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We consider a discrete – delay time, Kaldor non – linear business cycle model in income and capital. Given an investment function, resembling the one discussed by Rodano, we use the linear approximation analysis to state the local stability property and local bifurcations, in the parameter space. Finally, we will give some numerical examples to justify the theoretical results.

Key Words: business cycle, Neimark-Sacker bifurcation, discrete-delay time.

1. INTRODUCTION

Synchronization is a fundamental nonlinear phenomenon. The classical case of synchronization consists in an external periodic (usually harmonic)signal acting upon an auto-oscillating system with a stable cycle [3,6].

The first version of the analyzed model was proposed by Kaldor who discussed the model of business cycle in its most simple nonlinear form. However, we cannot take it into consideration when analyzing the case of actual dynamic economies.

The simplistic approach of Kaldor was furthermore developed by Chang and Smith who saw the model within the dynamic systems framework through introducing the notion of continuous time expressed by a two nonlinear differential equations system in income and capital. From this point on, the next step was to elaborate the case of discrete-time dynamic system expressed by a system of two nonlinear differential equations as showed by Dana and Malgrange in 1984, Hermann in 1985, Lorenz in 1992 and 1993.

The basic concepts of Kaldor model is that if the propensity to invest is greater than the propensity to save, then the system is unstable in a way which generates an onset of fluctuations due to the fact that if the system is far from the equilibrium point, the propensity to invest decreases until it becomes lower than propensity to save. Therefore, we have to take into consideration both speed of reaction to the excess demand and the propensity to save, knowing that the first parameter has a destabilizing effect and the second one has a stabilizing role.

Bischi [1] proposed the model to have a discrete dynamic system form, by assuming that the firms' investment decisions are based on a expected "normal" value of income, which is exogenously given. They analyze the joint dynamic effects of the two parameters above mentioned and show that "the exogenously given equilibrium is only stable for low values of the firms' speed of reaction and sufficiently high values of the propensity to save". Moreover, if the speed of adjustment is high enough, the dynamic scenario strongly depended on the values of the propensity to save; a low level of the propensity to save generates the situation of bi-stability.

The paper presents Kaldor model, which describes the income and capital on $n+1$ moment, taking into consideration income and capital in n, and also income on $n-m$, with $m \geq 0$. For $m=0$, the model represents the discrete-time Kaldor model, analyzed by Bischi [2]. In section 1 we will describe the discrete - delay Kaldor model, with respect to investment function, as presented by Rodano, and a saving function, as considered by Keynes. We establish Jacobian matrix in a fix point of the model, the characteristic equation and the eigenvectors $q \in \mathbb{R}^{m+2}$, $p \in \mathbb{R}^{m+2}$associated to the eigenvalues. In section 2 we analyze the roots of characteristic equation for $m=1$, function of the adjustment parameter. Using a variable transformation, we establish that there is a value which is Neimark – Sacker bifurcation. In section 3 we describe the normal form for $m=1$, as well as the orbits of state variables. Using the software Maple11, for p, q, r fixed, we verify the theoretical results. Finally, we present the conclusions concerning the obtained results' utility as well as the future possible analysis of this model.

2. THE DISCRETE-DELAY KALDOR MODEL

The discrete - delay Kaldor model describes the business cycle for the state variables characterized by income (national income) Y_n and capital stock K_n, where $n \in \mathbb{N}$. This model is represented by an equations system with discrete time and delay, given by:

$$\begin{aligned} Y_{n+1} &= Y_n + s\left[I(Y_n, K_n) - S(Y_n, K_n)\right] \\ K_{n+1} &= K_n + I(Y_{n-m}, K_n) - qK_n \end{aligned} \tag{1}$$

Where: Y_{n-m} represents income on moment $n-m$, with $m \geq 1$;

$I : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is the investment function;

$S : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is the savings function, both considered being differentiable functions.

The parameter s from the set of equations (1) is an adjustment parameter which measures the reaction of the system to the difference between investment and saving. We admit Keynes's hypothesis which states that the saving function is proportional with income, meaning that

$$S(Y, K) = pY$$

where $p \in (0,1)$ is the propensity to save, with respect to income.

The investment function I is defined by taking into consideration a certain normal level of income u, and a normal level of capital stock - $\frac{pu}{q}$, where $u > 0$, $q \in (0,1)$.

