=h_i^k h_j^l V_{[k;l]}. \qquad (20c)$$

Here σ_{ij} and ω_{ij} are spatial coordinates, as both of them are orthogonal to V^i, so that;

$$\sigma_{ij}V^i=\omega_{ij}V^i=0.$$

Again we have,

$$k^{ij}\sigma_{ij}=\sigma_i^i=h^{ij}\left(\theta_{ij}-\frac{1}{3}h_{ij}\theta\right)$$

$$=\theta-\frac{1}{3}h_{ij}h^{ij}\theta$$

$$=\theta-\frac{1}{3}\times 3\theta=0.$$

From equations (20 a, b, c) we get;

$$\sigma_{ij}+\frac{1}{3}h_{ij}\theta+\omega_{ij}$$

$$=\theta_{ij}+\omega_{ij}$$

$$=V_{(k;l)}h_i^k h_j^l+V_{[k;l]}h_i^k h_j^l$$

$$=\left(V_{(k;l)}+V_{[k;l]}\right)h_i^k h_j^l$$

$$=V_{i;j}. \qquad (21)$$

From geodesic equation (7) we get,

$$V^k\nabla_k\nabla_j V_i=V^k\nabla_j\nabla_k V_i+R_{ilkj}V^lV^k. \qquad (22)$$

Since V^k is the tangent to the geodesics, so that;

$$\nabla_j\left(V^k\nabla_k V_i\right)=0.$$

Equation (22) can be written as;

$$V^k\nabla_k\nabla_j V_i=-\left(\nabla_j V^k\right)\left(\nabla_k V_i\right)+\nabla_{ilkj}V^lV^k. \quad (23)$$

Taking trace to the equation (23) we get;

$$\frac{d\theta}{dt}=V^k\nabla_k V^i_{;i}=-V^k_{;i}V^i_{;k}-R_{lk}V^lV^k. \quad (24)$$

By the equation (21) and taking ω_{ij} anti-symmetric we get;

$$\frac{d\theta}{dt}=-R_{lk}V^lV^k-\left(\frac{1}{3}\theta\, h^k_i+\sigma^k_i+\omega^k_i\right)\left(\frac{1}{3}\theta\, h^i_k+\sigma^i_k+\omega^i_k\right)$$

$$\frac{d\theta}{dt}=-R_{lk}V^lV^k-\frac{1}{3}\theta^2-\sigma_{ij}\sigma^{ij}+\omega_{ij}\omega^{ij}$$

$$\frac{d\theta}{dt}=-R_{lk}V^lV^k-\frac{1}{3}\theta^2-2\sigma^2+2\omega^2. \quad (25)$$

For null case similar equations (25) hold with $\frac{1}{3}$ is replaced by $\frac{1}{2}$. These equations (25) are called the Raychaudhuri equations [6] which describe the rate of change of the volume expansion as one moves along the time like geodesic curves in the congruence.

5. Gravitational Focusing in the Raychaudhuri Equations

By Einstein equation (1a) we can write [4, 14];

$$R_{\mu\nu}V^\mu V^\nu=8\pi\left(T_{\mu\nu}V^\mu V^\nu+\frac{1}{2}T\right). \quad (26)$$

The term $T_{\mu\nu}V^\mu V^\nu$ is the energy density measured by a timelike observer with the unit tangent four velocity of the observer, V^μ. In classical physics;

$$T_{\mu\nu}V^\mu V^\nu\geq 0. \quad (27)$$

Such an assumption is called the weak energy condition. Now let us consider;

$$T_{\mu\nu}V^\mu V^\nu\geq -\frac{1}{2}T. \quad (28)$$

Such an assumption is called the strong energy condition which implies from (26) for all timelike vectors V^μ,

$$R_{\mu\nu}V^\mu V^\nu\geq 0. \quad (29)$$

Both the strong and weak energy condition will be valid for perfect fluid provided energy density $\rho \geq 0$ and there are no large negative pressures.

An additional energy condition required often by the singularity theorems is the dominant energy condition which states that in addition to the weak energy condition, the pressure of the medium must not exceed the energy density. Equation (26) implies that the effect of matter on space-time curvature causes a focusing effect in the congruence of timelike geodesics due to gravitational attraction.