We admit Rodano's hypothesis and consider the form of the investment function as follows:

$$I(Y, K) = pu + r\left(\frac{pu}{q} - K\right) + f(Y - u) \tag{2}$$

where: $r > 0$, and

$f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f(0) = 0$, $f'(0) \neq 0$ and $f'''(0) \neq 0$.

The dynamic system (1), with above mentioned hypothesis, can be written as:

$$\begin{aligned} Y_{n+1} &= (1-sp)Y_n - rsK_n + sf(Y_n - u) + spu\left(1 + \frac{r}{q}\right) \\ K_{n+1} &= (1-s-q)K_n + f(Y_{n-m} - u) + pu\left(1 + \frac{r}{q}\right) \end{aligned} \tag{3}$$

For $m = 0$ and $f(x) = \arctan(x)$, the model was proposed and analyzed by Bischi.

Using a change of variable

$$x^1 = Y_{n-m} ... x^m = Y_{n-1}, x^{m+1} = Y_n, x^{m+2} = K_n$$

the application associated to system (3) is as follows:

$$\begin{pmatrix} x^1 \\ ... \\ x^m \\ x^{m+1} \\ x^{m+2} \end{pmatrix} \to \begin{pmatrix} x^2 \\ ... \\ x^{m+1} \\ (1-sp)x^{m+1} - rsx^{m+2} + sf(x^{m+1} - u) + spu\left(1 + \frac{r}{q}\right) \\ (1-r-q)x^{m+2} + f(x^1 - u) + pu\left(1 + \frac{r}{q}\right) \end{pmatrix} \tag{4}$$

The fixed points of application (4) are the points of coordinates $(y_0, ... y_0, k_0) \in \mathbb{R}^{m+2}$ where (y_0, k_0) is a solution of the equations system :

$$\begin{aligned} py + rk - f(y-u) - pu\left(1 + \frac{r}{q}\right) &= 0 \\ (r+q)k - f(y-u) - pu\left(1 + \frac{r}{q}\right) &= 0 \end{aligned} \tag{5}$$

Let us consider (y_0, k_0) being a solution of system (5) and we note:

$$\begin{aligned} \rho_1 &= f'(y_0 - u), \rho_2 = f''(y_0 - u), \rho_3 = f'''(y_0 - u) \\ a_{10} &= s(\rho_1 - p), a_{01} = -rs, b_{10} = \rho_1, b_{01} = -q - r \end{aligned} \tag{6}$$

The following statements take place:

Proposition 1. *(i). The Jacobian matrix of application (4) in the fix point $(y_0, ...y_0, k_0)^T$ is as follows:*

$$A = \begin{pmatrix} 0 & 1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & ... & 1 & 0 \\ 0 & ... & ... & 1 + a_{10} & a_{01} \\ b_{10} & ... & ... & 0 & 1 + b_{01} \end{pmatrix} \tag{7}$$

(ii). The characteristic equation of matrix A, given by (7), is:

$$\lambda^{m+2} - a(s)\lambda^{m+1} + b(s)\lambda^m - c(s) = 0 \tag{8}$$

where $a(s) = 2 + a_{10} + b_{01}$, $b(s) = (1 + a_{10})(1 + b_{01})$, $c(s) = a_{01}b_{10}$

(iii). The eigenvector $q \in \mathbb{R}^{m+2}$*, that corresponds to the eigenvalue* μ *of matrix A has the components:*

$$q_i = \mu^{i-1}, i = 2, .., m+1, q_{m+2} = \frac{b_{10}}{\mu - 1 - b_{01}} \tag{9}$$

The eigenvector $p \in \mathbb{R}^{m+2}$*, that corresponds to the eigenvalue* $\overline{\mu}$ *of matrix* A^T *has the components:*

$$p_1 = \frac{(\overline{\mu} - 1 - a_{10})(\overline{\mu} - 1 - b_{10})}{m(\overline{\mu} - 1 - a_{10})(\overline{\mu} - 1 - b_{01}) + \overline{\mu}(2\overline{\mu} - 2 - a_{10} - b_{01})} \tag{10}$$

$p_i = \frac{1}{\overline{\mu}^{i-1}}p_1, \quad i = 2, .., m-2, \quad p_{m+1} = \frac{1}{\overline{\mu}^{m-1}(\overline{\mu} - 1 - a_{10})}p_1, \quad p_{m+2} = \frac{\overline{\mu}}{b_{10}}p_1$

The vectors q, p *given by (8) and (9) satisfy the relationship*

$$\sum_{i=1}^{m+2} q_i \overline{p_i} = 1$$

3. THE ANALYSIS OF THE CHARACTERISTIC EQUATION IN THE FIX POINT

We will analyze the roots of the characteristic equation (8), function of the adjustment parameter s, for $m = 1$. For the $m \geq 2$ analysis is quite difficult to perform.

PROPOSITION 2. *If* $m = 1$*, the following affirmations are true:*

(i). The equation (8) becomes :

$$\lambda^3 - a(s)\lambda^2 + b(s)\lambda - c(s) = 0 \tag{11}$$

(ii). The necessary and sufficient condition for equation (18) to admit two complex roots with their absolute value equal to 1 and one root with absolute value less than 1, is to exist $s_0 \in \mathbb{R}$ *so that:*

$$|c_0| < 1, |a_0 - c_0| < 2, b_0 = 1 + a_0c_0 - c_0^2 \tag{12}$$

where:

$$a_0 = a(s_0), b_0 = b(s_0), c_0 = c(s_0) \tag{13}$$

(iii). Let us consider:

$$\alpha_1 = \rho_1 - p, \beta_1 = 2 - q - r, \alpha_2 = (\rho_1 - p)(1 - q - r), \beta_2 = 1 - q - r, \alpha_3 = -r\rho_1$$

If, for $|\beta|$ *small enough, it is satisfied the expression:*

$$\left((a_0\alpha_3 + c_0\alpha_1)(1+\beta)^2 - \alpha_2(1+\beta)^4 - 2c_0\alpha_3\right)^2 - 4\left(\alpha_3\alpha_1(1+\beta)^2 - \alpha_3^2\right) *$$

$$* \quad \left(a_0c_0(1+\beta)^2 - c_0^2 - b_0(1+\beta)^4 + (1+\beta)^6\right) \geq 0 \tag{14}$$

then there is a function $g:\mathbb{R}\to\mathbb{R}$ with $g(0)=0$, $g'(0)=0$ so that the variable change:

$$s = s_0 + g(\beta) \tag{15}$$

transforms equation (11) in equation:

$$\lambda^3 - a(\beta)\lambda^2 + b(\beta)\lambda - c(\beta) = 0 \tag{16}$$

where

$$a(\beta) = a_0 + \alpha_1 g(\beta), b(\beta) = b_0 + \alpha_2 g(\beta), c(\beta) = c_0 + \alpha_3 g(\beta) \tag{17}$$

Function $g(\beta)$ is given by equation:

$$\begin{aligned} & \left(\alpha_3\alpha_1 (1+\beta)^2 - \alpha_3^2\right) g^2 + \left(a_0\alpha_3 (1+\beta)^2 + c_0\alpha_1 (1+\beta)^2 - 2c_0\alpha_3 - \alpha_2 (1+\beta)^4\right) g + \\ + \quad & a_0c_0 (1+\beta)^2 - c_0^2 - b_0 (1+\beta)^4 + (1+\beta)^6 = 0 \end{aligned} \tag{18}$$

Equation (16), has the roots :

$$\mu_{12}(\beta) = (1+\beta)\, e^{\pm i\theta(\beta)}, \lambda_1(\beta) = \frac{c(\beta)}{(1+\beta)^2} \tag{19}$$

with:

$$\theta(\beta) = \arccos \frac{a(\beta)(1+\beta)^2 - c(\beta)}{2(1+\beta)^3} \tag{20}$$

Following from Proposition 2, we can conclude that the analysis of the characteristic equation' roots it is made through the analysis of the equation transformed in function of β. From (19) results that $\beta = 0$ is the point of Neimark – Sacker bifurcation, and with (15), results that $s = s_0$ is point of Neimark – Sacker bifurcation for the given system.