Let us suppose γ is a timelike geodesic. Then two points *p* and *q* along γ are called conjugate points if there exists Jacobi field along γ which is not identically zero but vanishes at *p* and *q*. If

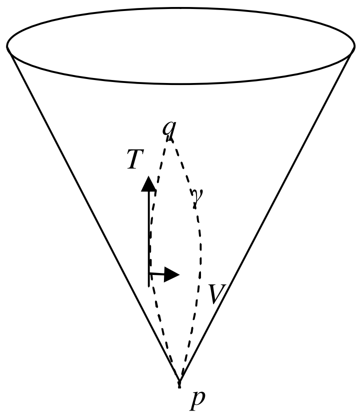

Figure 2. Infinitesimally separated null geodesics cross at *p* and *q*, which are conjugate points along the curve *γ*.

Infinitesimally nearby null geodesics of the congruence meet again at some other point q in future, and then *p* and *q* are called conjugate to each other, where $\theta \to -\infty$ at *q* (figure 2). We can define conjugate point another way as follows:

Let *S* be a smooth spacelike hypersurface in *M* which is an embedded three dimensional sub-manifold. Consider a congruence of timelike geodesics orthogonal to *S*. Then a point *p* along a timelike geodesic γ of the congruence is called conjugate to *S* along γ if there exists a Jacobi vector field along γ which is non-zero at *S* but vanishes at *p*, which means that there are two infinitesimally nearby geodesics orthogonal to *S* which intersect at *p* (figure 3). Again we face equivalent condition that the expansion for the congruence orthogonal to *S* tends to $-\infty$ at *p*. If V^{μ} denotes the normal to *S*, then the extrinsic curvature $\chi_{\mu\nu}$ of *S* is defined as;

$$\chi_{\mu\nu} = \nabla_{\mu} V_{\nu} \qquad (30)$$

which is evaluated at *S*.

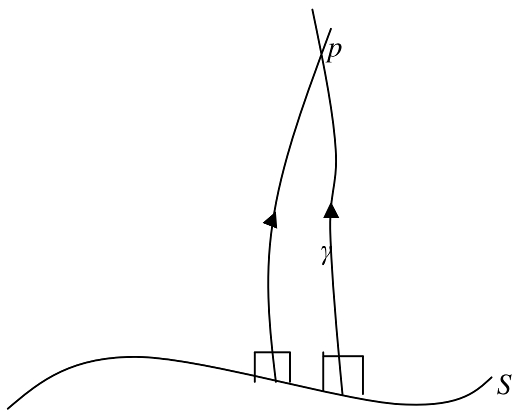

Figure 3. A point *p* conjugate to the spacelike hypersurface *S*. The timelike geodesic is orthogonal to *S*, which is intersected by another infinitesimally nearby timelike geodesic.

So, $\chi_{\mu\nu}V^{\mu} = \chi_{\mu\nu}V^{\nu} = 0$. Since *S* is orthogonal to the congruence this implies $\omega_{\mu\nu} = 0$, hence $\chi_{\mu\nu} = \chi_{\nu\mu}$. The trace of the extrinsic curvature, is denoted by χ, and is given by;

$$\chi = \chi^{\mu}_{\mu} = h^{\mu\nu}\chi_{\mu\nu} = \theta \qquad (31)$$

where θ is expansion of the congruence orthogonal to *S*.

Let us consider the situation when the space-time satisfies the strong energy condition and the congruence of timelike geodesics is hypersurface orthogonal, then $\omega_{\mu\nu} = 0$ implies $\omega^2 = 0$ then (25) gives;

$$\frac{d\theta}{d\tau} \leq -\frac{\theta^2}{3} \qquad (32)$$

which means that the volume expansion parameter must be necessarily decreasing along the timelike geodesics. Let us denote θ_0 initial expansion then integrating (32) we get;

$$\frac{1}{\theta} \geq \frac{\tau}{3} + c. \qquad (33)$$

Initially $\theta = \theta_0$ then (33) becomes;

$$\frac{1}{\theta} \geq \frac{\tau}{3} + \frac{1}{\theta_0}. \qquad (34)$$

By (34) we confirm that if the congruence is initially converging and θ_0 is negative then $\theta \to -\infty$ within a proper time distance $\tau \leq \frac{3}{\theta_0}$. Divergence of the expansion parameter does not imply singularity of space-time, but in certain cases it leads to space-time singularity.