4. THE NORMAL FORM OF SYSTEM (3) IF $M = 1$

If $m = 1$, Kaldor model for which it was made the translation $Y \to y + y_0$ and $K \to k + k_0$ is:

$$\begin{aligned} Y_{n+1} &= -spy_n - rsk_n + sf(y_n + y_0 - u) - f(y_0 - u) \\ K_{n+1} &= -(r+q)k_n + f(y_{n-1} + y_0 - u) - f(y_0 - u) \end{aligned} \tag{21}$$

Proposition 3. *The following affirmations are true:*
(i). The normal form associated to system (21) is as follows:

$$z_{n+1} = \mu(\beta)z_n + \frac{1}{2}\left(s(\beta)p_2 + p_3\right)\rho_2\left(z_n^2 + 2z_n\overline{z_n} + \overline{z_n}^2\right) + \frac{1}{2}g_{21}(\beta)z_n^2\overline{z_n} \tag{22}$$

where:

$$\begin{array}{rcl}
g_{21}(\beta) & = & \rho_2\left(s(\beta)p_2+p_3\right)\left(w_{20}\overline{q_2}+2q_2w_{11}\right)+\rho_3\left(s(\beta)p_2+p_3\right) \\
w_{11}^1 & = & v_{11}^1-\dfrac{1}{1-\mu(\beta)}p_2\rho_2-\dfrac{1}{1-\overline{\mu}(\beta)}\overline{p_2}\rho_2 \\
w_{11}^2 & = & v_{11}^1-\dfrac{1}{1-\mu(\beta)}p_2q_2\rho_2-\dfrac{1}{1-\overline{\mu}(\beta)}\overline{p_2q_2}\rho_2 \\
w_{11}^3 & = & v_{11}^1-\dfrac{1}{1-\mu(\beta)}p_2q_3\rho_2-\dfrac{1}{1-\overline{\mu}(\beta)}\overline{p_2q_3}\rho_2 \\
w_{02}^1 & = & \overline{w}_{20}^1, w_{02}^2=\overline{w}_{20}^2, w_{02}^3=\overline{w}_{20}^3 \\
v_{20}^1 & = & \dfrac{\rho_2}{\mu(\beta)^4-(1+\rho_1)\mu(\beta)^2-\rho_1}, v_{11}^1=-\dfrac{\rho_2}{2\rho_1} \\
w_{20}^1 & = & v_{20}^1-\dfrac{1}{\mu(\beta)^2-\mu(\beta)}p_2\rho_2-\dfrac{1}{\mu(\beta)^2-\overline{\mu}(\beta)}\overline{p_2}\rho_2 \\
w_{20}^2 & = & \mu(\beta)^2v_{20}^1-\dfrac{1}{\mu(\beta)^2-\mu(\beta)}p_2q_2\rho_2-\dfrac{1}{\mu(\beta)^2-\overline{\mu}(\beta)}\overline{p_2q_2}\rho_2 \\
w_{20}^3 & = & \mu(\beta)^4v_{20}^1-\dfrac{1}{\mu(\beta)^2-\mu(\beta)}p_2q_3\rho_2-\dfrac{1}{\mu(\beta)^2-\overline{\mu}(\beta)}\overline{p_2q_3}\rho_2 \\
q_2 & = & \mu(\beta), q_3=\dfrac{\rho_1}{\mu(\beta)-1-\rho_1} \\
p_1 & = & \dfrac{\left(\overline{\mu}(\beta)-1-s(\beta)\left(\rho_1-p\right)\right)\left(\overline{\mu}(\beta)-1-\rho_1\right)}{\begin{array}{c}[\left(\overline{\mu}(\beta)-1-s(\beta)\left(\rho_1-p\right)\right)\left(\overline{\mu}(\beta)-1+q+r\right)+ \\ +\overline{\mu}(\beta)\left(2\overline{\mu}(\beta)-2-s(\beta)\left(\rho_1-p\right)+q+r\right)]\end{array}} \\
p_2 & = & \dfrac{1}{\overline{\mu}(\beta)-1-s(\beta)\left(\rho_1-p\right)}p_1 \\
p_3 & = & \dfrac{\overline{\mu}(\beta)}{\rho_1}p_1
\end{array} \tag{23}$$

$s(\beta)$ is given by (15) and $\mu(\beta)$ is given by (19).