6. Space-Time Singularities in the Raychaudhuri Equations

Now let $\gamma(s)$ be any past directed null geodesic in *M* then,

$$\underset{s \to k}{Lim} \inf T_{\mu\nu} K^{\mu} K^{\nu} > 0 \qquad (35)$$

Must hold along γ where *k* is the limit of the affine parameter in the past. Such a condition arises when matter and radiation are present, for example, and the microwave background radiation, which should have higher densities in the past in view of the observed expansion of the universe [3].

Violation of any one of the higher-order causality condition in *M* implies that *M* is null geodesically incomplete, provided

1. the weak energy condition holds on *M* i.e., $T_{\mu\nu} V^{\mu} V^{\nu} \geq 0$ for any timelike vector V^{μ}, and
2. the matter tensor satisfies $\underset{s \to k}{Lim} \inf T_{\mu\nu} K^{\mu} K^{\nu} > 0$ on all null geodesics in *M*.

The Raychaudhuri equations (25) for null geodesic become,

$$\frac{d\theta}{dt} = -R_{\mu\nu} K^{\mu} K^{\nu} - \frac{1}{2}\theta^2 - 2\sigma^2 \, (\omega^2 = 0), \quad (36)$$

where K^{μ} is the tangent to the geodesic. The conjugacy of *p* and *q* for a function *y* is defined by $\theta \equiv \frac{1}{y} . \frac{dy}{ds}$ and is given as follows:

A new function *z* is defined by, $z^2 = y$, then $\theta \equiv \frac{2}{z} . \frac{dz}{ds}$, so (36) becomes;

$$\frac{d^2 z}{ds^2} + G(s) z = 0 \qquad (37)$$

where $G(s) = \frac{1}{2}\left(R_{\mu\nu} K^{\mu} K^{\nu} + 2\sigma^2\right)$.

Thus *y* will be zero at *p* and *q* iff *z* is zero at *p* and *q*. Now let γ be past complete, then condition (1) above implies $R_{\mu\nu} K^{\mu} K^{\nu} \geq 0$ for all null vector K^{μ}. Since $\sigma^2 > 0$ then (37) gives;

$$\int_0^{\infty} \frac{1}{2}\left(R_{\mu\nu} K^{\mu} K^{\nu} + 2\sigma^2\right) ds = \int_0^{\infty} G(s)\, ds = \infty . \qquad (38)$$

In this case the null trajectory γ must contain infinitely many points [15] and any two of such points can be timelike related.

Let us define a length scale *y* associated with the volume *V*(*t*) defined by $y^3 = V$. Now taking second derivative of Raychaudhuri equations (25) we get;

$$\frac{d^2 y}{dt^2}+\frac{1}{3}\left(R_{\mu\nu}V^{\mu}V^{\nu}+2\sigma^2\right)y=0. \qquad (39)$$

Let us choose $F(t)=\frac{1}{3}\left(R_{\mu\nu}V^{\mu}V^{\nu}+2\sigma^2\right)=\frac{B}{t^2}$ for $B>0$ then (39) becomes;

$$\frac{d^2 y}{dt^2}+\frac{B}{t^2}y=0. \qquad (40)$$

Let $y=t^{\beta}$ be a trial solution of (40), then we get;

$$B=\beta-\beta^2. \qquad (41)$$

Since $V(t)\to 0$ at $t = 0$ we must have $\beta>0$. Again since $B>0$ so that (41) gives;

$0<\beta<1$, and $B\le\frac{1}{4}$. Hence solution for y is given by;

$$y=t^{\left[1\pm(1-4B)^{1/2}\right]/2}. \qquad (42)$$

The volume $V(t)\to 0$ near the singularity at least as fast as $y\to t^{3/2}$. Hence at a strong curvature singularity, the gravitational tidal forces associated with this singularity are so strong that any object trying to cross it must be destroyed to zero size.

7. Concluding Remarks

In this article we present a brief review on the Raychaudhuri equations with the help of differential geometry, general relativity and topology. We first derived Raychaudhuri equations, then described gravitational focusing and finally described space-time singularities by these equations. Before explaining Raychaudhuri equations we briefly defined singularities and Einstein equation. We cannot do anything in general Relativity and Cosmology avoiding Einstein equation. Raychaudhuri predicts gravitational focusing and singularities about a decade ago of introducing space-time singularities by Stephen W. Hawking and Roger Penrose. Throughout the paper we have tried to avoid complicated mathematical calculations and difficult physical phenomenon.

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