(ii). The solution of the system (21) in the neighborhood of the fixed point $(y_0, y_0, k_0) \in \mathbb{R}^3$ is :

$$\begin{array}{rcl}
Y_n & = & y_0+q_2z_n+\overline{q_2z_n}+\dfrac{1}{2}w_{20}^2z_n^2+w_{11}^2z_n\overline{z_n}+\dfrac{1}{2}w_{02}^2\overline{z_n}^2 \\
K_n & = & k_0+q_3z_n+\overline{q_3z_n}+\dfrac{1}{2}w_{20}^3z_n^2+w_{11}^3z_n\overline{z_n}+\dfrac{1}{2}w_{02}^3\overline{z_n}^2 \\
U_n & = & y_0+q_1z_n+\overline{q_1z_n}+\dfrac{1}{2}w_{20}^1z_n^2+w_{11}^1z_n\overline{z_n}+\dfrac{1}{2}w_{02}^1\overline{z_n}^2
\end{array} \tag{24}$$

where $z_n \in \mathbb{C}^2$ is a solution of equation (22).

(iii). The coefficient $c_1(\beta)$ associated to the equation (22) is:

$$c_1(\beta)=\left(\frac{\left(s(\beta)p_2+p_3\right)^2\left(\overline{\mu}(\beta)-3+2\mu(\beta)\right)}{2\left(\mu(\beta)^2-\mu(\beta)\right)\left(\overline{\mu}(\beta)-1\right)}+\frac{\left|s(\beta)p_2+p_3\right|^2}{1-\overline{\mu}(\beta)}+\frac{\left|s(\beta)p_2+p_3\right|}{2\left(\mu(\beta)^2-\overline{\mu}(\beta)\right)}\right)\rho_2^2+\frac{g_{21}(\beta)}{2} \tag{25}$$

Let us consider

$$I_0=\mathrm{Re}(c_1(0)e^{-i\theta(0)})$$

where $\theta(0)$ is given by (20). If $I_0 < 0$, in the neighborhood of the fixed point (y_0, k_0) there is a a stable limit cycle (in a stable invariant closed curve).

Using the formulas from Proposition 3, using the Maple11, for fixed values of p, q, r we obtain the following maps:

Fig. 1.(n,Yn)

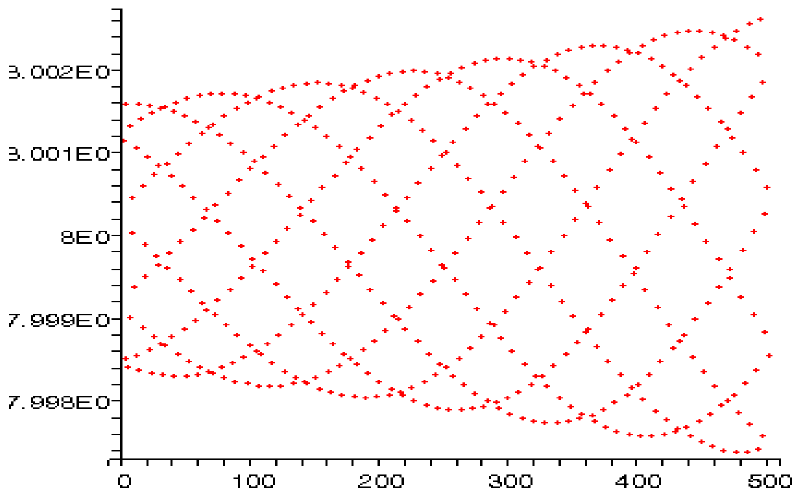


Fig.2.(n,Kn)

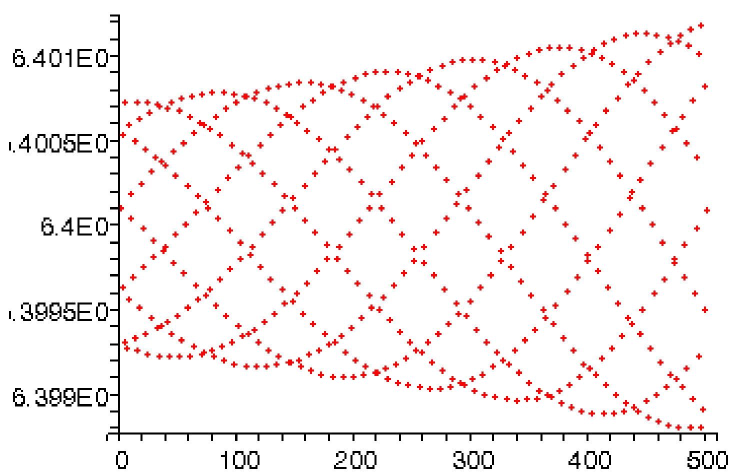


Fig.3.(Yn,Kn)

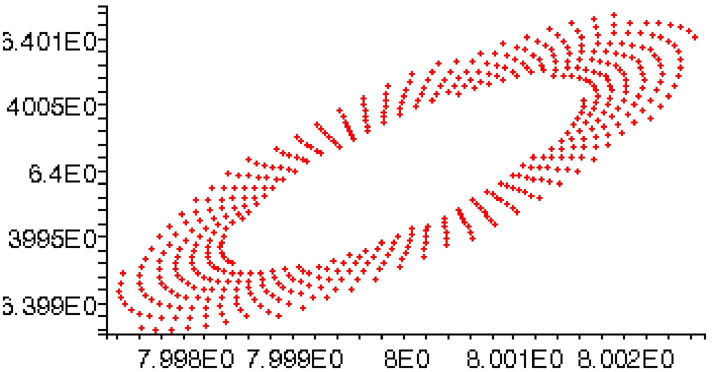


Fig.4.(Un,Kn)

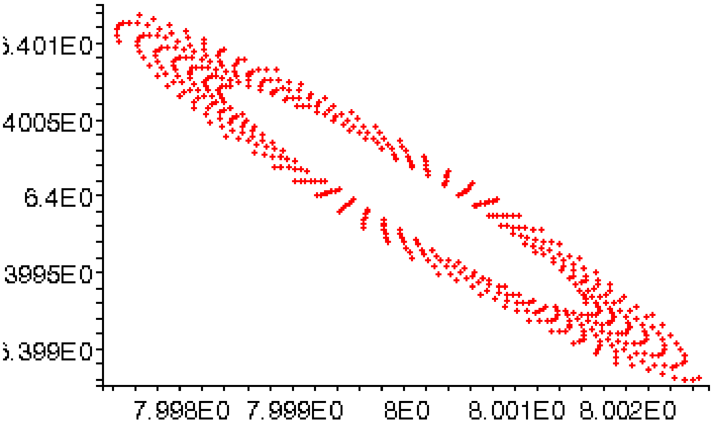
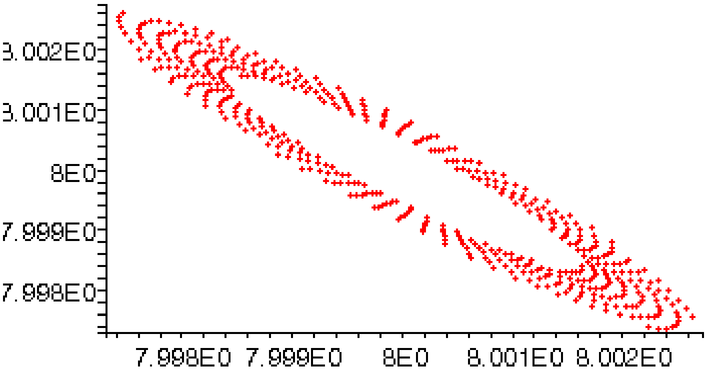


Fig 5. (Un,Yn)



5. CONCLUSIONS

In this paper it is described the discrete – delay Kaldor model, taking into consideration the fact that the variation of capital depends on the value of income on $n - m$ moment, where $n, m \in \mathbb{N}$ with $m \geq 0$. For $m = 1$, the model obtained is a dynamic system with discrete – time and delayed – argument. By taking as parameter s, the adjustment parameter, we determined the value s_0 for which the characteristic equation associated to the model in the equilibrium point has complex roots with absolute value equal to 1 and a root with absolute value less than 1, if $m = 1$. Using the method of normal forms, as showed by Kuznetsov [5], we obtained the equation which defines the stable limit cycle associated to the model. Through Maple11 we can visualize the orbits of the model's variables. Therefore, in this work, we establish for certain values of the parameters, the existence of business cycle. For $m \geq 2$, the analysis is more laborious and it will be considered in future works. The present analysis permits us to establish the behavior of state variables on different moments.

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